Shnirelman integral and p-adic L-functions associated to modular forms

Ha Huy Khoai

Institute of Mathematics
P.O.Box 631, Bo Ho
10000 Hanoi
Vietnam

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI / 92-39

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§1. Introduction.

It seems that at the present there is no in p-adic analysis a good analogue of the Cauchy integral. In many cases the Shnirelman integral is used in this role. The main applications of the Shnirelman integral can be found in the study of transcendent numbers in p-adic domains ([Ad]) and in construction of p-adic spectral theory ([Vi]). In an earlier paper ([Ha3]) we are interested in consideration of how the Shnirelman integral is convenient for an analogue of the Morera lemma. Namely, we considered the class of functions in the p-adic unit disc whose Shnirelman integrals are vanishing. The functions of this class have many properties analogous to one's of Krasner analytic functions, but this class is larger than the second.

In the present note we show some other situations where the above mentioned class appears. For example, p-adic L-functions associated to modular forms belong to this class with some "kernels".

In §2 we recall some basic facts about the Shnirelman integral and the class S of functions whose Shnirelman integrals are vanishing. Using the class S we give an inverse formula for the p-adic Mellin transform in §3. In §4 the functional equations satisfied by p-adic L-functions of modular forms are discribed in terms of class S. Some remarks and open questions are given in the last section.

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§2. Preliminaries.

Let p be a prime number, Q_p the field of p-adic number, and C_p the p-adic completion of the algebraic closure of C_p. The absolute value in Q_p is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the addition valuation on C_p which extends ord_p. Let D be the unit dics in C_p:

$$D = \{z \in C_p, |z| < 1\}$$

We donote by D_r the disc $\{z \in C_p, |z| < r\}$.

2.1.**Definition** . Let $f(z)$ be a C_p-valued function defined on all $z \in C_p$ such that $|z-a| = r$, where $a \in C_p$ and r is a positive real number (we shall always assume that r is in $|C_p|_p$ i.e. , a rational power of p). Let $\gamma \in C_p$ be such that $|\gamma| = r$. Then the *Shnirelman integral* is defined as the following limit if it exists:

$$\int_{a,\gamma} f(z)dz = \lim_{n\to\infty}{}' \frac{1}{n} \sum_{\xi^n=1} f(a+\xi\gamma), \tag{1}$$

where the ' indicates that the limit is only over n not divisible by p.

We recall that a function f in a domain M is said to be *Krasner analytic* if M is an union of open sets $D_i, D_i \subset D_{i+1}$ such that for each i , $f|_{D_i}$ is a uniform limit of rational functions having no poles in D_i. From properties of the Shnirelman integral we need the following.

2.2.**Theorem.** *If f is Krasner analytic in $D_a(r)$, and if $|\gamma| = r$, then for fixed $z \in C_p$ we have:*

$$\int_{a,\gamma} \frac{f(x)(x-a)}{x-z}dx = \begin{cases} f(z), & \text{if } |z-a| < r; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

2.3.**Theorem.** *Let f(z)=g(z)/h(z) , where g(z) is Krasner analytic in $D_a(r)$ and h(z) is a polynomial. Let $\{z_i\}$ be the roots of h(z) in $D_a(r)$, and suppose that for all $i, |z_i - a| < r$. Define $res_{z_i} f$ to be the coefficient of $(z-z_i)$ in the Laurent expansion of f(z) at z_i. Then:*

$$\int_{a,\gamma} f(z)(z-a)dz = \sum res_{z_i} f. \tag{3}$$

We refer the readers to [Ko] for more detail about the Shnirelman integral.

2.4.**Definition** . A function $f(z)$ in a domain M is said to be in class $S(M)$ if for all a, r such that $D_a(r) \subset M$

$$\int_{a,\gamma} f(z)(z-a)dz = 0. \tag{4}$$

2.5.**Remark** 1). Some basic properties of functions of class $S(M)$ can be found in [Ha3].

2). We denote by $H(M)$ and $A(M)$ respectively the class of Krasner analytic functions and locally analytic functions in M. Then $H(M) \subset S(M), A(M) \not\subset S(M)$ ([Ha3]).

§3. p adic Mellin transform.

3.1. **p-adic group of characters.** Let Δ_o be an integer prime number p and let

$$q = \begin{cases} 4, & \text{if } p = 2 \ ; \\ p, & \text{otherwise.} \end{cases}$$

We set $\Delta_o q = \Delta$ and denote:

$$Z^*_\Delta = \varprojlim (Z/\Delta p^m Z)^*$$

The group of p-adic characters is the group of continuous holomorphisms of Z^*_Δ into C^*_p:

$$X(Z^*_\Delta) = \mathrm{Hom}_{\mathrm{cont}}(Z^*_\Delta, C^*_p)$$

We set

$$U = 1 + qZ_p = \{z \in Z_p, v(z-1) \geq v(q)\}$$

Then, for every $g \in U$ such that $v(g-1) = v(q)$ the map $z \mapsto g^z$ is an isomorphism of Z_p onto U. We call g a topological generator of the group U. For each generator g of the group U the map

$$X(U) = \mathrm{Hom}_{\mathrm{cont}}(U, C^*_p) \longrightarrow C^*_p$$

transforming a continuous character χ of the group U into a point $\chi(g) - 1$ in the unit disc D of C_p. Also we have isomorphisms

$$Z^*_\Delta \simeq (Z/\Delta_o Z)^* \times Z^*_p$$

$$Z^*_p \simeq (Z/qZ)^* \times U \qquad (5)$$

From isomorphisms (5) it follows that $X(Z^*_\Delta)$ is a product of a finite group and $X(U)$, while the last is isomorphic to D. Since D is an open disc of C_p, this isomorphism makes $X(Z^*_\Delta)$ into an analytic group. A function $f(\chi)$ is said to be holomorphic function on the analytic group $X(Z^*_\Delta)$ if its restriction on each component isomorphic to D is a holomorphic function. Thus, we can regard every holomorphic function in the group $X(Z^*_\Delta)$ as a holomorphic function in the unit disc D. Note that each Dirichlet character χ of conductor Δp^m is an element of the group $\mathrm{Hom}((Z/\Delta p^m Z)^*, C^*_p)$ for each $m \geq n$, and is prolonged to an unique element of the group $X(Z^*_\Delta)$ which is again denoted by χ. Thus ,the torsion subgroup of $X(Z^*_\Delta)$ is identified with the group of Dirichlet characters of conductors dp^n where $d|\Delta_o$. Let χ be such a character and let $\chi = \chi_o.\chi_1$ be its decomposition in $\chi_o \in X(Z/\Delta Z)^*$ and $\chi_1 \in X(U)$. Then χ_1 takes values in the group $\mu_{p^\infty} = \cup_n \mu_{p^n}$ where by μ_{p^n} we denote the group of p^n-roots of unity in C_p. On the other hand, every $x \in Z_\Delta^*$ can be written in the form

$$x = \alpha(x)\theta(x)$$

with $\alpha(x) \in (Z/\Delta_o Z)^*$ and $\theta(x) \in Z_p^*$. The map $\theta : x \mapsto \theta(x)$ is an element of $X(Z_\Delta^*)$ which is called *the fundamental character.*

3.2.The Mellin transform. Let μ be a measure on Z_Δ^* , i.e. μ is a continuous linear functional with values in C_p on the space of continuous functions in Z_Δ^*. Then the restriction of μ on the analytic group $X(Z_\Delta^*)$ gives an analytic function:

$$L(\mu, \chi) = \int_{Z_\Delta^*} \chi d\mu. \quad (6)$$

The function $L(\mu, \chi)$ is called the *p-adic Mellin transform of the measure* μ.

Example. p-adic L-functions associated to modular forms are p-adic Mellin transforms of measures associated to modular forms (see [A-V], [Vi]). We will return to such functions in the next section.

In this section we give an inverse formula for the p-adic Mellin transform by using the Shnirelman integral . As an application we have an integral representation of Morita's p-adic Γ-function. Note that the Shnirelman integral is used by Vishik to find an inverse formula for the Stieltjes transform (see [Vi], [Ko]).

3.3.p-adic Mellin transforms as holomorphic functions in the unit disc.

Let

$$F(\chi) = \int_{Z_\Delta^*} \chi d\mu \quad (7)$$

be the p-adic Mellin transform of a measure μ. Then by the isomorphism (5) we can regard F as a holomorphic function on the analytic group $X(Z_\Delta^*)$. This means that for every character $\chi_o \in X(Z/\Delta Z)^*$ we have a holomorphic function $F_{\chi_o}(\chi_1)$ on the group $X(U)$. Thus every p-adic Mellin transform on $X(Z_\Delta^*)$ corresponds to a collection of holomorphic functions on the group $X(U)$. Now let $f(\chi_1)$ be such a "branch" of the function $F(\chi)$. Let g be a fixed topological generator of the group U. We set

$$z = \chi_1(g) - 1. \quad (8)$$

For each $x \in U$ we have $x = g^{\log x/\log g}$ and hence that

$$\chi_1(x) = (1+z)^{\log x/\log g}.$$

Then the p-adic Mellin transform of the measure μ corresponds to the function

$$f(z) = \int_U (1+z)^{\log x/\log g} d\mu(x). \quad (9)$$

Thus for every measure μ on U, the Mellin transform given by formula(9) is a holomorphic function in the unit disc D.

3.4.**Theorem.** *Let f(z) be a bounded holomorphic function in the unit disc D. For $x \in U$ and m=1,2,... we consider the following functions in D*

$$G_{m,x}(z) = \frac{1}{p^m - p^{m-1}} \sum_{\xi^{p^m}=1, \xi\neq 1} \xi^{-\log x/\log g} \frac{1}{(z+1-\xi)}. \tag{10}$$

Then the function $f(z)$ is p-adic Mellin transform of the following measure on U:

$$\mu\{x + (U_m)\} = \int_{a,\gamma} f(z) G_{m,x}(z)(z-a)dz \tag{11}$$

where $U_m = 1 + p^m U, a, \gamma$ are such that $D_a(|\gamma|) \subset D$ and the points $1 - \xi$ belong to this disc.

Proof. We first show formula (11) defines a bounded measure on U. We have:

$$\sum_{k=0}^{p-1} \mu\{x + kp^m + (U_{m+1})\} = \int_{a,\gamma} f(z) \sum_{k=o}^{p-1} G_{m+1,x+kp^m}(z)(z-a)dz$$

By an easy calculation we obtain :

$$\sum_{k=0}^{p-1} G_{m+1,x+kp^m}(z) = G_{m,x}(z).$$

From this it follows that

$$\mu\{x + (U_m)\} = \sum_{k=0}^{p-1} \mu\{x + kp^m + (U_{m+1})\}.$$

Obviously, the function $f(z)G_{m,x}(z)$ is a quotion of Krasner analytic functions in the disc $D_{a,|\gamma|}$. Then we have :

$$\mu\{x + (U_m)\} = \sum_{\xi^{p^m}=1, \xi\neq 1} res_{\xi-1}(f.G_{m,x}) =$$

$$\frac{1}{p^m - p^{m-1}} \sum f(\xi - 1)\xi^{-\log x/\log g} \tag{12}$$

By using formula (12) the boundnes of μ will be proved in 3.7. Indeed we will concern with more general situatons.

It remains to prove that $f(z)$ is the p-adic Mellin transform of the measure μ. We set

$$F(z) = \int_U (1+z)^{\log x/\log g} d\mu(x)$$

Since μ is bounded, $F(z)$ is a bounded holomorphic function in the unit disc. Then it suffices to show that the functions $f(z)$ and $F(z)$ are coincide on the set $\{\xi - 1\}$, where ξ are roots of unity of degree $p^m, m = 1, 2, \ldots$ (see [A-V], [Ha1], [Ha2]).

We have:

$$F(\xi - 1) = \int_U \xi^{\log x/\log g} d\mu(x) =$$

$$\sum_{\zeta^{p^m}=1, \zeta \neq 1} (\xi^{-1}\zeta)^{\log x/\log g} f(\zeta - 1) = f(\xi - 1)$$

Theorem is proved.

Thus, we have a correspondance between the set $H_o(D)$ of bounded holomorphic functions in D and the set $L(U)$ of continuous functionals on the space $C(U)$ of continuos functions on U. Namely, for any bounded measure μ on U, the p-adic Mellin transform

$$M\mu(z) = \int_U (1 + z)^{\log x/\log g} d\mu$$

defines a bounded holomorphic function in D. Conversely, let $f(z)$ be a bounded holomorphic function in D . Then we have a continuous functional $Nf \in L(U)$ which is defined by:

$$C(Z_p) \ni \Phi \mapsto \lim_{n \to \infty} \sum_{i=0}^{p^n - 1} \int_{a,\gamma} f(z)\Phi(x_i) G_{m,x_i}(z)(z - a) dz$$

where x_i runs on the set of representations of U/U_n.

3.5.**Theorem.**M *and* N *are mutually inverse topological isomorphism between* $H_o(D)$ and $L(U)$.

The proof is based on the formulas of operators M, N, and standard arguments.

3.6.**Morita's p-adic Γ-function.** We now apply Theorem 3.5. to Morita's p-adic Γ-function. In [Ba1] it is proved that we may consider the function $\Gamma_p(x)$ as the restriction on Z_p of a locally analytic function $\Gamma_p(z)$ of local analyticity ratio $\rho = p^{(-1/p)-(1/p-1)}$. This means for each point $x \in Z_p$ there exists ρ_x such that on $D(x, \rho_x) \cap Z_p$ the fuction $\Gamma_p(x)$ is the restriction of $F(x) = \sum_{n \geq 0} a_n (z - x)^n$ which is holomorphic on $D(x, \rho_x)$. The local analyticity ratio, by definition, is the number

$$\rho = \inf_{x \in Z_p} \rho_x > 0$$

Thus, on the disc $D(0, p^{(-1/p)-(1/p-1)})$ the function $\Gamma_p(x)$ is represented by a convergent power series. We set

$$f(z) = \Gamma_p(p^{(-1/p)-(1/p-1)}z)$$

then $f(z)$ is bounded holomorphic function on the unit disc D. We have an integral representation of the function $f(z)$:

$$f(z) = \int_U (1+z)^{\log x/\log g} d\mu$$

where the measure μ is defined by the formula (11). Hence, for Morita's p-adic Γ-function we have the follwing integral representation:

$$\Gamma_p(z) = \int_U (1+\alpha z)^{\log x/\log g} d\mu$$

where $\alpha = p^{(1/p-1)-(1/p)}$ and the measure μ is defined by the formula:

$$\mu\{x + (p^m U)\} = \frac{1}{p^m - p^{m-1}} \int_{a,\gamma} \Gamma_p(\alpha z) G_{m,x}(z)(z-a)dz.$$

3.7.**Mellin transform of non-bounded measures.** In [A-M], [Vi] Y. Amice and J. Vélu, and M. Vishik defined the p-adic Mellin transform of non-bounded measures and applied it to construct p-adic L-functions associated to modular forms. We recall here the definition and give an inverse formula for the p-adic Mellin transform of so-called *h-addmissible measures.* These measures are defined on Z_Δ^* and the corresponding Mellin transforms are holomorphic functions on the group $X(Z_\Delta^*)$. As in the case of bounded measures we consider the measures on U and corresponding holomorphic functions on the unit disc D.

Definition. Let f and g be two holomorphic functions in the unit disc D. We say f is of class $o(g)$if

$$\sup_{|z|\le r} |f(z)| = o(\sup_{|z|\le r} |g(z)|)$$

when $r \longrightarrow 1-0$.

Definition. A *h-addmissible measure* on U is a linear functional on the space of functions on U which are locally polynomials of degrees less than h and satisfy the following condition:

$$|\mu\{(x-b)^k \psi_{b,m}\}| = o(p^{(h-k)m}), k = 0, 1, ..., h-1,$$

where $\psi_{b,m}$ is the characteristic function of the set $b + (U_m)$.

It is proved in [Vi] that a h-addmissible measure is prolonged to a continuous linear functional on the space of $(h-1)$-differentiable functions whose derivates of oder $h-1$ satisfy the Lipschitz condition. The restriction of such a functional on the group $X(U)$ is a holomorphic function of class $o(\log^h)$ and is called *the Mellin transform* of the measure μ. The class of such measures contains, for example, the measures associated to modular forms.

Obviously, a h-addmissible measure is defined by giving its values on the set $\{x^k\psi_{b,m}\}$ with $b \in U, m = 1, 2, ...$ and $\psi_{b,m}$ is the characteristic function of the set $b + U_m$.

Theorem. *Let $f(z)$ be a holomorphic function of class $\log^h$ in the unit disc D. Then $f(z)$ is p-adic Mellin transform of the following h-addmissible measure on U:*

$$\mu\{x^k\psi_{b,m}\} = \int_{a,m} f(z)G_{m,b,k}(z)(z-a)dz \tag{13}$$

where $m = 1, 2, ..., k = 0, ..., h-1$ and $G_{m,b,k}$ are given by the following formula:

$$G_{m,b,k}(z) = \frac{1}{p^m - p^{m-1}} \sum_{\xi^{p^m=1}, \xi \neq 1} \frac{\xi^{-\log b/\log g}}{z + g^k\xi - 1}. \tag{14}$$

We recall that g is a fixed topological generator of the group $X(U)$, $v(g-1) \geq v(q)$.

Proof. It is easy to see that the formula (13) correctly defines a linear functional μ on the space of functions on Z_p which are locally polynomials of degrees less than h. To show that μ satisfies the conditions of h-addmissiblility we note that:

$$\mu\{(x^k\psi_{b,m}\} = \frac{1}{p^m - p^{m+1}} \sum_{\xi} \xi^{-\log b/\log g} f(g^k\xi - 1).$$

We first recall some notations. For each holomorphic functions $f(z)$ on D and each $t_o > 0$ we set

$$\|f\|_{t_o} = \sup_{v(z)=t_o} |f(z)|.$$

Then we obtain:

$$\|\log^h(1+z)\|_{t_m} = p^{mh},$$

where $t_m = 1/p^m - (p^{m-1}), m = 1, 2, ..., (|\log^h(1+z)|$ is calculated by the Newton polygon (see[Ha]). From the hypothesis we have:

$$\|f(z)\|_{t_m} = o(p^{mh})(m \to \infty).$$

Let u be the sequence $\{g^i\xi - 1\}, i = 0, 1, ..., h-1$, where $\{\xi\}$ is the sequence of primitive roots of unity of degree $p^m (m = 1, 2, ...)$. Since the function $f(z)$

is of class $o(\log^h)$, one infers u is an interpolating sequence of $f(z)$ (see [A-V], [Ha1],[Ha2]). We denote $\{S_m(z)\}$ the sequence of Lagrange's interpolation polynomials for the function $f(z)$ and the sequence u. Then $S_m(z)$ is defined by the following conditions:

$$\deg S_m(z) \le hp^m - 1$$

$$S_m(g^i\xi - 1) = f(g^i\xi - 1), i = 0, ..., h-1.$$

By Lazard's lemma ([La]) we may represent $f(z)$ in the form:

$$f(z) = \phi(z) \prod_{\gamma\in\mu_m, i=0,...,h-1} \frac{1-z}{g^i\xi - 1} + Q_m(z) \tag{15}$$

where μ_m is the set of primitive roots of unity of degree $p^m, Q_m(z)$ are polynomials of oder hp^m satisfying the condition:

$$\|Q_m\|_{t_m} \le \|f\|_{t_m}$$

Since the representation (15) is unique, we have $S_m(z) \equiv Q_m(z)$, and hence that

$$\|S_m\|_{t_m} \le \|f\|_{t_m}.$$

From this it follows

$$\|S_m\|_{t_m} = o(p^{mh}).$$

Supposing $S_m(z)$ is written in the form

$$S_m(z) = \sum_{l=0}^{hp^m-1} b_l(m)z^l$$

we have then

$$\|S_m\|_{t_m} = \max_{0\le l\le hp^m-1}\{|b_l(m)z^l|t_m\} =$$

$$\max_l\{|b_l(m)|p^{-1/(p^m-p^{m-1})}\} > p^{-hp/(p-1)} \max_l\{|b_l(m)|\}.$$

Thus we have $\max|b_l(m)| = o(p^{mh})(m \to \infty)$. Note that if we write

$$S_m(z-1) = \sum_{l=0}^{hp^m-1} a_l(m)z^l$$

then we obtain also $\max_l |a_l(m)| = o(p^{mh})$. By definition of the measure μ we have:

$$\mu\{(x-b)^k\psi_{b,m}\} = \sum_{j=0}^{k}(-b)^{k-j}\binom{k}{j}\frac{1}{p^m - p^{m-1}} \sum_{\xi^{p^m}=1,\xi\ne1} \xi^{-\log b/\log g} f(g^j\xi - 1) =$$

$$\sum_{j=0}^{k}(-b)^{k-j}\binom{k}{j}\frac{1}{p^m-p^{m-1}}\sum_{\xi^{p^m}=1,\xi\neq 1}\xi^{-\log b/\log g}S_m(g^j\xi-1)=$$

$$\sum_{j=0}^{k}(-b)^{k-j}\binom{k}{j}\frac{1}{p^m-p^{m-1}}\sum_{\xi^{p^m}=1,\xi\neq 1}\xi^{-\log b/\log g}\sum_{l=0}^{p^m-1}a_l(m)g^{jl}\xi^l=$$

$$\sum_{j=0}^{k}(-b)^{k-j}\binom{k}{j}\sum_{l=0}^{p^m-1}a_l(m)g^{jl}\frac{1}{p^m-p^{m-1}}\sum_{\xi^{p^m}=1,\xi\neq 1}\xi^{l-\log b/\log g}$$

$$\sum_{j=0}^{k}(-b)^{k-j}\binom{k}{j}\sum_{g^l\equiv b}a_l(m)g^{jl}=$$

$$\sum_{g^l\equiv b}a_l(m)(g^l-a)^j$$

Since $\max|a_l(m)|=o(p^{mh})$ from this it follows the h-addmissibility of μ.

§4.p-adic L-functions associated to modular forms.

4.1. **Definition.** Let

$$\phi(\tau)=\sum_{n=1}^{\infty}\lambda_n e^{2\pi in\tau}. \tag{16}$$

be a cusp form of weight $w+2$ on $SL(2,Z)$. Further, let ϕ is a normalized eigenfunction of the Hecke algebra, i.e. $\lambda_1=1$ and $T_n\phi=\lambda_n\phi$ for Hecke operators T_n. We set:

$$\Lambda_\phi(\chi,s)=d(\chi)^s\int_0^\infty \tau^{s-1}\phi_\chi(\tau)d\tau, \tag{17}$$

where χ is a Dirichlet character of conductor $d(\chi)$. A Manin's theorem ([Ma3], [Ma4]) asserts that there are the numbers $\omega_+\in iR$ and $\omega_-\in R$ such that for every entire $k\in[0,w]$ the numbers

$$A(\chi,k)=\frac{i^{k+1}\Lambda_\phi(\chi,k+1)}{G(\chi)\omega_{(-1)^k\chi(-1)}}, \tag{18}$$

are algebraic, where $G(\chi)$ is the Gauss sum

$$G(\chi)=\sum_{a \bmod d(\chi)}\chi(a)e^{2\pi ia/d(\chi)}. \tag{19}$$

Now let λ be a root of the equation:

$$x^2-\lambda_p x+p^{w+1}=0$$

with $v_p(\lambda) < w+1$. We set $h = [v_p(\lambda)]$. Then there exists an unique holomorphic function $f(\chi)$ on $X(Z_\Delta^*)$ such that for every $k \in [0,w]$ and every Dirichlet character $\chi = \chi_o.\chi_1$ with $d(\chi) \mid \Delta$ and $d(\chi_1) = p^n$ we have:

$$f(\chi\theta^k) = \frac{1}{\lambda^n}(-1)^k A(\chi,k). \tag{20}$$

The function $f(\chi)$ is called p-adic L-function associated to the cusp form $\phi(\tau)$ (see [Ma3, [Ma4], [M-S] , [Vi], [La], [A-V]).

Note that, f is $o(\log^{w+1})$.

4.2. p-adic L-functions as functions of class S.

We can regard $f(\chi)$ as a holomorphic function in the unit disc by using the isomorphism (5). We recall that p-adic L-functions $f(\chi)$ associated to modular forms satisfy the following functional equations:

$$f(\chi\theta^k) = \varepsilon(\chi_o)f(\chi^{-1}\theta^{w-k}) \tag{21}$$

for all characters $\chi \in X(Z_\Delta^*), k = 0,1,...,w$ and $\varepsilon(\chi_0) = \chi_o^{-1}(-1)(-1)^{k+1}$ if $\chi = \chi_o.\chi_1$ is the decomposition of χ with $\chi_o \in X((Z/\Delta Z)^*), \chi_1 \in X(U)$. For a given holomorphic function $f(\chi)$ on $X(Z_\Delta^*)$ we have a collection $\{f_{\chi_o}(\chi_1)\}$ of holomorphic functions on $X(U)$, where $\chi_o \in X((Z/\Delta Z)^*), \chi_1 \in X(U)$. Then we can write functional equations (21) in the form:

$$f_{\chi_o}(\chi_1\theta^k) = \varepsilon(\chi_o)f_{\chi_o^{-1}}(\chi_1^{-1}\theta^{w-k}) \tag{22}$$

Since the functions f_{χ_o} are in class $o(\log^{w+1})$ the functional equation (22) holds if it holds for all Dirichlet characters of conductors $p^n, n = 1,2,....$ Thus, the corresponding holomorphic functions $F_{\chi_o}(z)$ in D satisfy the following equations:

$$F_{\chi_o}[(\xi g^k - 1)] = \varepsilon(\chi_o)F_{\chi_o^{-1}}[(\xi^{-1}g^{w-k} - 1)] \tag{23}$$
$$k = 0,1,...,w, \varepsilon(\chi_o) = \chi_o^{-1}(-1)(-1)^{k-1}$$

where g is a topological generator of U, ξ is a root of unity of degree $p^n, n = 1,2,....$

By Theorem (2.2) we obtain:

$$F_{\chi_o}[(\xi g^k - 1)] = \int_{a,\gamma} F_{\chi_o}(z)\frac{(z-a)dz}{z-(\xi g^k - 1)}.$$

There is an analogous formula for the right hand side of (21). We set:

$$G_{F,\chi_o,\xi}(z) = F_{\chi_o}(z)\frac{1}{z-(\xi g^k - 1)} - \varepsilon_{\chi_o} F_{\chi_o^{-1}}(z)\frac{1}{z-(\xi^{-1}g^{w-k}-1)}. \qquad (24)$$

Thus we proved the following theorem:

4.3.**Theorem.** *For every p-adic L-function F associated to a modular form the functions $G_{F,\chi_o,\xi}$ are in class S.*

4.4. By using the Eichler-Shimura isomorphism we can show some new functional equations for p-adic L-functions associated to modular forms and some corresponding functions of class S. We need more detail about the values of p-adic L-functions of modular forms at the points $\chi\theta^k$. From (18), (19), and (20) it follows that:

$$f(\chi\theta^k) = \frac{1}{\lambda^n}(-1)^k \frac{i^{k+1}d(\chi)^k}{G(\chi)\omega_{(-1)^k\chi(-1)}} \int_0^\infty \phi_\chi(\tau)\tau^k d\tau. \qquad (25)$$

On the other hand we have

$$\phi_\chi(\tau) = \frac{G(\chi)}{d(\chi)} \sum_{b \bmod d(\chi)} \chi^{-1}(b)\phi(\tau + b/d(\chi)) \qquad (26)$$

and hence,

$$f(\chi\theta^k) = \frac{(-1)^k i^{k+1} d(\chi)^{k-1}}{\lambda^n \omega_{(-1)^k\chi(-1)}} \sum_{b \bmod d(\chi)} \chi^{-1}(-b) \int_0^\infty \phi(\tau + b/d(\chi))\tau^k d\tau. \qquad (27)$$

For any integer $j = 0, ..., w$ we define due to Eichler (see [La]) a *period* of the form $\phi(\tau)$

$$r_j(\phi) = \int_o^\infty \phi(\tau)\tau^j d\tau.$$

4.4.**Lemma.** *Let b/d be a fraction in lowest form, $0 < b/d < 1$. Then*

$$\int_0^{b/d} \phi(\tau)\tau^s d\tau$$

is a linear integral combination of the periods $r_j(\phi)$.

Note that the coefficients of the linear combination mentioned in Lemma(4.4) do not depend on ϕ, but depend on the weight w (see [La]).

By using Lemma (4.4) and the relations (27) we have:

4.5.Lemma. *There exist algebraic numbers $a_j(\chi, w, k)$ (not depending on ϕ) such that*

$$\lambda^n \omega_{(-1)^k\chi(-1)} f(\chi\theta^k) = \sum_j^w a_j(\chi, w, k) r_j(\phi). \tag{28}$$

We set

$$F_\phi(\chi\theta^k) = \lambda^n \omega_{(-1)^k\chi(-1)} f(\chi\theta^k). \tag{29}$$

Consider the following linear system of equations with variables x_j:

$$\sum_{j=0}^{w} a_j(\chi, w, k) x_j = F_\phi(\chi\theta^k) \tag{30.1}$$

$$x_s + (-1)^s x_{w-s} = 0 \tag{30.2}$$

$$x_s + (-1)^s \sum_{j=0, j\,even}^{s} \binom{s}{j} x_{w-s+j} + (-1)^s \sum_{j=0, j\equiv s\,mod 2}^{w-s} \binom{w-s}{j} x_j = 0 \tag{30.3}$$

$$\sum_{j=1, j\,odd}^{s} \binom{s}{j} x_{w-s+j} + \sum_{j=0, j\not\equiv s\,mod 2} \binom{w-s}{j} x_j = 0 \tag{30.4}$$

Note that (30.2)-(30.4) are the Eichler-Shimura relations . The system (30.1)-(30.4) with $w+1$ variables is an infinite system, where $s, k = 0, 1, ..., w$ and χ runs on the set of Dirichlet characters χ. The coefficients in the left hand side of this system do not depend on the form ϕ. We rewrite (30.1)-(30.4) in the following form:

$$AX = F$$

4.6.Conjecture. rank$A = w + 1$.

§5.Remarks and questions.

5.1. Let us now discuss about corollaries of Conjecture 4.6. Suppose that $A_1, A_2, ...$ are all the submatrices of A having maximal rank and the corresponding systems of equations are the following:

$$A_i X = F_i. \tag{31}$$

Then for every $i = 1, 2, ..$ we have a formula for the r_j's:

$$r_j(\phi) = \sum_{m=0}^{w} b_{m,i,j}(\chi_{m,i} \lambda^{n_{m,i}} f(\chi_{m,i}\theta^{k_{i,m}}), \tag{32}$$

where $\chi_{m,i}$ belong to a subset $X_i \in TorsX(Z_\Delta^*), n_{m,i} =$ conductor of $\chi_{m,i}$ and $k_{m,i} \in [0, w]$.

Thus,for all $i, l = 1, 2, ..., j = o, ..., w$ we have:

$$\sum_{m=0}^{w} b_{m,i,j}(\chi_{m,i})\lambda^{n_{m,i}} f(\chi_{m,i}\theta^{k_{m,i}}) =$$

$$\sum_{m=0}^{w} b_{m,l,j}(\chi_{m,l})\lambda^{n_{m,l}} f(\chi_{m,l}\theta^{k_{m,l}}) \tag{33}$$

Now we regard $f(\chi)$ as a function on D, as it is in previous sections. By using the Shnirelman integral we can represent the values $f(\chi\theta^k)$ in the following form:

$$f(\chi\theta^k) = \int_{a,\gamma} \frac{f(z)(z-a)}{z - \xi g^k + 1} dz \tag{34}$$

where $a \in D, |\gamma| \leq 1$. Note that the integral does not depend on the choice of a, γ.

Then (33) and (34) give us:

$$\int_{a,\gamma} f(z)\{\sum_{m=0}^{w}[\frac{b_{m,j,i}(\chi_{m,i})\lambda^{n_{m,i}}}{z - \xi_i g^{k_{m,i}} + 1} - \frac{b_{m,l,j}(\chi_{m,l})\lambda^{n_{m,l}}}{z - \xi_l g^{k_{m,l}} + 1}]\}(z-a)dz = 0. \tag{35}$$

for all $i, l = 1, 2, ..., j = 0, ..., w$.

Let $\{G_k(\lambda, z)\}$ be the set of functions:

$$\sum_{0}^{w}[\frac{b_{m,i,j}(\chi_{m,i})\lambda^{n_{m,i}}}{z - \xi_i g^{k_{m,i}} + 1} - \frac{b_{m,l,j}(\chi_{m,l})\lambda^{n_{m,l}}}{z - \xi_l g^{k_{m,l}} + 1}] \tag{36}$$

where i, l, j as above. Then it is proved the following

5.2.**Theorem.** *If Conjecture 4.6 is valid, the functions fG_k belong to class S.*

5.3. From the Eichler-Shimura isomorphism it follows that the rank of the coefficient matrix of the left hand side of systems (30.2), (30.3) with odd s, (30.4) with even s equal to the dimension ν of the space of cusp forms of weight $w+2$, and the similar rank of (30.2), (30.3) with even s, (30.4) with odd s is $\nu + 1$. Then Conjecture 4.3 says that:

$$rank\{a_j(\chi, w, k)\} \geq w - (2[\frac{w+2}{12}] + \varepsilon),$$

where

$$\varepsilon = \begin{cases} 0, & \text{if } w \neq 10 mod 12; \\ 2, & \text{if } w = 10 mod 12. \end{cases}$$

5.4. If Conjecture 4.6 holds, then the values of the periods $r_j(\phi)$ are defined by $w+1$ values of the p-adic L-function associated to ϕ. Then a cusp form of weight w which is an eigenfunction of the Hecke algebra is uniquely defined by values of its p-adic L-function at $w+1$ points $\chi\theta^k$.

5.5. Let $f(z)$ be a holomorphic function of class $o(\log^{w+1})$ in the unit disc D. Suppose that $f(z)$ satisfies the conditions $fG_k \in S$ for $k = 1, 2, ...$Then it is easy to show that the values of $f(z)$ on the set $\mu_n = \{g^k\xi - 1\}$ are defined by the values on a finite subset of μ_n. Since f is of class $o(\log^{w+1})$ this means that $f(z)$ is defined by the values on a finite set. Conjecture 4.6 says that p-adic L-functions associated to modular forms have this property.

Question. *Is it true that a function f(z) of class $o(\log^{w+1})$ is the p-adic L-function of a cusp form ϕ if the functions fG_k are of class S?*

Note that it is a p-adic analogue of the Hecke theorem on the Mellin transform of modular forms (see [Og], [We]).

5.6. Let ϕ be an eigenfunction of the Hecke algebra. Then for every Dirichlet character χ, $\phi_\chi(\tau)$ is an eigenfunction of the Hecke algebra relative to $\Gamma_o(d(\chi))$ (see [Og]). Then the relation (25) give us:

$$f(\chi\theta^k)\lambda^n\omega d(\chi)^k = r_k(\phi_\chi).$$

On the other hand we have the Eichler-Shimura relations for the periods of ϕ_χ (see ??):

$$\sum c_{k,j} r_k(\phi_\chi) = 0.$$

Hence, for every Dirichlet character χ and $k = 0, ..., w$ we have:

$$\sum_{k=0}^{w+1} c_{k,j} d(\chi)^k f(\chi\theta^k) = 0.$$

An argument analogous to those in the previous section implies that there exist the functions:

$$G_{j,\xi}(z) = \sum_{k=0}^{w+1} c_{k,j} d(\chi)^k \frac{1}{z - \xi g^k + 1}$$

such that

$$fG_{j,\xi} \in S \tag{37}$$

where ξ are roots of unity of degree $p^n, n = 1, 2, ...$

Question. *Let f(z) be a holomorphic function of class $o(\log^{w+1})$ in D satisfying the condition (37). Is f the p-adic L-function associated to an eigenfunction of the Hecke algebra?*

5.7. *When a function of class S is a Krasner analytic function? Is it true that*

$$A(D) \cap S(D) = H(D)?$$

REFERENCES

[Ad] W. Adams. Transcendental numbers in the p-adic domain. *Amer. J. Math 88(1966), 279-308.*

[Am] Y. Amice. Les nombres p-adiques. *SUP PUF, 1975.*

[A-V] Y. Amice and J. Vélu. Distributions p-adiques associées aux séries de Hecke. *Journées arithmétiques de Bordeaux, Astérisque 24-25, 1975, 119-131.*

[Ba1] D. Barsky. On Morita's p-adic Γ-function. *Groupe d'Etude ultemétrique, Paris, 1977.*

[Ba2] D. Barsky. Transformation de Cauchy p-adique et algèbre d'Iwasawa. *Math. Ann 232, 1978, 255-266.*

[Ha1] Ha Huy Khoai. On p-adic interpolation. *AMS translation: Math. Notes 26,1, 1980, 541-549.*

[Ha2] Ha Huy Khoai. On p-adic meromorphic functions. *Duke Math. J, 50, 1983, 695-711.*

[Ha3] Ha Huy Khoai. Sur le théorème de Morera p-adique. *Groupe d'Etude d'Analyse ultramétrique, 1987-88, 29-34.*

[Ha4] Ha Huy Khoai. Sur les séries L, associées aux formes modulaires. *Bull. Soc. Math. France, 120, 1992, 1-13.*

[Ko] N. Koblitz. p-adic Analysis: a Short Course on recent Work. *London Math. Society Lecture Note Series 46, 1980.*

[La] S. Lang. Introduction to Modular Forms. *Springer-Verlag : Berlin-Heidelberg-New York, 1976.*

[Laz] M. Lazard. Les zeros d'une fonction analytique d'une variable sur un corp valué complet. *Publ. Math. IHES, N.14.*

[Ma1] Y. Manin. Parabolic points and zeta functions of modular curves. *Izv. AN SSSR, vol. 6, 1, 1972, 19-64.*

[Ma2] Y. Manin. Explicit formulas for the eigenvalues of Hecke operators .*Acta Arith. 26 , 1973, 239-249.*

[Ma3] Y. Manin. Periods of parabolic forms and p-adic Hecke series . *Mat. Sb. 1973, 371-393.*

[Ma4] Y. Manin. The values of p-adic Hecke series at integer points of the critical strip. *mat. Sb. 1974, 631-637.*

[M-S] B. Mazur and H. Swinnerton-Dyer. Arithmetic of Weil curves. *Invent. Math. 25, 1974, 1-64*

[Mo] Y. Morita. A p-adic analogue of the Γ-function. *J. Fac. Sci. Univ. Tokyo, 22, 1975, 255-266.*

[Og] A. Ogg. Modular forms and Dirichlet series. *Benjamin, 1969.*

[Ra] M. Razar. Dirichlet series and Eichler cohomology. *Modular in one variable V, Springer Lecture Notes in Math.*

[Sh] L.G. Shnirelman. On functions in normed algebraically closed division rings. *Izv. AN SSSR 2, 1938, 487-498.*

[Vi] M.M. Vishik. Non-archimedean measures connected with Dirichlet series. *Mat. Sb. 99, 1976, 248-260.*

[We] A. Weil. Über der Bestimmung Dirichletscher Reihen durch Funktionelgleichungen. *Math. Ann., 168, 1967, 149-156.*

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SHNIRELMAN INTEGRAL AND p-ADIC L-FUNCTIONS ASSOCIATED TO MODULAR FORMS

Ha Huy Khoai

§1.Introduction.

It seems that at the present there is no in p-adic analysis a good analogue of the Cauchy integral. In many cases the Shnirelman integral is used in this role. The main applications of the Shnirelman integral can be found in the study of transcendent numbers in p-adic domains ([Ad]) and in construction of p-adic spectral theory ([Vi]). In an earlier paper ([Ha3]) we are interested in consideration of how the Shnirelman integral is convenient for an analogue of the Morera lemma. Namely, we considered the class of functions in the p-adic unit disc whose Shnirelman integrals are vanishing. The functions of this class have many properties analogous to one's of Krasner analytic functions, but this class is larger than the second.

In the present note we show some other situations where the above mentioned class appears. For example, p-adic L-functions associated to modular forms belong to this class with some "kernels".

In §2 we recall some basic facts about the Shnirelman integral and the class S of functions whose Shnirelman integrals are vanishing. Using the class S we give an inverse formula for the p-adic Mellin transform in §3. In §4 the functional equations satisfied by p-adic L-functions of modular forms are discribed in terms of class S. Some remarks and open questions are given in the last section.

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§2. Preliminaries.

Let p be a prime number, Q_p the field of p-adic number, and C_p the p-adic completion of the algebraic closure of C_p. The absolute value in Q_p is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the addition valuation on C_p which extends ord_p. Let D be the unit dics in C_p:

$$D = \{z \in C_p, |z| < 1\}$$

We donote by D_r the disc $\{z \in C_p, |z| < r\}$.

2.1.**Definition** . Let $f(z)$ be a C_p-valued function defined on all $z \in C_p$ such that $|z-a| = r$, where $a \in C_p$ and r is a positive real number (we shall always assume that r is in $|C_p|_p$ i.e. , a rational power of p). Let $\gamma \in C_p$ be such that $|\gamma| = r$. Then the *Shnirelman integral* is defined as the following limit if it exists:

$$\int_{a,\gamma} f(z)dz = \lim_{n\to\infty}{}' \frac{1}{n} \sum_{\xi^n=1} f(a+\xi\gamma), \tag{1}$$

where the ' indicates that the limit is only over n not divisible by p.

We recall that a function f in a domain M is said to be *Krasner analytic* if M is an union of open sets $D_i, D_i \subset D_{i+1}$ such that for each i , $f|_{D_i}$ is a uniform limit of rational functions having no poles in D_i. From properties of the Shnirelman integral we need the following.

2.2.**Theorem.***If f is Krasner analytic in $D_a(r)$, and if $|\gamma| = r$, then for fixed $z \in C_p$ we have:*

$$\int_{a,\gamma} \frac{f(x)(x-a)}{x-z}dx = \begin{cases} f(z), & \text{if } |z-a| < r; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

2.3.**Theorem.***Let f(z)=g(z)/h(z) , where g(z) is Krasner analytic in $D_a(r)$ and h(z) is a polynomial. Let $\{z_i\}$ be the roots of h(z) in $D_a(r)$, and suppose that for all $i, |z_i - a| < r$. Define $res_{z_i} f$ to be the coefficient of $(z-z_i)$ in the Laurent expansion of f(z) at z_i. Then:*

$$\int_{a,\gamma} f(z)(z-a)dz = \sum res_{z_i} f. \tag{3}$$

We refer the readers to [Ko] for more detail about the Shnirelman integral.

2.4.**Definition** . A function $f(z)$ in a domain M is said to be in class $S(M)$ if for all a, r such that $D_a(r) \subset M$

$$\int_{a,\gamma} f(z)(z-a)dz = 0. \tag{4}$$

2.5.**Remark** 1). Some basic properties of functions of class $S(M)$ can be found in [Ha3].

2). We denote by $H(M)$ and $A(M)$ respectively the class of Krasner analytic functions and locally analytic functions in M. Then $H(M) \subset S(M), A(M) \not\subset S(M)$ ([Ha3]).

§3. p adic Mellin transform.

3.1. **p-adic group of characters**. Let Δ_o be an integer prime number p and let
$$q = \begin{cases} 4, & \text{if } p = 2 \ ; \\ p, & \text{otherwise.} \end{cases}$$
We set $\Delta_o q = \Delta$ and denote:
$$Z_\Delta^* = \varprojlim (Z/\Delta p^m Z)^*$$
The group of p-adic characters is the group of continuous holomorphisms of Z_Δ^* into C_p^*:
$$X(Z_\Delta^*) = \mathrm{Hom}_{\mathrm{cont}}(Z_\Delta^*, C_p^*)$$
We set
$$U = 1 + qZ_p = \{z \in Z_p, v(z-1) \geq v(q)\}$$
Then, for every $g \in U$ such that $v(g-1) = v(q)$ the map $z \mapsto g^z$ is an isomorphism of Z_p onto U. We call g a topological generator of the group U. For each generator g of the group U the map
$$X(U) = \mathrm{Hom}_{\mathrm{cont}}(U, C_p^*) \longrightarrow C_p^*$$
transforming a continuous character χ of the group U into a point $\chi(g) - 1$ in the unit disc D of C_p. Also we have isomorphisms
$$Z_\Delta^* \simeq (Z/\Delta_o Z)^* \times Z_p^*$$
$$Z_p^* \simeq (Z/qZ)^* \times U \tag{5}$$
From isomorphisms (5) it follows that $X(Z_\Delta^*)$ is a product of a finite group and $X(U)$, while the last is isomorphic to D. Since D is an open disc of C_p, this isomorphism makes $X(Z_\Delta^*)$ into an analytic group. A function $f(\chi)$ is said to be holomorphic function on the analytic group $X(Z_\Delta^*)$ if its restriction on each component isomorphic to D is a holomorphic function. Thus, we can regard every holomorphic function in the group $X(Z_\Delta^*)$ as a holomorphic function in the unit disc D. Note that each Dirichlet character χ of conductor Δp^m is an element of the group $\mathrm{Hom}((Z/\Delta p^m Z)^*, C_p^*)$ for each $m \geq n$, and is prolonged to an unique element of the group $X(Z_\Delta^*)$ which is again denoted by χ. Thus ,the torsion subgroup of $X(Z_\Delta^*)$ is identified with the group of Dirichlet characters of conductors dp^n where $d|\Delta_o$. Let χ be such a character and let $\chi = \chi_o \cdot \chi_1$ be its decomposition in $\chi_o \in X(Z/\Delta Z)^*$ and $\chi_1 \in X(U)$. Then χ_1 takes values in the group $\mu_{p^\infty} = \cup_n \mu_{p^n}$ where by μ_{p^n} we denote the

group of p^n-roots of unity in C_p. On the other hand, every $x \in Z_\Delta^*$ can be written in the form

$$x = \alpha(x)\theta(x)$$

with $\alpha(x) \in (Z/\Delta_o Z)^*$ and $\theta(x) \in Z_p^*$. The map $\theta : x \mapsto \theta(x)$ is an element of $X(Z_\Delta^*)$ which is called *the fundamental character.*

3.2.The Mellin transform. Let μ be a measure on Z_Δ^* , i.e. μ is a continuous linear functional with values in C_p on the space of continuous functions in Z_Δ^*. Then the restriction of μ on the analytic group $X(Z_\Delta^*)$ gives an analytic function:

$$L(\mu, \chi) = \int_{Z_\Delta^*} \chi d\mu. \tag{6}$$

The function $L(\mu, \chi)$ is called the *p-adic Mellin transform of the measure μ.*

Example. p-adic L-functions associated to modular forms are p-adic Mellin transforms of measures associated to modular forms (see [A-V], [Vi]). We will return to such functions in the next section.

In this section we give an inverse formula for the p-adic Mellin transform by using the Shnirelman integral . As an application we have an integral representation of Morita's p-adic Γ-function. Note that the Shnirelman integral is used by Vishik to find an inverse formula for the Stieltjes transform (see [Vi], [Ko]).

3.3.p-adic Mellin transforms as holomorphic functions in the unit disc.

Let

$$F(\chi) = \int_{Z_\Delta^*} \chi d\mu \tag{7}$$

be the p-adic Mellin transform of a measure μ. Then by the isomorphism (5) we can regard F as a holomorphic function on the analytic group $X(Z_\Delta^*)$. This means that for every character $\chi_o \in X(Z/\Delta Z)^*$ we have a holomorphic function $F_{\chi_o}(\chi_1)$ on the group $X(U)$. Thus every p-adic Mellin transform on $X(Z_\Delta^*)$ corresponds to a collection of holomorphic functions on the group $X(U)$. Now let $f(\chi_1)$ be such a "branch" of the function $F(\chi)$. Let g be a fixed topological generator of the group U. We set

$$z = \chi_1(g) - 1. \tag{8}$$

For each $x \in U$ we have $x = g^{\log x/\log g}$ and hence that

$$\chi_1(x) = (1+z)^{\log x/\log g}.$$

Then the p-adic Mellin transform of the measure μ corresponds to the function

$$f(z) = \int_U (1+z)^{\log x/\log g} d\mu(x). \tag{9}$$

Thus for every measure μ on U, the Mellin transform given by formula(9) is a holomorphic function in the unit disc D.

3.4.**Theorem.** *Let f(z) be a bounded holomorphic function in the unit disc D. For $x \in U$ and m=1,2,... we consider the following functions in D*

$$G_{m,x}(z) = \frac{1}{p^m - p^{m-1}} \sum_{\xi^{p^m}=1,\xi\neq 1} \xi^{-\log x/\log g} \frac{1}{(z+1-\xi)}. \tag{10}$$

Then the function $f(z)$ is p-adic Mellin transform of the following measure on U:

$$\mu\{x + (U_m)\} = \int_{a,\gamma} f(z) G_{m,x}(z)(z-a)dz \tag{11}$$

where $U_m = 1 + p^m U, a, \gamma$ are such that $D_a(|\gamma|) \subset D$ and the points $1-\xi$ belong to this disc.

Proof. We first show formula (11) defines a bounded measure on U. We have:

$$\sum_{k=0}^{p-1} \mu\{x + kp^m + (U_{m+1})\} = \int_{a,\gamma} f(z) \sum_{k=o}^{p-1} G_{m+1,x+kp^m}(z)(z-a)dz$$

By an easy calculation we obtain :

$$\sum_{k=0}^{p-1} G_{m+1,x+kp^m}(z) = G_{m,x}(z).$$

From this it follows that

$$\mu\{x + (U_m)\} = \sum_{k=0}^{p-1} \mu\{x + kp^m + (U_{m+1})\}.$$

Obviously, the function $f(z)G_{m,x}(z)$ is a quotion of Krasner analytic functions in the disc $D_{a,|\gamma|}$. Then we have :

$$\mu\{x + (U_m)\} = \sum_{\xi^{p^m}=1,\xi\neq 1} res_{\xi-1}(f.G_{m,x}) =$$

$$\frac{1}{p^m - p^{m-1}} \sum f(\xi - 1)\xi^{-\log x/\log g} \tag{12}$$

By using formula (12) the boundnes of μ will be proved in 3.7. Indeed we will concern with more general situatons.

It remains to prove that $f(z)$ is the p-adic Mellin transform of the measure μ. We set

$$F(z) = \int_U (1+z)^{\log x/\log g} d\mu(x)$$

Since μ is bounded, $F(z)$ is a bounded holomorphic function in the unit disc. Then it suffices to show that the functions $f(z)$ and $F(z)$ are coincide on the set $\{\xi - 1\}$, where ξ are roots of unity of degree $p^m, m = 1, 2, ...$ (see [A-V], [Ha1], [Ha2]).

We have:

$$F(\xi - 1) = \int_U \xi^{\log x/\log g} d\mu(x) =$$

$$\sum_{\zeta^{p^m}=1, \zeta\neq 1} (\xi^{-1}\zeta)^{\log x/\log g} f(\zeta - 1) = f(\xi - 1)$$

Theorem is proved.

Thus, we have a correspondance between the set $H_o(D)$ of bounded holomorphic functions in D and the set $L(U)$ of continuous functionals on the space $C(U)$ of continuos functions on U. Namely, for any bounded measure μ on U, the p-adic Mellin transform

$$M\mu(z) = \int_U (1+z)^{\log x/\log g} d\mu$$

defines a bounded holomorphic function in D. Conversely, let $f(z)$ be a bounded holomorphic function in D . Then we have a continuous functional $Nf \in L(U)$ which is defined by:

$$C(Z_p) \ni \Phi \mapsto \lim_{n\to\infty} \sum_{i=0}^{p^n-1} \int_{a,\gamma} f(z)\Phi(x_i)G_{m,x_i}(z)(z-a)dz$$

where x_i runs on the set of representations of U/U_n.

3.5.**Theorem.**M *and* N *are mutually inverse topological isomorphism between* $H_o(D)$ and $L(U)$.

The proof is based on the formulas of operators M, N, and standard arguments.

3.6.**Morita's p-adic Γ-function.** We now apply Theorem 3.5. to Morita's p-adic Γ-function. In [Ba1] it is proved that we may consider the function $\Gamma_p(x)$ as the restriction on Z_p of a locally analytic function $\Gamma_p(z)$ of local analyticity ratio $\rho = p^{(-1/p)-(1/p-1)}$. This means for each point $x \in Z_p$ there exists ρ_x such that on $D(x, \rho_x) \cap Z_p$ the fuction $\Gamma_p(x)$ is the restriction of $F(x) = \sum_{n\geq 0} a_n(z-x)^n$ which is holomorphic on $D(x, \rho_x)$. The local analyticity ratio, by definition, is the number

$$\rho = \inf_{x\in Z_p} \rho_x > 0$$

Thus, on the disc $D(0, p^{(-1/p)-(1/p-1)})$ the function $\Gamma_p(x)$ is represented by a convergent power series. We set

$$f(z) = \Gamma_p(p^{(-1/p)-(1/p-1)}z)$$

then $f(z)$ is bounded holomorphic function on the unit disc D. We have an integral representation of the function $f(z)$:

$$f(z) = \int_U (1+z)^{\log x/\log g} d\mu$$

where the measure μ is defined by the formula (11). Hence, for Morita's p-adic Γ-function we have the follwing integral representation:

$$\Gamma_p(z) = \int_U (1+\alpha z)^{\log x/\log g} d\mu$$

where $\alpha = p^{(1/p-1)-(1/p)}$ and the measure μ is defined by the formula:

$$\mu\{x + (p^m U)\} = \frac{1}{p^m - p^{m-1}} \int_{a,\gamma} \Gamma_p(\alpha z) G_{m,x}(z)(z-a) dz.$$

3.7.**Mellin transform of non-bounded measures**. In [A-M], [Vi] Y. Amice and J. Vélu, and M. Vishik defined the p-adic Mellin transform of non-bounded measures and applied it to construct p-adic L-functions associated to modular forms. We recall here the definition and give an inverse formula for the p-adic Mellin transform of so-called *h-addmissible measures*. These measures are defined on Z^*_Δ and the corresponding Mellin transforms are holomorphic functions on the group $X(Z^*_\Delta)$. As in the case of bounded measures we consider the measures on U and corresponding holomorphic functions on the unit disc D.

Definition. Let f and g be two holomorphic functions in the unit disc D. We say f is of class $o(g)$if

$$\sup_{|z|\leq r} |f(z)| = o(\sup_{|z|\leq r} |g(z)|)$$

when $r \longrightarrow 1-0$.

Definition. A *h-addmissible measure* on U is a linear functional on the space of functions on U which are locally polynomials of degrees less than h and satisfy the following condition:

$$|\mu\{(x-b)^k \psi_{b,m}\}| = o(p^{(h-k)m}), k = 0, 1, ..., h-1,$$

where $\psi_{b,m}$ is the characteristic function of the set $b + (U_m)$.

It is proved in [Vi] that a h-addmissible measure is prolonged to a continuous linear functional on the space of $(h-1)$-differentiable functions whose derivates of oder $h-1$ satisfy the Lipschitz condition. The restriction of such a functional on the group $X(U)$ is a holomorphic function of class $o(\log^h)$ and is called *the Mellin transform* of the measure μ. The class of such measures contains, for example, the measures associated to modular forms.

Obviously, a h-addmissible measure is defined by giving its values on the set $\{x^k\psi_{b,m}\}$ with $b \in U, m = 1, 2, ...$ and $\psi_{b,m}$ is the characteristic function of the set $b + U_m$.

Theorem. *Let $f(z)$ be a holomorphic function of class* $\log^h$ *in the unit disc D. Then $f(z)$ is p-adic Mellin transform of the following h-addmissible measure on U:*

$$\mu\{x^k\psi_{b,m}\} = \int_{a,m} f(z)G_{m,b,k}(z)(z-a)dz \tag{13}$$

where $m = 1, 2, ..., k = 0, ..., h-1$ and $G_{m,b,k}$ are given by the following formula:

$$G_{m,b,k}(z) = \frac{1}{p^m - p^{m-1}} \sum_{\xi^{p^m}=1, \xi\neq 1} \frac{\xi^{-\log b/\log g}}{z + g^k\xi - 1}. \tag{14}$$

We recall that g is a fixed topological generator of the group $X(U)$, $v(g-1) \geq v(q)$.

Proof. It is easy to see that the formula (13) correctly defines a linear functional μ on the space of functions on Z_p which are locally polynomials of degrees less than h. To show that μ satisfies the conditions of h-addmissiblility we note that:

$$\mu\{(x^k\psi_{b,m}\} = \frac{1}{p^m - p^{m+1}} \sum_{\xi} \xi^{-\log b/\log g} f(g^k\xi - 1).$$

We first recall some notations. For each holomorphic functions $f(z)$ on D and each $t_o > 0$ we set

$$\|f\|_{t_o} = \sup_{v(z)=t_o} |f(z)|.$$

Then we obtain:

$$\| \log^h(1+z)\|_{t_m} = p^{mh},$$

where $t_m = 1/p^m - (p^{m-1}), m = 1, 2, ..., (|\log^h(1+z)|$ is calculated by the Newton polygon (see[Ha]). From the hypothesis we have:

$$\|f(z)\|_{t_m} = o(p^{mh})(m \to \infty).$$

Let u be the sequence $\{g^i\xi - 1\}, i = 0, 1, ..., h-1$, where $\{\xi\}$ is the sequence of primitive roots of unity of degree $p^m(m = 1, 2, ...)$. Since the function $f(z)$

is of class $o(\log^h)$, one infers u is an interpolating sequence of $f(z)$ (see [A-V], [Ha1],[Ha2]). We denote $\{S_m(z)\}$ the sequence of Lagrange's interpolation polynomials for the function $f(z)$ and the sequence u. Then $S_m(z)$ is defined by the following conditions:

$$\deg S_m(z) \leq hp^m - 1$$

$$S_m(g^i\xi - 1) = f(g^i\xi - 1), i = 0, ..., h-1.$$

By Lazard's lemma ([La]) we may represent $f(z)$ in the form:

$$f(z) = \phi(z) \prod_{\gamma\in\mu_m, i=0,...,h-1} \frac{1-z}{g^i\xi - 1} + Q_m(z) \tag{15}$$

where μ_m is the set of primitive roots of unity of degree p^m, $Q_m(z)$ are polynomials of oder hp^m satisfying the condition:

$$\|Q_m\|_{t_m} \leq \|f\|_{t_m}$$

Since the representation (15) is unique, we have $S_m(z) \equiv Q_m(z)$, and hence that

$$\|S_m\|_{t_m} \leq \|f\|_{t_m}.$$

From this it follows

$$\|S_m\|_{t_m} = o(p^{mh}).$$

Supposing $S_m(z)$ is written in the form

$$S_m(z) = \sum_{l=0}^{hp^m-1} b_l(m)z^l$$

we have then

$$\|S_m\|_{t_m} = \max_{0\leq l\leq hp^m-1}\{|b_l(m)z^l|t_m\} =$$

$$\max_l\{|b_l(m)|p^{-1/(p^m-p^{m-1})}\} > p^{-hp/(p-1)} \max_l\{|b_l(m)|\}.$$

Thus we have $\max |b_l(m)| = o(p^{mh})(m \to \infty)$. Note that if we write

$$S_m(z-1) = \sum_{l=0}^{hp^m-1} a_l(m)z^l$$

then we obtain also $\max_l |a_l(m)| = o(p^{mh})$. By definition of the measure μ we have:

$$\mu\{(x-b)^k\psi_{b,m}\} = \sum_{j=0}^{k}(-b)^{k-j}\binom{k}{j}\frac{1}{p^m - p^{m-1}}\sum_{\xi^{p^m}=1,\xi\neq 1} \xi^{-\log b/\log g} f(g^j\xi - 1) =$$

$$\sum_{j=0}^{k}(-b)^{k-j}\binom{k}{j}\frac{1}{p^m-p^{m-1}}\sum_{\xi^{p^m}=1,\xi\neq1}\xi^{-\log b/\log g}S_m(g^j\xi-1)=$$

$$\sum_{j=0}^{k}(-b)^{k-j}\binom{k}{j}\frac{1}{p^m-p^{m-1}}\sum_{\xi^{p^m}=1,\xi\neq1}\xi^{-\log b/\log g}\sum_{l=0}^{p^m-1}a_l(m)g^{jl}\xi^l=$$

$$\sum_{j=0}^{k}(-b)^{k-j}\binom{k}{j}\sum_{l=0}^{p^m-1}a_l(m)g^{jl}\frac{1}{p^m-p^{m-1}}\sum_{\xi^{p^m}=1,\xi\neq1}\xi^{l-\log b/\log g}$$

$$\sum_{j=0}^{k}(-b)^{k-j}\binom{k}{j}\sum_{g^l\equiv b}a_l(m)g^{jl}=$$

$$\sum_{g^l\equiv b}a_l(m)(g^l-a)^j$$

Since $\max|a_l(m)|=o(p^{mh})$ from this it follows the h-addmissibility of μ.

§4.p-adic L-functions associated to modular forms.

4.1. **Definition**. Let

$$\phi(\tau)=\sum_{n=1}^{\infty}\lambda_n e^{2\pi i n\tau}. \tag{16}$$

be a cusp form of weight $w+2$ on $SL(2,Z)$. Further, let ϕ is a normalized eigenfunction of the Hecke algebra, i.e. $\lambda_1=1$ and $T_n\phi=\lambda_n\phi$ for Hecke operators T_n. We set:

$$\Lambda_\phi(\chi,s)=d(\chi)^s\int_0^\infty\tau^{s-1}\phi_\chi(\tau)d\tau, \tag{17}$$

where χ is a Dirichlet character of conductor $d(\chi)$. A Manin's theorem ([Ma3], [Ma4]) asserts that there are the numbers $\omega_+\in iR$ and $\omega_-\in R$ such that for every entire $k\in[0,w]$ the numbers

$$A(\chi,k)=\frac{i^{k+1}\Lambda_\phi(\chi,k+1)}{G(\chi)\omega_{(-1)^k\chi(-1)}}, \tag{18}$$

are algebraic, where $G(\chi)$ is the Gauss sum

$$G(\chi)=\sum_{a\bmod d(\chi)}\chi(a)e^{2\pi ia/d(\chi)}. \tag{19}$$

Now let λ be a root of the equation:

$$x^2-\lambda_p x+p^{w+1}=0$$

with $v_p(\lambda) < w+1$. We set $h = [v_p(\lambda)]$. Then there exists an unique holomorphic function $f(\chi)$ on $X(Z_\Delta^*)$ such that for every $k \in [0,w]$ and every Dirichlet character $\chi = \chi_o.\chi_1$ with $d(\chi) \mid \Delta$ and $d(\chi_1) = p^n$ we have:

$$f(\chi\theta^k) = \frac{1}{\lambda^n}(-1)^k A(\chi,k). \tag{20}$$

The function $f(\chi)$ is called p-adic L-function associated to the cusp form $\phi(\tau)$ (see [Ma3, [Ma4], [M-S] , [Vi], [La], [A-V]).

Note that, f is $o(\log^{w+1})$.

4.2. p-adic L-functions as functions of class S.

We can regard $f(\chi)$ as a holomorphic function in the unit disc by using the isomorphism (5). We recall that p-adic L-functions $f(\chi)$ associated to modular forms satisfy the following functional equations:

$$f(\chi\theta^k) = \varepsilon(\chi_o) f(\chi^{-1}\theta^{w-k}) \tag{21}$$

for all characters $\chi \in X(Z_\Delta^*), k = 0,1,...,w$ and $\varepsilon(\chi_0) = \chi_o^{-1}(-1)(-1)^{k+1}$ if $\chi = \chi_o.\chi_1$ is the decomposition of χ with $\chi_o \in X((Z/\Delta Z)^*), \chi_1 \in X(U)$. For a given holomorphic function $f(\chi)$ on $X(Z_\Delta^*)$ we have a collection $\{f_{\chi_o}(\chi_1)\}$ of holomorphic functions on $X(U)$, where $\chi_o \in X((Z/\Delta Z)^*), \chi_1 \in X(U)$. Then we can write functional equations (21) in the form:

$$f_{\chi_o}(\chi_1\theta^k) = \varepsilon(\chi_o) f_{\chi_o^{-1}}(\chi_1^{-1}\theta^{w-k}) \tag{22}$$

Since the functions f_{χ_o} are in class $o(\log^{w+1})$ the functional equation (22) holds if it holds for all Dirichlet characters of conductors $p^n, n = 1,2,....$ Thus, the corresponding holomorphic functions $F_{\chi_o}(z)$ in D satisfy the following equations:

$$F_{\chi_o}[(\xi g^k - 1)] = \varepsilon(\chi_o) F_{\chi_o^{-1}}[(\xi^{-1} g^{w-k} - 1)] \tag{23}$$
$$k = 0,1,...,w, \varepsilon(\chi_o) = \chi_o^{-1}(-1)(-1)^{k-1}$$

where g is a topological generator of U, ξ is a root of unity of degree $p^n, n = 1,2,....$

By Theorem (2.2) we obtain:

$$F_{\chi_o}[(\xi g^k - 1)] = \int_{a,\gamma} F_{\chi_o}(z) \frac{(z-a)dz}{z-(\xi g^k - 1)}.$$

There is an analogous formula for the right hand side of (21). We set:

$$G_{F,\chi_o,\xi}(z) = F_{\chi_o}(z)\frac{1}{z-(\xi g^k - 1)} - \varepsilon_{\chi_o}F_{\chi_o^{-1}}(z)\frac{1}{z-(\xi^{-1}g^{w-k}-1)}. \tag{24}$$

Thus we proved the following theorem:

4.3.**Theorem.** *For every p-adic L-function F associated to a modular form the functions $G_{F,\chi_o,\xi}$ are in class S.*

4.4. By using the Eichler-Shimura isomorphism we can show some new functional equations for p-adic L-functions associated to modular forms and some corresponding functions of class S. We need more detail about the values of p-adic L-functions of modular forms at the points $\chi\theta^k$. From (18), (19), and (20) it follows that:

$$f(\chi\theta^k) = \frac{1}{\lambda^n}(-1)^k\frac{i^{k+1}d(\chi)^k}{G(\chi)\omega_{(-1)^k\chi(-1)}}\int_0^\infty \phi_\chi(\tau)\tau^k d\tau. \tag{25}$$

On the other hand we have

$$\phi_\chi(\tau) = \frac{G(\chi)}{d(\chi)}\sum_{b \bmod d(\chi)} \chi^{-1}(b)\phi(\tau + b/d(\chi)) \tag{26}$$

and hence,

$$f(\chi\theta^k) = \frac{(-1)^k i^{k+1}d(\chi)^{k-1}}{\lambda^n\omega_{(-1)^k\chi(-1)}}\sum_{b \bmod d(\chi)} \chi^{-1}(-b)\int_0^\infty \phi(\tau + b/d(\chi))\tau^k d\tau. \tag{27}$$

For any integer $j = 0, ..., w$ we define due to Eichler (see [La]) a *period* of the form $\phi(\tau)$

$$r_j(\phi) = \int_o^\infty \phi(\tau)\tau^j d\tau.$$

4.4.**Lemma.** *Let b/d be a fraction in lowest form, $0 < b/d < 1$. Then*

$$\int_0^{b/d} \phi(\tau)\tau^s d\tau$$

is a linear integral combination of the periods $r_j(\phi)$.

Note that the coefficients of the linear combination mentioned in Lemma(4.4) do not depend on ϕ, but depend on the weight w (see [La]).

By using Lemma (4.4) and the relations (27) we have:

4.5.**Lemma.** *There exist algebraic numbers $a_j(\chi, w, k)$ (not depending on ϕ) such that*

$$\lambda^n \omega_{(-1)^k \chi(-1)} f(\chi\theta^k) = \sum_j^w a_j(\chi, w, k) r_j(\phi). \tag{28}$$

We set

$$F_\phi(\chi\theta^k) = \lambda^n \omega_{(-1)^k \chi(-1)} f(\chi\theta^k). \tag{29}$$

Consider the following linear system of equations with variables x_j:

$$\sum_{j=0}^{w} a_j(\chi, w, k) x_j = F_\phi(\chi\theta^k) \tag{30.1}$$

$$x_s + (-1)^s x_{w-s} = 0 \tag{30.2}$$

$$x_s + (-1)^s \sum_{j=0, j\text{even}}^{s} \binom{s}{j} x_{w-s+j} + (-1)^s \sum_{j=0, j\equiv s \bmod 2}^{w-s} \binom{w-s}{j} x_j = 0 \tag{30.3}$$

$$\sum_{j=1, j\text{odd}}^{s} \binom{s}{j} x_{w-s+j} + \sum_{j=0, j\not\equiv s \bmod 2} \binom{w-s}{j} x_j = 0 \tag{30.4}$$

Note that (30.2)-(30.4) are the Eichler-Shimura relations . The system (30.1)-(30.4) with $w+1$ variables is an infinite system, where $s, k = 0, 1, ..., w$ and χ runs on the set of Dirichlet characters χ. The coefficients in the left hand side of this system do not depend on the form ϕ. We rewrite (30.1)-(30.4) in the following form:

$$AX = F$$

4.6.**Conjecture.** $\text{rank}A = w + 1$.

§5.Remarks and questions.

5.1. Let us now discuss about corollaries of Conjecture 4.6. Suppose that $A_1, A_2, ...$ are all the submatrices of A having maximal rank and the corresponding systems of equations are the following:

$$A_i X = F_i. \tag{31}$$

Then for every $i = 1, 2, ..$ we have a formula for the r_j's:

$$r_j(\phi) = \sum_{m=0}^{w} b_{m,i,j}(\chi_{m,i} \lambda^{n_{m,i}} f(\chi_{m,i} \theta^{k_{i,m}}), \tag{32}$$

where $\chi_{m,i}$ belong to a subset $X_i \in TorsX(Z_\Delta^*), n_{m,i} =$ conductor of $\chi_{m,i}$ and $k_{m,i} \in [0, w]$.

Thus,for all $i, l = 1, 2, ..., j = o, ..., w$ we have:

$$\sum_{m=0}^{w} b_{m,i,j}(\chi_{m,i})\lambda^{n_{m,i}} f(\chi_{m,i}\theta^{k_{m,i}}) =$$

$$\sum_{m=0}^{w} b_{m,l,j}(\chi_{m,l})\lambda^{n_{m,l}} f(\chi_{m,l}\theta^{k_{m,l}}) \tag{33}$$

Now we regard $f(\chi)$ as a function on D, as it is in previous sections. By using the Shnirelman integral we can represent the values $f(\chi\theta^k)$ in the following form:

$$f(\chi\theta^k) = \int_{a,\gamma} \frac{f(z)(z-a)}{z - \xi g^k + 1} dz \tag{34}$$

where $a \in D, |\gamma| \leq 1$. Note that the integral does not depend on the choice of a, γ.

Then (33) and (34) give us:

$$\int_{a,\gamma} f(z)\{\sum_{m=0}^{w} [\frac{b_{m,j,i}(\chi_{m,i})\lambda^{n_{m,i}}}{z - \xi_i g^{k_{m,i}} + 1} - \frac{b_{m,l,j}(\chi_{m,l})\lambda^{n_{m,l}}}{z - \xi_l g^{k_{m,l}} + 1}]\}(z-a)dz = 0. \tag{35}$$

for all $i, l = 1, 2, ..., j = 0, ..., w$.

Let $\{G_k(\lambda, z)\}$ be the set of functions:

$$\sum_0^w [\frac{b_{m,i,j}(\chi_{m,i})\lambda^{n_{m,i}}}{z - \xi_i g^{k_{m,i}} + 1} - \frac{b_{m,l,j}(\chi_{m,l})\lambda^{n_{m,l}}}{z - \xi_l g^{k_{m,l}} + 1}] \tag{36}$$

where i, l, j as above. Then it is proved the following

5.2.**Theorem.** *If Conjecture 4.6 is valid, the functions fG_k belong to class S.*

5.3. From the Eichler-Shimura isomorphism it follows that the rank of the coefficient matrix of the left hand side of systems (30.2), (30.3) with odd s, (30.4) with even s equal to the dimension ν of the space of cusp forms of weight $w+2$, and the similar rank of (30.2), (30.3) with even s, (30.4) with odd s is $\nu + 1$. Then Conjecture 4.3 says that:

$$rank\{a_j(\chi, w, k)\} \geq w - (2[\frac{w+2}{12}] + \varepsilon),$$

where

$$\varepsilon = \begin{cases} 0, & \text{if } w \neq 10 mod 12; \\ 2, & \text{if } w = 10 mod 12. \end{cases}$$

5.4. If Conjecture 4.6 holds, then the values of the periods $r_j(\phi)$ are defined by $w+1$ values of the p-adic L-function associated to ϕ. Then a cusp form of weight w which is an eigenfunction of the Hecke algebra is uniquely defined by values of its p-adic L-function at $w+1$ points $\chi\theta^k$.

5.5. Let $f(z)$ be a holomorphic function of class $o(\log^{w+1})$ in the unit disc D. Suppose that $f(z)$ satisfies the conditions $fG_k \in S$ for $k = 1, 2, \ldots$Then it is easy to show that the values of $f(z)$ on the set $\mu_n = \{g^k\xi - 1\}$ are defined by the values on a finite subset of μ_n. Since f is of class $o(\log^{w+1})$ this means that $f(z)$ is defined by the values on a finite set. Conjecture 4.6 says that p-adic L-functions associated to modular forms have this property.

Question. *Is it true that a function f(z) of class $o(\log^{w+1})$ is the p-adic L-function of a cusp form ϕ if the functions fG_k are of class S?*

Note that it is a p-adic analogue of the Hecke theorem on the Mellin transform of modular forms (see [Og], [We]).

5.6. Let ϕ be an eigenfunction of the Hecke algebra. Then for every Dirichlet character χ, $\phi_\chi(\tau)$ is an eigenfunction of the Hecke algebra relative to $\Gamma_o(d(\chi))$ (see [Og]). Then the relation (25) give us:

$$f(\chi\theta^k)\lambda^n\omega d(\chi)^k = r_k(\phi_\chi).$$

On the other hand we have the Eichler-Shimura relations for the periods of ϕ_χ (see ???):

$$\sum c_{k,j} r_k(\phi_\chi) = 0.$$

Hence, for every Dirichlet character χ and $k = 0, \ldots, w$ we have:

$$\sum_{k=0}^{w+1} c_{k,j} d(\chi)^k f(\chi\theta^k) = 0.$$

An argument analogous to those in the previous section implies that there exist the functions:

$$G_{j,\xi}(z) = \sum_{k=0}^{w+1} c_{k,j} d(\chi)^k \frac{1}{z - \xi g^k + 1}$$

such that

$$fG_{j,\xi} \in S \tag{37}$$

where ξ are roots of unity of degree $p^n, n = 1, 2, \ldots$

Question. *Let f(z) be a holomorphic function of class $o(\log^{w+1})$ in D satisfying the condition (37). Is f the p-adic L-function associated to an eigenfunction of the Hecke algebra?*

5.7. *When a function of class S is a Krasner analytic function? Is it true that*

$$A(D) \cap S(D) = H(D)?$$

REFERENCES

[Ad] W. Adams. Transcendental numbers in the p-adic domain. *Amer. J. Math 88(1966), 279-308.*

[Am] Y. Amice. Les nombres p-adiques. *SUP PUF, 1975.*

[A-V] Y. Amice and J. Vélu. Distributions p-adiques associées aux séries de Hecke. *Journées arithmétiques de Bordeaux, Astérisque 24-25, 1975, 119-131.*

[Ba1] D. Barsky. On Morita's p-adic Γ-function. *Groupe d'Etude ultemétrique, Paris, 1977.*

[Ba2] D. Barsky. Transformation de Cauchy p-adique et algèbre d'Iwasawa. *Math. Ann 232, 1978, 255-266.*

[Ha1] Ha Huy Khoai. On p-adic interpolation. *AMS translation: Math. Notes 26,1, 1980, 541-549.*

[Ha2] Ha Huy Khoai. On p-adic meromorphic functions. *Duke Math. J, 50, 1983, 695-711.*

[Ha3] Ha Huy Khoai. Sur le théorème de Morera p-adique. *Groupe d'Etude d'Analyse ultramétrique, 1987-88, 29-34.*

[Ha4] Ha Huy Khoai. Sur les séries L, associées aux formes modulaires. *Bull. Soc. Math. France, 120, 1992, 1-13.*

[Ko] N. Koblitz. p-adic Analysis: a Short Course on recent Work. *London Math. Society Lecture Note Series 46, 1980.*

[La] S. Lang. Introduction to Modular Forms. *Springer-Verlag : Berlin-Heidelberg-New York, 1976.*

[Laz] M. Lazard. Les zeros d'une fonction analytique d'une variable sur un corp valué complet. *Publ. Math. IHES, N.14.*

[Ma1] Y. Manin. Parabolic points and zeta functions of modular curves. *Izv. AN SSSR, vol. 6, 1, 1972, 19-64.*

[Ma2] Y. Manin. Explicit formulas for the eigenvalues of Hecke operators .*Acta Arith. 26 , 1973, 239-249.*

[Ma3] Y. Manin. Periods of parabolic forms and p-adic Hecke series . *Mat. Sb. 1973, 371-393.*

[Ma4] Y. Manin. The values of p-adic Hecke series at integer points of the critical strip. *mat. Sb. 1974, 631-637.*

[M-S] B. Mazur and H. Swinnerton-Dyer. Arithmetic of Weil curves. *Invent. Math. 25, 1974, 1-64*

[Mo] Y. Morita. A p-adic analogue of the Γ-function. *J. Fac. Sci. Univ. Tokyo, 22, 1975, 255-266.*

[Og] A. Ogg. Modular forms and Dirichlet series. *Benjamin, 1969.*

[Ra] M. Razar. Dirichlet series and Eichler cohomology. *Modular in one variable V, Springer Lecture Notes in Math.*

[Sh] L.G. Shnirelman. On functions in normed algebraically closed division rings. *Izv. AN SSSR 2, 1938, 487-498.*

[Vi] M.M. Vishik. Non-archimedean measures connected with Dirichlet series. *Mat. Sb. 99, 1976, 248-260.*

[We] A. Weil. Über der Bestimmung Dirichletscher Reihen durch Funktionelgleichungen. *Math. Ann., 168, 1967, 149-156.*