q-Index on braided non-commutative spheres

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Abstract

To some Hecke symmetries (i.e. Yang-Baxter braidings of Hecke type) we assign algebras called braided non-commutative spheres. For any such algebra, we introduce and compute a q-analog of the Chern-Connes index. Unlike the standard Chern-Connes index, ours is based on the so-called categorical trace specific for a braided category in which the algebra in question is represented.

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1 Introduction

The Chern-Connes index (pairing) was introduced and intensively used for needs of non-commutative geometry by A. Connes. Given an associative algebra A, the Chern-Connes index (in its simplest form, cf. [L]) can be defined via a pairing

$$\mathrm{Ind}\, : K_0(A) \times K^0(A) \to \mathbb{Z}, \tag{1.1}$$

where $K^0(A)$ is the Grothendieck group of the monoid of its finite dimensional representations and $K_0(A)$ is the Grothendieck group of classes of one-sided projective A-modules[1]. (Observe that according to the Serre-Swan approach such modules are considered as appropriate analogs of vector bundles on a variety.) We will only deal with the finite dimensional representations of algebras in question taking the category of finite dimensional $U(sl(n))$-modules as a pattern. In this sense our setting is purely algebraic. This is the main difference between our approach and that based on the Connes spectral triples in which a considerable amount of functional analysis is involved (cf. [C], where the function algebra on the quantum group $U_q(sl(2))$ is studied from this viewpoint).

Any projective module can be identified with an idempotent $e \in \mathrm{Mat}(A)$ where as usual $\mathrm{Mat}(A) = \oplus_n \mathrm{Mat}_n(A)$ and $\mathrm{Mat}_n(A)$ stands for the algebra of $n \times n$ matrices with entries from A.

Let us fix a representation $\pi_U : A \to \mathrm{End}\,(U)$ and an idempotent $e \in \mathrm{Mat}(A)$. Then the pairing (1.1) is defined by

$$\mathrm{Ind}\,(e, \pi_U) = \mathrm{tr}(\pi_U(\mathrm{tr}\, e)) = \mathrm{tr}(\pi_U(e)), \tag{1.2}$$

where π_U is naturally extended to $\mathrm{Mat}(A)$.

On defining $K_0(A)$ in the standard way (cf. [Ro]) it is not difficult to show that $\mathrm{Ind}\,(e, \pi_U)$ does not depend on a representative of a class from $K_0(A)$ and thus pairing (1.2) reduces to (1.1).

[1]Throughout the paper all projective modules are supposed to be finitely generated.

In this paper we introduce a "braided version" of the Chern-Connes index. This version is based on the so-called "categorical trace" (see section 2) and motivated by the "braided nature" of the algebras considered. These algebras are quotients of some braided analogs of enveloping algebras $U(gl(n))$ and $U(sl(n))$ and are thought of as braided non-commutative counterparts of orbits in $sl(n)^*$. As a result we compute our index on a class of orbits of this type called "braided spheres", in particular, for the "quantum sphere" related to the quantum group $U_q(sl(2))$.

First of all, let us briefly describe the braided categories we are working with. The detailed consideration is presented in section 2. Any such category is generated by a finite dimensional vector space V equipped with a map to be called *a braiding (morphism)*

$$R : V^{\otimes 2} \to V^{\otimes 2} \tag{1.3}$$

which satisfies the quantum Yang-Baxter equation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}, \quad R_{12} = R \otimes \mathrm{id}\,, \quad R_{23} = \mathrm{id} \otimes R. \tag{1.4}$$

Besides, we will suppose R to be of the Hecke type. This means that the braiding R satisfies the following *Hecke condition*

$$(q\,\mathrm{id} - R)(q^{-1}\mathrm{id} + R) = 0, \quad q \in \mathbb{K}. \tag{1.5}$$

Hereafter $\mathbb{K}$ stands for the basic field (usually $\mathbb{C}$ but sometimes $\mathbb{R}$ is allowed) and the parameter $q \in \mathbb{K}$ is assumed to be generic (but $q = 1$ is permitted). The braidings of the Hecke type will be also called *Hecke symmetries.*

Let $\mathcal{C} = \mathcal{C}(V)$ be the category generated by the space V. The sets of its objects and categorical morphisms will be denoted respectively by $\mathrm{Ob}\,(\mathcal{C})$ and $\mathrm{Mor}\,(\mathcal{C})$. The category $U_q(sl(n)) - \mathrm{Mod}$ of all finite dimensional modules over the quantum group $U_q(sl(n))$ serves as an example of $\mathcal{C}(V)$. In this case the space V is the fundamental (vector) module, the braiding R is the Drinfeld-Jimbo R-matrix and the categorical morphisms are linear maps commuting with the action of the quantum group $U_q(sl(n))$.

Under some additional conditions on R (see section 2) the braided categories in question are *rigid* (for the terminology the reader is referred to [CP]). This means that for any $U \in \mathrm{Ob}\,(\mathcal{C})$ there exists $U_r^* \in \mathrm{Ob}\,(\mathcal{C})$ (resp. $U_l^* \in \mathrm{Ob}\,(\mathcal{C})$) for which one can define a non-degenerate pairing

$$U \otimes U_r^* \to \mathbb{K} \quad (\text{resp. } U_l^* \otimes U \to \mathbb{K})$$

and this map is a categorical morphism. The space U_r^* (resp. U_l^*) is called the *right (left) dual* space to U. Therefore, for any $U \in \mathrm{Ob}\,(\mathcal{C})$ the space of its right (resp. left) "internal endomorphisms"

$$\mathrm{End}_r(U) \stackrel{\mathrm{def}}{=} U_r^* \otimes U, \quad (\text{resp.} \quad \mathrm{End}_l(U) \stackrel{\mathrm{def}}{=} U \otimes U_l^*) \tag{1.6}$$

is also contained in $\mathrm{Ob}\,(\mathcal{C})$.

Then, in $\mathcal{C}$ we define an important categorical morphism

$$\mathrm{tr}_R : \quad \mathrm{End}_\varepsilon(U) \to \mathbb{K}, \quad \varepsilon = l, r$$

called *the categorical trace.* The super-trace is an example of such a categorical trace. Namely, in super-algebra and super-geometry this trace replaces the classical one. For a similar reason, dealing with a braided category, we make use of the categorical trace specific for this category.

Now, let us pass to algebras in question. Assume for a moment that $q = 1$. This means that our braiding R becomes involutive: $R^2 = \mathrm{id}$. For such a braiding there exists a natural way to define *a generalized Lie bracket*:

$$[\ ,\ ] : \quad \mathrm{End}_\varepsilon(V)^{\otimes 2} \to \mathrm{End}_\varepsilon(V), \quad \varepsilon = l, r \tag{1.7}$$

(cf. [G] for detail). Being equipped with such a bracket, the space $\mathrm{End}_l(V)$ (for definiteness we set $\varepsilon = l$) becomes *a generalized Lie algebra*. It will be denoted $gl_R(V)$. For instance, a super-Lie algebra is a particular case of generalized one.

Moreover, for the aforementioned categorical trace the subspace $sl_R(V)$ of all traceless elements is closed with respect to this bracket. Thus, the space $sl_R(V)$ is also *a generalized Lie algebra*. Then their enveloping algebras $U(gl_R(V))$ and $U(sl_R(V))$ can be defined by systems of quadratic-linear equations. Furthermore, they become *braided Hopf algebras*, being equipped with an appropriate coproduct, antipode and counit. On generators $X \in gl_R(V)$ (or $X \in sl_R(V)$) this coproduct has the classical form:

$$\Delta(X) = X \otimes 1 + 1 \otimes X. \tag{1.8}$$

By means of the coproduct (which gives rise to a braided version of the Leibniz rule) we can construct an embedding

$$\mathrm{End}_l(V) \to \mathrm{End}_l(V^{\otimes m}),\ \forall m.$$

Restricting these maps to the subspaces associated with the Young diagrams (the corresponding Young projectors can be constructed for any involutive braiding) we get a family of irreducible representations of the generalized Lie algebra $sl_R(V)$. Then considering all their direct sums we get a category of finite dimensional representations of the algebra in question similar to that of $sl(n)$, $n = \mathrm{dim}V$. The main difference between the later category and a braided one consists in traces. When considering the algebra $U(gl_R(V))$ (or $U(sl_R(V))$) with the aforementioned category of finite dimensional representations it is natural to use the corresponding categorical trace in order to define all numerical characteristics (dimensions, indexes, etc).

It is this scheme that is realized in the present paper. However, here we deal with a more interesting (and more difficult) case of algebras and categories associated to some non-involutive Hecke symmetries. In this case it is not evident which algebras should be taken as $U(gl_R(V))$ and $U(sl_R(V))$. The matter is that a direct generalization of bracket (1.7) to a non-involutive R leads to "enveloping algebras" which are not flat deformations of the classical ones even if R is a deformation of the usual flip. Otherwise stated, the dimensions of the homogeneous components of the corresponding graded algebra differ from their classical analogs.

Nevertheless, there exist algebras (denoted below as $\mathcal{L}_{\hbar,q}$ and $\mathcal{SL}_{\hbar,q}$) possessing good deformational properties and playing the role of braided analogs of enveloping algebras $U(gl(n))$ and $U(sl(n))$ respectively (though apparently they do not look like the usual enveloping algebras). They can be described in terms of the *modified reflection equation* (mRE). This equation can be defined for any braiding; for involutive R it leads to the aforementioned enveloping algebras $U(gl_R(V))$ and $U(sl_R(V))$. The algebras $\mathcal{L}_{\hbar,q}$ and $\mathcal{SL}_{\hbar,q}$ are described in the next section.

In the case related to the quantum group $U_q(sl(n))$ (in the $U_q(sl(n))$ case for short) these algebras are one-sided $U_q(sl(n))$–modules unlike the $U_q(sl(n))$ themself which is a two-sided $U_q(sl(n))$–module. Using the results of [LS] it is possible to show that any $U_q(sl(n))$–module becomes an $\mathcal{SL}_{\hbar,q}$–one. For the Hecke symmetry coming from $U_q(sl(n))$ the corresponding category $\mathcal{C}(V)$ is the representation category of this quantum group and all its objects can be equipped with an action of the algebra $\mathcal{SL}_{\hbar,q}$. The family of all representations of the algebra $\mathcal{SL}_{\hbar,q}$ is, however, larger than that of $U_q(sl(n))$. Nevertheless, we only need finite dimensional $\mathcal{SL}_{\hbar,q}$–representations which are $U_q(sl(n))$–modules and are equivariant (covariant) with respect to the action of the quantum group. (The problem of describing other representations of the algebras in question will be considered elsewhere. Here we only consider the representation theory in the category $\mathcal{C}(V)$.)

But in general case we do not have such a useful tool as the quantum group. So, we modify the notion of equivariant representation in order to adapt it to a new setting.

Futhermore, restricting ourselves to the case $\mathrm{rk}\,(R) = 2$ (see section 2) we equip the category $\mathcal{C}(V)$ with an equivariant action of the algebra $\mathcal{SL}_{\hbar,q}$. Then we introduce "a braided non-commutative sphere" which is a quotient of this algebra (we get it by fixing a value of the quadratic braided Casimir element). In the $U_q(sl(2))$ case the braided sphere is also called "the quantum non-commutative sphere".

Our quantum sphere is close to the known Podles sphere. However, while the Podles sphere is $U_q(su(2))$–homogeneous space and is introduced via some reduction from its dual, our quantum sphere is defined via the mRE algebra as its approriate quotient. Consequently, the representation theory of the Podles quantum sphere constructed in [P] differs drastically from that of $\mathcal{SL}_{\hbar,q}$.

The Chern-Connes index on the Podles sphere was computed in the work [H] with the use of the representation theory from [P], the trace defined in [MNW] and idempotents introduced in [HM]. These idempotenets are labeled by $n \in \mathbb{Z}$ and are presented via generators of the function algebra on the quantum group $U_q(su(2))$.

In contrast, our method of constructing projective modules over the braided (in particular, quantum) non-commutative spheres makes use of a braided version of the Cayley-Hamilton identity for some matrices from $\mathrm{Mat}(\mathcal{L}_{\hbar,q})$. This allows us to construct a larger set of projective modules. However, this set of modules turns out to be "too large" and this leads us to the problem of defining a reasonable equivalence between modules in order to get the group K_0 of the classical size. This problem is discussed in the last section.

In the $U_q(sl(2))$ case all ingredients of our construction (representation theory, projective modules, traces, indices) have the classical limits at $q \to 1$. The corresponding algebra was earlier considered in [GS2] under the name of non-commutative sphere.

The paper is organized as follows. In the next section we define the categories and algebras we are dealing with. In section 3 we introduce and compute the braided version of the Chern-Connes index. In section 4 we consider the quantum sphere as an example of our general construction. In section 5 some problems are discussed which arise in connection with our approach.

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2 Categories $\mathcal{C}(V)$ and related algebras

We begin this section with a short description of the category $\mathcal{C} = \mathcal{C}(V)$ generated by a finite dimensional vector space V equipped with a Hecke symmetry R. This category forms a base of all our considerations, for its detailed description see [GLS].

Given a Hecke symmetry R, one can connect with it a "symmetric" (resp. "skew-symmetric") algebra $\Lambda_+(V)$ (resp. $\Lambda_-(V)$) of the space V defined as the quotient

$$\Lambda_+(V) = T(V)/\{\mathrm{Im}\,(q\,\mathrm{id}\, - R)\} \quad (\text{resp. } \Lambda_-(V) = T(V)/\{\mathrm{Im}\,(q^{-1}\,\mathrm{id}\, + R)\}).$$

Here $T(V)$ stands for the free tensor algebra. Let $\Lambda^k_\pm(V)$ be the homogeneous component of $\Lambda_\pm(V)$ of degree k. If there exists an integer p such that $\Lambda^k_-(V)$ is trivial for $k > p$ and $\dim(\Lambda^p_-(V)) = 1$, then R is called *an even symmetry* and p is called *the rank* of R: $p = \mathrm{rk}\,(R)$. Hereafter the symbol "dim" stands for the classical dimensions. In what follows all Hecke symmetries are assumed to be even.

Using the Yang-Baxter equation (1.4) we can extend braiding (1.3) onto any tensor powers of V

$$R: \quad V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes n} \otimes V^{\otimes m} \tag{2.1}$$

(as usual, we put $V^{\otimes 0} = \mathbb{K}$ and $(1 \otimes x) \triangleleft R = x \otimes 1,\ (x \otimes 1) \triangleleft R = 1 \otimes x,\ \forall x \in V^{\otimes m}$).

For an arbitrary fixed integer $m \geq 2$ we consider partitions $\lambda \vdash m$

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_k), \quad \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k > 0, \quad \lambda_1 + ... + \lambda_k = m$$

(k is called *the height* of λ). There exists a natural way to assign a space V_λ (equipped with a set of embeddings $V_\lambda \hookrightarrow V^{\otimes m}$) to any partition λ (cf. [GLS]). By definition the spaces V_λ are simple objects of the category $\mathcal{C}$ (we will motivate this definition below). All other objects are the direct sums of the simple ones.

Let us describe *the categorical morphisms* in $\mathcal{C}$. The term *categorical* emphasizes the difference among morphisms from $\mathrm{Mor}(\mathcal{C})$ and "internal endomorphisms" which are elements of $\mathrm{End}_{\,\varepsilon}(U) \in \mathrm{Ob}(\mathbb{C})$, $\epsilon = r, l$.

We distinguish categorical morphisms of two kinds. Categorical morphisms of *the first kind* are the linear maps $V^{\otimes m} \to V^{\otimes m}$ $m \geq 0$ coming from the Hecke algebra, as well as their restrictions to any object embedded into $V^{\otimes m}$. Recall, that the Hecke algebra H_m can be represented in $V^{\otimes m}$ by means of the Hecke symmetry R.

The categorical morphisms *of the second kind* arise from a procedure of cancelling columns of height p in the Young diagram corresponding to a given partition λ. The procedure is as follows. Denote by v a generator of the one dimensional space $\Lambda^p_-(V)$ (we call v *the associated determinant*). If one fixes a basis $\{x_i\}$ in the space V, then v can be represented in the form

$$v = v^{i_1\dots i_p} x_{i_1} \otimes \dots \otimes x_{i_p}.$$

Hereafter the summation over repeated indices is always understood. The tensor $v^{i_1\dots i_p}$ is one of the two structure tensors that define the matrix of the highest order antisymmetrizer (projector) $A^{(p)} : V^{\otimes p} \to \Lambda^p_-(V)$ in the basis $x_{i_1} \otimes \dots \otimes x_{i_p}$ of $V^{\otimes p}$

$$A^{(p)}(R)_{i_1\dots i_p}^{\;j_1\dots j_p} = u_{i_1\dots i_p} v^{j_1\dots j_p}. \tag{2.2}$$

As one can show, the associated determinant v possesses the property

$$(x \otimes v) \triangleleft R = (v \otimes x') \quad \text{and} \quad (v \otimes x) \triangleleft R = (x'' \otimes v) \quad \forall x \in V$$

where the correspondences $x \to x'$ and $x \to x''$ are some linear maps $V \to V$. Demand them be scalar (multiple of the identity map) and equal to each other. In other words, let there exists a nonzero $a \in \mathbb{K}$ such that

$$(x \otimes v) \triangleleft R = a(v \otimes x) \quad \text{and} \quad (v \otimes x) \triangleleft R = a(x \otimes v).$$

The Hecke symmetry R satisfying such a requirement will be called *admissible*. In this case, by setting $\bar{R} = a^{-1}R$ we have

$$(x \otimes v) \triangleleft \bar{R} = (v \otimes x), \quad (v \otimes x) \triangleleft \bar{R} = (x \otimes v). \tag{2.3}$$

The *cancelling a column* is defined as a map ψ

$$\Lambda^p_-(V) \xrightarrow{\psi} \mathbb{K} : \quad \psi(v) = 1.$$

Due to (2.3) ψ obeys the following condition

$$\bar{R} \circ (\mathrm{id} \otimes \psi) = (\psi \otimes \mathrm{id}\,) \circ \bar{R}. \tag{2.4}$$

The above relation is valid also for the map ψ^{-1} (inverse to ψ). By definition ψ and ψ^{-1} are the morphisms of the second kind. Any product (composition) $f \cdot g$ of morphisms of the both kinds gives *a categorical morphism.* Also, any tensor product $f \otimes g$ of categorical morphisms will be a categorical morphism by definition.

Remark 1 Condition (2.4) (in a litlle bit more general form) is sometimes included in the system of axioms for braided categories. Thus, in [T] a braiding satisfying such a condition is called *natural.*

Let us emphasize that for an object V_λ with a fixed embedding $V_\lambda \hookrightarrow V^{\otimes m}$ a map $V_\lambda \to V_\lambda$ is a categorical morphism if and only if it is a scalar map. This is the reason to call these objects *simple*. Such an observation plays an important role in what follows. If a map $\xi : U \to W$ is proved to be a categorical morphism, then on each simple component of U it is a scalar map.

The category thus introduced is a monoidal and quasitensor one whose braidings are restrictions of maps (2.1) onto the simple spaces V_λ and their direct sums. We call such a category *braided.* Note that its Grothendieck (semi)ring is isomorphic to that of $sl(p)$-modules where $p = \mathrm{rk}\,(R)$.

Moreover, if a Hecke symmetry R is admissible then the corresponding category $\mathcal{C}$ is rigid that is $\forall U \in \mathrm{Ob}(\mathcal{C})$ the dual spaces $U^*_\epsilon, \epsilon = r, l$ (right and left) are also contained in $\mathrm{Ob}(\mathcal{C})$. In particular, one can show that $\Lambda^{p-1}_-(V)$ is the dual (right and left) of V. This means that there exist non-degenerate pairings

$$\Lambda^{p-1}_-(V) \otimes V \to \mathbb{K} \quad \text{and} \quad V \otimes \Lambda^{p-1}_-(V) \to \mathbb{K} \tag{2.5}$$

which are categorical morphisms.

Let $\{x^i_r\}$ (resp. $\{x^i_l\}$) be the dual basis in the right (resp. left) dual space to V

$$< x_i, x^j_r >= \delta^j_i, \quad < x^j_l, x_i >= \delta^j_i. \tag{2.6}$$

The dual bases $\{x^i_r\}$ and $\{x^i_l\}$ can be expressed in the form

$$x^i_r = v^{a_1 \dots a_{p-1} i} x_{a_1} \otimes \dots \otimes x_{a_{p-1}}, \quad x^i_l = v^{i a_1 \dots a_{p-1}} x_{a_1} \otimes \dots \otimes x_{a_{p-1}}, \tag{2.7}$$

hence $\{x^i_\epsilon\} \in \Lambda^{p-1}_-(V)$, $\epsilon = r, l$. Now, pairings (2.6) can be explicitly constructed by means of the categorical morphism ψ. As a consequence, pairings (2.6) become categorical morphisms justifying the identification of $\Lambda^{p-1}_-(V)$ with the dual space of V (see (2.5)).

Introduce now a *categorical trace* which will play the central role in all our subsequent considerations. It is defined as a properly normalized categorical morphism $\mathrm{tr}_R : \mathrm{End}_\varepsilon(U) \to \mathbb{K}$. For details the reader is referred to [GLS] and we only briefly outline this construction.

For any admissible Hecke symmetry R there exists an operator Q that we call "inverse to R by column", i.e.,

$$R^{jb}_{ia} Q^{al}_{bk} = \delta^l_i \delta^j_k \quad \Leftrightarrow \quad Q^{jb}_{ia} R^{al}_{bk} = \delta^l_i \delta^j_k,$$

where R^{jb}_{ia} is the matrix of the Hecke symmetry in the basis $x_i \otimes x_j$:

$$(x_i \otimes x_j) \triangleleft R = R^{kl}_{ij}\, x_k \otimes x_l.$$

Consider the matrices

$$B_i^{\;j} = Q^{\;aj}_{ai}, \quad C_i^{\;j} = Q^{\;ja}_{ia}.$$

Evidently, they satisfy

$$B_b^{\;a} R^{bj}_{ai} = \delta^j_i, \quad R^{jb}_{ia} C_b^{\;a} = \delta^j_i. \tag{2.8}$$

Extending these matrices to any objects V_λ in a proper way we get the matrices B_λ and C_λ such that *the categorical trace* tr_R on the space $\mathrm{End}_r(V_\lambda)$ (resp. $\mathrm{End}_l(V_\lambda)$, see (1.6)) is defined as follows

$$\mathrm{tr}_R X = tr(B_\lambda \cdot \hat{X}),\ \forall X \in \mathrm{End}_r(V_\lambda) \quad (\text{resp. } \mathrm{tr}_R Y = tr(C_\lambda \cdot \hat{Y}),\ \forall Y \in \mathrm{End}_l(V_\lambda)). \tag{2.9}$$

Here tr is the usual matrix trace and $\hat{X}$ (resp. $\hat{Y}$) is the matrix of the linear operator $V_\lambda \to V_\lambda$ corresponding to an element $X \in \mathrm{End}_r(V_\lambda)$ (resp. $Y \in \mathrm{End}_l(V_\lambda)$).

The matrices B_λ and C_λ are constructed in such a way that the map $X \to \mathrm{tr}_R X$ is a categorical morphism and, besides, the *categorical dimension*

$$\dim_R(U) = \mathrm{tr}_R\, \mathrm{id}_U \quad \forall U \in \mathrm{Ob}\,(\mathcal{C})$$

is an additive-multiplicative functional on the Grothendieck (semi)ring. Then for any simple object V_λ one gets

$$\dim_R(V_\lambda) = s_\lambda(q^{p-1}, q^{p-3}, ..., q^{3-p}, q^{1-p}),$$

where s_λ is the Schur function in p variables. A proof of this fact is given in [H], [GLS] (in another setting an equivalent formula can be also found in [KW]).

A consequence of (2.7) is the fact that the maps

$$x_r^i \to x_l^a B_a^{\ i}, \quad x_l^i \to x_r^a C_a^{\ i} \tag{2.10}$$

belong to Mor$\,(\mathcal{C})$. Therefore, the same is true for the maps

$$1 \to x_l^a B_a^{\ i} \otimes x_i, \quad 1 \to x_i \otimes x_r^a C_a^{\ i}$$

since they are compositions of the map ψ^{-1}

$$1 \xrightarrow{\psi^{-1}} v = x_r^i \otimes x_i = x_i \otimes x_l^i.$$

and morphisms (2.10).

Our next aim is to introduce some associative algebras naturally connected to the categories involved. We consider these algebras as braided analogs of the enveloping algebras $U(gl(n))$ and $U(sl(n))$. Motivation will be given later.

As a starting point of our construction we introduce elements $l_i^{\,j} = x_i \otimes x_r^j$ and form the matrix

$$L = \|l_i^{\,j}\|, \quad 1 \le i,\, j \le n = \dim(V) \tag{2.11}$$

where the lower index enumerates rows and the upper one enumerates columns. Assume R to be an admissible Hecke symmetry and impose the following relations on the free algebra, generated by all the elements $l_i^{\,j}$:

$$R\,L_1\,R\,L_1 - L_1\,R\,L_1\,R - \hbar(R\,L_1 - L_1\,R) = 0, \quad \text{where} \quad L_1 = L \otimes \mathrm{id}\,, \ \hbar \in \mathbb{K} \tag{2.12}$$

or, explicitly,

$$R_{i_1 i_2}^{a_1 b_2}\, l_{a_1}^{\,b_1}\, R_{b_1 b_2}^{c_1 j_2}\, l_{c_1}^{\,j_1} - l_{i_1}^{\,a_1}\, R_{a_1 i_2}^{b_1 c_2}\, l_{b_1}^{\,c_1}\, R_{c_1 c_2}^{j_1 j_2} - \hbar(R_{i_1 i_2}^{j_1 a}\, l_a^{\,j_2} - l_{i_1}^{\,a}\, R_{a\, i_2}^{j_1 j_2}) = 0.$$

We call this relation the *modified reflection equation* (mRE) and the corresponding algebra, $\mathcal{L}_{\hbar,q}$, *the modified reflection equation algebra.* For any Hecke symmetry R with $q \neq 1$ this algebra can be obtained from the non-modified one (corresponding to $\hbar = 0$) by a shift of generators $l_i^{\,j} \to l_i^{\,j} - a\,\delta_i^{\,j}\,\mathrm{id}$ with $a = \hbar(q - q^{-1})^{-1}$.

For the matrix L we consider the maps

$$L \to \mathrm{id} \quad (l_i^{\,j} \to \delta_i^{\,j}) \quad \text{and} \quad L \to L^{\otimes k} \quad (l_i^{\,j} \to l_i^{\,a_1} \otimes l_{a_1}^{\,a_2} \otimes \ldots \otimes l_{a_{k-1}}^{\,j}) \tag{2.13}$$

which evidently belong to Mor$\,(\mathcal{C})$. In what follows the matrices $L^{\otimes k}$ whose entries are considered as elements of the space $\mathcal{L}_{\hbar,q}$ (or $\mathcal{SL}_{\hbar,q}$ defined below) will be denoted L^k.

Proposition 2 *Let us set $l_i^{\,j} \triangleright x_k = x_i B_k^{\ j}$. Then the image of the left hand side of (2.12) under this map is equal to 0 if we put $\hbar = 1$. Hence, we have a representation*

$$\pi_1 : \ \mathcal{L}_{\hbar,q} \to \mathrm{End}_{\,l}(V), \ \hbar = 1.$$

Proof Straightforward calculations. Suffice it to apply the left hand side of (2.12) to an arbitrary element of V and use property (2.8) of matrix B. ■

In $\mathrm{End}_l(V)$ we can choose the natural basis $\{h_i^{\,j}\} = x_i \otimes x_l^j$

$$h_i^{\,j} \triangleright x_k = \delta_k^j\, x_i$$

and identify the element $l_i^{\,j}$ with $h_i^{\,a} B_a^{\,j}$. Since matrix B is non-degenerate, we can consider $\{l_i^{\,j}\}$ as another basis in this space. In the basis $\{h_i^{\,j}\}$ the representation π_1 becomes tautological: $\pi_1(h_i^{\,j}) = h_i^{\,j}$.

It is worth emphasizing a difference between these bases. For the set $\{h_i^{\,j}\}$, the product

$$h_i^{\,j} \otimes h_k^{\,l} \to \delta_k^j\, h_i^{\,l}$$

is a categorical morphism while for $\{l_i^{\,j}\}$ the maps given in (2.13) are categorical morphisms.

Thus, we have identified the vector space $\mathrm{Span}(l_i^j)$ with $\mathrm{End}_l(V)$. We denote this space by $gl_R(V)$ too and describe its subspace of traceless elements as follows.

For any rank p, the space $gl_R(V)$ decomposes into the direct sum of two simple spaces, one of them is one-dimensional. This one-dimensional component is generated by the element $h_i^{\,i} = x_i \otimes x_l^i = v$ or, equivalently, by $\mathbf{1} = C_i^{\,j}\, l_j^{\,i}$. The elements of other simple component of $\mathrm{End}_l(V)$ are called *the traceless elements*. In the sequel such a space will be denoted $sl_R(V)$. Moreover, we define the algebra $\mathcal{SL}_{\hbar,q}$ as the quotient $\mathcal{SL}_{\hbar,q} = \mathcal{L}_{\hbar,q}/\{\mathbf{1}\}$.

Propositions 2 and 4, 6 below suggest a new way of constructing the representation theory of the algebras $\mathcal{L}_{\hbar,q}$ and $\mathcal{SL}_{\hbar,q}$. In contrast with the usual method valid in the case related to the quantum group when the triangle decomposition of L into the product of L^+ and L^- is used, our approach works in the general setting (for arbitrary admissible Hecke symmetry). In more detail this approach and the proof of propositions 4 and 6 will be presented elsewhere.

Observe that all representations in question are equivariant in the sense of the following definition.

Definition 3 Let A be either $\mathcal{L}_{\hbar,q}$ or $\mathcal{SL}_{\hbar,q}$ and $U \in \mathcal{C}(V)$ be an object with an associative product $U \otimes U \to U$ which is a categorical morphism (for example, $U = \mathrm{End}_\varepsilon(W)$, $\varepsilon = r, l$ or a direct sum of tensor products of similar spaces). We say that a map

$$\pi_U :\ A \to U$$

is *an equivariant representation* if it is a representation (i.e., an algebra morphism) and its restriction to $gl_R(V)$ (resp. $sl_R(V)$) is a categorical morphism.

Our next step is to define representations of $\mathcal{L}_{\hbar,q}$ in all simple spaces V_λ via a "truncated coproduct" defined below. We constrain ourselves to the simplest case $\mathrm{rk}(R) = 2$. In this situation the simple objects of $\mathcal{C}$ are labelled by partitions of height 1: $\lambda = (m)$. The corresponding Young diagram has only one row of length m. For brevity, we will write $V_{(m)}$ instead of $V_\lambda,\ \lambda = (m)$.

Describe now the truncated coproduct. As we said above, for an involutive R, coproduct (1.8) together with an antipode and a counit converts the enveloping algebra into a braided Hopf algebra. Unfortunately, this coproduct is not "Hopf compatible" with the algebraic structure of $\mathcal{L}_{\hbar,q}$ at $q \neq 1$. So, such a coproduct does not allow us to define maps $\mathrm{End}_l(V) \to \mathrm{End}_l(V^{\otimes m})$ which would give rise to the higher representations of the algebra $\mathcal{L}_{\hbar,q}$.

However, we define categorical morphisms which are q-analogs of restricted maps

$$\mathrm{End}_l(V) \to \mathrm{End}_l(V_{(m)}) \hookrightarrow \mathrm{End}_l(V^{\otimes m}).$$

We will refer to the family of these categorical morphisms as *the truncated coproduct.*

To construct them explicitly we take into account that $V_{(m)}$ is the image of the projection $V^{\otimes m}\to\Lambda_+^m(V)$ where the corresponding projector P_+^m is a polynomial in R. Then to an arbitrary element $X\in \mathrm{End}_l(V)$ we assign the element $X_{(m)}\in \mathrm{End}_l(V_{(m)})$ by the following rule

$$X_{(m)}\triangleright g=q^{1-m}\,[m]_q\,P_+^m(X_{(1)}\triangleright g),\quad \forall\, g\in\Lambda_+^m(V). \tag{2.14}$$

Here $X_{(1)}=X\otimes \mathrm{id}_{(m-1)}$ and $[m]_q=\frac{q^m-q^{-m}}{q-q^{-1}}$ is the q-analog of the integer m. Note that the map

$$\Delta_m:\ \mathrm{End}_l(V)\to \mathrm{End}_l(V_{(m)}),\quad X\to X_{(m)} \tag{2.15}$$

is a categorical morphism due to the structure of P_+^m. Composing π_1 with Δ_m we get the map

$$\pi_m:\ l_i^j\to \mathrm{End}_l(V_{(m)}).$$

It is worth emphasizing that unlike $\Delta_m(l_i^j)\in \mathrm{End}_l(V_{(m)})$ the elements $\pi_m(l_i^j)$ are considered to be operators (see section 3).

Proposition 4 *The image of the left hand side of (2.12) under the map π_m is equal to 0 at $\hbar=1$. So, we get a representation*

$$\pi_m:\ \mathcal{L}_{\hbar,q}\to \mathrm{End}_l(V_{(m)}),\quad \hbar=1.$$

Remark 5 If $\mathrm{rk}\,(R)=2$, it is easy to introduce a braided analog of the Lie bracket in the space $sl_R(V)$. Taking into account the decomposition

$$sl_R(V)^{\otimes 2}=V_{(4)}\oplus V_{(2)}\oplus V_{(0)}$$

we set $[\ ,\ ]:\ V_{(4)}\oplus V_{(0)}\to 0$ and require the map $[\ ,\ ]:\ V_{(2)}\to sl_R(V)$ to be a categorical morphism. This requirement defines the map $[\ ,\ ]$ uniquely, up to a factor. This bracket can be naturally extended to $gl_R(V)$ by the requirement $[\mathbf{1},\,x]=0$ for any $x\in gl_R(V)$. (A q-counterpart of the Lie algebra $sl(n)$ has been introduced in [LS].)

Having introduced such a bracket, we can treat the algebra $\mathcal{SL}_{\hbar,q}$ as the universal enveloping algebra of the corresponding "q-Lie algebra" in a standard manner (the parameter $\hbar$ depends on a normalization of the bracket). Similarly, the algebra $\mathcal{L}_{\hbar,q}$ can be treated as the enveloping algebra of $gl_R(V)$.

However, we prefer to do without any "q-Lie algebra" structure. Similarly to the usual enveloping algebra, our algebra $\mathcal{SL}_{\hbar,q}$ has the following properties. It is generated by the space $sl_R(V)$ (more precisely, it is the quotient of the algebra $T(sl_R(V))$ modulo an ideal generated by some quadratic-linear terms). Moreover, its representation theory resembles that of $U(sl(2))$ and, being constructed via the truncated coproduct, is equivariant. This is the reason for considering the algebra $\mathcal{SL}_{\hbar,q}$ as a proper "braided analog" of the enveloping algebra $U(sl(2))$ (and similarly, the algebra $\mathcal{L}_{\hbar,q}$ is treated as the enveloping algebra of $gl_R(V)$).

The representations of the algebra $\mathcal{SL}_{\hbar,q}$ can be easily deduced from those of $\mathcal{L}_{\hbar,q}$. To construct them, we set

$$\mathbf{1}\triangleright x=0\ x\in V_{(m)}$$

and preserve prolongation (2.14) for the elements of the traceless component $sl_R(V)$.

Proposition 6 *Thus defined maps are representations of the algebra $\mathcal{SL}_{\hbar,q}$ with some $\hbar\neq 0$.*

We will refer to these representations of the algebra $\mathcal{SL}_{\hbar,q}$ as *sl-representations* and and keep the same notation π_m for them: $\pi_m : \mathcal{SL}_{\hbar,q} \to \mathrm{End}\,(V_{(m)})$.

The exact value of $\hbar$ in proposition 6 is not important. Given a representation of $\mathcal{SL}_{\hbar,q}$ with some $\hbar \neq 0$, we can get a representation with another $\hbar$ renormalizing the generators in an appropriate way. Note that in the $U_q(sl(2))$ case this method of constructing representation theory of the algebra $\mathcal{SL}_{\hbar,q}$ was suggested in [DGR].

Up to now we considered the "left" representations of the algebras in question but we need also the "right" ones. Such representations are given by appropriate maps

$$l_i^j \to \mathrm{End}_{\,r}(V^*_{(m)}).$$

Note that we do not specify which dual space — right or left — is used in the formula above. In fact, it is of no importance due to (iso)morphisms (2.10). The right representation $\bar\pi_m$ of $\mathcal{L}_{\hbar,q}$ (and $\mathcal{SL}_{\hbar,q}$) in the space $\Lambda^m_+(V^*_l)$ is introduced in the standard way as the map $\bar\pi_m : \mathcal{L}_{\hbar,q} \to \mathrm{End}_{\,r}(\Lambda^m_+(V^*_l))$ given by the formula

$$< g \triangleleft \bar\pi_m(X),\, f >=< g,\, \pi_m(X) \triangleright f > \quad \text{for any} \quad f \in \Lambda^m_+(V),\ g \in \Lambda^m_+(V^*_l).$$

This construction is valid for an admissible Hecke symmetry of any rank. The case $\mathrm{rk}\,(R) = 2$ which we are dealing with leads to additional technical simplifications. The point is that in this case we can equip the space V with a non-degenerate bilinear form $V^{\otimes 2} \to \mathbb{K}$ which is a categorical morphism. This form allows us to identify V with V^*_ε, $\varepsilon = r, l$ and, therefore, to define representations $\bar\pi_m : \mathcal{L}_{\hbar,q} \to \mathrm{End}_{\,r}(V_{(m)})$ and their sl-counterparts $\bar\pi_m : \mathcal{SL}_{\hbar,q} \to \mathrm{End}_{\,r}(V_{(m)})$. These representations are categorical morphisms also. Explicitly, such a bilinear form can be written as follows

$$< x_i,\, x_j >= v^{-1}_{ij}. \tag{2.16}$$

Here $\|v^{-1}_{ij}\|$ is the matrix inverse to $\|v^{ij}\|$ which is invertible as has been shown in [G].

Now, we define our main object — the braided non-commutative sphere — as a quotient of $\mathcal{SL}_{\hbar,q}$.

Definition 7 Let $\mathrm{rk}\,(R) = 2$ and $\sigma \in \mathcal{SL}_{\hbar,q}$ be a nontrivial quadratic central element (e.g., take $\mathrm{Tr}_R L^2$, where Tr_R is defined in (3.10)). Fix $\alpha \in \mathbb{K}$. The quotient $\mathcal{SL}_{\hbar,q}/\{\sigma - \alpha\}$ will be called *the braided non-commutative sphere.*

Here α is assumed to be generic. Below we consider some polynomial (called Cayley-Hamilton) identities whose coefficients depend on α. By demanding their roots to be distinct we get more concrete restrictions on α.

In the particular case of the quantum sphere the element σ will be specified in section 4.

Note that as in the classical case this quotient has the following spectral decomposition:

$$\mathcal{SL}_{\hbar,q}/\{\sigma - \alpha\} = \oplus_i V_{(2i)}.$$

3 Index via braided Casimir element

In [GS1] we suggested a way to construct a family of projective modules over the RE algebra (modified or not) by means of the Cayley-Hamilton identity. As was shown in [GPS], the matrix L satisfying (2.12) with any even Hecke symmetry R obeys to a polynomial relation

$$L^p + \sum_{i=0}^{p-1} \sigma_{p-i}(L)\, L^i = 0, \quad p = \mathrm{rk}\,(R), \tag{3.1}$$

where the coefficients $\sigma_i(L)$ belong to the center $Z(\mathcal{L}_{\hbar,q})$ of the algebra $\mathcal{L}_{\hbar,q}$. This relation is called *the Cayley-Hamilton identity.*

Let us consider the quotient algebra $\mathcal{L}^{\chi}_{\hbar,q} = \mathcal{L}_{\hbar,q}/\{I^{\chi}\}$, where $\{I^{\chi}\}$ is the ideal generated by the elements

$$z - \chi(z), \quad z \in Z(\mathcal{L}_{\hbar,q}), \tag{3.2}$$

where

$$\chi: \; Z(\mathcal{L}_{\hbar,q}) \to \mathbb{K}$$

is a character of $Z(\mathcal{L}_{\hbar,q})$. After factorization to the algebra $\mathcal{L}^{\chi}_{\hbar,q}$ the coefficients in (3.1) become numerical

$$L^p + \sum_{i=0}^{p-1} a_i\, L^i = 0, \quad a_i = \chi(\sigma_{p-i}(L)) \tag{3.3}$$

(putting $\chi(\mathbf{l}) = 0$ we obtain a quotient of $\mathcal{SL}_{\hbar,q}$ denoted as $\mathcal{SL}^{\chi}_{\hbar,q}$).

Assuming the roots of the equation

$$\mu^p + \sum_{i=0}^{p-1} a_i \mu^i = 0$$

to be distinct, one can introduce p idempotents in the usual way

$$e_i = \prod_{j\neq i} \frac{(L-\mu_j)}{(\mu_i - \mu_j)}, \quad 0 \le i \le p-1. \tag{3.4}$$

If no character χ is fixed, then the roots μ_i can be treated as elements of the algebraic closure $\overline{Z(\mathcal{L}_{\hbar,q})}$ of the center $Z(\mathcal{L}_{\hbar,q})$ (or $\overline{Z(\mathcal{SL}_{\hbar,q})}$ if $\chi(\mathbf{l}) = 0$).

Besides the basic Cayley-Hamilton identity (3.1), we are interested in the so-called *derived* ones which are valid for some extensions of the matrix L [GS1]. A regular way to introduce these extensions can be realized via "a (split) braided Casimir element". A particular case of such a Casimir element corresponding to $q = 1$ was used in [K], [R] in the study of the family algebras.

The *braided Casimir element* is defined to be

$$\mathbf{Cas} = \sum_{i,j} l_i^{\,j} \otimes h_j^{\,i} = \sum_{i,j} l_i^{\,j} \otimes l_j^{\,k} C_k^{\,i}. \tag{3.5}$$

Its crucial property is that the map

$$1 \to \mathbf{Cas} \tag{3.6}$$

belongs to Mor $(\mathcal{C})$. Therefore, $\mathbf{Cas}$ is a central element of the category $\mathcal{C}$ in the following sense

$$(\mathbf{Cas} \otimes U) \triangleleft \overline{R} = U \otimes \mathbf{Cas} \quad \forall U \in \mathrm{Ob}(\mathcal{C})$$

with $\overline{R}$ defined in the previous section. To prove this it suffices to observe that the above relation is obviously valid for the unity of $\mathbb{K}$, hence, for $\mathbf{Cas}$ due to the fact that (3.6) is a categorical morphism.

The braided Casimir element is a very useful tool in constructing some extensions of the "quantum matrix" L. In order to get the initial matrix L as well as its higher analogs, one should replace the elements $h_j^{\,i} \in \mathrm{End}_l(V)$ in (3.5) with their images $\pi_m(h_j^{\,i})$ realizing them as matrices. For example, the matrix L defined in (2.11) can be represented as follows

$$L^t \stackrel{\text{def}}{=} \sum_{i,j} l_i^{\,j} \otimes \pi_1(h_j^{\,i}), \tag{3.7}$$

where L^t stands for the transposed matrix. The numerical matrix $\pi_1(h_j^{\,i})$ has the only nonzero (i,j)-th entry equal to 1.

Now, consider the set of maps

$$\Delta_m^{(2)} = \mathrm{id} \otimes \Delta_m : \ \mathbf{Cas} \to \mathcal{L}_{\hbar,q} \otimes \mathrm{End}_{\,l}(V_{(m)}) \quad m = 1, 2 \ldots$$

where $\Delta_m : \mathrm{End}_{\,l}(V) \to \mathrm{End}_{\,l}(V_{(m)})$ are defined by (2.15), and another set

$$\pi_m^{(2)} = \mathrm{id} \otimes \pi_m : \ \mathbf{Cas} \to \mathcal{L}_{\hbar,q} \otimes \mathrm{Mat}_{m+1}(\mathbb{K}) = \mathrm{Mat}_{m+1}(\mathcal{L}_{\hbar,q}), \quad m = 1, 2 \ldots$$

where the elements of $\mathrm{End}_{\,l}(V)$ are represented by the corresponding matrices.

An extension $L_{(m)}$ of the matrix L is defined to be the image of **Cas** under the map $\pi_m^{(2)}$

$$L^t_{(m)} = \pi_m^{(2)}(\mathbf{Cas}) = \sum_{i,j} l_i^{\,j} \otimes \pi_m(h_j^{\,i}) \in \mathcal{L}_{\hbar,q} \otimes \mathrm{Mat}_{m+1}(\mathbb{K}) = \mathrm{Mat}_{m+1}(\mathcal{L}_{\hbar,q}).$$

An explicit example of such an extension related to the quantum sphere will be given in section 4.

Below we will use the Cayley-Hamilton identity for the braided Casimir element which follows from (3.1); therefore, we need powers of **Cas**. We define the product of Casimir elements as the following composition of morphisms

$$l_{i_1}^{\,j_1} \otimes h_{j_1}^{\,i_1} \longrightarrow l_{i_1}^{\,j_1} \otimes 1 \otimes h_{j_1}^{\,i_1} \stackrel{(3.6)}{\longrightarrow} l_{i_1}^{\,j_1} l_{i_2}^{\,j_2} \otimes h_{j_2}^{\,i_2} h_{j_1}^{\,i_1} \longrightarrow l_{i_1}^{\,a} l_a^{\,j_2} \otimes h_{j_2}^{\,i_1}.$$

That is

$$\mathbf{Cas}^2 = l_i^{\,a} l_a^{\,j} \otimes h_j^{\,i}, \quad \mathbf{Cas}^3 = l_i^{\,a} l_a^{\,b} l_b^{\,j} \otimes h_j^{\,i}, \quad \text{etc.} \tag{3.8}$$

It is worth explaining the meaning of such a definition in more detail. For matrices from $\mathrm{Mat}(\mathcal{L}_{\hbar,q})$ one generally has $L^t \cdot L^t \neq (L^2)^t$. Therefore, a care should be taken to define the product of the Casimir elements in such a way as to preserve the proper correspondence between the powers of **Cas** and those of the matrix L. In particular, the definition might not imply $\mathbf{Cas}^2 \to (L^t)^2$. Our definition (3.8) is suitable in this sense because it gives the following correspondence

$$\mathbf{Cas} \to L^t, \quad \mathbf{Cas}^2 \to (L^2)^t, \quad \mathbf{Cas}^3 \to (L^3)^t, \quad \text{etc.}$$

This means that if L satisfies a Cayley-Hamilton identity, then **Cas** obeys the same relation but transposed as *a whole*, which does not affect the identity.

Definition (3.8) is directly transferred to the product of extensions $L_{(m)}$ leading to the following result

$$(L^k_{(m)})^t \stackrel{\mathrm{def}}{=} l_{i_1}^{\,j_1} \ldots l_{i_k}^{\,j_k} \otimes \pi_m(h_{j_k}^{\,i_k}) \ldots \pi_m(h_{j_1}^{\,i_1}) \equiv \left(\pi_m^{(2)}(\mathbf{Cas})\right)^k, \tag{3.9}$$

where the last equality is merely a conventional notation for the preceding expression.

Define now a map Tr_R in the following way

$$\mathrm{Tr}_R : \ \mathcal{L}_{\hbar,q} \otimes \mathrm{Mat}_{m+1}(\mathbb{K}) = \mathrm{Mat}_{m+1}(\mathcal{L}_{\hbar,q}) \to \mathcal{L}_{\hbar,q}, \quad \mathrm{Tr}_R \stackrel{\mathrm{def}}{=} \mathrm{id} \otimes \mathrm{tr}_R, \tag{3.10}$$

where tr_R is the categorical trace (2.9) and the space $\mathrm{Mat}_{m+1}(\mathbb{K})$ is identified with $\mathrm{End}_{\,l}(V_{(m)})$. In particular, we have

$$\mathrm{Tr}_R L = \mathrm{Tr}_R\, \pi_1^{(2)}(\mathbf{Cas}) = l_i^{\,j} \otimes \mathrm{tr}_R(h_j^{\,i}) = l_i^{\,j} C_j^{\,i} = \mathbf{l}.$$

In the $U_q(sl(n))$ case the trace $\mathrm{Tr}_R L$ coincides with the *quantum trace* (cf. [FRT]) which plays an important role in the theory of the RE algebra.

Let us summarize the above construction once more. Given an admissible Hecke symmetry R, we introduce the category $\mathcal{C}$ as was shortly described in section 2 and construct the morphism (categorical trace) $\mathrm{tr}_R : \mathrm{End}_{\,l}(U) \to \mathbb{K}$, $U \in \mathrm{Ob}(\mathcal{C})$ which is defined by R. Then with the category

$\mathcal{C}$ we associate the algebra $\mathcal{L}_{\hbar,q}$ defined by system (2.12) and use this categorical trace in order to define the map Tr_R sending *matrices* with entries from $\mathcal{L}_{\hbar,q}$ treated as elements of $\mathcal{L}_{\hbar,q}\otimes\mathrm{End}_l(V_{(m)})$ into the algebra $\mathcal{L}_{\hbar,q}$.

As we said above, the matrices $L_{(m)}$ also satisfy Cayley-Hamilton identities which we will call derived ones. Namely, there exists a monic polynomial $\mathcal{CH}_{(m)}(t)$ of degree $m+1$ (recall that $\mathrm{rk}\, R=2$) whose coefficients belong to $Z(\mathcal{L}_{\hbar,q})$ such that

$$\mathcal{CH}_{(m)}(L_{(m)})=0, \quad m=1,2,... \tag{3.11}$$

Pass now to the algebra $\mathcal{L}^{\chi}_{\hbar,q}$ (see (3.2)) and consider the image $\mathcal{CH}^{\chi}{}_{(m)}(t)$ of the polynomial $\mathcal{CH}_{(m)}(t)$ in this algebra. Relation (3.11) transforms into a corresponding one in the algebra $\mathcal{L}^{\chi}_{\hbar,q}$:

$$\mathcal{CH}^{\chi}{}_{(m)}(L_{(m)})=0,\ m=1,2,... \tag{3.12}$$

the coefficients of the polynomial $\mathcal{CH}^{\chi}{}_{(m)}(t)$ being numerical. An explicit form of $\mathcal{CH}^{\chi}{}_{(m)}(L_{(m)})$ is determined by the following proposition (the existence of identity (3.11) and the next proposition will be proved in [GS3]).

Proposition 8 *Let μ_0 and μ_1 be roots of the polynomial $\mathcal{CH}^{\chi}{}_{(1)}(t)$ (3.3) (at $p=2$ this polynomial is quadratic). Then for each $m\geq 2$ the polynomial $\mathcal{CH}^{\chi}{}_{(m)}(t)$ is of the degree $(m+1)$ and its roots $\mu_i(m)$ are given by the formula*

$$q^{m-1}\mu_i(m)=q^{-i}\,[m-i]_q\,\mu_0+q^{i-m}[i]_q\,\mu_1+[i]_q\,[m-i]_q\,\hbar, \quad i=0,1,...,m. \tag{3.13}$$

Assuming the roots $\mu_i(m)$, $0\leq i\leq m$, of the polynomial $\mathcal{CH}^{\chi}{}_{(m)}(t)$ $(m\geq 2)$ to be distinct we can introduce idempotents $e_i(m)\in\mathcal{L}^{\chi}_{\hbar,q}\otimes\mathrm{End}_l(V_{(m)})$ analogously to (3.4) (to get uniform notations, we put $e_i=e_i(1)$).

If upon fixing some $m\geq 2$ one multiplies (3.12) by $L^n_{(m)}$, $n\geq 0$, and then applies Tr_R to the resulting equalities, one obtains a recurrence for $\alpha_n(m)=\mathrm{Tr}_R L^n_{(m)}$, $n\geq 0$. The general solution for such a recurrence is of the form

$$\alpha_n(m)=\sum_{i=0}^{m}\mu_i^n(m)\,d_i(m),$$

where $\mu_i(m)$ are the roots of the polynomial $\mathcal{CH}^{\chi}{}_{(m)}(t)$ (distinct by assumption) and the quantities $d_i(m)$ are defined by the initial conditions, i.e., by the values $\mathrm{Tr}_R L^k_{(m)}$, $k=0,1,...,m$. Thus, we have the following proposition.

Proposition 9 *If the roots $\mu_i(m)$ of the polynomial $\mathcal{CH}^{\chi}{}_{(m)}(t)$ are all distinct, then there exist $d_i(m)$ such that*

$$\mathrm{Tr}_R L^n_{(m)}=\sum_{i=0}^{m}\mu_i^n(m)\,d_i(m), \quad n=0,1,2,...$$

The coefficients $d_i(m)$ in the above expansion are functions in the roots $\mu_i(m)$. These functions are singular if there are coinciding roots. If we treat the roots $\mu_i(m)$ as elements of $\overline{Z(\mathcal{L}_{\hbar,q})}$ (or $\overline{Z(\mathcal{SL}_{\hbar,q})}$), then the quantities $d_i(m)$ becomes elements of the field of fractions of this algebra. (Note that the case of multiple roots can be studied by the method of the paper [DM].)

Now, let us consider a representation $\bar\pi_k:\mathcal{L}_{\hbar,q}\to\mathrm{End}_r(V_{(k)})$ of the algebra $\mathcal{L}_{\hbar,q}$ (or $\mathcal{SL}_{\hbar,q}$) defined at the end of section 2. It is easy to see that for a generic q the map $\bar\pi_k$ is surjective and hence for any $z\in Z(\mathcal{L}_{\hbar,q})$ the operator $\bar\pi_k(z)$ is scalar. Therefore, we can define a character $\chi_k: Z(\mathcal{L}_{\hbar,q})\to\mathbb{K}$ by taking

$$\chi_k(z)=a_k(z), \quad \text{where} \quad \bar\pi_k(z)=a_k(z)\,\mathrm{id}\,, \quad \forall\, z\in Z(\mathcal{L}_{\hbar,q})$$

and denote $\mathcal{CH}_{k,m}(t) = \mathcal{CH}^{\chi}{}_{(m)}$ with $\chi = \chi_k$. In what follows we will also use the notation

$$L^t_{(k,m)} = \bar{\pi}_k(l_i^{\,j}) \otimes \pi_m(h_j^{\,i}). \tag{3.14}$$

We emphasize that $L^t_{(k,m)}$ which is the image of the matrix $L^t_{(m)}$ under the representation $\bar{\pi}_k$ is treated as an element of $\mathrm{Mat}_{m+1}(\mathrm{Mat}_{k+1}(\mathbb{K}))$. Also, $L^t_{(k,m)}$ can be treated as an operator acting in the space $V_{(k)} \otimes V_{(m)}$. Indeed, if in the formula (3.14) we consider $\bar{\pi}_k(l_i^{\,j})$ and $\pi_m(h_j^{\,i})$ as operators we get an operator acting in the space $V_{(k)} \otimes V_{(m)}$. More precisely, we put the Casimir element **Cas** between the factors $V_{(k)}$ and $V_{(m)}$ and apply it to these spaces via the representations $\bar{\pi}_k$ and π_m respectively. This operator generated by **Cas** and acting in the product $V_{(k)} \otimes V_{(m)}$ will be denoted $\mathbf{Cas}_{(k,m)}$.

It is evident that the matrix $L_{(k,m)}$ satisfies the Cayley-Hamilton identity

$$\mathcal{CH}_{k,m}(L_{(k,m)}) = 0 \tag{3.15}$$

which is a specialization of (3.12) with $\chi = \chi_k$. If the roots of the polynomial $\mathcal{CH}_{(k,m)}(t)$ are distinct, one can introduce idempotents $e_i(k,m)$ similarly to $e_i(m)$.

Applying the morphism

$$\mathbf{tr} = \mathrm{tr}^{(1)}_R \otimes \mathrm{tr}^{(2)}_R : \ \ \mathrm{End}_{\,r}(V_{(k)}) \otimes \mathrm{End}_{\,l}(V_{(m)}) \to \mathbb{K}$$

to all powers of the matrix $L_{(k,m)}$ and using the Cayley-Hamilton identity for this matrix we can prove the following proposition (similarly to proposition 9).

Proposition 10 *Let $\mu_i(k,m)$ be all the roots of the polynomial $\mathcal{CH}_{(k,m)}(t)$. Let them be distinct. Then there exist numbers $d_i(k,m)$, $0 \le i \le m$ such that*

$$\mathbf{tr}\, L^n_{(k,m)} = \sum_{i=0}^{m} \mu_i(k,m)^n \, d_i(k,m), \quad n = 0,1,2,...$$

They are uniquely defined by the values of $\mathbf{tr}\, L^l_{(k,m)}$, $l = 0,...,m$.

Definition 11 The quantities $\mu_i(m)$ and $d_i(m)$ (or $\mu_i(k,m)$ and $d_i(k,m)$) will be called respectively *eigenvalues* and *braided* multiplicities of the matrix $L_{(m)}$ (or $L_{(k,m)}$).

Corollary 12 *Let $f(t)$ be a polynomial (or a convergent series) in t. Then*

$$\begin{aligned} \mathrm{Tr}_R f(L_{(m)}) &= \sum f(\mu_i(m)) d_i(m), \\ \mathbf{tr}\, f(L_{(k,m)}) &= \sum f(\mu_i(k,m)) d_i(k,m). \end{aligned}$$

In particular, taking as f the polynomial in the right hand side of (3.4) and its higher analogs we get the following proposition.

Proposition 13 *If the eigenvalues $\mu_i(m)$ (resp. $\mu_i(k,m)$) are distinct, then*

$$\mathrm{Tr}_R\, e_i(m) = d_i(m), \tag{3.16}$$
$$\mathbf{tr}\, e_i(k,m) = d_i(k,m). \tag{3.17}$$

Definition 14 The quantity $\mathbf{tr}\, e_i(k,m)$ will be called *the q-index* and denoted $\mathrm{Ind}\,(e_i(m), \bar{\pi}_k)$.

Remark 15 Multiplying the trace by a factor results in a modification of the eigenvalues μ_i but does not affect the multiplicities d_i. We are only interested in the latter quantities and therefore can disregard a normalization of the trace. Similarly, the multiplicities d_i are stable under changes of the numeric factor in (2.14). However, only for the factor $q^{1-m}[m]_q$ in the definition of $X_{(m)}$ we get (3.13).

Remark 16 On restricting to the $U_q(sl(2))$ case we emphasize that similarly to [K], [R] we deal with elements from $(\mathcal{L}_{\hbar,q}\otimes \mathrm{End}\,(V_{(m)}))^{U_q(sl(2))}$, i.e., we consider $U_q(sl(2))$-invariant elements of this tensor product. Representing $\mathcal{L}_{\hbar,q}$ (or $\mathcal{SL}_{\hbar,q}$) in the space $V_{(k)}$ we obtain the space

$$(\mathrm{End}\,(V_{(k)})\otimes \mathrm{End}\,(V_{(m)}))^{U_q(sl(2))}.$$

Thus, all our constructions are "equivariant" with respect to the action of the quantum group (at least at the $U_q(sl(2))$ case).

Taking into account that $e_i(m)\in \mathcal{L}^{\chi}_{\hbar,q}\otimes \mathrm{End}_{\,l}(V_{(m)})$ and putting $\chi=\chi_k$ we get

$$e_i(k,m)=\bar{\pi}_k^{(1)}(e_i(m)). \tag{3.18}$$

Finally, we have

$$\mathrm{Ind}\,(e_i(m),\bar{\pi}_k)=\mathbf{tr}\,e_i(k,m)=\mathbf{tr}\,\bar{\pi}_k^{(1)}(e_i(m))=\mathrm{tr}_R\,\bar{\pi}_k(\mathrm{Tr}_R\,e_i(m)). \tag{3.19}$$

This justifies our treatment of the quantity $\mathbf{tr}\,e_i(k,m)$ as a braided (or q-)analog of the Chern-Connes index. We would like to emphasize that (3.18) and (3.19) are valid provided that the eigenvalues $\mu_i(k,m)$ are pairwise distinct.

Remark 17 Speaking about the braided sphere, we are actually dealing with a family of such spheres depending on the value of the character $\chi=\chi_k$. So, if we treat the entries of the idempotent $e_i(m)$ as elements of $\mathcal{L}^{\chi}_{\hbar,q}$ the q-index $\mathrm{Ind}\,(e_i(m),\bar{\pi}_k)$ is well-defined only for a special value of χ depending on k.

Thus, due to (3.17) and (3.19) the computation of our q-index reduces to calculation of the braided multiplicity $d_i(k,m)$. Now we will show how the latter can be computed with the help of the operators $\mathbf{Cas}_{(k,m)}$ defined above as images of the braided Casimir element:

$$\mathbf{Cas}_{(k,m)}:\ \ V_{(k)}\otimes V_{(m)}\ \to\ V_{(k)}\otimes V_{(m)}.$$

Since map (3.6) is a categorical morphism and since the representations $\bar{\pi}_k$ and π_m are equivariant we can conclude that each operator $\mathbf{Cas}_{(k,m)}$ belongs to $\mathrm{Mor}\,(\mathcal{C})$. This implies that it is scalar on any simple component of the product $V_{(k)}\otimes V_{(m)}$.

Assuming $k\ge m$ one gets the following decomposition

$$V_{(k)}\otimes V_{(m)}=V_{(k+m)}\oplus V_{(k+m-2)}\oplus ...\oplus V_{(k-m)}.$$

Here we use that fact that the Grothendieck (semi)ring of the category in question is isomorphic to that of $sl(2)$-modules. Now, compute the trace of the operator $\mathbf{Cas}_{(k,m)}$ using this decomposition. However, before doing it we would like to make the following observation.

Having fixed an object $U\in \mathrm{Ob}\,(\mathcal{C})$, consider an arbitrary linear operator $\mathcal{F}:U\to U$. What is its trace? The answer depends on the way this operator is realized. To $\mathcal{F}$, we can assign two elements: $F_l\in \mathrm{End}_{\,l}(U)$ and $F_r\in \mathrm{End}_{\,r}(U)$. In general, $\mathrm{tr}_R\,F_l\neq \mathrm{tr}_R\,F_r$. However, under the additional assumption $\mathcal{F} \in \mathrm{Mor}\,(\mathcal{C})$, the operator $\mathcal{F}$ should be scalar on any simple component of U. For such an operator the categorical trace is uniquely defined:

$$\mathrm{tr}_R \mathcal{F} \stackrel{\mathrm{def}}{=} \mathrm{tr}_R\, F_l = \mathrm{tr}_R\, F_r.$$

This follows from the trivial fact that $\mathrm{tr}_R\,\mathrm{id}$ is the same for the right and left realization of the identity operator. To sum up, if a linear operator belongs to $\mathrm{Mor}(\mathcal{C})$, then it is scalar on simple objects and its categorical trace is uniquely defined. This observation enables us to calculate $d_i(k,m)$.

Let μ_i be the eigenvalue of $\mathbf{Cas}_{(k,m)}$ corresponding to the component $V_{(k+m-2i)}$ $(0 \le i \le m)$. Then we have

$$\mathrm{tr}_R\, \mathbf{Cas}^n_{(k,m)} = \sum_{i=0}^{m} \mu_i^n\, d_i, \quad \text{where} \quad d_i = \dim_R(V_{(k+m-2i)}). \tag{3.20}$$

Since $\dim_R(V_{(m)}) = [m+1]_q$ in $\mathcal{C}$, we have the final result.

Proposition 18 *Let $k \ge m$ and let the eigenvalues $\mu_i(k,m)$ be all distinct. Then we have*

$$\mathrm{Ind}\,(e_i(m), \bar{\pi}_k) = [m+k-2i+1]_q, \;\; 0 \le i \le m. \tag{3.21}$$

Proof Under the hypothesis formulae (3.16)–(3.19) are valid and, therefore, the family of multiplicities $d_i(k,m)$ coincides with that of d_i from (3.20). ■

In the next section we will see that if $k \ge m$ and $\bar{\pi}_k$ are sl-representations in the $U_q(sl(2))$ case, then the eigenvalues $\mu_i(k,m)$ are automatically distinct.

In the case of "classical" non-commutative sphere (that is $q = 1$, $\hbar \ne 0$) we get

$$\mathrm{Ind}\,(e_i(m), \bar{\pi}_k) = m+k-2i+1$$

which proves the formula given in [GS2].

Observe that in Proposition 18 the eigenvalues $\mu_i(k,m)$ are numbered according to decreasing (at $q=1$) dimensions of the components whereas above their numeration was arbitrary.

4 Example: quantum non-commutative sphere

Let us consider a particular case of the previous construction, namely, the "quantum non-commutative sphere". In the framework of our general approach we will introduce it using only the corresponding Hecke symmetry, without any quantum group.

Let V be a two dimensional vector space with a fixed base $\{x_1,\, x_2\}$. Represent the Hecke symmetry by the following matrix

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \qquad \lambda = q - q^{-1}.$$

The matrices B and C can be computed directly and after multiplying by q^2 (which is just renormalization for the future convenience) take the form

$$B = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \quad C = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}.$$

We can choose the associated determinant as $v = x_1 \otimes x_2 - q\, x_2 \otimes x_1$. Thus, we have

$$\|v^{ij}\| = \begin{pmatrix} v^{11} & v^{12} \\ v^{21} & v^{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}, \quad \|v^{ij}\|^{-1} = \begin{pmatrix} 0 & -q^{-1} \\ 1 & 0 \end{pmatrix}.$$

Set

$$l_1^1 = a, \quad l_1^2 = b, \quad l_2^1 = c, \quad l_2^2 = d.$$

In these generators the mRE algebra given by (2.12) takes the form

$$\begin{array}{ll} qab - q^{-1}ba = \hbar b & q(bc - cb) = (\lambda a - \hbar)(d - a) \\ qca - q^{-1}ac = \hbar c & q(cd - dc) = c(\lambda a - \hbar) \\ ad - da = 0 & q(db - bd) = (\lambda a - \hbar)b. \end{array} \tag{4.1}$$

Represent the matrix L in accordance with (3.7)

$$L^t = l_i^j \otimes \pi_1(h_j^i) = a \otimes \pi_1(h_1^1) + b \otimes \pi_1(h_2^1) + c \otimes \pi_1(h_1^2) + d \otimes \pi_1(h_2^2) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

(Recall that $\pi_1(h_i^j) \triangleright x_k = \delta_k^j\, x_i$.) Taking (3.10) into account we find

$$\mathbf{l} = \mathrm{Tr}_R L = l_i^j C_j^i = q^{-1}\, a + q\, d.$$

It is straightforward to check that $\mathbf{l}$ is a central element in the mRE algebra. Now, let us consider the traceless component $V_{(2)} = sl_R(V)$ of the space

$$gl_R(V) = \mathrm{span}\,(a, b, c, d).$$

For a basis in $sl_R(V)$ we take $\{b,\, c,\, g = a - d\}$. Being reduced onto the traceless component of $gl_R(V)$, system (4.1) becomes

$$\begin{array}{c} q^2 gb - bg = \hbar(q + q^{-1})b \\ gc - q^2 cg = -\hbar(q + q^{-1})c \\ (q^2 + 1)(bc - cb) + (q^2 - 1)g^2 = \hbar(q + q^{-1})g. \end{array} \tag{4.2}$$

Let us explicitly write the vector (two dimensional) representations of $\mathcal{SL}_{\hbar,q}$ generated by (4.2). Written respectively in the bases $\{x_1,\, x_2\}$ and $\{x_r^1,\, x_r^2\}$ the representations π_1 and $\bar{\pi}_1$ read on the generators:

$$\pi_1(g) = \kappa \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix} \quad \pi_1(b) = \kappa \begin{pmatrix} 0 & q^{-1} \\ 0 & 0 \end{pmatrix} \quad \pi_1(c) = \kappa \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} \quad \kappa \equiv \hbar \frac{q^2 + 1}{q^4 + 1}$$

$$\bar{\pi}_1(g) = \kappa \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix} \quad \bar{\pi}_1(b) = \kappa \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \quad \bar{\pi}_1(c) = \kappa \begin{pmatrix} 0 & 0 \\ q^{-1} & 0 \end{pmatrix} \quad .$$

In order to get the quantum non-commutative sphere we fix a value of a nontrivial quadratic central element. As such an element we take the coefficient σ entering the Cayley-Hamilton identity (4.3). Then the *quantum sphere* is obtained as the quotient of algebra (4.2) modulo the ideal $\{\sigma - \alpha\}$, for some $\alpha \in \mathbb{K}$.

An explicit form of the matrices L and $L_{(2)}$ for $\mathcal{SL}_{\hbar,q}$ is as follows. Taking the sl-representation π_2 to construct $L_{(2)}$ we get

$$L = L_{(1)} = \begin{pmatrix} q[2]_q^{-1} g & b \\ c & -q^{-1}[2]_q^{-1} g \end{pmatrix}, \qquad L_{(2)} = q^{-1} \begin{pmatrix} qg & [2]_q b & 0 \\ q^{-1}c & (q - q^{-1})g & b \\ 0 & q[2]_q c & -q^{-1} g \end{pmatrix}$$

(the latter matrix is calculated in the basis $\{x_1^2,\, qx_1x_2+x_2x_1,\, x_2^2\}$).

One can directly check that the matrix L satisfies the Cayley-Hamilton identity of the form

$$L^2 - q^{-1}\hbar L + \sigma\,\mathrm{id} = 0 \tag{4.3}$$

where

$$\sigma = -[2]_q^{-1}\,\mathrm{Tr}_R L^2 = -[2]_q^{-1}\,([2]_q^{-1}g^2 + q^{-1}bc + qcb) \in Z(\mathcal{SL}_{\hbar,q}). \tag{4.4}$$

The corresponding identity for the matrix $L_{(2)}$ reads

$$L_{(2)}^3 - 2\hbar\frac{[2]_q}{q^2}L_{(2)}^2 + \frac{[2]_q^2}{q^2}(q^{-2}\hbar^2+\sigma)L_{(2)} - \hbar\frac{[2]_q^3}{q^4}\sigma = 0.$$

This was shown in [GS1] for a different normalization of $L_{(2)}$.

So, setting $\sigma = \alpha \in \mathbb{K}$ we come to the equation for L with numerical coefficients

$$L^2 - q^{-1}\hbar L + \alpha\,\mathrm{id} = 0$$

with the roots

$$\mu_0 = \mu_0(1) = (q^{-1}\hbar - \sqrt{q^{-2}\hbar^2 - 4\alpha})/2, \quad \mu_1 = \mu_1(1) = (q^{-1}\hbar + \sqrt{q^{-2}\hbar^2 - 4\alpha})/2.$$

The corresponding multiplicities (which coincide with $\mathrm{Tr}_R\, e_i(1)$ due to (3.16)) are

$$d_0(1) = \mathrm{Tr}_R\, e_0(1) = \mathrm{Tr}_R(L - \mu_1\,\mathrm{id})(\mu_0-\mu_1)^{-1} = \frac{[2]_q}{2} + \frac{[2]_q\hbar}{2\sqrt{\hbar^2 - 4\alpha q^2}},$$

$$d_1(1) = \mathrm{Tr}_R\, e_1(1) = \mathrm{Tr}_R(L - \mu_0\,\mathrm{id})(\mu_1-\mu_0)^{-1} = \frac{[2]_q}{2} - \frac{[2]_q\hbar}{2\sqrt{\hbar^2 - 4\alpha q^2}}.$$

As for the matrix $L_{(2)}$, its eigenvalues can be found by means of (3.13) with $m=2$.

Our next aim is to compute the value of α corresponding to the representation $\bar\pi_k$ or, in other words, the value of $\chi_k(\sigma)$. Clearly, this value does not change if we replace $\bar\pi_k$ by π_k. Such a value (for a Casimir element being a multiple of (4.4)) was computed in [DGR]. Using this result we get

$$\alpha = \chi_k(\sigma) = -\frac{\hbar^2\,[k]_q\,[k+2]_q}{q^2(\,[k+2]_q - [k]_q)^2}. \tag{4.5}$$

This implies that

$$\sqrt{q^{-2}\hbar^2 - 4\alpha} = \pm\frac{q^{-1}\,[2]_q\,[k+1]_q}{[k+2]_q - [k]_q}\hbar. \tag{4.6}$$

Choosing the positive sign in the right hand side of this formula we get

$$\mu_0(k,1) = \frac{-q^{-1}\hbar[k]_q}{[k+2]_q - [k]_q}, \qquad \mu_1(k,1) = \frac{q^{-1}\hbar[k+2]_q}{[k+2]_q - [k]_q},$$

$$d_0(k,1) = [k+2]_q, \qquad d_1(k,1) = [k]_q.$$

Note that the eigenvalues $\mu_i(k,1)$, $i=0,1$, are distinct for all $k \ge 1$.

Proposition 19 *On the quantum non-commutative sphere we have*

$$d_i(m) = \frac{q^{m-2i} + q^{-m+2i}}{2} + \frac{[m-2i]_q[2]_q\hbar}{2\sqrt{\hbar^2 - 4\alpha q^2}},\ 0 \le i \le m.$$

Proof Suffice it to check that

$$\bar{\pi}_k(d_i(m))\,[k+1]_q = [k+m+1-2i]_q.$$

It can be easily done with the help of the following formula

$$[k+m]_q + [k-m]_q = k_q\,(q^m + q^{-m}).$$

■

In fact, this proposition is valid for any braided sphere since (4.5) can be shown to be true for any admissible Hecke symmetry of rank 2. Also, note that for $q = 1$ we get formula (32) from [K].

5 Concluding remarks

1. If R is an involutive symmetry, then there exists a natural generalization of the notion of cyclic (co)homology, and categorical trace becomes an "R-cyclic cocycle". However, we do not know any generalization of this notion to the case of non-involutive R. Thus, we do not know what should be a proper generalization of the "highest Chern-Connes indices" to this case.

2. As we said in Introduction, in the classical case, pairing (1.2) reduces to (1.1) thanks to a suitable definition of the group $K_0(A)$ based on the notion of *stable isomorphic modules*. Two modules M_1 and M_2 are called *stable isomorphic* (and are identified in $K_0(A)$) if the corresponding idempotents e_1 and e_2 (to be extended by 0 if necessary) become "similar": $e_1 = Pe_2P^{-1}$ with some invertible $P \in \mathrm{Mat}_n(A)$ (cf. [Ro] for detail).

However, in the braided case this equivalence implies neither $\mathrm{Tr}_R\, e_1 = \mathrm{Tr}_R\, e_2$ nor $\mathbf{tr}\ \pi(e_1) = \mathbf{tr}\ \pi(e_2)$ for any representation π of the algebra in question. Let us point out that for the usual trace the latter relation is satisfied because of the particular property of the classical trace: it is stable under the change $M \to PMP^{-1}$, $M, P \in \mathrm{Mat}_n(\mathbb{K})$, whereas the categorical trace is not.

Nevertheless, as follows from our computation (at least for the quantum non-commutative sphere)

$$\mathrm{Tr}_R\, e_i(m) = \mathrm{Tr}_R\, e_{i+1}(m+2), \quad 0 \le i \le m, \quad \forall\, m \ge 0. \tag{5.1}$$

In order to justify once more the use of the categorical trace we want to emphasize that if we replace Tr_R by the usual trace, then formula (5.1) becomes wrong.

The modules for which the corresponding idempotents have equal traces Tr_R will be called *trace-equivalent*. It is easy to see that, for a generic q, the modules from the sequence

$$e(0),\ e_0(m),\ e_m(m), \quad m = 1, 2, ...$$

are not trace-equivalent. Thus, the set of classes of trace-equivalent modules is labelled by $n = m - 2i \in \mathbb{Z}$. This looks like the Picard group of the usual sphere.

The problem whether the projective modules related to the idempotents $e_i(m)$ and $e_{i+1}(m+2)$ are equivalent in the conventional sense or not seems, however, to be somewhat difficult even for $q = 1$. Fortunately, the notion of trace-equivalent modules suffices for our purposes. In terms of $n = m - 2i$ we can represent (3.21) as follows

$$\mathrm{Ind}\,(e_i(m), \bar{\pi}_k) = [n+k+1]_q. \tag{5.2}$$

Thus, the q-index depends only on n labelling classes of trace-equivalent modules and k labelling classes of irreducible representations of the algebra $\mathcal{SL}_{\hbar,q}$.

3. Consider a "q-commutative" analog of our algebras, i.e., set $\hbar = 0$. In this case the algebra $\mathcal{SL}_{\hbar,q}$ does not have meaning of an enveloping algebra and we do not consider its representations. Now we take the classical commutative counterpart as a pattern.

Let us realize the usual sphere as a complex projective variety. Then the line bundles $\mathcal{O}(n)$ and $\mathcal{O}(-n)$, $n \geq 0$, become analogs of our modules corresponding to the idempotents $e_0(n)$, $e_n(n)$. Which line bundle corresponds to which projective module depends on the holomorphic structure on the sphere (in our setting the result depends on the sign of the root in (4.6)).

Let us consider the Euler characteristic

$$\chi(\mathcal{O}(n)) = \dim H^0(\mathcal{O}(n)) - \dim H^1(\mathcal{O}(n))$$

of the bundle $\mathcal{O}(n)$ (for $n \geq 0$ it gives the dimension of the space of global sections). Due to the Riemann-Roch theorem we have $\chi(\mathcal{O}(n)) = n + 1$ which coincides with the above quantity $[n+k+1]_q$ at $k = 0$ and $q = 1$. So, for $q = 1$ we consider index (5.2) as a non-commutative analog of the Euler characteristic of the class of the idempotents $e_i(m)$ with $n = m - 2i$. Similarly, as the q-analog of the Euler characteristic for a generic q (in the q-commutative case) we consider the specialization of the q-Chern-Connes index at $k = 0$, that is $[n+1]_q$.

4. As we said above, our quantum sphere is close to the Podles one but it is defined without any quantum group as a quotient of the mRE algebra. So, a question arises: whether it is possible to equip the RE algebra (modified or not) or its quotient (the quantum sphere) with an involution? The answer is positive: it is easy to see that the involution operator $*$ given by

$$*b = c, \quad *c = b, \quad *g = g$$

possesses the classical property $*(x\,y) = (*y)\,(*x)$ and is compatible with (4.2) (here we assume $\hbar$ and q to be real).

This fact is not surprising: such an involution exists for any mRE algebra with the so-called real type R (cf. [M]). This involution, considered as an operator in $\mathrm{End}_{\varepsilon}(V)$, however, is not a categorical morphism since the Euclidean pairing in the space V is not. The only (up to a factor) pairing in V which is a categorical morphism is given by (2.16). Using this paring it is possible to define an involution in $\mathrm{End}_{\varepsilon}(V)$ which does not satisfy the above classical property (it looks like an involution in a super-algebra) but we do not need it at all (cf. [DGR] for a discussion).

5. Let us mention the Poisson structures corresponding to the quantum sphere and, more generally, to quantum orbits related to the quantum group $U_q(sl(n))$. On a generic orbit in $sl(n)^*$, there exists a family of the so-called Poisson-Lie structures (cf. [DGS]). Their quantization (in general, formal deformational) leads to algebras covariant with respect to $U_q(sl(n))$. But in this family only one bracket (up to a numerical factor) is compatible with the Kirillov one. Namely, the simultaneous quantization of the corresponding "Poisson pencil" gives rise to the quantum algebras which are appropriate quotients of $\mathcal{SL}_{\hbar,q}$ ("quantum orbits"). They depend on two parameters and the particular case $\hbar = 0$ is considered as "q-commutative" (the reader is referred to [GS1] for detail).

However, the properties of quantum algebras arising from the Kirillov bracket alone and those arising from the above pencil are different. The Kirillov structure is symplectic and for it there exists an invariant (Liouville) measure. It gives rise to the classical trace in the corresponding quantum algebra. On the contrary, the other brackets from the Poisson pencil are not symplectic and they have no invariant mesure. Their quantization leads to the algebras with trace but this trace is braided. It is these algebras and their "non-quasiclassical" analogs which are the main objects of the present paper.

Also note that the Poisson-Lie structures non-compatible with the Kirillov bracket give rise to one-parameter quantum algebras. It seems that for these algebras there is no reasonable means to construct meaningful projective modules. On the usual sphere such structures do not exist due to its low dimension.

6. The scheme presented in this paper is valid for the "braided orbits" related to the Hecke symmetries of higher rank. Being braided counterparts of semisimple but not necessary generic

orbits in $sl(n)^*$, such orbits can be defined (at least for the $U_q(sl(n))$ case) by methods of the paper [DM]. Thus, the "easy part", namely, the fact that the q-index is nothing but a q-dimension of a component in some tensor product, can be straightforwardly generalized. The proof of an analog of formula (3.13) is, however, much harder. Nevertheless, our low dimensional computations make the following conjecture very plausible.

Conjecture 20 *Let μ_i $1 \le i \le p$ be roots of the polynomial $\mathcal{CH}^\chi{}_{(1)}(t)$ (3.3). Then for $\forall\, m \ge 2$ the degree of the polynomial $\mathcal{CH}^\chi{}_{(m)}(t)$ reads*

$$\deg(\mathcal{CH}^\chi{}_{(m)}(t)) = \binom{m+p-1}{m}$$

and its roots are given by the formula

$$q^{m-1}\mu_{k_1\dots k_p}(m) = \sum_{i=1}^{p} \frac{[k_i]_q}{q^{m-k_i}}\,\mu_i + \xi_p(k_1,\dots,k_p)\,\hbar, \quad k_i \ge 0, \;\; k_1+\ldots+k_p = m, \tag{5.3}$$

where $\xi_p(k_1,\dots,k_p)$ is the symmetric function defined as follows

$$\xi_p(k_1,\dots,k_p) = \sum_{s=2}^{p} q^{k_1+k_2+\ldots+k_s-m}[k_s]_q[k_1+k_2+\ldots+k_{s-1}]_q.$$

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