Max-Planck-Institut für Mathematik
Bonn

Special tilting modules for algebras with positive dominant dimension

by

Matthew Pressland
Julia Sauter

Max-Planck-Institut für Mathematik
Preprint Series 2017 (40)

Special tilting modules for algebras with positive dominant dimension

Matthew Pressland
Julia Sauter

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Institut für Algebra und Zahlentheorie
Universität Stuttgart
Pfaffenwaldring 57
70569 Stuttgart
Germany

Fakultät für Mathematik
Universität Bielefeld
Postfach 100 131
33501 Bielefeld
Germany

MPIM 17-40

SPECIAL TILTING MODULES FOR ALGEBRAS WITH POSITIVE DOMINANT DIMENSION

MATTHEW PRESSLAND AND JULIA SAUTER

Dedicated to Idun Reiten on the occasion of her 75th birthday.

Abstract. We study a set of uniquely determined tilting and cotilting modules for an algebra with positive dominant dimension, with the property that they are generated or cogenerated (and usually both) by projective-injectives. These modules have various interesting properties, for example that their endomorphism algebras always have global dimension at most that of the original algebra. We characterise d-Auslander–Gorenstein algebras and d-Auslander algebras via the property that the relevant tilting and cotilting modules coincide. By the Morita–Tachikawa correspondence, any algebra of dominant dimension at least 2 may be expressed (essentially uniquely) as the endomorphism algebra of a generator-cogenerator for another algebra, and we also study our special tilting and cotilting modules from this point of view, via the theory of recollements and intermediate extension functors.

1. Introduction

In [11], Crawley-Boevey and the second author associated to each Auslander algebra a distinguished tilting-cotilting module T, with the property that it is both generated and cogenerated by a projective-injective module. In this paper, we study more general instances of tilting modules generated by projective-injectives, and cotilting modules cogenerated by projective-injectives. In contrast to the case of Auslander algebras, we consider here tilting and cotilting modules of arbitrary finite projective or injective dimension.

More precisely, let Γ be a finite-dimensional algebra with dominant dimension d (see Definition 2.1). Then for every $0<k<d$, we explain how to uniquely determine a 'shifted' k-tilting module T_k and a 'coshifted' k-cotilting module C^k (usually distinct, unlike the case of Auslander algebras) that are generated and cogenerated by projective-injectives. The construction also allows for $k=0$ or $k=d$, although in this case the relevant module is either generated or cogenerated by projective-injectives, but usually not both. We are also interested in the resulting shifted and coshifted algebras $B_k=\operatorname{End}_\Gamma(T_k)^{\mathrm{op}}$ and $B^k=\operatorname{End}_\Gamma(C^k)^{\mathrm{op}}$.

Finite-dimensional algebras with dominant dimension at least 2 are of particular interest. Any such algebra is isomorphic to an endomorphism algebra $\operatorname{End}_A(E)^{\mathrm{op}}$ for a generating-cogenerating module E over a finite-dimensional algebra A. In fact, assuming for simplicity that all objects are basic, the assignment $(A,E)\mapsto\operatorname{End}_A(E)^{\mathrm{op}}$ induces a bijection

$$\{(A,E) : E \text{ generating-cogenerating } A\text{-module}\} \xrightarrow{\sim} \{\Gamma : \operatorname{domdim}\Gamma\geq 2\},$$

with objects considered up to isomorphism on each side [21,27]. This result is sometimes known [13,26] as the *Morita–Tachikawa correspondence*. The following definition will be convenient throughout the paper.

Definition 1.1. A *Morita–Tachikawa triple* (A,E,Γ) consists of a finite-dimensional algebra A, a generating-cogenerating A-module E, and $\Gamma\cong\operatorname{End}_A(E)^{\mathrm{op}}$.

Thus, assuming as we usually will that all objects are basic, the set of Morita–Tachikawa triples is the graph of the Morita–Tachikawa correspondence. Given a basic algebra Γ of dominant dimension at least 2, it appears in the (unique up to isomorphism) Morita–Tachikawa triple

$$(A=\operatorname{End}_\Gamma(\Pi)^{\mathrm{op}}, E=\mathrm{D}\Pi, \Gamma),$$

Date: June 25, 2017.

2010 *Mathematics Subject Classification.* 16G10, 18E30, 18G05.

Key words and phrases. tilting, dominant dimension, higher Auslander–Reiten theory, recollement.

where Π is a maximal projective-injective summand of Γ, and D is the usual duality over the base field. The pair (A,E) in the above triple plays an important role in some of our results on the shifted and coshifted algebras of Γ.

The structure of the paper is as follows. We give the definitions and preliminary observations in Section 2, in which we also prove (Corollary 2.16) that

$$\operatorname{gldim} B \leq \operatorname{gldim} \Gamma$$

whenever B is one of the algebras B_k or B^k associated to Γ. In Section 3, we investigate the modules T_k and C^k in the context of higher Auslander–Reiten theory, which provides a wealth of examples of algebras with high dominant dimension. The main result of this section is the following.

Theorem 1 (Theorem 3.9)**.** *Let Γ be a finite-dimensional algebra, and let $d \geq 1$. Assume* $\operatorname{domdim}\Gamma \geq d+1$*, and write*

$$T_* = \{T_k : 0 \leq k \leq d+1\},$$
$$C^* = \{C^k : 0 \leq k \leq d+1\}$$

for the sets of (isomorphism classes of) shifted and coshifted modules of Γ. Then the following are equivalent:

(i) *Γ is a d-Auslander–Gorenstein algebra,*
(ii) *$T_* = C^*$, and*
(iii) *$T_* \cap C^*$ is non-empty.*

The definition of a d-Auslander–Gorenstein algebra, due to Iyama and Solberg [18], is given in Definition 3.1. One may replace 'd-Auslander–Gorenstein' in this theorem by 'd-Auslander' by assuming additionally that Γ has finite global dimension, and so this result generalises [11, Lem. 1.1] for Auslander algebras.

If Π is the maximal projective-injective summand of Γ, it is a summand of every tilting or cotilting Γ-module. Thus if B is the endomorphism algebra of such a module, it has an idempotent given by projection onto Π, yielding a recollement involving the categories B-mod and $\operatorname{End}_\Gamma(\Pi)^{\mathrm{op}}$-mod; note that if $\operatorname{domdim}\Gamma \geq 2$ then $\operatorname{End}_\Gamma(\Pi)^{\mathrm{op}}$ is the algebra A from the Morita–Tachikawa triple involving Γ. In Section 4, we study these recollements for the shifted and coshifted algebras. In particular, we give in Theorems 4.9 and 4.12 an explicit formula for the intermediate extension functor in such a recollement; this functor is by definition the image of the universal map from the restriction functor's left adjoint to its right adjoint.

To obtain this description, we show that just as in [11], each shifted and coshifted algebra of Γ can be described in terms of its Morita–Tachikawa partner (A,E). We construct for each $0 \leq k \leq d$ explicit objects E_k and E^k in the bounded homotopy category $\mathcal{K}^{\mathrm{b}}(A)$, and prove the following.

Theorem 2 (Theorem 4.4)**.** *Let (A,E,Γ) be a Morita–Tachikawa triple, with* $\operatorname{domdim}\Gamma = d$*. Then for all $0 \leq k \leq d$, there are isomorphisms*

$$B_k \cong \operatorname{End}_{\mathcal{K}^{\mathrm{b}}(A)}(E_k)^{\mathrm{op}}, \qquad B^k \cong \operatorname{End}_{\mathcal{K}^{\mathrm{b}}(A)}(E^k)^{\mathrm{op}},$$

where B_k and B^k are the coshifted algebras of Γ.

In other words, we have for any algebra A and generator-cogenerator E the schematic

$$\begin{tikzcd}
(A,E) \arrow[r] \arrow[dr] & \operatorname{End}_A(E)^{\mathrm{op}} \arrow[d, "\text{tilt at } T_k"] \\
 & \operatorname{End}_{\mathcal{K}^{\mathrm{b}}(A)}(E_k)^{\mathrm{op}}
\end{tikzcd}$$

and a similar picture for the coshifted module C^k.

A k-tilting or k-cotilting Γ-module with endomorphism algebra B defines $k+1$ pairs of equivalent subcategories in Γ-mod and B-mod; in the classical case $k=1$, the two subcategories on each side form torsion pairs. In Section 5, we give various descriptions of the relevant subcategories associated to shifted or coshifted modules, many of which can be characterised in terms of generation or cogeneration by certain projective or injective modules.

In Section 6, we consider again the recollements involving B-mod and A-mod, where B is one of the shifted or coshifted algebras of an algebra Γ in a Morita–Tachikawa triple (A, E, Γ). Recall from general tilting theory that B_k, as a tilt of Γ by T_k, has a preferred cotilting module $\mathrm{D}T_k$. Similarly B^k has the preferred tilting module $\mathrm{D}C^k$. We prove the following.

Theorem 3 (Theorems 6.5, 6.6)**.** *Let (A, E, Γ) be a Morita–Tachikawa triple and $0 < k < \operatorname{domdim}\Gamma$. Denoting by c_k and c^k the intermediate extension functors in the recollements relating B_k-*mod *and B^k-*mod *respectively with A-*mod, *we have*

$$c_k(E) = \mathrm{D}T_k, \qquad c^k(E) = \mathrm{D}C^k.$$

We note that the shifted modules T_k appear briefly in a recent paper of Chen–Xi [9], where they are called canonical tilting modules, and some results on the dominant dimensions of the shifted algebras are obtained. Some of these ideas have also been studied independently in very recent work of Nguyen, Reiten, Todorov and Zhu [23]. The second author presented results of this paper at ICRA 2016 in Syracuse and at the workshop *Representation Theory of Quivers and Finite Dimensional Algebras* at Oberwolfach in February 2017.

Throughout the paper, all algebras are finite-dimensional $\mathbb{K}$-algebras over some field $\mathbb{K}$, and, without additional qualification, 'module' is taken to mean 'finitely-generated left module'. Morphisms are composed from right-to-left.

2. Shifted modules and algebras

Throughout this section, we fix a finite-dimensional algebra Γ, assumed for simplicity to be basic, over a field $\mathbb{K}$. The goal of this section is to characterise certain special tilting and cotilting Γ-modules in the case that Γ has positive dominant dimension. We begin with the relevant definitions.

Definition 2.1. Let k be a non-negative integer. We say that Γ has *dominant dimension* at least k and write $\operatorname{domdim}\Gamma \geq k$ if the regular module ${}_\Gamma\Gamma$ has an injective resolution

$$0 \longrightarrow \Gamma \longrightarrow \Pi_0 \longrightarrow \cdots \longrightarrow \Pi_{k-1} \longrightarrow \cdots$$

with $\Pi_0, \ldots, \Pi_{k-1}$ projective-injective; when $k = 0$, this condition is taken to be empty. As the notation suggests, we write $\operatorname{domdim}\Gamma = d$ if $\operatorname{domdim}\Gamma \geq d$ and $\operatorname{domdim}\Gamma \not\geq d + 1$.

Remark 2.2. As always, we refer to left Γ-modules in our definition of dominant dimension. However, Müller [22, Thm. 4] has shown that the analogous definition using right modules is equivalent to this one. As a consequence, a finite-dimensional algebra Γ has dominant dimension at least k if and only if $\mathrm{D}\Gamma$ has a projective resolution

$$\cdots \longrightarrow \Pi^{k-1} \longrightarrow \cdots \longrightarrow \Pi^0 \longrightarrow \mathrm{D}\Gamma \longrightarrow 0$$

with $\Pi^0, \ldots, \Pi^{k-1}$ projective-injective.

Definition 2.3. Let $k \geq 0$. We say that $T \in \Gamma$-mod is a *k-tilting* module if

(T1) $\operatorname{pd} T \leq k$,
(T2) $\operatorname{Ext}^j_\Gamma(T, T) = 0$ for $j \geq 1$, and
(T3) there is an $\operatorname{add} T$-coresolution of Γ of length k, i.e. an exact sequence

$$0 \longrightarrow \Gamma \longrightarrow t_0 \longrightarrow \cdots \longrightarrow t_k \longrightarrow 0$$

with $t_j \in \operatorname{add} T$ for $0 \leq j \leq k$.

We say a k-tilting module T is *P-special* for a projective module P if there is a sequence as in (T3) with $t_j \in \operatorname{add} P$ for $0 \leq j \leq k-1$, in which case (T1) is superfluous.

Dually, we say that C is a *k-cotilting module* if

(C1) $\operatorname{id} C \leq k$,
(C2) $\operatorname{Ext}^j_\Gamma(C, C) = 0$ for $j \geq 1$, and
(C3) there is an $\operatorname{add} C$-resolution of $\mathrm{D}\Gamma$ of length k, i.e. an exact sequence

$$0 \longrightarrow c^k \longrightarrow \cdots \longrightarrow c^0 \longrightarrow \mathrm{D}\Gamma \longrightarrow 0$$

with $c^j \in \operatorname{add} C$ for $0 \leq j \leq k$.

We say a k-cotilting module C is *I-special* for an injective module I if there is a sequence as in (C3) with $c^j \in \operatorname{add} I$ for $0 \leq j \leq k-1$, in which case (C1) is superfluous.

Proposition 2.4. *Assume* $\operatorname{domdim} \Gamma \geq k$*, and let* Π *be a maximal projective-injective summand of* Γ*. Then there is a basic* Π*-special* k*-tilting* Γ*-module* T_k*, and a basic* Π*-special* k*-cotilting* Γ*-module* C^k*. These modules are unique up to isomorphism.*

Proof. We prove the statements involving T_k, those for C^k being dual. Since $\operatorname{domdim} \Gamma \geq k$, there is an exact sequence

$$0 \longrightarrow \Gamma \longrightarrow \Pi_0 \longrightarrow \cdots \longrightarrow \Pi_{k-1} \longrightarrow T \longrightarrow 0 \tag{2.1}$$

with Π_i projective-injective for $0 \leq i \leq k-1$. Let T_k be a basic module with $\operatorname{add} T_k = \operatorname{add}(T \oplus \Pi)$. Then T_k satisfies (T1) and (T3) by (2.1). A standard homological argument, involving the application of the functors $\operatorname{Hom}_\Gamma(T_k, -)$ and $\operatorname{Hom}_\Gamma(-, T_k)$ to the short exact sequences coming from (2.1), shows that $\operatorname{Ext}^i_\Gamma(T_k, T_k) = \operatorname{Ext}^i_\Gamma(\Gamma, \Gamma) = 0$ for $i > 0$, so T_k satisfies (T2).

Any two Π-special k-tilting Γ-modules are, by definition, k-th cosyzygies of the regular module Γ. Thus if T' is an arbitrary k-th cosyzygy of Γ, it differs from T_k only by the possible removal of projective-injective summands and addition of injective summands, so $T \in \operatorname{add} T'$. If T' is tilting then we must also have $\Pi \in \operatorname{add} T'$, so $T_k \in \operatorname{add} T'$. If T' is basic, it then follows that $T' \cong T_k$ since all tilting modules have the same number of indecomposable summands up to isomorphism. □

To give a slightly different characterisation of the modules T_k and C^k, we introduce the following definitions, which will also be useful in Section 5.

Definition 2.5. Let $\mathcal{A}$ be an abelian category, and $X \in \mathcal{A}$ an object. For $k \geq 0$, define $\operatorname{gen}_k(X)$ to be the full subcategory of $\mathcal{A}$ on objects M such that there exists an exact sequence

$$X^k \longrightarrow \cdots \longrightarrow X^0 \longrightarrow M \longrightarrow 0$$

with $X^i \in \operatorname{add} X$ for $0 \leq i \leq k$. Dually, $\operatorname{cogen}^k(X)$ is the full subcategory of $\mathcal{A}$ on objects N such that there exists an exact sequence

$$0 \longrightarrow N \longrightarrow X_0 \longrightarrow \cdots \longrightarrow X_k$$

with $X_i \in \operatorname{add} X$ for all $0 \leq i \leq k$. When $k = 0$, we omit it from the notation and refer simply to $\operatorname{gen}(X)$ and $\operatorname{cogen}(X)$. It is both natural and convenient to define

$$\operatorname{gen}_{-1}(X) = \Gamma\text{-mod} = \operatorname{cogen}^{-1}(X).$$

Proposition 2.6. *Let* Π *be a maximal projective-injective summand of* Γ*, and* $k \geq 0$*.*

(a) *The subcategory* $\operatorname{gen}_{k-1}(\Pi) \subseteq \Gamma\text{-mod}$ *contains a* k*-tilting object if and only if* $\operatorname{domdim} \Gamma \geq k$*. When it exists, a basic such* k*-tilting object is isomorphic to the* Π*-special* k*-tilting module* T_k *from Proposition 2.4.*

(b) *The subcategory* $\operatorname{cogen}^{k-1}(\Pi) \subseteq \Gamma\text{-mod}$ *contains a* k*-cotilting object if and only if* $\operatorname{domdim} \Gamma \geq k$*. When it exists, a basic such* k*-cotilting object is isomorphic to the* Π*-special* k*-cotilting module* C^k *from Proposition 2.4.*

Proof. We prove only (a), since (b) is dual. If $\operatorname{domdim} \Gamma \geq k$, then the module T_k from Proposition 2.4 lies in $\operatorname{gen}_{k-1}(\Pi)$. Conversely, if $T \in \operatorname{gen}_{k-1}(\Pi)$ is k-tilting, it has projective dimension at most k, and the minimal projective resolution of T is of the form

$$0 \longrightarrow P \longrightarrow \Pi_{k-1} \longrightarrow \cdots \longrightarrow \Pi_0 \longrightarrow T \longrightarrow 0$$

for $\Pi_i \in \operatorname{add} \Pi$ and P projective. Without loss of generality, we may assume T, like Γ, is basic. Then the number of indecomposable summands of P is the number of non-projective-injective summands of T, which is the number of non-projective-injective summands of Γ. Thus there is an exact sequence

$$0 \longrightarrow \Gamma \longrightarrow \Pi_{k-1} \oplus \Pi \longrightarrow \cdots \longrightarrow \Pi_0 \longrightarrow T \longrightarrow 0,$$

from which it follows simultaneously that $\operatorname{domdim} \Gamma \geq k$ and that T is Π-special, hence isomorphic to T_k by Proposition 2.4. □

Definition 2.7. We call the module T_k (respectively C^k) from Proposition 2.4 the *k-shifted* (respectively *k-coshifted*) module of Γ, and the algebras

$$B_k = \operatorname{End}_\Gamma(T_k)^{\mathrm{op}}, \quad B^k = \operatorname{End}_\Gamma(C^k)^{\mathrm{op}},$$

are called respectively the k-shifted and k-coshifted algebras of Γ.

Remark 2.8. If $\operatorname{domdim}\Gamma \geq k$, then $\operatorname{domdim}\Gamma^{\mathrm{op}} \geq k$ by Remark 2.2. The $\mathbb{K}$-dual of the k-coshifted Γ^{op}-module is the k-shifted Γ-module.

It is well-known that if T is a k-tilting Γ-module, then the right derived functor of $\operatorname{Hom}_\Gamma(T,-)$ and the left derived functor of $\mathrm{D}\operatorname{Hom}_\Gamma(-,T)$ are quasi-inverse triangle equivalences between the bounded derived categories $\mathcal{D}^{\mathrm{b}}(\Gamma)$ and $\mathcal{D}^{b}(\operatorname{End}_\Gamma(T)^{\mathrm{op}})$, cf. [10, Thm. 2.1]. In particular, an algebra of positive dominant dimension is derived equivalent to all of its k-shifted and k-coshifted algebras.

We use the adjective ‘shifted’ by analogy with properties of self-injective algebras. If Γ is self-injective, so any projective module is also injective, then the syzygy and cosyzygy Ω and Ω^- induce mutually inverse equivalences of the stable module category of Γ, with Ω^- being the shift, or suspension, functor on this triangulated category. The crucial property here is that, trivially by the assumption on Γ, a projective cover of any Γ-module is injective, and similarly an injective hull is projective. For more general algebras, there will still be some modules which have such projective covers or injective hulls, and on such modules the syzygy and cosyzygy operations have many of the same properties as for self-injective algebras.

By definition, $\operatorname{domdim}\Gamma \geq 1$ precisely when Γ itself has a projective injective hull, or equivalently when $\mathrm{D}\Gamma$ has an injective projective cover. The proof of Proposition 2.4 illustrates that the shifted and coshifted modules are related to Γ and $\mathrm{D}\Gamma$ analogously to the way in which an arbitrary module over a selfinjective algebra is related to its shifts in the stable module category. Despite this analogy, the case in which Γ is selfinjective does not provide any interesting examples of our constructions.

Remark 2.9. If Γ is selfinjective, then $T_k \cong \Gamma \cong C^k$ for all $k \geq 0$, since there are no other tilting or cotilting Γ modules.

More interestingly, selfinjective algebras may even be characterised by the property that their shifted modules fail to be pairwise non-isomorphic; cf. [4, Prop. 1.3].

Proposition 2.10. *Let $k \geq 0$ and $k' > 0$, and let Γ be a finite-dimensional algebra of dominant dimension at least $k+k'$. If $T_k \cong T_{k+k'}$, then Γ is selfinjective. Dually, if $C^k \cong C^{k+k'}$, then Γ is selfinjective.*

Proof. Let $T_k^\circ \cong T_{k+k'}^\circ$ be the maximal non-projective-injective summand of $T_k \cong T_{k+k'}$. Let P be the maximal non-injective summand of Γ. By the characterisation of T_k from Proposition 2.4, taking the minimal injective resolution of P and truncating yields an exact sequence

$$0 \longrightarrow P \longrightarrow \Pi_0 \longrightarrow \Pi_1 \longrightarrow \cdots \longrightarrow \Pi_{k-1} \longrightarrow T_k^\circ \longrightarrow 0$$

with $\Pi_j \in \operatorname{add}\Pi$ projective for all j, so this sequence is a minimal projective resolution of T_k°. Continuing this minimal injective resolution of P, we obtain a second exact sequence

$$0 \longrightarrow T_k^\circ \longrightarrow \Pi_k \longrightarrow \cdots \longrightarrow \Pi_{k+k'-1} \longrightarrow T_{k+k'}^\circ \longrightarrow 0,$$

again with $\Pi_j \in \operatorname{add}\Pi$ for all j. Since $T_{k+k'}^\circ \cong T_k^\circ$, taking the Yoneda product of the two sequences yields another minimal projective resolution

$$0 \longrightarrow P \longrightarrow \Pi_0 \longrightarrow \Pi_1 \longrightarrow \cdots \longrightarrow \Pi_{k+k'-1} \longrightarrow T_k^\circ \longrightarrow 0$$

of T_k°. Since minimal projective resolutions are unique up to isomorphism, we must have $P = 0$, and so Γ is selfinjective. The dual statement is proved similarly. □

Remark 2.11. Just as in [4], an easy consequence of Proposition 2.10 is that the Nakayama conjecture, that $\operatorname{domdim}\Gamma = \infty$ if and only if Γ is selfinjective, holds for representation-finite algebras. One also sees from the proof that the projective dimension of T_k is exactly k unless Γ is selfinjective, in which case $T_k = \Gamma$ has projective dimension zero, and similarly for the injective dimension of C^k. Combining these observations, one sees that while selfinjective algebras have $T_k \cong \Gamma$ for all $k \geq 0$, any counterexample to the Nakayama conjecture would behave very differently, with $T_k \ncong T_{k'}$ for any $k \neq k'$.

It is possible to identify those algebras that may be obtained as k-shifted or k-coshifted algebras intrinsically, via the existence of cotilting or tilting modules with special properties. As usual, we write $\nu = \mathrm{D}\operatorname{Hom}_A(-, A)$ and $\nu^- = \operatorname{Hom}_A(\mathrm{D}A, -)$ for the Nakayama functors on A.

Lemma 2.12. *Let T be a k-tilting Γ-module with endomorphism algebra B. By the Brenner–Butler tilting theorem [7], $C = \mathrm{D}T$ is a k-cotilting B-module with endomorphism algebra Γ.*

1. *If T is P-special for some projective Γ-module P, then C is I_P-special for $I_P = \mathrm{D}\operatorname{Hom}_\Gamma(P, T)$. Dually, if C is I-special for some injective B-module I, then T is P^I-special for $P^I = \operatorname{Hom}_B(C, I)$.*
2. *Let $\Pi \in \operatorname{add} T$ be projective-injective. Then the projective B-module $P_\Pi = \operatorname{Hom}_\Gamma(T, \Pi)$ and the injective B-module $I_\Pi = \mathrm{D}\operatorname{Hom}_\Gamma(\Pi, T)$ satisfy $I_\Pi = \nu P_\Pi$. Dually, if $\Pi \in \operatorname{add} C$ is projective-injective, then the Γ-modules $P^\Pi = \operatorname{Hom}_B(C, \Pi)$ and $I^\Pi = \mathrm{D}\operatorname{Hom}_B(\Pi, C)$ satisfy $I^\Pi = \nu P^\Pi$.*
3. *If P is a projective Γ-module with $P, \nu P \in \operatorname{add} T$, then $I_P := \mathrm{D}\operatorname{Hom}_\Gamma(P, T)$ is a projective-injective B-module. Dually, if I is an injective B-module with $I, \nu^- I \in \operatorname{add} C$, then $P^I := \operatorname{Hom}_B(C, I)$ is a projective-injective Γ-module.*

Proof. As usual, we give the proof only for the first item in each pair of dual statements.

1. This follows by applying $\mathrm{D}\operatorname{Hom}_\Gamma(-, T)$ to the exact sequence from (T3), using that T is P-special.
2. Since $\operatorname{Hom}_\Gamma(T, -)\colon \operatorname{add} T \to B\text{-proj}$ is fully faithful, we have
$$\nu P = \mathrm{D}\operatorname{Hom}_B(\operatorname{Hom}_\Gamma(T, \Pi), \operatorname{Hom}_\Gamma(T, T)) = \mathrm{D}\operatorname{Hom}_\Gamma(\Pi, T) = I.$$
3. Since $\nu P \in \operatorname{add} T$, the module $\operatorname{Hom}_\Gamma(T, \nu P)$ is projective. Since $P \in \operatorname{add} T$, the Nakayama formula implies that $\operatorname{Hom}_\Gamma(T, \nu P) \cong \mathrm{D}\operatorname{Hom}_\Gamma(P, T)$ is also injective. □

Proposition 2.13. *A finite-dimensional basic algebra B is isomorphic to a k-shifted algebra if and only if there is an injective B-module I and an I-special k-cotilting B-module C with $\nu^- I \in \operatorname{add} C$. Under this isomorphism, C is the dual of the k-shifted module.*

Dually, a finite-dimensional basic algebra B is isomorphic to a k-coshifted algebra if and only if there exists a projective B-module P and a P-special k-tilting B-module T with $\nu P \in \operatorname{add} T$. Under this isomorphism, T is the dual of the k-coshifted module.

Proof. Let T_k be the k-shifted module of some algebra Γ with maximal projective-injective summand Π. Then by Lemma 2.12(1), $\mathrm{D}T_k$ is an I_Π-special k-cotilting B_k-module, where $I_\Pi = \mathrm{D}\operatorname{Hom}_\Gamma(\Pi, T)$. By Lemma 2.12(2), $\nu^- I_\Pi = \operatorname{Hom}_\Gamma(T_k, \Pi)$ lies in $\operatorname{add} \mathrm{D}T$, since $\Pi \in \operatorname{add} \mathrm{D}\Gamma$.

Conversely, assume B, C and I are as in the statement, replacing C and I by basic modules with the same additive hull if necessary. Then $\Gamma = \operatorname{End}_B(C)^{\mathrm{op}}$ has a basic k-tilting module $T = \mathrm{D}C$, which is $P^I = \operatorname{Hom}_B(C, I)$-special by Lemma 2.12(1). By Lemma 2.12(3), P^I is projective-injective. If Π is the maximal projective-injective summand of Γ, then Π is a summand of T since T is k-tilting, so $\Pi \in \operatorname{gen}(P^I)$ since T is P^I-special. It follows that $\operatorname{add} P^I = \operatorname{add} \Pi$, and so $T \cong T_k$ is the k-shifted module of Γ by Proposition 2.4.

The second statement is proved dually, reversing the roles of Γ and B in Lemma 2.12. □

To close this section, we will show that if B_k is the k-shifted algebra of Γ, then $\operatorname{gldim} B_k \leq \operatorname{gldim} \Gamma$. Thus we obtain a tighter bound on this global dimension than would be possible if B_k were replaced by the endomorphism algebra of an arbitrary tilting Γ-module.

We use the following technical lemma, mildly generalising a result of Happel [14, Lem. III.2.7]. Given $\mathcal{C} \subseteq \Gamma\text{-mod}$ a full subcategory, we write $\mathcal{K}^{-,\mathrm{b}}(\mathcal{C})$ for the homotopy category of complexes with terms in $\mathcal{C}$, bounded below, with finitely many non-zero cohomology groups. We write $\mathcal{K}^{\mathrm{b}}(\mathcal{C})$ for the homotopy category of bounded complexes with terms in $\mathcal{C}$.

Lemma 2.14. *Assume Γ has finite global dimension. Let T be a Γ-module such that $\operatorname{Ext}^i_\Gamma(T, T) = 0$ for all $i > 0$ and $\operatorname{id} T = m$. Then for any $T^\bullet \in \mathcal{K}^{-,\mathrm{b}}(\operatorname{add} T)$ with no non-negative cohomology, we have $T^\bullet \cong T^\bullet_1 \oplus T^\bullet_2$ such that $T^\bullet_2$ is acyclic and $T^i_1 = 0$ for all $i < 1 - m$.*

Proof. For $j \leq 0$, write $K^j = \ker(d^j)$. (As the upper index notation suggests, we use cohomological conventions, so the differentials in T are $d^i \colon T^i \to T^{i+1}$.) Since $T^\bullet$ has no non-negative cohomology, we have exact sequences

$$0 \longrightarrow K^j \longrightarrow T^j \longrightarrow K^{j+1} \longrightarrow 0$$

for all $j < 0$. By writing $K^1 = T^0/K^0 \cong \operatorname{im} d^0$, and $K^2 = T^1/\operatorname{im} d^0$, even though these spaces are not kernels of the differential, we also get exact sequences above for $j = 0$ and $j = 1$.

Happel [14, Lem. III.2.7] proves, without the assumption on $\operatorname{id} T$, that we can decompose $T^\bullet$ almost as required, except with $T_1^i = 0$ for $i < 1 - n$. The key step in this argument is to show that $\operatorname{Ext}^1_\Gamma(K^{2-n}, K^{1-n}) = 0$, so that the sequence

$$0 \longrightarrow K^{1-n} \longrightarrow T^{1-n} \longrightarrow K^{2-n} \longrightarrow 0$$

splits, meaning $K^{1-n} \oplus K^{2-n} \in \operatorname{add} T$. With our additional assumption that $\operatorname{id} T = m$, we will in fact show that all of these statements hold with n replaced by m. Then construction of our desired $T_1^\bullet$, $T_2^\bullet$ and isomorphism $T^\bullet \xrightarrow{\sim} T_1^\bullet \oplus T_2^\bullet$ is exactly as in [14, Lem. III.2.7], so we simply refer the reader to Happel's proof. The rest of the argument given here is devoted to showing that $\operatorname{Ext}^1_\Gamma(K^{2-m}, K^{1-m}) = 0$.

Since $\operatorname{Ext}^i_\Gamma(T, T) = 0$ for all $i > 0$, applying $\operatorname{Hom}_\Gamma(-, T)$ to the sequences

$$0 \longrightarrow K^j \longrightarrow T^j \longrightarrow K^{j-1} \longrightarrow 0$$

yields isomorphisms

$$\operatorname{Ext}^i_\Gamma(K^j, T) \xrightarrow{\sim} \operatorname{Ext}^{i+1}_\Gamma(K^{j+1}, T)$$

for all $i > 0$ and $j \leq 1$. Since $\operatorname{id} T = m$, it follows that

$$\operatorname{Ext}^i_\Gamma(K^j, T) \xrightarrow{\sim} \operatorname{Ext}^{m+1}_\Gamma(K^{j+m+1-i}, T) = 0$$

whenever $i > 0$ and $j \leq 1 + i - m$.

Now pick $t \leq 2$. Applying $\operatorname{Hom}_\Gamma(K^t, -)$ to our sequences we get exact sequences

$$\operatorname{Ext}^i_\Gamma(K^t, T^j) \longrightarrow \operatorname{Ext}^i_\Gamma(K^t, K^{j+1}) \longrightarrow \operatorname{Ext}^{i+1}_\Gamma(K^t, K^j) \longrightarrow \operatorname{Ext}^{i+1}_\Gamma(K^t, T^j)$$

for all $i \geq 0$ and $j \leq 1$. It follows that we have isomorphisms

$$\operatorname{Ext}^i_\Gamma(K^t, K^{j+1}) \xrightarrow{\sim} \operatorname{Ext}^{i+1}_\Gamma(K^t, K^j)$$

whenever $i > 0$, $j \leq 1$, and $t \leq 1 + i - m$. In particular

$$\operatorname{Ext}^1_\Gamma(K^{2-m}, K^{1-m}) \xrightarrow{\sim} \operatorname{Ext}^{n+1}_\Gamma(K^{2-m}, K^{1-m-n}) = 0$$

since $\operatorname{gldim} \Gamma = n$. □

Theorem 2.15. *Assume* $\operatorname{gldim} \Gamma = n$*, and let* $T \in \Gamma\text{-mod}$ *be a* k*-tilting object with injective dimension* m*. Let* $B = \operatorname{End}_\Gamma(T)^{\mathrm{op}}$*. Then*

$$n - k \leq \operatorname{gldim} B \leq m + k.$$

Proof. It is well-known, see for example [14, Prop. III.3.4], that

$$n - k \leq \operatorname{gldim} B \leq n + k,$$

so $\operatorname{gldim} B$ is finite, and we need only prove that $\operatorname{gldim} B \leq m + k$.

Let $M \in B\text{-mod}$, and let $P^\bullet$ be a minimal projective resolution of M. Since $\operatorname{gldim} B$ is finite, $P^\bullet \in \mathcal{K}^{\mathrm{b}}(\operatorname{proj} B)$. Precisely, the width of $P^\bullet$ is $\operatorname{pd} M + 1$, which we want to bound. In fact we will, equivalently, bound the width of the complex $T \otimes_B P^\bullet \in \mathcal{K}^{\mathrm{b}}(\operatorname{add} T)$.

By the general theory of tilting modules [14, Lem. 2.8], we have mutually inverse triangle equivalences $T \otimes_B - \colon \mathcal{K}^{-,\mathrm{b}}(\operatorname{proj} B) \to \mathcal{K}^{-,\mathrm{b}}(\operatorname{add} T)$ and $\operatorname{Hom}_\Gamma(T, -) \colon \mathcal{K}^{-,\mathrm{b}}(\operatorname{add} T) \to \mathcal{K}^{-,\mathrm{b}}(\operatorname{proj} B)$. Since $P^\bullet$ was chosen to be minimal, $P^\bullet$ has no non-zero acyclic summands, and it follows from the above equivalences that the same is true of $T \otimes_B P^\bullet$.

We have $\mathrm{H}^i(T \otimes_B P^\bullet) = \operatorname{Tor}^B_{-i}(T, M) = 0$ for $i < -k$, since T has projective dimension at most k as a right B-module by [14, Lem. III.2.4]. Thus $T^\bullet = T \otimes_B P^\bullet[-k-1]$ has no non-negative cohomology. By construction, $T^i = 0$ for $i > k + 1$. By Lemma 2.14, we can write $T^\bullet = T_1^\bullet \oplus T_2^\bullet$, with $T_2^\bullet$ acyclic and $T_1^i = 0$ for $i < 1 - m$. But $T^\bullet$ has no non-zero acyclic summands, so $T_2^\bullet = 0$ and $T^\bullet = T_1^\bullet$. Since

$T^i = 0$ for $i > k+1$ and $i < 1-m$, we conclude that $T^\bullet$ has width at most $m+k+1$. Thus the same is true of $P^\bullet$, and so $\operatorname{pd} M \leq m+k$. □

Corollary 2.16. *Assume* $\operatorname{domdim}\Gamma = d$, *let* $0 \leq k \leq d$ *and let* B_k *be the corresponding shifted algebra. Then*

$$\operatorname{gldim}\Gamma - k \leq \operatorname{gldim} B_k \leq \operatorname{gldim}\Gamma.$$

Proof. Writing $n = \operatorname{gldim}\Gamma$, we have $\operatorname{id}\Gamma \leq n$. Since T_k is a k-th cosyzygy of Γ, it follows that $\operatorname{id} T_k \leq n-k$. Thus, by Theorem 2.15

$$n-k \leq \operatorname{gldim} B_k \leq n-k+k = n.$$ □

A dual argument, using that C^k is k-cotilting with $\operatorname{pd} C^k \leq n-k$, shows that if B^k is the k-coshifted algebra of Γ, then $\operatorname{gldim}\Gamma - k \leq \operatorname{gldim} B^k \leq \operatorname{gldim}\Gamma$.

3. SHIFTING AND COSHIFTING FOR d-AUSLANDER–GORENSTEIN ALGEBRAS

In the context of [11], Crawley-Boevey and the second author considered the 1-shifted and 1-coshifted modules for Auslander algebras, and noted that these two modules in fact coincide. In this section, we will consider a more general situation in which the families of shifted and coshifted modules of Γ coincide with each other, namely when Γ is a d-Auslander–Gorenstein algebra, as defined by Iyama–Solberg in [18] and recalled below. We will in fact show that the property of shifted and coshifted modules coinciding leads to another characterisation of such algebras, generalising [11, Lem. 1.1] for Auslander algebras.

Definition 3.1. Let Γ be a finite-dimensional $\mathbb{K}$-algebra, and let $d \geq 1$. We say Γ is *d-Auslander–Gorenstein* if

$$\operatorname{id}\Gamma \leq d+1 \leq \operatorname{domdim}\Gamma,$$

and that it is a *d-Auslander algebra* if

$$\operatorname{gldim}\Gamma \leq d+1 \leq \operatorname{domdim}\Gamma.$$

Remark 3.2. Our definition of d-Auslander–Gorenstein agrees with Iyama–Solberg's definition of minimal d-Auslander–Gorenstein [18, Defn. 1.1], but we will follow their convention in the bulk of their paper and drop the word 'minimal'. The definition of a d-Auslander algebra is due to Iyama [17] (see also [15, Defn. 4.1] for more general versions), generalising Auslander for $d = 1$ [2].

Note that any d-Auslander algebra is d-Auslander–Gorenstein, and a d-Auslander–Gorenstein algebra is a d-Auslander algebra if and only if it has finite global dimension [18, Prop. 4.8]. A selfinjective algebra is d-Auslander–Gorenstein for all d, and so is a d-Auslander algebra for all d if and only if it is semisimple. On the other hand, by [18, Prop. 4.1], any d-Auslander–Gorenstein algebra Γ that is not selfinjective satisfies $\operatorname{id}\Gamma = d+1 = \operatorname{domdim}\Gamma$, so d is uniquely determined. Similarly, any d-Auslander algebra Γ that is not semisimple has $\operatorname{gldim}\Gamma = d+1 = \operatorname{domdim}\Gamma$.

If Γ is a d-Auslander–Gorenstein algebra for some $d \geq 1$, then in particular $\operatorname{domdim}\Gamma \geq 2$, and so Γ is part of a Morita–Tachikawa triple (A, E, Γ) (recall Definition 1.1). We can translate the conditions on Γ from Definition 3.1 into conditions on the A-module E. Given a subcategory $\mathcal{C}$ of A-mod, write

$$\begin{aligned}\mathcal{C}^{\perp_n} &= \{X \in A\text{-mod} : \operatorname{Ext}^i_A(C, X) = 0 \ \forall\ 1 \leq i \leq n,\ C \in \mathcal{C}\},\\ {}^{\perp_n}\mathcal{C} &= \{X \in A\text{-mod} : \operatorname{Ext}^i_A(X, C) = 0 \ \forall\ 1 \leq i \leq n,\ C \in \mathcal{C}\}.\end{aligned}$$

Definition 3.3. Let $d \geq 2$, and let A be a finite-dimensional algebra. A subcategory $\mathcal{C}$ of A-mod is called *d-precluster-tilting* if

(i) $\mathcal{C}$ is generating and cogenerating,
(ii) $\mathcal{C}^{\perp_{d-1}} = {}^{\perp_{d-1}}\mathcal{C}$,
(iii) $\operatorname{Ext}^i_A(C, C) = 0$ for all $1 \leq i \leq d-1$ and $C \in \mathcal{C}$, and
(iv) $\mathcal{C}$ is functorially finite.

The subcategory $\mathcal{C}$ is *d-cluster-tilting* if the two subcategories in (ii) are also equal to $\mathcal{C}$, in which case conditions (i) and (iii) follow automatically. An A-module E is called d-precluster-tilting or d-cluster-tilting if the corresponding property holds for the subcategory $\operatorname{add} E$.

For $d = 1$, we replace condition (ii) by the requirement that $\mathcal{C}$ is closed under the Auslander–Reiten translations τ and τ^-, and (iii) becomes vacuous. The unique 1-cluster-tilting subcategory is A-mod.

Remark 3.4. This definition of a d-precluster-tilting subcategory is equivalent to Iyama–Solberg's [18, Defn. 3.2], by [18, Prop. 3.7(a)]. The definition of a d-cluster-tilting subcategory is due to Iyama [16, Defn. 2.2], who originally referred to such subcategories as *maximal* $(d-1)$*-orthogonal.*

Similar to our observation for d-Auslander–Gorenstein algebras, it follows from condition (i) that any d-precluster-tilting module is part of a Morita–Tachikawa triple. We are now able to relate Definitions 3.1 and 3.3 via these triples.

Theorem 3.5 ([18, Thm. 4.5], [17, Thm. 2.6])**.** *Let (A,E,Γ) be a Morita–Tachikawa triple. Then Γ is d-Auslander–Gorenstein if and only if E is a d-precluster-tilting A-module, and Γ is a d-Auslander algebra if and only if E is a d-cluster-tilting A-module.*

Remark 3.6. The statement that if Γ is d-Auslander–Gorenstein then E is d-precluster tilting also follows from a more general result of Chen–Koenig [8, Thm. 1.3], giving properties of E whenever Γ is a Gorenstein algebra with $d+1\leq\operatorname{domdim}\Gamma$ and $\operatorname{id}\Gamma\leq d+1+m$. In their language, they show that in this case E is a $(d-1)$-rigid, $(d-1,m)$-orthosymmetric generator-cogenerator, which reduces to E being d-precluster-tilting when $m=0$.

We now give the first part of our characterisation of d-Auslander–Gorenstein algebras via shifted and coshifted modules.

Proposition 3.7. *Let Γ be a d-Auslander–Gorenstein algebra. Then the shifted and coshifted modules of Γ coincide; more precisely, $T_k=C^{d+1-k}$ for all $0\leq k\leq d+1$.*

Proof. By assumption, $\operatorname{id}\Gamma\leq d+1$. By the assumption on $\operatorname{domdim}\Gamma$, a minimal injective resolution of Γ has the form

$$0\longrightarrow\Gamma\longrightarrow\Pi_0\longrightarrow\cdots\longrightarrow\Pi_d\longrightarrow I\longrightarrow 0$$

with each Π_j projective-injective. Then the number of indecomposable summands of I is equal to the number of non-injective indecomposable summands of Γ. Without loss of generality, we may assume Γ is basic, and so I has as summands one copy of each indecomposable non-projective injective Γ-module. It follows that we have $\mathrm{D}\Gamma=I\oplus\Pi$ for Π the maximal projective-injective summand of Γ. Thus, by adding the identity map $\Pi\to\Pi$ to the right-hand end of the above injective resolution, we obtain a sequence

$$0\longrightarrow\Gamma\longrightarrow\Pi_0\longrightarrow\cdots\longrightarrow\Pi_d\longrightarrow\mathrm{D}\Gamma\longrightarrow 0$$

in which each Π_j is projective-injective. This is simultaneously an injective resolution of Γ and a projective resolution of $\mathrm{D}\Gamma$ with the appropriate number of projective-injective terms for computing shifted and coshifted modules, so these modules must coincide as claimed. ☐

Remark 3.8. Note that if Γ is selfinjective, then all shifted and coshifted modules are equal to Γ, as we observed in Remark 2.9. Thus the ambiguity of d in this case does not cause any issues with the identification $T_k=C^{d+1-k}$, and indeed the proof given remains valid. As already remarked, in all other cases, d is uniquely determined [18, Prop. 4.1].

Our characterisation of d-Auslander–Gorenstein algebras may now be stated as follows.

Theorem 3.9. *Let Γ be a finite-dimensional algebra, and let $d\geq 1$. Assume* $\operatorname{domdim}\Gamma\geq d+1$*, and write*

$$T_*=\{T_k:0\leq k\leq d+1\},$$
$$C^*=\{C^k:0\leq k\leq d+1\}$$

for the sets of (isomorphism classes of) shifted and coshifted modules of Γ. Then the following are equivalent:

(i) *Γ is a d-Auslander–Gorenstein algebra,*
(ii) *$T_*=C^*$, and*
(iii) *$T_*\cap C^*$ is non-empty.*

Proof. Assume Γ is d-Auslander–Gorenstein, let Π be the maximal projective-injective summand, and pick $m,n\geq 0$ with $d=m+n-1$. By construction, the m-th shifted module T_m is m-tilting and lies in $\operatorname{gen}_{m-1}(\Pi)$, and the n-th coshifted module C^n is n-cotilting and lies in $\operatorname{cogen}^{n-1}(\Pi)$. By

Proposition 3.7, we have $T_m = C^n$, and so we see that (i) implies (ii). Since (ii) trivially implies (iii), it remains to show that (iii) implies (i).

Assume there is some $T \in \operatorname{gen}_{m-1}(\Pi) \cap \operatorname{cogen}^{n-1}(\Pi)$ that is m-tilting and n-cotilting. Note that Π is a summand of every tilting and cotilting module, and so in particular of T, and that the number of indecomposable summands of T is equal to that of Γ. Since $T \in \operatorname{gen}_{m-1}(\Pi)$ and $\operatorname{pd} T \leq m$, a minimal projective resolution of T has the form

$$0 \longrightarrow P \longrightarrow \Pi_{m-1} \longrightarrow \cdots \longrightarrow \Pi_0 \longrightarrow T \longrightarrow 0,$$

where $\Pi_j \in \operatorname{add} \Pi$ for each $j < m$. By minimality, P has no injective summands, and so by counting we see that P has as summands one copy of each indecomposable non-injective projective Γ-module, and hence $\Gamma = P \oplus \Pi$. Thus by adding the identity map $\Pi \to \Pi$ to the left-hand end of the above resolution, we obtain a sequence

$$0 \longrightarrow \Gamma \longrightarrow \Pi_0 \longrightarrow \cdots \longrightarrow \Pi_{m-1} \longrightarrow T \longrightarrow 0$$

with $\Pi_j \in \operatorname{add} \Pi$ for each j. Similarly, using that $T \in \operatorname{cogen}^{n-1}(\Pi)$ and that T is n-cotilting, we obtain a sequence

$$0 \longrightarrow T \longrightarrow \Pi_m \longrightarrow \cdots \longrightarrow \Pi_{n+m-1} \longrightarrow I \longrightarrow 0,$$

with $\Pi_j \in \operatorname{add} \Pi$ for each j and I injective. Taking the Yoneda product of these two sequences produces a sequence

$$0 \longrightarrow \Gamma \longrightarrow \Pi_0 \longrightarrow \cdots \longrightarrow \Pi_{n+m-1} \longrightarrow I \longrightarrow 0,$$

which shows that $\operatorname{id} \Gamma \leq n + m \leq \operatorname{domdim} \Gamma$, i.e. that Γ is $(m + n - 1)$-Auslander–Gorenstein. ☐

Corollary 3.10. *An algebra Γ is d-Auslander–Gorenstein if and only if for some (or equivalently every) $m, n \geq 0$ such that $d = m + n - 1$, there is a Γ-module in $\operatorname{gen}_{m-1}(\Pi) \cap \operatorname{cogen}^{n-1}(\Pi)$ that is m-tilting and n-cotilting, where Π is the maximal projective-injective summand of Γ.*

Proof. This follows from the equivalence of (i) and (iii) in Theorem 3.9, using the characterisation of shifted and coshifted modules from Proposition 2.6. ☐

Remark 3.11. In the proof of Theorem 3.9, we could have arranged that $I = \mathrm{D}\Gamma$, just as we were able to replace P by Γ. However, unlike the replacement of P, this was not necessary for the argument; the asymmetry arises from that in the definitions of dominant dimension and d-Auslander–Gorenstein, which favour properties of the projective generator Γ over, for example, equivalent dual properties of the injective generator $\mathrm{D}\Gamma$. One viewpoint on Corollary 3.10 is that it provides a more symmetric definition of d-Auslander–Gorenstein, without such favouritism. Indeed, to recover the usual definition, one can set $m = 0$, forcing T to be a projective generator. Setting $n = 0$ forces T to be an injective cogenerator and recovers the dual definition of Remark 2.2.

As an additional corollary, we get a characterisation of d-Auslander algebras, both generalising and strengthening a characterisation of (1-)Auslander algebras due to Crawley-Boevey and the second author [11, Lem. 1.1].

Corollary 3.12. *Let Γ be a finite-dimensional algebra, and let $d \geq 1$. Assume* $\operatorname{domdim} \Gamma \geq d + 1$ *and* $\operatorname{gldim} \Gamma < \infty$. *In the notation of Theorem 3.9, the following are equivalent:*

(i) *Γ is a d-Auslander algebra,*
(ii) *$T_* = C^*$, and*
(iii) *$T_* \cap C^*$ is non-empty.*

Proof. This follows from Theorem 3.9 together with the previously noted fact that d-Auslander algebras are precisely d-Auslander–Gorenstein algebras of finite global dimension [18, Prop. 4.8]. ☐

Corollary 3.13. *An algebra Γ is a d-Auslander algebra if and only if* $\operatorname{gldim} \Gamma < \infty$ *and for some (or equivalently every) $m, n \geq 0$ such that $d = m + n - 1$, there is a Γ-module in* $\operatorname{gen}_{m-1}(\Pi) \cap \operatorname{cogen}^{n-1}(\Pi)$ *that is m-tilting and n-cotilting, where Π is the maximal projective-injective summand of Γ.*

4. Recollements and homotopy categories

4.1. **Idempotent recollements.** Let B be a finite-dimensional algebra, let $e \in B$ be an idempotent element and write $A = eBe$ for the corresponding idempotent subalgebra (sometimes called the *corner* or *boundary algebra*). We obtain from e a diagram

$$\begin{tikzcd}
B/BeB\text{-mod} \arrow[r, "i"] & B\text{-mod} \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "e"] & A\text{-mod} \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "r"]
\end{tikzcd} \tag{4.1}$$

of six functors, defined by

$$\begin{aligned}
q &= B/BeB \otimes_B -, & \ell &= Be \otimes_A -,\\
i &= B/BeB \otimes_{B/BeB} - & e &= \operatorname{Hom}_B(Be, -)\\
&= \operatorname{Hom}_{B/BeB}(B/BeB, -), & &= eB \otimes_B -,\\
p &= \operatorname{Hom}_B(B/BeB, -), & r &= \operatorname{Hom}_A(eB, -).
\end{aligned}$$

Such data is known as a *recollement* of abelian categories, and can be defined in abstract, but we will only consider recollements of module categories determined by idempotents as above (cf. [24]). For a Γ-module M, one obtains the same A-module eM either by applying the functor e in this diagram, or by multiplying on the left by the idempotent e, hence the abuse of notation.

Since ℓ and r are left and right adjoints of e respectively, and $e\ell \cong er \cong 1$, there is a natural isomorphism

$$\operatorname{Hom}_\Gamma(\ell M, rM) \xrightarrow{\sim} \operatorname{Hom}_A(M, M),$$

functorial in M, and so determining a canonical map of functors $\ell \to r$. This map is equivalently described as the composition of the counit of the adjunction (ℓ, e) with the unit of the adjunction (e, r). Taking its image yields a seventh functor $c\colon A\text{-mod} \to \Gamma\text{-mod}$, called the intermediate extension [19], which, like ℓ and r, is fully faithful. In the sequel, we will implicitly use the natural epimorphism $\ell \to c$ and monomorphism $c \to r$ composing to the natural map $\ell \to r$.

Since ℓ, r and c are fully faithful and $er \cong 1 \cong e\ell$, we also have $ec \cong 1$, and we obtain three induced equivalences of categories

$$\begin{tikzcd}
\operatorname{im}\ell & \\
\operatorname{im}c & A\text{-mod} \arrow[ul, "\ell"'] \arrow[l, "c"] \arrow[dl, "r"] \\
\operatorname{im}r &
\end{tikzcd}$$

with quasi-inverses given by the respective restrictions of the functor e. On the other side of the recollement, the functor i embeds $\Gamma/\Gamma e\Gamma$-mod into Γ-mod, and since $pi \cong 1 \cong qi$ we see that the restrictions of q and of p to $\operatorname{im} i$ are both quasi-inverse to i.

The recollement (4.1) determines a *TTF-triple* in B-mod, meaning a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of subcategories such that both $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are torsion pairs, by

$$\operatorname{TTF}(e) = (\mathcal{X}(e), \mathcal{Y}(e), \mathcal{Z}(e)) := (\ker q, \ker e, \ker p).$$

We now give some alternative descriptions of the kernels and images of the functors in our recollement (4.1), including the categories $\ker q$ and $\ker p$ appearing in this TTF-triple, in terms of the categories $\operatorname{gen}_k(X)$ and $\operatorname{cogen}^k(X)$ associated to $X \in B$-mod as in Definition 2.5.

Lemma 4.1. *For B and e as in (4.1), write $P = Be$ and $I = \nu P = \mathrm{D}(eB)$. We have*

$$\begin{aligned}
\ker q &= \operatorname{gen}(P), & \operatorname{im}\ell &= \operatorname{gen}_1(P),\\
\ker p &= \operatorname{cogen}(I), & \operatorname{im} r &= \operatorname{cogen}^1(I).
\end{aligned}$$

Moreover, the image of the intermediate extension $c = \operatorname{im}(\ell \to r)$ is given by

$$\operatorname{im} c = \ker p \cap \ker q = \operatorname{gen}(P) \cap \operatorname{cogen}(I).$$

Proof. For the equalities $\operatorname{im}\ell = \operatorname{gen}_1(P)$ and $\operatorname{im} r = \operatorname{cogen}^1(I)$, see [3, Lem. 3.1]. By [11, Lem/Def. 2.4], if $X \in \ker q$ then the counit map $\ell eX \to X$ is an epimorphism. Take a projective cover $Q \to eX$; since ℓ preserves epimorphisms we obtain an epimorphism $\ell Q \to \ell eX \to X$. Since $\ell A = P$, we have $\ell Q \in \operatorname{add} P$ and thus $X \in \operatorname{gen}(P)$. Conversely, $\operatorname{gen}(P) \subseteq \ker q$ since $qP = q\ell A = 0$ and q preserves epimorphisms. Using instead [11, Lem/Def. 2.3], one similarly proves that $\ker p = \operatorname{cogen}(I)$. Finally, the equality $\operatorname{im} c = \ker p \cap \ker q$ is the first statement of [12, Prop 4.11]. □

Now let (A, E, Γ) be a Morita–Tachikawa triple, with Π the maximal projective-injective summand of Γ. Recall from the Morita–Tachikawa correspondence that $A \cong \operatorname{End}_\Gamma(\Pi)^{\mathrm{op}}$. If T is any tilting (or cotilting) Γ-module, we must have $\Pi \in \operatorname{add} T$. It follows that there is an idempotent $e \in B = \operatorname{End}_\Gamma(T)^{\mathrm{op}}$, given by projection onto the summand Π of T, such that

$$eBe = \operatorname{End}_\Gamma(\Pi)^{\mathrm{op}} \cong A.$$

Thus we get a recollement as in (4.1). In particular, this holds for the shifted and coshifted algebras B_k and B^k of Γ. In this section, we explain how these different recollements are related, for different values of k, and give an explicit formula for the intermediate extension functor in each case.

4.2. Recollements for shifted and coshifted algebras.

We first introduce some notation for our preferred idempotents. Let Γ be a finite-dimensional algebra with dominant dimension d and maximal projective-injective summand Π, and let $0 \le k \le d$. We denote by e_k the idempotent of the k-th shifted algebra B_k of Γ given by projection onto $\Pi \in \operatorname{add} T_k$, and by e^k the idempotent of the k-th coshifted algebra B^k given by projection onto Π.

Remark 4.2. The reader is warned that while we have natural isomorphisms $B_0 \cong \Gamma \cong B^0$, the idempotents e_0 and e^0 are typically not equal. Rather, e_0 is the idempotent indicated by the top of Π, and e^0 that indicated by the socle, so that $\Gamma e_0 \cong \Pi \cong \mathrm{D}(e^0\Gamma)$.

The algebras $e_kB_ke_k$ and $e^kB^ke^k$ are all isomorphic to $A := \operatorname{End}_\Gamma(\Pi)^{\mathrm{op}}$, so A-mod appears on the right-hand side of all of our recollements. In the case of the quotient algebras $B_k/B_ke_kB_k$ and $B^k/B^ke^kB^k$ appearing on the other side of the recollements, we have the following.

Lemma 4.3. *For all $0 \le k \le d$ we have isomorphisms*

$$B_k/B_ke_kB_k \cong \Gamma/\Gamma e_0\Gamma,$$
$$B^k/B^ke^kB^k \cong \Gamma/\Gamma e^0\Gamma,$$

induced by taking syzygies and cosyzygies.

Proof. The idempotents e_k are chosen such that there is an isomorphism

$$B_k/B_ke_kB_k \cong \operatorname{End}_{\Gamma\text{-mod}/\operatorname{add}\Pi}(T_k).$$

Moreover, since Π is projective-injective, [5, Thm 5.2] provides mutually inverse equivalences

$$\Omega\colon \operatorname{gen}(\Pi)/\operatorname{add}\Pi \overset{\sim}{\longleftrightarrow} \operatorname{cogen}(\Pi)/\operatorname{add}\Pi\colon \Omega^-,$$

where $\Omega(X)$ is the kernel of a minimal projective cover of X, and $\Omega^-(Y)$ is the cokernel of a minimal injective hull of Y; when $X \in \operatorname{gen}(\Pi)$, a minimal projective cover coincides with a minimal left $\operatorname{add}\Pi$-approximation as referred to in [5, Thm. 5.2], and the corresponding statement holds for $Y \in \operatorname{cogen}(\Pi)$.

We now prove the first set of isomorphisms, involving the shifted algebras, by induction on k, noting that when $k = 0$ there is nothing to prove. Let $1 \le k \le d$. Then, by construction, T_{k-1} lies in $\operatorname{cogen}(\Pi)$ and $\Omega^-(T_{k-1})$ agrees with T_k up to a projective-injective summand, i.e. an object of $\operatorname{add}\Pi$. We therefore obtain isomorphisms

$$\begin{aligned} B_k/B_ke_kB_k &\cong \operatorname{End}_{\Gamma\text{-mod}/\operatorname{add}\Pi}(T_k) \\ &\cong \operatorname{End}_{\Gamma\text{-mod}/\operatorname{add}\Pi}(\Omega(T_k)) \\ &\cong \operatorname{End}_{\Gamma\text{-mod}/\operatorname{add}\Pi}(T_{k-1}) \cong \Gamma/\Gamma e_0\Gamma, \end{aligned}$$

the last by the induction hypothesis. The second statement is proved similarly, using that $\Omega^-(C^k) = C^{k-1}$ in $\operatorname{cogen}(\Pi)/\operatorname{add}\Pi$ for $1 \le k \le d$. □

It follows from Lemma 4.3 that the families of shifted and coshifted modules each provide a family of recollements, such that the left-hand side of the recollement is constant in each family, and the right-hand side is constant across both families. More precisely, for each $0 \leq k \leq d = \operatorname{domdim}\Gamma$, we get a pair of recollements as follows.

$$\begin{tikzcd}
\Gamma/\Gamma e_0\Gamma\text{-mod} \arrow[r, "i_k" description] & B_k\text{-mod} \arrow[l, shift right=3, "q_k"'] \arrow[l, shift left=3, "p_k"] \arrow[dr, "e_k" description] & \\
& & A\text{-mod} \arrow[ul, shift right=3, "\ell_k"'] \arrow[ul, shift left=3, "r^k"] \arrow[dl, shift right=3, "\ell_k"'] \arrow[dl, shift left=3, "r^k"] \\
\Gamma/\Gamma e^0\Gamma\text{-mod} \arrow[r, "i^k" description] & B^k\text{-mod} \arrow[l, shift right=3, "q^k"'] \arrow[l, shift left=3, "p^k"] \arrow[ur, "e^k" description] &
\end{tikzcd} \tag{4.2}$$

4.3. **Homotopy categories.** We now turn to the problem of computing the intermediate extension functor in each recollement from (4.2). To do this, it will be useful to give a new description of the shifted and coshifted algebras as endomorphism algebras in the bounded homotopy category of A-modules, rather than in the category of Γ-modules, generalising a result of Crawley-Boevey and the second author in the case that Γ is an Auslander algebra.

We begin with the following very general considerations. Let A be a finite-dimensional algebra, $E \in A$-mod, and $\Gamma = \operatorname{End}_A(E)^{\mathrm{op}}$. The bounded homotopy categories $\mathcal{K}^{\mathrm{b}}(\Gamma\text{-proj})$ and $\mathcal{K}^{\mathrm{b}}(\Gamma\text{-inj})$ of complexes of projective and injective Γ modules respectively admit tautological functors to $\mathcal{D}^{\mathrm{b}}(\Gamma)$, equivalences onto their images, which we treat as identifications. These subcategories may be characterised intrinsically as the full subcategories of $\mathcal{D}^{\mathrm{b}}(\Gamma)$ on the compact and cocompact objects (in the context of additive categories) respectively. Extending the Yoneda equivalences

$$\operatorname{Hom}_A(E, -)\colon \operatorname{add} E \xrightarrow{\sim} \Gamma\text{-proj},$$
$$\operatorname{D}\operatorname{Hom}_A(-, E)\colon \operatorname{add} E \xrightarrow{\sim} \Gamma\text{-inj}$$

to complexes, one sees that both of these subcategories of $\mathcal{D}^{\mathrm{b}}(\Gamma)$ are equivalent to the full subcategory $\operatorname{thick}(E)$ of $\mathcal{K}^{\mathrm{b}}(A)$, i.e. the smallest triangulated subcategory of the homotopy category $\mathcal{K}^{\mathrm{b}}(A)$ closed under direct summands and containing (the stalk complex) E.

Now let $F\colon \mathcal{T} \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\Gamma)$ be any equivalence of triangulated categories. It follows from the intrinsic description of $\mathcal{K}^{\mathrm{b}}(\Gamma\text{-proj})$ and $\mathcal{K}^{\mathrm{b}}(\Gamma\text{-inj})$ above that F induces respective equivalences from the subcategories of compact and cocompact objects of $\mathcal{T}$ to these subcategories of $\mathcal{D}^{\mathrm{b}}(\Gamma)$, and thus allows us to realise $\operatorname{thick} E$ as a full subcategory of $\mathcal{T}$ (in two ways). This holds in particular when $\mathcal{T} = \mathcal{D}^{\mathrm{b}}(B)$ for some algebra B derived equivalent to Γ, such as the endomorphism algebra of a tilting or cotilting Γ-module.

Whenever B is derived equivalent to Γ, it follows from Rickard's Morita theory for derived categories [25] that the image in $\mathcal{K}^{\mathrm{b}}(\Gamma\text{-proj})$ of the stalk complex $B \in \mathcal{K}^{\mathrm{b}}(B\text{-proj})$ is a tilting complex with endomorphism algebra B, inducing the derived equivalence. The preimage of this tilting complex under the Yoneda equivalence is an object of $\operatorname{thick} E \subseteq \mathcal{K}^{\mathrm{b}}(A)$, again with endomorphism algebra B. Similarly, the image of $\mathrm{D}B \in \mathcal{K}^{\mathrm{b}}(B\text{-inj})$ in $\mathcal{K}^{\mathrm{b}}(\Gamma\text{-inj})$ is a cotilting complex, and its preimage under the dual Yoneda equivalence is another object of $\operatorname{thick} E$ with endomorphism algebra B. Our conclusion is that when Γ is the endomorphism algebra of an A-module E (or more generally an object $E \in \mathcal{K}^{\mathrm{b}}(A)$), any algebra B derived equivalent to Γ must also appear as an endomorphism algebra in $\operatorname{thick} E \subseteq \mathcal{K}^{\mathrm{b}}(A)$. In general, B need not be an endomorphism algebra in A-mod.

When E is a generator-cogenerator and B is one of the shifted or coshifted algebras of Γ, we may compute the relevant objects of $\operatorname{thick} E$ explicitly, and obtain a particularly straightforward answer.

Proposition 4.4. *Let (A, E, Γ) be a Morita–Tachikawa triple with all objects basic, and let $0 \leq k \leq \operatorname{domdim}\Gamma$. We denote by B_k and B^k the k-th shifted and coshifted algebras of Γ respectively.*

(a) *Write*

$$E^k = (P_{k-1} \to \cdots \to P_0 \to E) \oplus A[k] \in \mathcal{K}^{\mathrm{b}}(A),$$

where the first summand denotes the complex whose non-zero part is given by the first k terms of a minimal projective resolution of E, with E in degree 0, and the second denotes the stalk complex with A in degree $-k$. Then

$$B^k \cong \operatorname{End}_{\mathcal{K}^{\mathrm{b}}(A)}(E^k)^{\mathrm{op}},$$

with the idempotent $e \in B^k$ given by projection onto Π corresponding under this isomorphism to projection onto the summand $A[k]$.

(b) *Write*

$$E_k = (E \to Q_0 \to \cdots \to Q_{k-1}) \oplus \mathrm{D}A[-k],$$

where the first summand denotes the complex whose non-zero part is given by the first k terms of a minimal injective resolution of E, with E in degree 0, and the second denotes the stalk complex with $\mathrm{D}A$ in degree k. Then

$$B_k \cong \operatorname{End}_{\mathcal{K}^{\mathrm{b}}(A)}(E_k)^{\mathrm{op}},$$

with the idempotent $e \in B_k$ given by projection onto Π corresponding under this isomorphism to projection onto the summand $\mathrm{D}A[-k]$.

Proof. As usual, we only prove (a), since (b) is dual. By definition, B^k is the endomorphism algebra of the k-cotilting Γ-module C^k, so that the image of $\mathrm{D}B^k$ in $\mathcal{K}^{\mathrm{b}}(\Gamma\text{-inj})$ is given by an injective resolution of C^k. By construction, there is such an injective resolution of the form

$$0 \longrightarrow C^k \longrightarrow \Pi^{k-1} \oplus \Pi \longrightarrow \Pi^{k-2} \longrightarrow \cdots \longrightarrow \Pi^0 \longrightarrow \mathrm{D}\Gamma \longrightarrow 0, \tag{4.3}$$

where

$$\Pi^{k-1} \longrightarrow \Pi^{k-2} \longrightarrow \cdots \longrightarrow \Pi^0 \longrightarrow \mathrm{D}\Gamma \longrightarrow 0,$$

begins a minimal projective resolution of $\mathrm{D}\Gamma$, and Π is as usual the maximal projective-injective summand of Γ. Recall that $\Pi = \mathrm{D}E = \mathrm{D}\operatorname{Hom}_A(A, E)$, and $\Pi_i \in \operatorname{add}\Pi$. Thus the representative of $C^k \in \mathcal{K}^{\mathrm{b}}(\Gamma\text{-inj})$ given by the resolution (4.3) has as preimage under the (dual) Yoneda equivalence $\mathrm{D}\operatorname{Hom}_A(-, E)$ the complex E_k (up to a degree shift), and the desired isomorphism follows. The claimed relationship between idempotents follows since the summand Π of C^k contributes the summand consisting of the stalk complex Π to its representative in $\mathcal{K}^{\mathrm{b}}(\Gamma\text{-inj})$, and this summand has preimage given by the stalk complex A (in the correct degree). □

Remark 4.5. The assumptions of minimality of the projective and injective resolutions in Proposition 4.4 are necessary since B^k and B_k are, by construction, basic algebras. However, one can remove these assumptions from the statement at the cost of replacing the isomorphisms by Morita equivalences.

Remark 4.6. When Γ is an Auslander algebra, so A is representation-finite and $\operatorname{add} E = A\text{-mod}$, the category $\operatorname{add} E^1$ is equivalent to the category $\mathcal{H}$ from [11, §3], and so Proposition 4.4(a) recovers [11, Prop. 5.5] in this case.

Recall that e_k and e^k denote the idempotents of B_k and B^k given by projection onto Π. As a consequence of Theorem 4.4, we may identify the corresponding recollements with

$$\begin{tikzcd}[column sep=huge] (\operatorname{add} E_k/\mathrm{D}A[-k])\text{-mod} \arrow[r, "i_k" description] & (\operatorname{add} E_k)\text{-mod} \arrow[l, bend right, "q_k"'] \arrow[l, bend left, "p_k"] \arrow[r, "e_k" description] & A\text{-mod} \arrow[l, bend right, "\ell_k"'] \arrow[l, bend left, "r_k"] \end{tikzcd}$$

in which e_k is given by restriction of functors from $\operatorname{add} E_k$ to $\operatorname{add}\mathrm{D}A[-k]$, and

$$\begin{tikzcd}[column sep=huge] (\operatorname{add} E^k/A[k])\text{-mod} \arrow[r, "i^k" description] & (\operatorname{add} E^k)\text{-mod} \arrow[l, bend right, "q^k"'] \arrow[l, bend left, "p^k"] \arrow[r, "e^k" description] & A\text{-mod} \arrow[l, bend right, "\ell^k"'] \arrow[l, bend left, "r^k"] \end{tikzcd}$$

in which e^k is given by restriction of functors from $\operatorname{add} E^k$ to $\operatorname{add} A[k]$, and e_k by restriction from $\operatorname{add} E_k$ to $\operatorname{add}\mathrm{D}A[-k]$. Note that both $A[k]$ and $\mathrm{D}A[-k]$ have endomorphism algebra A in $\mathcal{K}^{\mathrm{b}}(A)$.

4.4. **Intermediate extensions.** We now describe ℓ^k, r^k and c^k in the preceding recollement, and state the dual results for ℓ_k, r_k and c_k. By using the identification of B^k-mod with $(\operatorname{add} E^k)$-mod, we are able to give a particularly clean formula for c^k.

Lemma 4.7. *For $X = (X_k \xrightarrow{f} X_{k-1} \to \cdots \to X_0)$ in $\operatorname{add} E^k$ and $M \in A$-mod, we have*

$$\begin{aligned} \ell^k(M)(X) &= \operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, A[k]) \otimes_A M, \\ r^k(M)(X) &= \operatorname{Hom}_A(\ker f, M) \\ &= \operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(A)}(X, M[k]), \end{aligned}$$

where $\mathcal{D}^{\mathrm{b}}(A)$ is the bounded derived category of A.

Proof. Since the functor e^k is given by restriction to the subcategory $\operatorname{add} A[k]$ of $\operatorname{add} E^k$, we may use the general form of adjoints to this restriction (see [11, Lem. 2.6] and the discussion preceding this lemma) to see that

$$\ell^k(M)(X) = \operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, -[k]) \otimes_{A\text{-proj}} M$$

and

$$r^k(M)(X) = \operatorname{Hom}_{(A\text{-proj})\text{-mod}}(\operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(-[k], X), M),$$

where we abuse notation somewhat, and use M to denote both an A-module and the equivalent data of a functor in $(A$-proj$)$-mod.

Converting the functors M, $\operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, -[k])$ and $\operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(-[k], X)$ on the right hand side of these expressions into more traditional A-modules by evaluating on A, we see in the first case that

$$\ell^k(M)(X) = \operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, A[k]) \otimes_A M,$$

as claimed. In the second case we may compute $\operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(A[k], X) = \ker f$, and so

$$r^k(M)(X) = \operatorname{Hom}_A(\ker f, M)$$

as required. Since $X \in \operatorname{add} E^k$, it follows from the definition of this object that $X \cong \ker(f)[k]$ in the bounded derived category $\mathcal{D}^{\mathrm{b}}(A)$, and so we may also compute that $\operatorname{Hom}_A(\ker f, M) = \operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(A)}(X, M[k])$ as claimed. □

Proposition 4.8. *In the notation of Lemma 4.7, if $k \geq 2$ then*

$$\ell^k(M)(X) = \operatorname{coker}(\operatorname{Hom}_A(X_{k-1}, M) \xrightarrow{f^*} \operatorname{Hom}_A(X_k, M)) = \operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, M[k]).$$

Proof. Let $P_1 \to P_0 \to M \to 0$ be a projective presentation of M, from which we obtain the exact sequence

$$\operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, A[k]) \otimes_A P_1 \longrightarrow \operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, A[k]) \otimes_A P_0 \longrightarrow \operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, A) \otimes_A M \longrightarrow 0.$$

We have $\operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, A) \otimes_A M = \ell^k(M)(X)$ by Lemma 4.7, and there are natural isomorphisms $\operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, A[k]) \otimes_A P_i \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, P_i[k])$ since the P_i are projective, and the right A-module structure on $\operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, A[k])$ comes from the identification $A \cong \operatorname{End}_{\mathcal{K}^{\mathrm{b}}(A)}(A[k])^{\mathrm{op}}$. Thus $\ell^k(M)(X)$ may be identified with the cokernel of the map $\operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, P_1[k]) \to \operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, P_0[k])$.

For any $N \in A$-mod, we may compute $\operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, N[k])$ via the exact sequence

$$\operatorname{Hom}_A(X_{k-1}, N) \longrightarrow \operatorname{Hom}_A(X_k, N) \longrightarrow \operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, N[k]) \longrightarrow 0.$$

From this observation and our projective presentation of M, we may construct the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_A(X_{k-1},P_1) \arrow[r] \arrow[d] & \operatorname{Hom}_A(X_{k-1},P_0) \arrow[r] \arrow[d] & \operatorname{Hom}_A(X_{k-1},M) \arrow[r] \arrow[d] & 0 \\
\operatorname{Hom}_A(X_k,P_1) \arrow[r] \arrow[d] & \operatorname{Hom}_A(X_k,P_0) \arrow[r] \arrow[d] & \operatorname{Hom}_A(X_k,M) \arrow[r] \arrow[d] & 0 \\
\operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X,P_1[k]) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X,P_0[k]) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X,M[k]) \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

with exact columns. The second row is exact since X_k is projective, by the definition of E^k. Moreover, X_{k-1} is also projective by the assumption that $k \geq 2$, so the first row is also exact. Now a variant of the snake lemma implies that the third row is exact, and so $\ell^k(M)(X) = \operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, M[k])$ as claimed. ☐

Theorem 4.9. *Keeping the notation of Lemma 4.7, the intermediate extension* $c^k(M)$ *is given by*

$$\begin{aligned} c^k(M)(X) &= \operatorname{im}(\operatorname{Hom}_A(X_k, M) \to \operatorname{Hom}_A(\ker f, M)) \\ &= \operatorname{coker}(\operatorname{Hom}_A(\operatorname{im} f, M) \to \operatorname{Hom}_A(X_k, M)) \\ &= \ker(\operatorname{Hom}_A(\ker f, M) \to \operatorname{Ext}^1_A(\operatorname{im} f, M)). \end{aligned}$$

Proof. Applying $\operatorname{Hom}_A(-, M)$ to the exact sequence $0 \to \ker f \to X_k \to \operatorname{im} f \to 0$ gives an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(\operatorname{im} f, M) \longrightarrow \operatorname{Hom}_A(X_k, M) \longrightarrow \operatorname{Hom}_A(\ker f, M) \longrightarrow \operatorname{Ext}^1_A(\operatorname{im} f, M) \longrightarrow 0,$$

from which we obtain canonical isomorphisms between the three spaces on the right hand side of the statement. Denote by f^* the map $\operatorname{Hom}_A(X_{k-1}, M) \to \operatorname{Hom}_A(X_k, M)$ induced by f.

By [11, Lem. 4.2], $c^1(M)(X) = \operatorname{coker}(\operatorname{Hom}_A(X_0, M) \to \operatorname{Hom}_A(X_1, M))$, and $\operatorname{im} f = X_0$ in this case by the definition of E^1, giving the desired result. Assume now that $k \geq 2$, so that $\ell^k(M)(X) = \operatorname{coker} f^*$ by Proposition 4.8.

The map $f^* \colon \operatorname{Hom}_A(X_{k-1}, M) \to \operatorname{Hom}_A(X_k, M)$ factors through the inclusion $\operatorname{Hom}_A(\operatorname{im} f, M) \to \operatorname{Hom}_A(X_k, M)$. Therefore the canonical map $\ell^k \to r^k$ factors as

$$\ell^k(M)(X) = \operatorname{coker} f^* \twoheadrightarrow \operatorname{Hom}_A(X_k, M)/\operatorname{Hom}_A(\operatorname{im} f, M) \hookrightarrow \operatorname{Hom}_A(\ker f, M) = r^k(M)(X),$$

and the image is given by $\operatorname{im}(\operatorname{Hom}_A(X_k, M) \to \operatorname{Hom}_A(\ker f, M))$, as required. ☐

The corresponding dual results, for ℓ_k, r_k and c_k, are as follows.

Lemma 4.10. *For* $Y = (Y_0 \to \cdots \to Y_{k-1} \xrightarrow{g} Y_k)$ *in* $\operatorname{add} E_k$ *and* $M \in A\text{-mod}$, *we have*

$$\begin{aligned} \ell_k(M)(Y) &= \mathrm{D}\operatorname{Hom}_A(M, \operatorname{coker} g) \\ &= \mathrm{D}\operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(A)}(M[-k], Y), \\ r_k(M)(Y) &= \operatorname{Hom}_A(\operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(\mathrm{D}A[-k], Y), M). \end{aligned}$$

Proposition 4.11. *In the notation of Lemma 4.10, if* $k \geq 2$ *then*

$$r_k(M)(X) = \ker(\mathrm{D}\operatorname{Hom}_A(M, Y_k) \xrightarrow{\mathrm{D}g_*} \mathrm{D}\operatorname{Hom}_A(M, Y_{k-1})) = \mathrm{D}\operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(M[-k], Y).$$

Theorem 4.12. *Keeping the notation of Lemma 4.10, the intermediate extension* $c_k(M)$ *is given by*

$$\begin{aligned} c_k(M)(X) &= \operatorname{im}(\mathrm{D}\operatorname{Hom}_A(M, \operatorname{coker} g) \to \mathrm{D}\operatorname{Hom}_A(M, Y_k)) \\ &= \ker(\mathrm{D}\operatorname{Hom}_A(M, Y_k) \to \mathrm{D}\operatorname{Hom}_A(M, \operatorname{im} g)) \\ &= \operatorname{coker}(\mathrm{D}\operatorname{Ext}^1_A(M, \operatorname{im} g) \to \mathrm{D}\operatorname{Hom}_A(M, \operatorname{coker} g)). \end{aligned}$$

Using the descriptions

$$\begin{aligned}\ell^k(M)(X) &= \operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, M[k]),\\ r^k(M)(X) &= \operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(A)}(X, M[k])\end{aligned}$$

of ℓ^k and r^k when $k \geq 2$, we see that the canonical map $\ell^k \to r^k$ agrees with that coming from the Verdier localisation functor $\mathcal{K}^{\mathrm{b}}(A) \to \mathcal{D}^{\mathrm{b}}(A)$. Indeed, the isomorphism of $\operatorname{Hom}_{\mathcal{K}^{\mathrm{b}}(A)}(X, M[k])$ with $\operatorname{Hom}_A(X_k, M)/\operatorname{im} f^*$ identifies the set of maps factoring through an acyclic complex, which is the kernel of the Verdier localisation functor, with $\operatorname{Hom}_A(\operatorname{im} f, M)/\operatorname{im} f^*$. In the dual case, the canonical map $\ell_k \to r_k$ agrees with the dual of that from Verdier localisation.

5. Subcategories associated to shifted modules

In the context of tilting theory, and of recollements, it is natural to consider various subcategories of the relevant module categories. In this section we give alternative descriptions of some of these subcategories associated to shifted and coshifted modules, using the highly explicit construction of these modules.

We start by considering a finite-dimensional algebra Γ with idempotent e, yielding the recollement

$$\begin{tikzcd}[column sep=large] \Gamma/\Gamma e\Gamma\text{-mod} \arrow[r, "i" description] & \Gamma\text{-mod} \arrow[l, bend right, "q"'] \arrow[l, bend left, "p"] \arrow[r, "e" description] & A\text{-mod} \arrow[l, bend right, "\ell"'] \arrow[l, bend left, "r"] \end{tikzcd} \tag{5.1}$$

analogous to (4.1), in which $A = e\Gamma e$. We also use this idempotent to fix a projective module $P = \Gamma e$ and an injective module $I = \mathrm{D}(e\Gamma)$.

5.1. k-idempotents and isomorphisms on Ext-groups.

Since i is an exact functor, we have induced linear maps

$$\operatorname{Ext}^j_{\Gamma/\Gamma e\Gamma}(X, Y) \to \operatorname{Ext}^j_{\Gamma}(i(X), i(Y))$$

for all $X, Y \in \Gamma/\Gamma e\Gamma$-mod and $j \geq 0$. We recall the following definition and results of Auslander–Platzek–Todorov [3], indicating that the categories $\operatorname{gen}_k(X)$ and $\operatorname{cogen}^k(X)$ from Definition 2.5 play an important role in our discussion.

Definition 5.1. Let $0 \leq k \leq \infty$. The idempotent e is called a $(k+1)$-*idempotent* if the maps $\operatorname{Ext}^j_{\Gamma/\Gamma e\Gamma}(X, Y) \to \operatorname{Ext}^j_{\Gamma}(i(X), i(Y))$ are isomorphisms for all $X, Y \in \Gamma/\Gamma e\Gamma$-mod and $0 \leq j \leq k+1$.

Theorem 5.2 ([3, Thm. 2.1']). *The idempotent e is a $(k+1)$-idempotent if and only if $\Gamma e\Gamma \in \operatorname{gen}_k(P)$.*

The proof of this theorem involves the following characterisations of $\operatorname{gen}_k(P)$ and $\operatorname{cogen}^k(I)$.

Proposition 5.3 ([3, Prop. 2.4, Prop. 2.6]). *Let $1 \leq k \leq \infty$. Then*

$$\operatorname{gen}_k(P) = \bigcap_{j=0}^{k} \ker \operatorname{Ext}^j_{\Gamma}(-, i(\mathrm{D}\Gamma/\Gamma e\Gamma)) \quad \textit{and} \quad \operatorname{cogen}^k(I) = \bigcap_{j=0}^{k} \ker \operatorname{Ext}^j_{\Gamma}(i(\Gamma/\Gamma e\Gamma), -).$$

Theorem 5.4 ([3, Lem. 3.1, Thm. 3.2]). *Let $X \in \operatorname{gen}_k(P)$ and $Y \in \operatorname{cogen}^\ell(I)$ for some $k, \ell \geq -1$. Then for every $j \leq k + \ell$, the natural map*

$$\rho^j_{X,Y} \colon \operatorname{Ext}^j_{\Gamma}(X, Y) \to \operatorname{Ext}^j_A(eX, eY)$$

is an isomorphism. Furthermore, if $X \in \operatorname{gen}(P)$ or $Y \in \operatorname{cogen}(I)$, then $\rho^0_{X,Y}$ is a monomorphism.

We have already shown, via Lemma 4.1, that $e\colon \operatorname{gen}_1(P) = \operatorname{im}\ell \to A$-mod and $e\colon \operatorname{cogen}^1(I) = \operatorname{im} r \to A$-mod are equivalences. The following result describes $e(\operatorname{gen}_k(P))$ and $e(\operatorname{cogen}^k(I))$ for higher values of k, in terms of the A-modules $E = e\Gamma = \mathrm{D}I$ and $\mathcal{E} = \mathrm{D}(\Gamma e) = \mathrm{D}P$.

Proposition 5.5 ([3, Prop. 3.7]). *For $k \geq 1$, we have*

$$e(\operatorname{gen}_k(P)) = \bigcap_{j=1}^{k-1} \ker \operatorname{Ext}^j_A(-, \mathcal{E}) \quad \textit{and} \quad e(\operatorname{cogen}^k(I)) = \bigcap_{j=1}^{k-1} \ker \operatorname{Ext}^j_A(E, -).$$

5.2. **Equivalences of subcategories induced by tilting and cotilting modules.** Now let $T \in \Gamma\text{-mod}$ be k-tilting. We set $B := \operatorname{End}_\Gamma(T)^{\mathrm{op}}$ and note that $\mathrm{D}T$ is a k-cotilting left B-module. Consider the functors

$$\Phi := \operatorname{Hom}_\Gamma(T,-),\ \Psi := \mathrm{D}\operatorname{Hom}_\Gamma(-,T)\colon \Gamma\text{-mod} \to B\text{-mod}.$$

Note that Φ is right adjoint to $\Phi' = \mathrm{D}\operatorname{Hom}_B(-,\mathrm{D}T) = T\otimes_B -$, and Ψ, which can also be written as $\mathrm{D}T\otimes_\Gamma -$, is left adjoint to $\Psi' = \operatorname{Hom}_B(\mathrm{D}T,-)$. Moreover, we may compute

$$\begin{aligned} \Phi(\mathrm{D}\Gamma) &= \mathrm{D}T, & \qquad \Psi(\Gamma) &= \mathrm{D}T,\\ \Phi(T) &= B, & \qquad \Psi(T) &= \mathrm{D}B. \end{aligned}$$

We also define subcategories

$$\mathcal{T}_i(T) := \bigcap_{j\neq i} \ker \operatorname{Ext}^j_\Gamma(T,-) \quad\text{and}\quad \mathcal{C}_i(\mathrm{D}T) := \bigcap_{j\neq i} \ker \operatorname{Ext}^j_B(-,\mathrm{D}T),$$

which we refer to collectively as the *tilting subcategories* associated to T. It is immediate from the definition that $\mathcal{T}_i(T) = \mathcal{C}_i(\mathrm{D}T) = 0$ if $i > k$. The following result is due to Miyashita.

Theorem 5.6 ([20, Thm. 1.16])**.** *For $0 \leq i \leq k$, the functor $\operatorname{Ext}^i_\Gamma(T,-)\colon \mathcal{T}_i(T) \to \mathcal{C}_i(\mathrm{D}T)$ is an equivalence of categories with quasi-inverse $\mathrm{D}\operatorname{Ext}^i_B(-,\mathrm{D}T)\colon \mathcal{C}_i(\mathrm{D}T) \to \mathcal{T}_i(T)$.*

When T is a classical tilting module, i.e. when $k = 1$, we obtain a torsion pair $(\mathcal{T}_0(T), \mathcal{T}_1(T))$ (cf. [6, Ch. 1]) in Γ-mod, where

$$\mathcal{T}_0(T) = \operatorname{gen}(T) = \ker \operatorname{Ext}^1_\Gamma(T,-)$$

is the torsion class, and

$$\mathcal{T}_1(T) = \ker \operatorname{Hom}_\Gamma(T,-)$$

is the torsion-free class. Similarly, $(\mathcal{C}_1(DT), \mathcal{C}_0(DT))$ is a torsion pair in B-mod, where

$$\mathcal{C}_1(\mathrm{D}T) = \ker \operatorname{Hom}_B(-,\mathrm{D}T)$$

is the torsion class, and

$$\mathcal{C}_0(\mathrm{D}T) = \operatorname{cogen}(\mathrm{D}T) = \ker \operatorname{Ext}^1_B(-,\mathrm{D}T)$$

is the torsion-free class. The torsion class in each torsion pair is equivalent to the torsion-free class in the other, with equivalences given by $\operatorname{Hom}_\Gamma(T,-)\colon \mathcal{T}_0(T) \to \mathcal{C}_0(\mathrm{D}T)$, with quasi-inverse $\mathrm{D}\operatorname{Hom}_B(-,DT)$, and $\operatorname{Ext}^1_\Gamma(T,-)\colon \mathcal{T}_1(T) \to \mathcal{C}_1(\mathrm{D}T)$, with quasi-inverse $\mathrm{D}\operatorname{Ext}^1_B(-,\mathrm{D}T)$. These are the four equivalences from the Brenner–Butler tilting theorem [7] (see also [1, §VI.3]).

For arbitrary k, we observe that $T, \mathrm{D}\Gamma \in \mathcal{T}_0(T)$, $\mathrm{D}T, B \in \mathcal{C}_0(\mathrm{D}T)$, and there are inclusions

$$\begin{aligned} \Omega^{-k}(\Gamma\text{-mod}) &\subseteq \mathcal{T}_0(T),\\ \Omega^{k}(B\text{-mod}) &\subseteq \mathcal{C}_0(\mathrm{D}T). \end{aligned}$$

5.3. **The four torsion pairs for 1-(co)shifted modules and their duals.** We now focus on the special case of k-shifted and k-coshifted modules, so assume $\operatorname{domdim}\Gamma = d > 0$. As usual, denote by Π a maximal projective-injective summand of Γ.

Since $\operatorname{pd} T_1 \leq 1$ and $\operatorname{id} C^1 \leq 1$, we obtain torsion pairs $(\mathcal{T}_0(T_1), \mathcal{T}_1(T_1))$ and $(\mathcal{C}_1(C^1), \mathcal{C}_0(C^1))$ in Γ-mod, and their Brenner–Butler equivalent counterparts $(\mathcal{C}_1(\mathrm{D}T_1), \mathcal{C}_0(\mathrm{D}T_1))$ and $(\mathcal{T}_0(\mathrm{D}C^1), \mathcal{T}_1(\mathrm{D}C^1))$ in in B_1-mod and B^1-mod respectively.

For higher values of k, since Π is a summand of Γ, $\mathrm{D}\Gamma$ and T_k, we see that $I_k := \mathrm{D}\operatorname{Hom}_\Gamma(\Pi, T_k)$ is both an injective B_k-module and a summand of $\mathrm{D}T_k$, and that $P_k := \operatorname{Hom}_\Gamma(T_k, \Pi)$ is both a projective B_k-module and a summand of $\mathrm{D}T_k$. Similarly, $I^k := \mathrm{D}\operatorname{Hom}_\Gamma(\Pi, C^k)$ is an injective B^k-module and $P^k := \operatorname{Hom}_\Gamma(C^k, \Pi)$ a projective B^k-module, and both are summands of $\mathrm{D}C^k$.

Fix k, and let $\Phi = \operatorname{Hom}_\Gamma(T_k,-)$ and $\Psi = \mathrm{D}\operatorname{Hom}_\Gamma(-,T_k)$ be the tilting functors associated to T_k in Section 5.2. Then by definition and the Nakayama formula, we have

$$\begin{aligned} \Phi(\Pi) &= P_k, & \qquad \Psi(\Pi) &= I_k,\\ \Phi(\nu\Pi) &= I_k, & \qquad \Psi(\nu^-\Pi) &= P_k. \end{aligned}$$

Similar identities, involving P^k and I^k, hold for the corresponding functors associated to C^k.

Recall from Remark 4.2 that in this context we have two preferred idempotents e_0 and e^0 of Γ, depending on whether we view Γ as a the 0-shifted or 0-coshifted algebra, defined by

$$\Pi = \Gamma e_0 = \mathrm{D}(e^0\Gamma).$$

We also have preferred idempotents e_k of B_k and e^k of B^k given by projection onto the summand Π of T_k and C^k respectively; this means that $P_k = B_k e_k$, $I_k = \mathrm{D}(e_k B_k) = \nu P_k$, $P^k = B^k e^k$ and $I^k = \mathrm{D}(e^k B^k) = \nu P^k$. In particular, $P_0 = \Pi = I^0$, and hence $I_0 = \nu\Pi$ and $P^0 = \nu^-\Pi$. The rest of Section 5 is concerned with applying the preceding general results on idempotent recollements and (co)tilting modules to the case of the (co)shifted modules and algebras, with these preferred idempotents.

We begin by considering the TTF-triples of the recollements. Recall from Lemma 4.1 that

$$\begin{aligned}\mathrm{TTF}(e_k) &= (\mathrm{gen}(P_k), \ker\mathrm{Hom}_{B_k}(P_k,-), \mathrm{cogen}(I_k)),\\ \mathrm{TTF}(e^k) &= (\mathrm{gen}(P^k), \ker\mathrm{Hom}_{B^k}(P^k,-), \mathrm{cogen}(I^k)).\end{aligned}$$

By the Nakayama formula, we may also write

$$\begin{aligned}\ker\mathrm{Hom}_{B_k}(P_k,-) &= \ker\mathrm{Hom}_{B_k}(-,I_k),\\ \ker\mathrm{Hom}_{B^k}(P^k,-) &= \ker\mathrm{Hom}_{B^k}(-,I^k).\end{aligned}$$

We see that Φ (being left exact) maps the second torsion pair in $\mathrm{TTF}(e_0)$ to the second torsion pair in $\mathrm{TTF}(e_k)$, that is

$$\Phi(\ker\mathrm{Hom}_\Gamma(\Pi,-)) \subseteq \ker\mathrm{Hom}_{B_k}(P_k,-), \quad \Phi(\mathrm{cogen}(\nu\Pi)) \subseteq \mathrm{cogen}(I_k).$$

We also see that Ψ (being right exact) maps the first torsion pair in $\mathrm{TTF}(e_0)$ to the first torsion pair in $\mathrm{TTF}(e_k)$, that is

$$\Psi(\mathrm{gen}(\nu^-\Pi)) \subseteq \mathrm{gen}(P_k), \quad \Psi(\ker\mathrm{Hom}_\Gamma(-,\Pi)) \subseteq \ker\mathrm{Hom}_{B_k}(-,I_k).$$

We may as usual obtain similar dual results involving $\mathrm{TTF}(e^k)$.

For T_1 and C^1, the tilting subcategories from Section 5.2 have the following descriptions, which, at least for the subcategories of Γ-mod, may also be expressed in terms of higher shifted or coshifted modules.

Lemma 5.7. *For $1 \leq k \leq d$ one has*

$$\begin{aligned}\mathcal{T}_0(T_1) &= \mathrm{gen}(\Pi) = \mathrm{gen}(T_k), & \mathcal{C}_1(\mathrm{D}T_1) &= \ker\mathrm{Hom}_{B_1}(-,I_1),\\ \mathcal{T}_1(T_1) &= \ker\mathrm{Hom}_\Gamma(\Pi,-) = \ker\mathrm{Hom}_\Gamma(T_k,-), & \mathcal{C}_0(\mathrm{D}T_1) &= \mathrm{cogen}(I_1),\end{aligned}$$

$$\begin{aligned}\mathcal{C}_1(C^1) &= \ker\mathrm{Hom}_\Gamma(-,\Pi) = \ker\mathrm{Hom}_\Gamma(-,C^k), & \mathcal{T}_0(\mathrm{D}C^1) &= \mathrm{gen}(P^1),\\ \mathcal{C}_0(C^1) &= \mathrm{cogen}(\Pi) = \mathrm{cogen}(C^k), & \mathcal{T}_1(\mathrm{D}C^1) &= \ker\mathrm{Hom}_{B^1}(P^1,-).\end{aligned}$$

Proof. We give the proof only for the first torsion pair, the other three cases being similar. First we describe $\mathcal{T}_0(T_1)$. By construction, Π is a summand of $T_k \in \mathrm{gen}(\Pi)$, so we have $\mathrm{gen}(\Pi) = \mathrm{gen}(T_k)$ for all k. Since T_1 is 1-tilting, we also have $\mathcal{T}_0(T_1) = \mathrm{gen}(T_1)$, and our claimed equalities follow.

Just as for the first pair of equalities, since T_1 is 1-tilting, we have $\mathcal{T}_1(T_1) = \ker\mathrm{Hom}_\Gamma(T_1,-)$, and it is only necessary to show that $\ker\mathrm{Hom}_\Gamma(\Pi,-) = \ker\mathrm{Hom}_\Gamma(T_k,-)$ for all $1 \leq k \leq d$. As Π is a summand of T_k, we have $\ker\mathrm{Hom}_\Gamma(T_k,-) \subseteq \ker\mathrm{Hom}_\Gamma(\Pi,-)$. For the converse, since $T_k \in \mathrm{gen}(\Pi)$ there is an epimorphism $\Pi^N \to T_k$ for some N, yielding a monomorphism $\mathrm{Hom}_\Gamma(T_k,-) \to \mathrm{Hom}_\Gamma(\Pi^N,-)$. It follows that

$$\ker\mathrm{Hom}_\Gamma(\Pi,-) = \ker\mathrm{Hom}_\Gamma(\Pi^N,-) \subseteq \ker\mathrm{Hom}_\Gamma(T_k,-). \qquad \square$$

Miyashita's result, stated here as Theorem 5.6, provides equivalences involving the tilting subcategories $\mathcal{T}_j(T_k)$ for higher values of j and k. Hence we would also like to give easier descriptions of these categories, which is the content of the next subsection.

5.4. **Tilting subcategories for higher shifted modules.** As in the previous section, we assume that $\operatorname{domdim}\Gamma = d > 0$, so we have a family of shifted modules T_k for $0 \leq k \leq d$. Define $\mathcal{K}_i = \ker \operatorname{Ext}^1_\Gamma(T_i, -)$, so in particular $\mathcal{K}_1 = \mathcal{T}_0(T_1)$.

Proposition 5.8. *For $1 \leq k \leq d$, we have*

$$\mathcal{T}_j(T_k) = \begin{cases} \bigcap_{i=1}^k \mathcal{K}_i, & j = 0, \\ \ker \operatorname{Hom}_\Gamma(\Pi, -), & j = k, \\ \{0\}, & \textit{otherwise.} \end{cases}$$

Proof. By construction, for any $1 \leq i \leq k$ we have an exact sequence

$$0 \longrightarrow T_{i-1} \longrightarrow \Pi_i \longrightarrow T_i \longrightarrow 0$$

with $\Pi_i \in \operatorname{add}\Pi$. Passing to the long exact sequences of functors, and using that Π_i is projective, we see that $\operatorname{Ext}^i_\Gamma(T_k, -) = \operatorname{Ext}^1_\Gamma(T_{k-i+1}, -)$, so $\ker \operatorname{Ext}^i_\Gamma(T_k) = \mathcal{K}_{k-i+1}$. Our description of $\mathcal{T}_j(T_k)$ now follows directly from the definition of this subcategory.

The above calculation also shows that

$$\mathcal{T}_k(T_k) = \ker \operatorname{Hom}_\Gamma(T_k, -) \cap \bigcap_{i=2}^k \mathcal{K}_i.$$

By Lemma 5.7, we have $\ker \operatorname{Hom}_\Gamma(T_k, -) = \ker \operatorname{Hom}_\Gamma(\Pi, -)$, so $\mathcal{T}_k(T_k) \subseteq \ker \operatorname{Hom}_\Gamma(\Pi, -)$. Conversely, we show that $\ker \operatorname{Hom}_\Gamma(\Pi, -) \subseteq \mathcal{K}_i$ for $2 \leq i \leq k$. Assume $\operatorname{Hom}_\Gamma(\Pi, X) = 0$, and apply $\operatorname{Hom}_\Gamma(-, X)$ to the sequence

$$0 \longrightarrow T_{i-1} \longrightarrow \Pi_i \longrightarrow T_i \longrightarrow 0$$

above to obtain

$$\operatorname{Hom}(T_{i-1}, X) \xrightarrow{\sim} \operatorname{Ext}^1(T_i, X).$$

Since $i \geq 2$, there is an epimorphism $\Pi_{i-1} \to T_{i-1}$ with Π_{i-1} projective injective, and hence a monomorphism $\operatorname{Hom}(T_{i-1}, X) \to \operatorname{Hom}(\Pi_{i-1}, X) = 0$. Thus $\operatorname{Ext}^1(T_i, X) \cong \operatorname{Hom}_\Gamma(T_{i-1}, X) = 0$, so $X \in \mathcal{K}_i$, completing the proof that $\mathcal{T}_k(T_k) = \ker \operatorname{Hom}_\Gamma(\Pi, -)$.

As calculated at the start of the proof, we have

$$\operatorname{Ext}^k_\Gamma(T_k, -) = \operatorname{Ext}^1_\Gamma(T_1, -),$$

and so $\mathcal{T}_j(T_k) \subseteq \ker \operatorname{Ext}^1_\Gamma(T_1, -) = \mathcal{T}_0(T_1)$ when $j \neq k$. Using again that

$$\mathcal{T}_j(T_k) \subseteq \ker \operatorname{Hom}(T_k, -) = \ker \operatorname{Hom}(\Pi, -) = \mathcal{T}_1(T_1)$$

for $j \neq 0$, we see that $\mathcal{T}_j(T_k) \subseteq \mathcal{T}_0(T_1) \cap \mathcal{T}_1(T_1)$ for j different from 0 and k. However, since $(\mathcal{T}_0(T_1), \mathcal{T}_1(T_1))$ is a torsion pair, this intersection is $\{0\}$. □

By combining the calculation in Proposition 5.8 with Miyashita's theorem [20, Thm. 1.16] (stated above as Theorem 5.6), we obtain the following.

Corollary 5.9. *Let $1 \leq k \leq d$, and let $\mathrm{D}T_k$ be the k-cotilting B_k-module induced by the k-shifted module T_k. Then $\mathcal{C}_j(\mathrm{D}T_k) = 0$ for j different from 0 and k, and there is an equivalence of categories $\mathcal{C}_k(\mathrm{D}T_k) \xrightarrow{\sim} \mathcal{C}_1(\mathrm{D}T_1)$. Furthermore, if $k \geq 2$ there is a fully faithful functor $\mathcal{C}_0(\mathrm{D}T_k) \to \mathcal{C}_0(\mathrm{D}T_{k-1})$ sending $\mathrm{D}T_k$ to $\mathrm{D}T_{k-1}$.*

Proof. The first two statements are immediate from Theorem 5.6 and Proposition 5.8. For the third, we obtain the desired fully faithful functor from the inclusion

$$\mathcal{T}_0(T_k) = \mathcal{T}_0(T_{k-1}) \cap \mathcal{K}_k \subseteq \mathcal{T}_0(T_{k-1})$$

which follows from Proposition 5.8, by applying the equivalences $\operatorname{Hom}_\Gamma(T_i, -) \colon \mathcal{T}_0(T_i) \xrightarrow{\sim} \mathcal{C}_0(\mathrm{D}T_i)$ for $i = k, k-1$. The resulting functor maps $\mathrm{D}T_k$ to $\mathrm{D}T_{k-1}$, since these are the images of $\mathrm{D}\Gamma$ under the preceding equivalences. □

We now give another, more direct, description of the tilting subcategories $\mathcal{T}_0(T_k)$. Recall that we write $I_k = \mathrm{D}\operatorname{Hom}_\Gamma(\Pi, T_k) \in B_k$-mod, and $A = \operatorname{End}_\Gamma(\Pi)^{\mathrm{op}}$.

Proposition 5.10. *For $1 \leq k \leq d$, we have*

(i) $\mathcal{T}_0(T_k) = \operatorname{gen}_{k-1}(\Pi)$, *and*
(ii) $\mathcal{C}_0(\mathrm{D}T_k) = \operatorname{cogen}^{k-1}(I_k)$.

Combining this with Theorem 5.6 yields an equivalence $\operatorname{Hom}_\Gamma(T_k, -)\colon \operatorname{gen}_{k-1}(\Pi) \to \operatorname{cogen}^{k-1}(I_k)$.

Proof. (i) Assume $X \in \operatorname{gen}_{k-1}(\Pi)$, so we have an exact sequence

$$0 \longrightarrow Y \longrightarrow \Pi^{k-1} \longrightarrow \cdots \longrightarrow \Pi^0 \longrightarrow X \longrightarrow 0$$

with $\Pi^i \in \operatorname{add}\Pi$. The standard homological argument with long exact sequences shows that for $j \geq 1$ we have

$$\operatorname{Ext}^j_\Gamma(T_k, X) = \operatorname{Ext}^{k+j}_\Gamma(T_k, Y) = 0,$$

so $X \in \mathcal{T}_0(T_k)$.

We prove the converse by induction on k. The case $k = 1$ is already dealt with in Lemma 5.7, so assume $k \geq 2$ and $\operatorname{gen}_{k-2}(\Pi) = \mathcal{T}_0(T_{k-1})$. Let $X \in \mathcal{T}_0(T_k)$. Since $\mathcal{T}_0(T_k) \subseteq \mathcal{T}_0(T_{k-1}) = \operatorname{gen}_{k-2}(\Pi)$, we have an exact sequence

$$0 \longrightarrow Z \longrightarrow \Pi^{k-2} \longrightarrow \cdots \longrightarrow \Pi^0 \longrightarrow X \longrightarrow 0,$$

with $\Pi^i \in \operatorname{add}\Pi$, and so we only need to see that $Z \in \operatorname{gen}(\Pi) = \mathcal{T}_0(T_1)$. We claim

$$\operatorname{Ext}^1_\Gamma(T_1, Z) \cong \operatorname{Ext}^k_\Gamma(T_k, Z) \cong \operatorname{Ext}^1_\Gamma(T_k, X) = 0.$$

The first isomorphism follows from the construction of the shifted modules, and the second follows from the above long exact sequence connecting X and Z. Finally $\operatorname{Ext}^1_\Gamma(T_k, X) = 0$ since $X \in \mathcal{T}_0(T_k)$. Thus $Z \in \ker \operatorname{Ext}^1(T_1, -) = \mathcal{T}_0(T_1)$, as required.

(ii) The case $k = 1$ was dealt with in Lemma 5.7, so we may assume $k \geq 2$. Consider the functors

$$\begin{aligned} e_0 &= \operatorname{Hom}_\Gamma(\Pi, -)\colon \Gamma\text{-mod} \to A\text{-mod}, \\ e_k &= \operatorname{Hom}_{B_k}(P_k, -)\colon B_k\text{-mod} \to A\text{-mod}, \\ \Phi &= \operatorname{Hom}_\Gamma(T_k, -)\colon \Gamma\text{-mod} \to B_k\text{-mod}. \end{aligned}$$

Recalling that $P_k = \Phi\Pi$, we have $e_k \circ \Phi = e_0$, and so we have the following commutative diagram.

$$\begin{tikzcd} \mathcal{T}_0(T_k) \arrow[dd, "\Phi"'] \arrow[dr, "e_0"] & \\ & e_0\mathcal{T}_0(T_k) \\ \mathcal{C}_0(\mathrm{D}T_k) \arrow[ur, "e_k"'] & \end{tikzcd}$$

By [20, Thm. 1.16] (stated here as Theorem 5.6), Φ is an equivalence. By part (i) of this proposition, $\mathcal{T}_0(T_k) = \operatorname{gen}_{k-1}(\Pi) \subseteq \operatorname{gen}_1(\Pi)$, so e_0 is fully faithful by [3, Lem. 3.1] (see Theorem 5.4 for $\ell = -1$, noting that we use $k \geq 2$ at this point). It follows that e_k is also fully faithful, with image $e_0\mathcal{T}_0(T_k)$. Recalling that $\ell = \Pi \otimes_A -\colon A\text{-mod} \to \Gamma\text{-mod}$ is the left adjoint of e_0, and that $e_0 \circ \ell = 1$, we see that the restriction of $\Phi \circ \ell$ to $e_0\mathcal{T}_0(T_k)$ is quasi-inverse to e_k.

Since $P_k = \operatorname{Hom}_\Gamma(T_k, \Pi)$, it is naturally a right A-module. Moreover, the functors $\ell_k = P_k \otimes_A -\colon A\text{-mod} \to B_k\text{-mod}$ and $\Phi \circ \ell$ are naturally isomorphic when restricted to $\operatorname{add} A$. Note that ℓ_k is right exact, and $\Phi \circ \ell$, being an equivalence, is right exact when restricted to $e_0(\mathcal{T}_0(T_k))$. Moreover, $A = e_0\Pi \in e_0\mathcal{T}_0(T_k)$, so we may use projective presentations to see that ℓ_k and $\Phi \circ \ell$ are isomorphic on this subcategory. It follows that $\mathcal{C}_0(\mathrm{D}T_k) = \ell_k e_0 \mathcal{T}_0(T_k)$ is a full subcategory of $\operatorname{im} \ell_k = \operatorname{cogen}^1(I_k)$ (Lemma 4.1).

By [3, Lem. 3.1], e_k is fully faithful on $\operatorname{cogen}^1(I_k)$, which contains both $\operatorname{cogen}^{k-1}(I_k)$, since $k \geq 2$, and $\mathcal{C}_0(\mathrm{D}T_k)$, by the above argument. Thus to see that $\mathcal{C}_0(\mathrm{D}T_k) \subseteq \operatorname{cogen}^{k-1}(I_k)$, we can use e_k to transport the problem to A-mod and instead show that

$$e_k\mathcal{C}_0(\mathrm{D}T_k) = e_0\mathcal{T}_0(T_k) \subseteq e_k(\operatorname{cogen}^{k-1}(I_k)) = \bigcap_{j=1}^{k-2} \ker \operatorname{Ext}^j_A(e_k B_k, -),$$

with the last equality following from [3, Prop. 3.7], stated here as Proposition 5.5. Observe that $e_0T_k = e_k\Phi T_k = e_kB_k$ as A-modules. If $X \in \mathcal{T}_0(T_k) = \operatorname{gen}_{k-1}(\Pi)$ then by [3, Lem. 3.1, Thm. 3.2] (see Theorem 5.4), we have

$$0 = \operatorname{Ext}^j_\Gamma(T_k, X) \cong \operatorname{Ext}^j_A(e_0T_k, e_0X) = \operatorname{Ext}^j_A(e_kB_k, e_0X)$$

for $1 \leq j \leq k-2$, so $e_0X \in \bigcap_{j=1}^{k-2} \ker \operatorname{Ext}^j_A(e_kB_k, -)$.

Conversely, the inclusion $\operatorname{cogen}^{k-1}(I_k) \subseteq \mathcal{C}_0(\mathrm{D}T_k)$ can be seen directly, as follows. Let $M \in \operatorname{cogen}^{k-1}(I_k)$, so there exists an exact sequence

$$0 \longrightarrow M \longrightarrow J_0 \longrightarrow \cdots \longrightarrow J_{k-1} \longrightarrow N \longrightarrow 0$$

with each $J_i \in \operatorname{add} I_k$. Recalling that $\mathrm{D}T_k$ is I_k-special, so $\operatorname{Ext}^j_{B_k}(J_i, \mathrm{D}T_k) = 0$ for all i and all $j \geq 1$, it follows by a standard homological argument that

$$\operatorname{Ext}^j_{B_k}(M, \mathrm{D}T_k) \cong \operatorname{Ext}^{j+k}_{B_k}(N, \mathrm{D}T_k) = 0,$$

for $j \geq 1$ since $\operatorname{id} \mathrm{D}T_k \leq k$. Thus $M \in \mathcal{C}_0(\mathrm{D}T_k)$, as required. □

Naturally, one can make dual arguments for the coshifted modules, and obtain the following dual description, where $P^k = \operatorname{Hom}(C^k, \Pi) \in B^k$-mod.

Proposition 5.11. *For $1 \leq k \leq d$, we have*

- (i) $\mathcal{C}_0(C^k) = \operatorname{cogen}^{k-1}(\Pi)$*, and*
- (ii) $\mathcal{T}_0(\mathrm{D}C^k) = \operatorname{gen}_{k-1}(P^k)$.

Combining this with Theorem 5.6 yields an equivalence $\mathrm{D}\operatorname{Hom}_\Gamma(-, C^k)\colon \operatorname{cogen}^{k-1}(\Pi) \to \operatorname{gen}_{k-1}(P^k)$.

6. Tilting modules as intermediate extensions

As usual, let Γ be a finite-dimensional algebra with $\operatorname{domdim} \Gamma = d > 0$, let Π be a maximal projective-injective summand, and let $A = \operatorname{End}_\Gamma(\Pi)^{\mathrm{op}}$. In this section, we consider the intermediate extensions in our preferred recollements involving the shifted and coshifted algebras B_k and B^k, which we denote by c_k and c^k respectively.

Our main result is that when (A, E, Γ) is a Morita–Tachikawa triple, and $0 < k < d$, the distinguished cotilting module $\mathrm{D}T_k$ for the k-th shifted algebra B_k of Γ is the intermediate extension c_kE. Similarly, $c^kE = \mathrm{D}C^k$ is the distinguished tilting module for the coshifted algebra B^k.

We first give some general results, for arbitrary tilting or cotilting modules.

Proposition 6.1. *Let Γ be a finite-dimensional algebra with tilting module T, cotilting module C and maximal projective summand Π, and write $B = \operatorname{End}_\Gamma(T)^{\mathrm{op}}$ and $B' = \operatorname{End}_\Gamma(C)^{\mathrm{op}}$. Let e and e' be the idempotents of B and B' given in each case by projection onto Π. Then*

$$e\mathrm{D}T = \mathrm{D}\Pi = e'\mathrm{D}C.$$

In particular, if Γ is part of a Morita–Tachikawa triple (A, E, Γ), then

$$e\mathrm{D}T = E = e'\mathrm{D}C.$$

Proof. Writing $\Phi = \operatorname{Hom}_\Gamma(T, -)$, we have $Be = \Phi(\Pi)$ and $\mathrm{D}T = \Phi(\mathrm{D}\Gamma)$. It follows that

$$e(\mathrm{D}T) = \operatorname{Hom}_B(\Phi(\Pi), \Phi(\mathrm{D}\Gamma)) = \operatorname{Hom}_\Gamma(\Pi, \mathrm{D}\Gamma) = \mathrm{D}\Pi,$$

since by [20, Thm. 1.16] (here Theorem 5.6) Φ is fully faithful on the subcategory $\mathcal{T}_0(T)$, which contains all injective Γ-modules. Writing $\Phi' = \mathrm{D}\operatorname{Hom}_\Gamma(-, C)$, we have $\mathrm{D}(e'B') = \Phi'(\Pi)$ and $\mathrm{D}C = \Phi'(\Gamma)$. It follows that

$$e'(\mathrm{D}C) = \mathrm{D}\operatorname{Hom}_{B'}(\Phi'(\Gamma), \Phi'(\Pi)) = \mathrm{D}\operatorname{Hom}_\Gamma(\Gamma, \Pi) = \mathrm{D}\Pi,$$

since by [20, Thm. 1.16] again, Φ' is fully faithful on the subcategory $\mathcal{C}_0(C)$, which contains all projective Γ-modules. The final statement follows since the module E in a Morita–Tachikawa triple is always given by $\mathrm{D}\Pi$, where Π is a maximal projective-injective summand of Γ. □

Maintaining the notation of Proposition 6.1, consider the B-modules

$$P := \operatorname{Hom}_\Gamma(T,\Pi),$$
$$I := \mathrm{D}\operatorname{Hom}_\Gamma(\Pi,T),$$

noting that P is projective, I is injective and $\nu P = I$. Furthermore, since Π is a summand of both Γ and $\mathrm{D}\Gamma$, we have $P\oplus I\in\operatorname{add}\mathrm{D}T$. In terms of the idempotent e, we have $P = Be$ and $I = \mathrm{D}(eB)$. Our aim is now to characterise when the cotilting B-module $\mathrm{D}T$ is in the image of the intermediate extension functor c associated to this idempotent.

Proposition 6.2. *In the context of the preceding paragraph, let $m,n\geq 0$, and denote by Ω and Ω^- the usual syzygy functors associated to Γ.*

- (i) *The following are equivalent:*
  - (a) $\Gamma\in\operatorname{cogen}^{m-1}(\Pi)$ *and* $\operatorname{Ext}^1_\Gamma(\Omega^{-i}\Gamma,T)=0$ *for* $1\leq i\leq m$, *and*
  - (b) $\mathrm{D}T\in\operatorname{cogen}^{m-1}(I)$.
- (ii) *The following are equivalent:*
  - (a) $\mathrm{D}\Gamma\in\operatorname{gen}_{n-1}(\Pi)$ *and* $\operatorname{Ext}^1_\Gamma(T,\Omega^i\mathrm{D}\Gamma)=0$ *for* $1\leq i\leq n$, *and*
  - (b) $\mathrm{D}T\in\operatorname{gen}_{n-1}(P)$.

Moreover the conditions in (i) and (ii) both hold for some $m,n\geq 1$ if and only if $\mathrm{D}T$ is in the image of the intermediate extension functor c associated to e, and in this case $\mathrm{D}T = c(\mathrm{D}\Pi)$.

Proof. Since conditions (a) and (b) are vacuous for $m=0$ and $n=0$ respectively, we may assume $m,n\geq 1$. We will also use the following straightforward observations, which hold for modules over an arbitrary algebra. Given an exact sequence

$$X_\bullet = (\ \cdots \longrightarrow X_{i-1}\longrightarrow X_i\longrightarrow X_{i+1}\longrightarrow\cdots\ ),$$

let $Z_i=\ker(X_i\to X_{i+1})$ for each $i\in\mathbb{Z}$. Then for any module Y,

1. if $\operatorname{Ext}^1(Y,Z_{i-1})=0$, then $\operatorname{Hom}(Y,X_\bullet)$ is exact at $\operatorname{Hom}(Y,X_i)$, and
2. if $\operatorname{Ext}^1(Z_{i+2},Y)=0$, then $\operatorname{Hom}(X_\bullet,Y)$ is exact at $\operatorname{Hom}(X_i,Y)$.

The proof now proceeds as follows.

(i) Assume $\Gamma\in\operatorname{cogen}^{m-1}(\Pi)$, so there is an exact sequence

$$0\longrightarrow\Gamma\longrightarrow\Pi_0\longrightarrow\cdots\longrightarrow\Pi_{m-1}\longrightarrow X\longrightarrow 0$$

with $\Pi_i\in\operatorname{add}\Pi$. Thinking of this as an infinite complex with Π_i in degree i and defining Z_i as above, we can apply the functor $\Psi=\mathrm{D}\operatorname{Hom}_\Gamma(-,T)$ and use observation (2) above to see that the resulting sequence

$$0\longrightarrow\Psi\Gamma\longrightarrow\Psi\Pi_0\longrightarrow\cdots\longrightarrow\Psi\Pi_{k-1}$$

is exact, since $\operatorname{Ext}^1_\Gamma(Z_i,T)=\operatorname{Ext}^1_\Gamma(\Omega^{-i}\Gamma,T)=0$ for $1\leq i\leq m$. Since $\Psi(\Gamma)=\mathrm{D}T$ and $\Psi(\Pi)=I$, it follows that $\mathrm{D}T\in\operatorname{cogen}^{m-1}(I)$.

Conversely, assume $\mathrm{D}T\in\operatorname{cogen}^{m-1}(I)$, and take an exact sequence

$$0\longrightarrow\mathrm{D}T\longrightarrow J_0\longrightarrow J_1\longrightarrow\cdots\longrightarrow J_{m-1}\longrightarrow Y\longrightarrow 0$$

with each $J_i\in\operatorname{add}I$, viewed as an infinite complex with J_i in degree i, and define Z_i as above. Then a standard homological argument using the above sequence shows that

$$\operatorname{Ext}^1_B(\mathrm{D}T,Z_i)=\operatorname{Ext}^{i+1}_B(\mathrm{D}T,\mathrm{D}T)=0$$

for $0\leq i\leq m-1$. So by observation (1) we can apply the right adjoint $\Psi'=\operatorname{Hom}_B(\mathrm{D}T,-)$ of Ψ, which satisfies $\Psi'(\mathrm{D}T)=\Gamma$ and $\Psi'(I)=\Pi$, to get an exact sequence

$$0\longrightarrow\Gamma\longrightarrow\Pi_0\longrightarrow\cdots\longrightarrow\Pi_{m-1}\longrightarrow\Psi'Y\longrightarrow 0.$$

It follows that $\Gamma\in\operatorname{cogen}^{m-1}(\Pi)$. This is also a projective resolution of $\Psi'Y$, so we can use it to compute $\mathrm{D}\operatorname{Ext}^i(\Psi'Y,T)$ by applying the right exact functor Ψ. However, on applying $\Psi'Y$ we recover the part

$$0\longrightarrow\mathrm{D}T\longrightarrow J_0\longrightarrow J_1\longrightarrow\cdots\longrightarrow J_{m-1}$$

of the original exact sequence, since the natural map $\Psi\Psi'(\mathrm{D}T) \to \mathrm{D}T$ is an isomorphism and $I \in \operatorname{add}\mathrm{D}T$, so the cohomology of this complex vanishes in degrees $i \leq m-2$. On the other hand, the cohomology in degree $-1 \leq i \leq m-2$ computes

$$\mathrm{D}\operatorname{Ext}^{m-1-i}_{\Gamma}(\Psi'Y, T) = \mathrm{D}\operatorname{Ext}^{m-1-i}_{\Gamma}(\Omega^{-m}\Gamma, T) = \mathrm{D}\operatorname{Ext}^{1}_{\Gamma}(\Omega^{-(i+2)}\Gamma, T),$$

and so $\operatorname{Ext}^1_\Gamma(\Omega^{-i}\Gamma, T) = 0$ for $1 \leq i \leq m$ as required.

(ii) Recall that the functor $\Phi = \operatorname{Hom}_\Gamma(T, -)$ satisfies $\Phi(\mathrm{D}\Gamma) = \mathrm{D}T$ and $\Phi(\Pi) = P$, and its left adjoint $\Phi' = \mathrm{D}\operatorname{Hom}_B(-, \mathrm{D}T)$ satisfies $\Phi'(\mathrm{D}T) = \mathrm{D}\Gamma$ and $\Phi'(P) = \Pi$. The rest is analogous to (i).

If $\mathrm{D}T = c\mathrm{D}\Pi$, then $\mathrm{D}T \in \operatorname{im} c = \operatorname{gen}(P) \cap \operatorname{cogen}(I)$ (Lemma 4.1), so the conditions in (i) and (ii) hold for $m = n = 1$. For the converse, note that these conditions become stronger as m and n increase, so it suffices to show that if they hold for $m = n = 1$ then $\mathrm{D}T = c\mathrm{D}\Pi$. In this case we have $\mathrm{D}T \in \operatorname{gen}(P) \cap \operatorname{cogen}(I) = \operatorname{im} c$, so $\mathrm{D}T = ce\mathrm{D}T$. But by Proposition 6.1, we have $e\mathrm{D}T = \mathrm{D}\Pi$, and the result follows. ☐

Remark 6.3. For the conditions of Proposition 6.2(i) to hold, it is necessary that $P \in \operatorname{add}\mathrm{D}T \subseteq \operatorname{cogen}(I)$. Since $I = \nu P$, this means that the support of the top of P is contained in the support of its socle. Similarly, the conditions of Proposition 6.2(ii) may only hold if the support of the socle of I is contained in the support of its top.

Similarly, again in the context of Proposition 6.1, we can describe when the tilting B'-module $\mathrm{D}C$ is in the image of the intermediate extension c' associated to the idempotent e'. We define

$$P' := \operatorname{Hom}_\Gamma(C, \Pi),$$
$$I' := \mathrm{D}\operatorname{Hom}_\Gamma(\Pi, C),$$

noting again that P' is projective, $I' = \nu P'$ is injective, and $P' \oplus I' \in \operatorname{add}\mathrm{D}C$. We then have the following analogous statement to Proposition 6.2, by swapping the roles of the two algebras.

Proposition 6.4. *In the context of the preceding paragraph, let $m, n \geq 0$.*

(i) *The following are equivalent:*
  (a) $\mathrm{D}\Gamma \in \operatorname{gen}_{m-1}(\Pi)$ *and* $\operatorname{Ext}^1_\Gamma(C, \Omega^i\mathrm{D}\Gamma) = 0$ *for* $1 \leq i \leq m$, *and*
  (b) $\mathrm{D}C \in \operatorname{gen}_{m-1}(P')$.

(ii) *The following are equivalent:*
  (a) $\Gamma \in \operatorname{cogen}^{n-1}(\Pi)$ *and* $\operatorname{Ext}^1_\Gamma(\Omega^{-i}\Gamma, C) = 0$ *for* $1 \leq i \leq n$, *and*
  (b) $\mathrm{D}C \in \operatorname{cogen}^{n-1}(I')$.

Moreover the conditions in (i) and (ii) both hold for some $m, n \geq 1$ if and only if $\mathrm{D}C$ is in the image of the intermediate extension functor c' associated to e', and in this case $\mathrm{D}C = c'(\mathrm{D}\Pi)$.

We now apply these results to the shifted and coshifted modules and algebras of an algebra Γ with dominant dimension d and maximal projective-injective summand Π, to show that for $0 < k < d$ we have

$$c_k(\mathrm{D}\Pi) = \mathrm{D}T_k, \quad \text{and} \quad c^k(\mathrm{D}\Pi) = \mathrm{D}C^k,$$

where c_k and c^k are the intermediate extensions in the recollements involving B_k and B^k respectively (4.2). Note that for our result to have any content we must assume $d \geq 2$, so Γ is part of a Morita–Tachikawa triple (A, E, Γ) and $\mathrm{D}\Pi = E$.

Theorem 6.5. *Let Γ be a finite-dimensional algebra of dominant dimension $d \geq 2$, with maximal projective-injective summand Π, and let $0 < k < d$. Write T_k and B_k for the k-th shifted module and algebra of Γ, and let c_k be the intermediate extension functor from the recollement in (4.2) involving B_k. Then for $P_k = \operatorname{Hom}_\Gamma(T_k, \Pi)$ and $I_k = \mathrm{D}\operatorname{Hom}_\Gamma(\Pi, T_k)$ we have*

$$\mathrm{D}T_k \in \operatorname{gen}_{d-k-1}(P_k) \cap \operatorname{cogen}^{k-1}(I_k).$$

In particular,

$$\mathrm{D}T_k = c_k E,$$

where E is the A-module from the Morita–Tachikawa triple (A, E, Γ).

Proof. Since $0 < k < d$, we have

$$\Gamma \in \operatorname{cogen}^{d-1}(\Pi) \subseteq \operatorname{cogen}^{k-1}(\Pi),$$
$$\mathrm{D}\Gamma \in \operatorname{gen}_{d-1}(\Pi) \subseteq \operatorname{gen}_{d-k-1}(\Pi).$$

To apply Proposition 6.2, it is therefore enough to check that

$$\operatorname{Ext}^1_\Gamma(\Omega^{-i}\Gamma, T_k) = 0 = \operatorname{Ext}^1_\Gamma(T_k, \Omega^j \mathrm{D}\Gamma), \quad 1 \le i \le k,\ 1 \le j \le d-k,$$

so fix i and j satisfying these constraints. Since $0 < i, j < d$, the standard homological argument shows that

$$\operatorname{Ext}^n_\Gamma(\Omega^{-i}\Gamma, -) = \operatorname{Ext}^{n-i}_\Gamma(\Gamma, -) = 0,$$
$$\operatorname{Ext}^m_\Gamma(-, \Omega^j \mathrm{D}\Gamma) = \operatorname{Ext}^{m-j}_\Gamma(-, \mathrm{D}\Gamma) = 0$$

for all $n > i$ and $m > j$, using that the relevant syzygy and cosyzygy can be computed using projective-injective covers and envelopes. By the construction of T_k from Proposition 2.4, we then have

$$\operatorname{Ext}^1_\Gamma(\Omega^{-i}\Gamma, T_k) = \operatorname{Ext}^{1+k}_\Gamma(\Omega^{-i}\Gamma, \Gamma) = 0,$$
$$\operatorname{Ext}^1_\Gamma(T_k, \Omega^j \mathrm{D}\Gamma) = \operatorname{Ext}^1_\Gamma(\Omega^{-k}\Gamma, \Omega^j \mathrm{D}\Gamma) = \operatorname{Ext}^{1+d-k}_\Gamma(\Omega^{-d}\Gamma, \Omega^j \mathrm{D}\Gamma) = 0$$

by the above calculations, noting that $1+k > i$ and $1+d-k > j$. Our desired conclusions now follow directly from Proposition 6.2. □

Dually, we obtain the following result for coshifted algebras from Proposition 6.4.

Theorem 6.6. *Let Γ be a finite-dimensional algebra of dominant dimension $d \ge 2$, with maximal projective-injective summand Π, and let $0 < k < d$. Write C^k and B^k for the k-th coshifted module and algebra of Γ, and let c^k be the intermediate extension functor from the recollement in (4.2) involving B^k. Then for $P^k = \operatorname{Hom}_\Gamma(C^k, \Pi)$ and $I^k = \mathrm{D}\operatorname{Hom}_\Gamma(\Pi, C^k)$ we have*

$$\mathrm{D}C^k \in \operatorname{gen}_{d-k-1}(P^k) \cap \operatorname{cogen}^{k-1}(I^k).$$

In particular,

$$\mathrm{D}C^k = c^k E,$$

where E is the A-module from the Morita–Tachikawa triple (A, E, Γ).

We close by characterising, in the context of Theorem 6.5, when $\mathrm{D}T_k$ additively generates the category $\operatorname{gen}_{d-k-1}(P_k) \cap \operatorname{cogen}^{k-1}(I_k)$ that it must be contained in. The characterisation is in terms of the module E in the Morita–Tachikawa triple, and we have a similar dual result in the context of Theorem 6.6.

Proposition 6.7. *Keep all the notation and assumptions of Theorems 6.5 and 6.6, and let E be the module from the Morita–Tachikawa triple (A, E, Γ) involving Γ. Then*

(i) $\operatorname{add}\mathrm{D}T_k = \operatorname{gen}_{d-k-1}(P_k) \cap \operatorname{cogen}^{k-1}(I_k)$ *if and only if*

$$\operatorname{add} E = \bigcap_{j=1}^{k-2} \ker \operatorname{Ext}^j_A(-, E) \cap \bigcap_{j=k+1}^{d-2} \ker \operatorname{Ext}^j_A(-, E),$$

and

(ii) $\operatorname{add}\mathrm{D}C^k = \operatorname{gen}_{k-1}(P^k) \cap \operatorname{cogen}^{d-k-1}(I^k)$ *if and only if*

$$\operatorname{add} E = \bigcap_{j=1}^{k-2} \ker \operatorname{Ext}^j_A(E, -) \cap \bigcap_{j=k+1}^{d-2} \ker \operatorname{Ext}^j_A(E, -).$$

Proof. We prove only (i), the proof of (ii) being completely analogous. Recall that $I_k = \mathrm{D}(e_k B_k)$ and $P_k = B_k e_k$. By [3, Prop. 3.7] (stated here as Proposition 5.5) we have

$$e_k(\operatorname{gen}_{d-k-1}(P_k)) = \bigcap_{j=1}^{d-k-2} \ker \operatorname{Ext}^j_A(-, e_k \mathrm{D}B_k),$$

and

$$e_k \mathrm{D}B_k = \mathrm{D}\operatorname{Hom}_\Gamma(T_k, \Pi) = e^0 T_k.$$

Since e^0 is an exact functor, we can use the definition of T_k to obtain an exact sequence

$$0 \longrightarrow e^0\Gamma \longrightarrow e^0\Pi_0 \longrightarrow \cdots \longrightarrow e^0\Pi_{k-1} \longrightarrow e^0T_k \longrightarrow 0$$

with $\Pi_i \in \operatorname{add}\Pi$. As $e^0\Pi = \mathrm{D}(e^0\Gamma e^0) = \mathrm{D}A$ is injective and $e^0\Gamma = E$ we have $\operatorname{Ext}^j_A(-, e^0T_k) \cong \operatorname{Ext}^{j+k}_A(-, E)$ for all $j \geq 1$. It follows that

$$e_k(\operatorname{gen}_{d-k-1}(P_k)) = \bigcap_{j=1}^{d-k-2} \ker \operatorname{Ext}^{j+k}_A(-, E) = \bigcap_{j=k+1}^{d-2} \ker \operatorname{Ext}^j(-, E).$$

On the other hand $e_k \circ \Phi = e_0$, for $\Phi = \operatorname{Hom}_\Gamma(T_k, -)$, as in the proof of Proposition 5.10. This proposition together with [20, Thm. 1.16] (here Theorem 5.6) shows that $\Phi\colon \operatorname{gen}_{k-1}(\Pi) \to \operatorname{cogen}^{k-1}(I_k)$ is an equivalence, so by [3, Prop. 3.7] again we see that

$$e_k(\operatorname{cogen}^{k-1}(I_k)) = e_0(\operatorname{gen}_{k-1}(\Pi)) = \bigcap_{j=1}^{k-2} \ker \operatorname{Ext}^j_A(-, E),$$

here using that $\Pi = \Gamma e_0$ and $e_0\mathrm{D}\Gamma = E$. We conclude that

$$\bigcap_{j=1}^{k-2} \ker \operatorname{Ext}^j_A(-, E) \cap \bigcap_{j=k+1}^{d-2} \ker \operatorname{Ext}^j_A(-, E) = e_k(\operatorname{gen}_{d-k-1}(P_k) \cap \operatorname{cogen}^{k-1}(I_k)).$$

As calculated in Theorem 6.5, we have $\mathrm{D}T_k \in \operatorname{gen}_{d-k-1}(P_k) \cap \operatorname{cogen}^{k-1}(I_k)$. Moreover, by Proposition 6.1 we have $e_k(\mathrm{D}T_k) = E$. Since $d \geq 2$, the functor e_k is fully faithful on $\operatorname{gen}_{d-k-1}(P_k) \cap \operatorname{cogen}^{k-1}(I_k)$ by [3, Lem. 3.1], and so is an equivalence onto its image, which is thus equal to $\operatorname{add} E$ if and only if $\operatorname{gen}_{d-k-1}(P_k) \cap \operatorname{cogen}^{k-1}(I_k) = \operatorname{add} \mathrm{D}T_k$. □

7. Examples

Example 7.1. Let A be the path algebra of a linearly-oriented quiver of type A_3, and take E basic with $\operatorname{add} E = \operatorname{add}(A \oplus \mathrm{D}A)$. Then $\Gamma = \operatorname{End}_A(E)^{\mathrm{op}}$ is isomorphic to the quotient of the path algebra of the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5$$

by paths of length 3, and has global dimension 3. We have

$$\Pi = \Gamma(e_1 + e_2 + e_3) = \begin{smallmatrix}1&&\\&2&\\&&3\end{smallmatrix} \oplus \begin{smallmatrix}2&&\\&3&\\&&4\end{smallmatrix} \oplus \begin{smallmatrix}3&&\\&4&\\&&5\end{smallmatrix},$$

and

$$T_1 = \Pi \oplus 3 \oplus \begin{smallmatrix}3&\\&4\end{smallmatrix}$$

so we may compute B_1 to be the path algebra of

$$\begin{tikzcd}
& 2 \arrow[dr] & & \\
1 \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & 5 \arrow[r] & 3 \\
& 4 \arrow[ur] & &
\end{tikzcd}$$

modulo the commutativity relation on the square. We see that $\operatorname{gldim} B_1 = 2$ (cf. Corollary 2.16). We can compute that, as B_1-modules, we have

$$\ell_1(E) = 1 \oplus \begin{smallmatrix}&1&\\4&&2\\&5&\end{smallmatrix} \oplus \begin{smallmatrix}&1&\\4&&2\\&5&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&2\\5&\\3&\end{smallmatrix} \oplus 3,$$

$$r_1(E) = 1 \oplus \begin{smallmatrix}1&\\&2\end{smallmatrix} \oplus \begin{smallmatrix}&1&\\4&&2\\&5&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}4&&2\\&5&\\&3&\end{smallmatrix} \oplus 3,$$

so the image of the universal map is

$$c_1(E) = 1 \oplus \begin{smallmatrix}1&\\&2\end{smallmatrix} \oplus \begin{smallmatrix}&1&\\4&&2\\&5&\\&3&\end{smallmatrix} \oplus \begin{smallmatrix}&2\\5&\\3&\end{smallmatrix} \oplus 3 = \mathrm{D}T_1,$$

as claimed in Theorem 6.5.

Example 7.2. A simple but instructive family of examples is the following. Let A be the path algebra of a linearly oriented A_n quiver modulo the radical squared, and take E basic with $\operatorname{add} E = \operatorname{add}(A \oplus \mathrm{D}A)$. Then E is $(n-1)$-cluster-tilting, and so $\Gamma = \operatorname{End}_A(E)^{\mathrm{op}}$ is an $(n-1)$-Auslander algebra, with dominant and global dimension n, and its families of shifted and coshifted algebras coincide by Theorem 3.9.

We compute that the k-th coshifted algebra B^k may be presented as the path algebra of a linearly oriented A_{n+1} quiver, with vertices labelled

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n+1$$

modulo all paths of length 2 except that passing through the vertex $k+1$. It follows that

$$\operatorname{gldim} B^k = \max\{n-k, k\}.$$

Since A is an $(n-1)$-Auslander algebra, the shifted and coshifted algebras coincide, with

$$B_k \cong B^{n-k}.$$

Example 7.3. The following example demonstrates that the conclusion of Theorem 6.5 may not hold when $k = d$. Consider the algebra

$$\Gamma = \begin{tikzcd} 1 \arrow[r, "b"] \arrow[d, "d"'] & 2 \arrow[d, "a"] \\ 3 \arrow[r, "c"'] & 4 \end{tikzcd} \quad \text{with relation } ab = cd,$$

defined over a field $\mathbb{K}$. Then $\operatorname{domdim} \Gamma = 1$, the maximal projective injective summand is $\Pi = P(1) = I(4)$, and

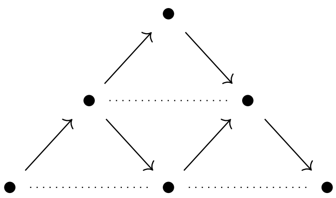
$$T_1 = I(2) \oplus I(3) \oplus I(4) \oplus \begin{smallmatrix} & 1 & \\ 3 & & 2 \end{smallmatrix} \qquad \text{and} \qquad C^1 = P(1) \oplus P(2) \oplus P(3) \oplus \begin{smallmatrix} 3 & & 2 \\ & 4 & \end{smallmatrix}$$

$$B_1 = \begin{tikzcd} & 2 \arrow[d] & \\ 3 \arrow[r] & 1' \arrow[r] & 4 \end{tikzcd} \qquad \text{and} \qquad B^1 = \begin{tikzcd} 1 \arrow[r] & 4' \arrow[r] \arrow[d] & 2 \\ & 3 & \end{tikzcd}$$

Now, projection onto the summand Π in T_1 (resp. in C^1) corresponds to $e_4 \in B_1$ (resp. $e_1 \in B^1$). Since we have $e_4 B_1 e_4 = \mathbb{K}$ we conclude that $\operatorname{im} c_1 = \operatorname{add} S(4)$ for the corresponding intermediate extension c_1. Since $\mathrm{D}T_1 \notin \operatorname{add} S(4)$, the conclusion of Theorem 6.5 does not hold in this case.

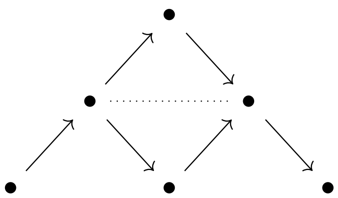
Example 7.4. It can happen that the dominant dimension of a shifted algebra is again positive, allowing us to iterate sequences of shifts and coshifts. We illustrate this on the Auslander algebra

$\Gamma =$



of the path algebra of a linearly oriented A_3 quiver. We may compute that the first shifted algebra is

$B_1 =$



(noting the absence of relations in the lowest row) and then use Theorem 3 to see that $B^1 \cong B_1$, $B_2 \cong B^0 \cong \Gamma$ and $B^2 \cong B_0 \cong \Gamma$. Since $\operatorname{domdim} B_1 = 1$, we can shift again to obtain

$$B_{1,1} = \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\ \bullet \arrow[r] \arrow[u] \arrow[ur, dotted, no head] & \bullet \arrow[u] & & \end{tikzcd}$$

which also has dominant dimension 1; note in particular that $B_{1,1} \ncong B_2$. Shifting once more, we find

$$B_{1,1,1} = \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\ & \bullet \arrow[u] & & & \end{tikzcd}$$

which has dominant dimension 0, so the sequence ends.

Acknowledgements

We would like to thank William Crawley-Boevey, Sondre Kvamme and Greg Stevenson for helpful comments. This project began while the first author visited Universität Bielefeld with funding from the Deutsche Forschungsgemeinschaft grant SFB 701—he thanks Henning Krause and the rest of the Bielefeld representation theory group for their hospitality. Subsequently, the first author was supported by a postdoctoral fellowship from the Max-Planck-Gesellschaft. The second author is supported by the Alexander von Humboldt-Stiftung in the framework of an Alexander von Humboldt Professorship endowed by the Federal Ministry of Education and Research.

References

[1] I. Assem, D. Simson, and A. Skowroński, *Elements of the representation theory of associative algebras. Vol. 1*, London Mathematical Society Student Texts, vol. 65, Cambridge University Press, Cambridge, 2006.

[2] M. Auslander, *Representation dimension of artin algebras* (1971). Reprinted in *Selected works of Maurice Auslander. Part 1*, American Mathematical Society, Providence, RI, 1999. Pages 505–575.

[3] M. Auslander, M. I. Platzeck, and G. Todorov, *Homological theory of idempotent ideals*, Trans. Amer. Math. Soc. *332* (1992), no. 2, 667–692.

[4] M. Auslander and I. Reiten, *On a generalized version of the Nakayama conjecture*, Proc. Amer. Math. Soc. *52* (1975), 69–74.

[5] ———, *Cohen-Macaulay and Gorenstein Artin algebras*, Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991), 1991, pp. 221–245.

[6] A. Beligiannis and I. Reiten, *Homological and homotopical aspects of torsion theories*, Mem. Amer. Math. Soc. *188* (2007), no. 883, viii+207.

[7] S. Brenner and M. C. R. Butler, *Generalizations of the Bernstein–Gel'fand–Ponomarev reflection functors*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), 1980, pp. 103–169.

[8] H. Chen and S. Koenig, *Ortho-symmetric modules, Gorenstein algebras and derived equivalences*, Int. Math. Res. Not. IMRN *23* (2016), 6979–7037. arXiv:1506.03337 [math.RT].

[9] H. Chen and C. Xi, *Dominant dimensions, derived equivalences and tilting modules*, Israel J. Math. *215* (2016), no. 1, 349–395. arXiv:1503.02385 [math.RT].

[10] E. Cline, B. Parshall, and L. Scott, *Derived categories and Morita theory*, J. Algebra *104* (1986), no. 2, 397–409.

[11] W. Crawley-Boevey and J. Sauter, *On quiver Grassmannians and orbit closures for representation-finite algebras*, Math. Z. *285* (2017), no. 1-2, 367–395. arXiv:1509.03460 [math.RT].

[12] V. Franjou and T. Pirashvili, *Comparison of abelian categories recollements*, Doc. Math. *9* (2004), 41–56. arXiv:math/0403081 [math.CT].

[13] N. Gao and S. Koenig, *Grade, dominant dimension and Gorenstein algebras*, J. Algebra *427* (2015), 118–141.

[14] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.

[15] O. Iyama, *Auslander correspondence*, Adv. Math. *210* (2007), no. 1, 51–82. arXiv:math/0411631 [math.RT].

[16] ———, *Higher-dimensional Auslander–Reiten theory on maximal orthogonal subcategories*, Adv. Math. *210* (2007), no. 1, 22–50. arXiv:math/0407052 [math.RT].

[17] ———, *Auslander–Reiten theory revisited*, Trends in representation theory of algebras and related topics, 2008, pp. 349–397. arXiv:0803.2841 [math.RT].

[18] O. Iyama and Ø. Solberg, *Auslander–Gorenstein algebras and precluster tilting*, 2016. arXiv:1608.04179v1 [math.RT].

[19] N. J. Kuhn, *Generic representations of the finite general linear groups and the Steenrod algebra. II*, *K*-Theory *8* (1994), no. 4, 395–428.

[20] Y. Miyashita, *Tilting modules of finite projective dimension*, Math. Z. *193* (1986), no. 1, 113–146.

[21] K. Morita, *Flat modules, injective modules and quotient rings*, Math. Z. *120* (1971), 25–40.

[22] B. J. Müller, *The classification of algebras by dominant dimension*, Canad. J. Math. *20* (1968), 398–409.

[23] V. C Nguyen, I. Reiten, G. Todorov, and S. Zhu, *Dominant dimension and tilting modules*, 2017. arXiv:1706.00475v1 [math.RT].

[24] C. Psaroudakis and J. Vitória, *Recollements of module categories*, Appl. Categ. Structures *22* (2014), no. 4, 579–593. arXiv:1304.2692 [math.RT].

[25] J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. (2) *39* (1989), no. 3, 436–456.

[26] C. M. Ringel, *Artin algebras of dominant dimension at least 2*, 2007. Seminar notes, Bielefeld. `http://www.math.uni-bielefeld.de/~ringel/opus/domdim.pdf`.

[27] H. Tachikawa, *On left* QF − 3 *rings*, Pacific J. Math. *32* (1970), 255–268.

MP: Institut für Algebra und Zahlentheorie, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany
E-mail address: `presslmw@mathematik.uni-stuttgart.de`

JS: Fakultät für Mathematik, Universität Bielefeld, Postfach 100 131, D-33501 Bielefeld, Germany
E-mail address: `jsauter@math.uni-bielefeld.de`