Polynomial Retracts and the Jacobian Conjecture

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Abstract. Let $K[x,y]$ be the polynomial algebra in two variables over a field K of characteristic 0. A subalgebra R of $K[x,y]$ is called a retract if there is an idempotent homomorphism (a *retraction*, or *projection*) $\varphi : K[x,y] \to K[x,y]$ such that $\varphi(K[x,y]) = R$. The presence of other, equivalent, definitions of retracts provides several different methods of studying them, and brings together ideas from combinatorial algebra, homological algebra, and algebraic geometry. In this paper, we characterize all the retracts of $K[x,y]$ up to an automorphism, and give several applications of this characterization, in particular, to the well-known Jacobian conjecture. Notably, we prove that if a polynomial mapping φ of $K[x,y]$ has invertible Jacobian matrix *and* fixes a non-constant polynomial, then φ is an automorphism.

1. Introduction

Let $K[x,y]$ be the polynomial algebra in two variables over a field K of characteristic 0. A subalgebra R of $K[x,y]$ is called a *retract* if it satisfies any of the following equivalent conditions:

(R1) There is an idempotent homomorphism (a *retraction*, or *projection*) $\varphi : K[x,y] \to K[x,y]$ such that $\varphi(K[x,y]) = R$.

(R2) There is a homomorphism $\varphi : K[x,y] \to R$ that fixes every element of R.

(R3) $K[x,y] = R \oplus I$ for some ideal I of the algebra $K[x,y]$.

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(R4) $K[x,y]$ is a projective extension of R in the category of K-algebras. In other words, there is a split exact sequence $1 \to I \to K[x,y] \to R \to 1$, where I is the same ideal as in (R3) above.

Examples: K; $K[x,y]$; any subalgebra of the form $K[p]$, where $p \in K[x,y]$ is a *coordinate* polynomial (i.e., $K[p,q] = K[x,y]$ for some polynomial $q \in K[x,y]$). There are other, less obvious, examples of retracts: if $p = x + x^2y$, then $K[p]$ is a retract of $K[x,y]$, but p is not coordinate since it has a fiber $\{p = 0\}$ which is reducible, and therefore is not isomorphic to a line.

The very presence of several equivalent definitions of retracts shows how natural these objects are. Later on, we shall also comment on a very natural geometric meaning of retracts.

In [6], Costa has proved that every proper retract of $K[x,y]$ (i.e., a one different from K and $K[x,y]$) has the form $K[p]$ for some polynomial $p \in K[x,y]$, i.e., is isomorphic to a polynomial K-algebra in one variable. A natural problem now is to characterize somehow those polynomials $p \in K[x,y]$ that generate a retract of $K[x,y]$. Since the image of a retract under any automorphism of $K[x,y]$ is again a retract, it would be reasonable to characterize retracts up to an automorphism of $K[x,y]$, i.e., up to a "change of coordinates". We give an answer to this problem by proving the following

Theorem 1.1. Let $K[p]$ be a retract of $K[x,y]$. There is an automorphism ψ of $K[x,y]$ that takes the polynomial p to $x + y \cdot q$ for some polynomial $q = q(x,y)$. A retraction for $K[\psi(p)]$ is given then by $x \to x + y \cdot q$; $y \to 0$.

Our proof of this result is based on the well-known Abhyankar-Moh theorem [1]. We note in passing that Theorem 1.1 also yields a characterization of retracts of a free associative algebra $K\langle x,y\rangle$ (see Theorem 2.1 in the next Section 2) if one uses a natural lifting. In Section 2, we also make an observation on retracts of a polynomial algebra in arbitrarily many variables (Proposition 2.2).

Theorem 1.1 yields another useful characterization of retracts of $K[x,y]$:

Corollary 1.2. A polynomial $p \in K[x,y]$ generates a retract of $K[x,y]$ if and only if there is a polynomial mapping of $K[x,y]$ that takes p to x. The "if" part is actually valid for a polynomial algebra in arbitrarily many variables.

Although the form to which any retract can be reduced by Theorem 1.1 might seem rather general, it is in fact quite restrictive, and has several interesting applications, in particular, to the notorious

Jacobian conjecture.[11] If for a pair of polynomials $p,q \in K[x,y]$, the corresponding Jacobian matrix is invertible, then $K[p,q] = K[x,y]$.

For a survey and background on this problem, the reader is referred to [3].

To explain how retracts of $K[x,y]$ and the Jacobian conjecture are connected, we need one more

Definition. A polynomial $p \in K[x,y]$ has a *unimodular gradient* if there are $u, v \in K[x,y]$ such that $\frac{\partial p}{\partial x} \cdot u + \frac{\partial p}{\partial y} \cdot v = 1$.

If the ground field K is algebraically closed, then this is equivalent, by Hilbert's Nullstellensatz, to the gradient being nowhere-vanishing.

Now we are ready to establish a link between retracts of $K[x,y]$ and the Jacobian conjecture by means of the following

Conjecture "R". If a polynomial $p \in K[x,y]$ has a unimodular gradient, then $K[p]$ is a retract of $K[x,y]$.

Theorem 1.3. Conjecture "R" implies the Jacobian conjecture.

In fact, Conjecture "R" is stronger than the Jacobian conjecture since there are polynomials that have a unimodular gradient, but do not have a Jacobian mate. The simplest example is $p = x + x^2y$; this polynomial has a unimodular gradient, but this gradient cannot be completed to an invertible 2×2 matrix by the gradient of any other polynomial.

Therefore, a statement *equivalent* to the Jacobian conjecture would be: if (p,q) is a Jacobian pair (i.e., the corresponding Jacobian matrix is invertible), then $K[p]$ is a retract of $K[x,y]$. This statement is formally weaker than the Jacobian conjecture, but is actually equivalent to it by our Theorem 1.3.

There are several different ways of proving Theorem 1.3 using the characterization of retracts given in Theorem 1.1; we believe that the proof we give here (using Newton polygons) is particularly elegant.

As a corollary, we show that the Jacobian conjecture is equivalent to other (formally) much weaker statements:

Corollary 1.4. Either of the following claims is equivalent to the Jacobian conjecture. Suppose (p,q) is a Jacobian pair. Then:

(i) for some coordinate polynomial g, $K[x,y] = K[p] + < g >$, where $< g >$ is the ideal of $K[x,y]$ generated by the polynomial g.

(ii) for some coordinate polynomial g, $K[x,y] = K[g] + < p >$.

Corollary 3.1 in Section 3 also seems interesting since it improves several known partial results on two-variable Jacobian conjecture.

Conjecture "R" has the following geometric interpretation. Suppose we have a polynomial $p \in \mathbf{C}[x, y]$ with nowhere-vanishing gradient. Then every fiber $\{p = c,\ c \in \mathbf{C}\}$ is a non-singular curve on the complex plane. Fix a particular fiber, say, $\{p = c_0\}$. If there is another polynomial $q \in \mathbf{C}[x, y]$ such that every fiber of q intersects the curve $\{p = c_0\}$ transversally (i.e., without common tangent line) at exactly one point, then we can arrange a geometric projection of the plane onto the curve $\{p = c_0\}$ by sliding a point on a fiber $\{q = c\}$ toward the intersection point of $\{q = c\}$ with $\{p = c_0\}$. This geometric projection will also be algebraic, i.e., $\mathbf{C}[p]$ will be a retract of $\mathbf{C}[x, y]$ in this case.

As far as the Jacobian conjecture is concerned, the Jacobian condition ensures transversal intersection of any fiber of p with any fiber of q if (p, q) is a Jacobian pair. Moreover, by a result of Kaliman [10], we can restrict our attention to the situation where all the fibers of p and q are irreducible, i.e., are connected curves. The only problem is to show that there is a fiber of p which intersects every fiber of q at *exactly* one point, i.e., that for some $c_0 \in \mathbf{C}$, the system of equations $p(x, y) = c_0;\ q(x, y) = c$ has a unique solution for any $c \in \mathbf{C}$. Thus, we have:

Corollary 1.5. (cf. [8]). Suppose φ is a polynomial mapping of $\mathbf{C}[x, y]$ with invertible Jacobian matrix. If φ is injective on some line, then φ is an automorphism.

We also note here that the aforementioned result of Kaliman [10] calls for an example of a non-coordinate polynomial $p \in \mathbf{C}[x, y]$ that has a unimodular gradient and all of whose fibers are irreducible. If there is no polynomial like that, then the two-variable Jacobian conjecture is true.

Another application of retracts to the Jacobian conjecture (somewhat indirect though) is based on the "φ^∞-trick" familiar in combinatorial group theory (see [14]). For a polynomial mapping $\varphi : K[x, y] \to K[x, y]$, denote by $\varphi^\infty(K[x, y]) = \bigcap_{k=1}^{\infty} \varphi^k(K[x, y])$ the *stable image* of φ. Then we have:

Theorem 1.6. Let φ be a polynomial mapping of $K[x, y]$. If the Jacobian matrix of φ is invertible, then either φ is an automorphism, or $\varphi^\infty(K[x, y]) = K$.

Our proof of Theorem 1.6 is based on recent results of Formanek [9] and Connell-Zweibel [5].

Obviously, if φ fixes a polynomial $p \in K[x, y]$, then $p \in \varphi^\infty(K[x, y])$. Therefore, we have:

Corollary 1.7. Suppose φ is a polynomial mapping of $K[x,y]$ with invertible Jacobian matrix. If $\varphi(p) = p$ for some non-constant polynomial $p \in K[x,y]$, then φ is an automorphism.

This yields the following re-formulation of the Jacobian conjecture: if φ is a polynomial mapping of $K[x,y]$ with invertible Jacobian matrix, then for some automorphism α, the mapping $\alpha \cdot \varphi$ fixes a non-constant polynomial.

2. Retracts of $\mathbf{K}[\mathbf{x},\mathbf{y}]$

We start with

Proof of Theorem 1.1. Let $K[p]$, $p \in K[x,y]$, be a retract of $K[x,y]$ (note that by a result of Costa [6], every retract of $K[x,y]$ has this form). Without loss of generality, we may assume that p has zero constant term.

Let the corresponding retraction be given by $\phi : x \to q_1(p);\ y \to q_2(p)$ for some one-variable polynomials q_1, q_2. Again, we may assume that both q_1 and q_2 have zero constant term.

Then, since ϕ is a retraction, polynomials $q_1(p), q_2(p)$ should generate $K[p]$. Suppose q_1 and q_2 have degree $n \geq 1$ and $m \geq 1$, respectively. Then, by Abhyankar-Moh theorem [1], either n divides m, or m divides n. Suppose $deg(q_1) = k \cdot deg(q_2)$ for some integer $k \geq 1$.

Then we make the following change of coordinates: $x \to \widetilde{x} = x - c \cdot y^k;\ y \to \widetilde{y} = y$, where the coefficient $c \in K^*$ is chosen so that in the polynomial $q_1 - c \cdot q_2^k$, the leading terms would cancel out.

In these new coordinates, our retraction acts as follows: $\phi : \widetilde{x} \to q_1 - c \cdot q_2^k = \widetilde{q_1};\ \widetilde{y} \to q_2 = \widetilde{q_2}$. Polynomials $\widetilde{q_1}$ and $\widetilde{q_2}$ are easily seen to be another generating set of $K[p]$, but the sum of degrees of $\widetilde{q_1}$ and $\widetilde{q_2}$ is less than that of q_1 and q_2.

Continuing this process, we shall eventually arrive at a pair of polynomials one of which is zero. Denote the other one by h; then we must have $K[h] = K[p]$, i.e., $h = c \cdot p$ for some $c \in K^*$.

Thus, we have shown that for some automorphism $\psi \in Aut(K[x,y])$ ("change of coordinates"), the composition $\phi\psi$ takes x to p, and y to 0. It follows that in these coordinates, $p(x,y) = x + y \cdot q(x,y)$ for some polynomial q. This completes the proof. $\Box$

Proof of Corollary 1.2. (1) Suppose $p \in K[x,y]$ generates a retract of $K[x,y]$. Then, by Theorem 1.1, for some automorphism $\psi \in Aut(K[x,y])$, the polynomial $\psi(p)$ has the form $x + y \cdot q(x,y)$. Let ϕ be a mapping of $K[x,y]$ that takes x to x; y to 0. Then $\phi(\psi(p)) = x$.

(2) We are going to prove the "if" part for a polynomial algebra $K[x_1, ..., x_n]$ in arbitrarily many variables.

Let $\varphi(p) = x_1$. Consider the following mapping of $K[x_1, ..., x_n]$: $\psi : x_1 \to p;\ x_i \to 0,\ \ i = 1, ..., n$. Then

$$\psi(\varphi(p)) = p. \tag{1}$$

Denote $\varrho = \psi\varphi$. Then, by (1), $\varrho(p) = p$, which means ϱ fixes every element of $K[p]$. Also, it is clear that $\varrho(K[x_1, ..., x_n]) = K[p]$. Therefore, ϱ is a retraction of $K[x_1, ..., x_n]$, and $K[p]$ a retract. $\Box$

Now we give a characterization of retracts of a free associative algebra $K\langle x, y\rangle$:

Theorem 2.1. Let R be a proper retract of $K\langle x, y\rangle$. There is an automorphism ψ of $K\langle x, y\rangle$ that takes R to $K\langle v\rangle = K[v]$ for some element v of the form $x + y \cdot q(x, y) + w(x, y)$, where $w(x, y)$ belongs to the commutator ideal of $K\langle x, y\rangle$.

Proof. First of all, every element of the given form generates a retract of $K\langle x, y\rangle$. Indeed, the corresponding retraction is given by $x \to v;\ y \to 0$ (this mapping will take w to 0 as well).

Now we are going to show that every retract of $K\langle x, y\rangle$ has the form $K\langle v\rangle = K[v]$ for some element $v \in K\langle x, y\rangle$. From the definition (R1) of a retract, we see that every retract of $K\langle x, y\rangle$ can be generated by two elements ($\varphi(x)$ and $\varphi(y)$).

Another easy observation is that if R is a retract of $K\langle x, y\rangle$ with the corresponding retraction ϕ, then R^α is a retract of $K[x, y]$, and the corresponding retraction is ϕ^α. Here R^α denotes the image of R under the natural abelianization mapping $\alpha : K\langle x, y\rangle \longrightarrow K[x, y]$ (note that the kernel of this mapping is the commutator ideal of $K\langle x, y\rangle$), and ϕ^α is the natural abelianization of ϕ. The best way to see it is to apply the definition (R2) of a retract.

Upon combining these two observations with what we know about retracts of $K[x, y]$, we see that generators of our retract R must be of the form $v_1(x, y) = q_1(x, y) + w_1(x, y)$ and $v_2(x, y) = q_2(x, y) + w_2(x, y)$, where w_1, w_2 belong to the commutator ideal of $K\langle x, y\rangle$, and $q_1^\alpha = h_1(p)$, $q_2^\alpha = h_2(p)$, where $p = p(x, y) \in K[x, y]$ is a polynomial that generates a retract of $K[x, y]$.

Moreover, since R^α should be equal to $K[p]$, we should have $K[h_1(p), h_2(p)] = K[p]$, so that we are in a position to apply Abhyankar-Moh theorem. Repeating the argument from the proof of Theorem 1.1, we see that after applying an appropriate automorphism $\psi \in Aut(K[x,y])$ ("change of coordinates"), our pair $(h_1(p), h_2(p)$ becomes $(\psi(h_1(p)), 0)$.

Since every automorphism of $K[x,y]$ can be lifted to an automorphism of $K\langle x,y\rangle$ (see [4], Theorem 8.5), we have shown that, after applying an automorphism if necessary, our retract R can be generated by two elements, one of which (call it u) belongs to the commutator ideal of $K\langle x,y\rangle$, and the other one (call it v) is of the form $x + y \cdot q(x,y) + w(x,y)$, where $w(x,y)$ belongs to the commutator ideal of $K\langle x,y\rangle$, and the corresponding retraction is $\phi : x \to v;\ y \to u$.

Then, since ϕ^α annihilates y^α (see above), we should have $u = \phi(y)$ belong to the commutator ideal of $K\langle x,y\rangle$.

On the other hand, ϕ should fix every element of R, in particular, the element u. Suppose $u \neq 0$. Write u in the form $u = m_1 + \widetilde{u}$, where m_1 is the sum of lowest degree terms, i.e., any monomial of $\widetilde{u}$ has degree greater than that of monomials in m_1. Then, since u belongs to the commutator ideal of $K\langle x,y\rangle$, every monomial in m_1 depends on y. Therefore, the image of any monomial in m_1 under the endomorphism ϕ is a sum of monomials whose degree is greater than that of monomials in m_1 (this sum might be equal to zero, but it does not affect the argument). The same applies to monomials of $\widetilde{u}$.

Thus, there is no way we can have $\phi(u) = u$; this contradiction shows that $u = 0$, which completes the proof of the theorem. $\Box$

We conclude this section with an observation on retracts of a polynomial algebra in arbitrarily many variables.

Proposition 2.2. Let R be a proper retract of $K[x_1, ..., x_n]$ generated by polynomials $p_1, ..., p_n$, $n \geq 2$. Then $p_1, ..., p_n$ are algebraically dependent.

Proof. Let $\phi : K[x_1, ..., x_n] \to R$ be a retraction, so that $\phi(R) = R$. In particular, ϕ restricted to R is an automorphism of R. If polynomials $p_1, ..., p_n$ were algebraically independent, then a result of [5] would imply that ϕ is an automorphism of $K[x_1, ..., x_n]$ as well. In that case, we would have $R = K[x_1, ..., x_n]$, which means R is not a proper retract, hence a contradiction. $\Box$

We note that it is an open problem whether or not any retract of $K[x_1, ..., x_n]$, $n \geq 3$, can be generated by algebraically independent polynomials. If this is the case, it would imply the positive solution of the well-known *cancellation problem* – see [6] for discussion.

3. The Jacobian conjecture

First we prove that our Conjecture "R" implies the Jacobian conjecture.

Proof of Theorem 1.3. Let $\varphi : x \to p(x,y);\ y \to q(x,y)$ be a polynomial mapping of $K[x,y]$ with invertible Jacobian matrix. Then, in particular, the polynomial $p(x,y)$ has a unimodular gradient.

If we assume that our Conjecture "R" is true, then $p(x,y)$ generates a retract of $K[x,y]$, hence by Theorem 1.1, for some automorphism $\psi \in Aut(K[x,y])$, polynomial $\psi(p)$ has the form $x + y \cdot h(x,y)$. Therefore, upon combining φ with ψ if necessary, we may assume that $p(x,y)$ itself has this form.

Now we appeal to a result of [13] concerning Newton polygons of a Jacobian pair $(p(x,y), q(x,y))$. The Newton polygon of a polynomial $f = f(x,y) = \sum a_{ij}x^iy^j$ is the convex hull of $\{(i,j)|\ a_{ij} \neq 0\} \cup \{(0,0)\}$.

The result of [13] we need is that if $(p(x,y), q(x,y))$ is a Jacobian pair, but $\varphi : x \to p(x,y);\ y \to q(x,y)$ is not an automorphism, then Newton polygons of $p(x,y)$ and $q(x,y)$ are radially similar.

Now, by way of contradiction, suppose φ is not an automorphism. Look at the Newton polygon of $p(x,y)$. We see that it has an edge of length 1, namely, the one between the vertices (0,0) and (1,0). It follows that the similarity ratio for Newton polygons of $p(x,y)$ and $q(x,y)$ is an integer; in particular, $q(x,y)$ has the form $\sum_{i=1}^{k} c_ix^i + y \cdot f(x,y)$ for some $c_i \in K,\ \ c_k \neq 0$.

Then replace the pair (p,q) with $(p,\ q - c_kp^k)$. This new pair clearly has the same properties as (p,q) does: it is a Jacobian pair, but the corresponding mapping is not an automorphism. However, the highest degree of monomials of the form x^m in the second polynomial has been decreased.

Therefore, we can repeatedly apply our argument (note that $p(x,y)$ does not change), until we get a pair (p,g), where g has no monomials of the form x^m. But in that case, the Newton polygon of g has no edges along the x-axis, hence Newton polygons of p and g cannot be radially similar. This contradiction completes the proof of Theorem 1.3. $\square$

Proof of Corollary 1.4. (i) Without loss of generality, we may assume that both p and g have zero constant term. Moreover, upon applying an automorphism to all polynomials under consideration if necessary, we may assume $g = x$.

First we show that if $K[x,y] = K[p] + < x >$, then the sum is actually direct.

By way of contradiction, suppose we have

$$x \cdot u = \sum_{i=1}^{m} c_i \cdot p^i \tag{2}$$

for some non-zero polynomial $u = u(x,y)$ and constants $c_i \in K$. Since the left-hand side of (2) is divisible by x, the right-hand side should be divisible by x, too. This is only possible if p itself is divisible by x, but in that case, the Newton polygon of p would not have an edge along the y-axis, which contradicts p having a Jacobian mate (see e.g. [13]).

Thus, we have shown that $K[x,y] = K[p] \oplus \langle x \rangle$. By the definition (R3) of a retract, this implies $K[p]$ is a retract of $K[x,y]$. Therefore, by our Theorem 1.3, $K[p,q] = K[x,y]$. $\square$

(ii) Arguing as in (i), we get $p \cdot u = \sum_{i=1}^{m} c_i \cdot x^i$. In this case, p cannot depend on y, so $p = p(x)$; but then the equality $K[x,y] = K[x] + \langle p \rangle$ is not possible.

Thus, $K[x,y] = K[x] \oplus \langle p \rangle$. This clearly implies $p = c \cdot y$ for some $c \in K^*$. The result follows. $\square$

Corollary 3.1. Let K be an algebraically closed field, and let (p,q), $p,q \in K[x,y]$, be a Jacobian pair. Suppose p has the form $p = x + g$ for some $g \in K[x,y]$, which is divisible by a homogeneous polynomial. Then $K[p,q] = K[x,y]$.

Proof. If p is linear, then we are done. Suppose p is non-linear, and suppose g is divisible by a non-constant homogeneous polynomial h.

If h is divisible by x, then p itself is divisible by x. In that case, the Newton polygon of p does not have an edge along the y-axis, which contradicts p having a Jacobian mate (see [13]).

If h is divisible by y, then $K[p]$ is a retract of $K[x,y]$. If not, then there is $c \in K^*$ such that the homomorphism $x \to p$; $y \to cp$ takes h to 0. This is obviously a retraction, so again, $K[p]$ is a retract of $K[x,y]$.

Applying our Theorem 1.3 yields the result. $\square$

Van den Essen and Tutaj [7] have shown that if a Jacobian pair (p,q) is of the form $(x+h_1,\ y+h_2)$, where *both* h_1 and h_2 are homogeneous polynomials, then $K[p,q] = K[x,y]$. It is notable that our Corollary 3.1 not only relaxes the condition on the form of polynomials, but, most importantly, our condition is imposed on *one* polynomial only.

To prove Theorem 1.5, we need the following result of independent interest:

Lemma 3.2. Let $\varphi : x \to p(x,y);\ y \to q(x,y)$ be a polynomial mapping of $K[x,y]$ with invertible Jacobian matrix. Suppose $\varphi(K[x,y])$ contains a coordinate polynomial. Then φ is an automorphism.

Proof. Upon composing φ with an automorphism if necessary, we may assume that $x \in \varphi(K[x,y])$. This implies $K[p,q,x] = K[p,q]$, and, therefore, $K(p,q,x) = K(p,q)$, where $K(p,q)$ is the quotient field of $K[p,q]$.

On the other hand, by a result of [9], (p,q) being a Jacobian pair implies $K(p,q,x) = K(x,y)$.

Therefore, we have $K(p,q) = K(x,y)$, which by Keller's theorem [11] implies $K[p,q] = K[x,y]$. $\Box$

Proof of Theorem 1.6. Suppose φ is *not* an automorphism. Then, by Lemma 3.2, $\varphi(K[x,y])$ contains no coordinate polynomials. In particular, the degree of any polynomial in $\varphi(K[x,y])$ is at least 2. By using inductive argument, we are going to show now that the degree of any polynomial in $\varphi^k(K[x,y])$ is at least $(k+1)$; this will imply $\varphi^\infty(K[x,y]) = K$.

Consider an algebra $\varphi^k(K[x,y])$ for some $k \ge 1$. Since φ is injective (this is ensured by the Jacobian condition), polynomials $\varphi^k(x)$ and $\varphi^k(y)$ are algebraically independent.

If φ restricted to $\varphi^k(K[x,y])$ were an automorphism of $\varphi^k(K[x,y])$, then, by a result of [5], φ would be an automorphism of $K[x,y]$, contrary to our assumption. (Note that $\varphi^k(K[x,y])$ is invariant under φ since, by induction, $\varphi^k(K[x,y]) \subseteq \varphi^{k-1}(K[x,y])$ implies $\varphi^{k+1}(K[x,y]) \subseteq \varphi^k(K[x,y])$). Hence, $\varphi\mid_{\varphi^k(K[x,y])}$ is not an automorphism of $\varphi^k(K[x,y])$, and our previous argument yields that $\varphi^{k+1}(K[x,y])$ contains no coordinate polynomials of $\varphi^k(K[x,y])$. In particular, the degree of any polynomial in $\varphi^{k+1}(K[x,y])$ is greater than that of any coordinate polynomial in $\varphi^k(K[x,y])$. This completes the induction. $\Box$

A. van den Essen has pointed out to us that in the case $K = \mathbf{C}$, a more complicated (geometric) proof of the result of Theorem 1.6 was given by Kraft [12], who also proved that if a polynomial mapping φ of $\mathbf{C}[x,y]$ is *not* birational (i.e., φ does not induce an automorphism of the quotient field $\mathbf{C}(x,y)$), then the stable image of φ is a retract of $\mathbf{C}[x,y]$. This yields a natural question – is the same true for *any* polynomial mapping? (cf. [14]).

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