Billiards and Hilbert modular surfaces

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In this talk we discuss a connection between billiards in polygons and algebraic curves in the moduli space of Riemann surfaces. In genus two, we find these *Teichmüller curves* lie on Hilbert modular surfaces parameterizing Abelian varieties with real multiplication. Explicit examples give L-shaped billiard tables with optimal dynamical properties.

Let $P \subset \mathbb{C}$ be a polygon whose angles are rational multiples of π. Then a billiard ball bouncing off the sides of P moves in only finitely many directions. A typical billiard trajectory in a regular pentagon is shown in Figure 1.

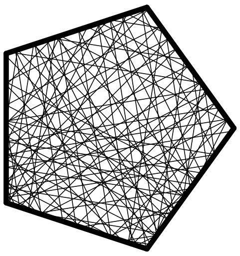

Figure 1. Billiards in a regular pentagon.

By gluing together copies of (P, dz) reflected through its sides, one obtains a compact Riemann surface X equipped with a holomorphic 1-form ω. Billiard trajectories in P then correspond to geodesics on the surface $(X, |\omega|)$. The metric $|\omega|$ is flat apart from isolated singularities coming from the vertices of P.

The space of all pairs (X, ω) forms a bundle $\Omega\mathcal{M}_g \to \mathcal{M}_g$ over the moduli space of Riemann surface of genus g, and admits a natural action of $\mathrm{SL}_2(\mathbb{R})$. Upon projection to $\mathcal{M}_g$, the $\mathrm{SL}_2(\mathbb{R})$-orbit of a given point (X, ω) determines a 'complex geodesic'

$$f : \mathbb{H} \to \mathcal{M}_g,$$

i.e. a holomorphic and isometric immersion of the hyperbolic plane into moduli space.

Usually the image of f is dense in $\mathcal{M}_g$. On rare occasions, however, f may cover an algebraic curve $V \subset \mathcal{M}_g$. This happens exactly when the stabilizer $\mathrm{SL}(X, \omega)$ of (X, ω) is a lattice in $\mathrm{SL}_2(\mathbb{R})$. In this case V is a *Teichmüller curve* and P is a *lattice polygon*.

Using renormalization and Teichmüller theory, Veech showed that billiards in a lattice polygon is dynamically optimal:

- every billiard trajectory is either periodic or uniformly distributed, and
- the number of classes of closed trajectories of length $\leq L$ is asymptotic to $c(P) \cdot L^2$.

It is classical that a square is a lattice polygon, with $\mathrm{SL}(X,\omega) = \mathrm{SL}_2(\mathbb{Z})$. Only a handful of additional examples are known: these include the regular n-gons [V], certain triangles [Wa], [KS], [Pu], and other polygons derived from these. Similarly, until recently only finitely many 'primitive' Teichmüller curves were known in each $\mathcal{M}_g$. (A Teichmüller curve is *primitive* if it does not arise from a curve in a moduli space of lower genus via a branched covering construction.)

In [Mc1] we give a synthetic construction of infinitely many primitive Teichmüller curves $V \subset \mathcal{M}_2$.

We begin by observing that if $(X,\omega) \in \Omega\mathcal{M}_2$ generates a primitive Teichmüller curve V, then the Jacobian of X admits real multiplication by the trace field $K \cong \mathbb{Q}(\sqrt{d})$ of $\mathrm{SL}(X,\omega)$. This observation shows V lies on a certain Hilbert modular surface: we have

$$V \subset \Sigma \cong (\mathbb{H} \times \mathbb{H})/\Gamma \subset \overline{\mathcal{M}_2},$$

where Γ is commensurable to $\mathrm{SL}_2(\mathcal{O}_K)$, and Σ parameterizes those X admitting real multiplication by a given order in K.

Let us say ω is a *Weierstrass form* if its zero divisor is concentrated at a single point. By imposing this additional condition, we reduce from surfaces to curves and obtain:

Theorem 1 *The locus*

$$\begin{aligned} \mathcal{W}_2 \quad = \quad & \{X : \mathrm{Jac}(X) \textit{ admits real multiplication with} \\ & \quad \textit{a Weierstrass eigenform}\} \subset \mathcal{M}_2 \end{aligned}$$

is a union of infinitely many primitive Teichmüller curves.

As a concrete application, consider the L-shaped billiard table $P(a)$ with sides of lengths 1 and a as in Figure 2. Then we find:

Theorem 2 *$P(a)$ is a lattice polygon if and only if $a = (1+\sqrt{d})/2$ for some $d \in \mathbb{Q}_+$.*

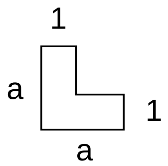

Figure 2. An L-shaped billiard table determines a Riemann surface of genus two.

The geometry of the Teichmüller curve $V \subset \Sigma \subset \mathcal{M}_2$ generated by $P(a)$ is also interesting. After passing to the universal cover, the lift

$$\widetilde{V} \subset \mathbb{H} \times \mathbb{H} = \widetilde{\Sigma}$$

becomes the graph of a holomorphic function $F : \mathbb{H} \to \mathbb{H}$, intertwining the action of $\Gamma = \mathrm{SL}(X, \omega) \subset \mathrm{SL}_2(K)$ with its Galois conjugate Γ'. When $d = 5$, F is the 'pentagon-star' map shown in Figure 3; it can be constructed by mapping an ideal pentagon to an ideal star, and then analytically continuing to the whole of $\mathbb{H}$ by Schwarz reflection. (In particular F is transcendental, and V is *not* a Shimura curve on Σ.)

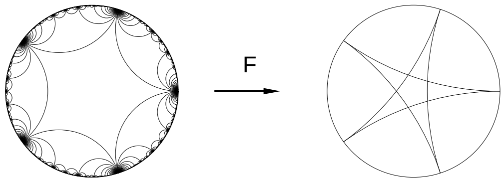

Figure 3. A holomorphic map from an ideal pentagon to an ideal star.

To place these results in context, recall that Ratner has establish a powerful classification theorem for the orbits and invariant measures of unipotent flows on homogeneous spaces [Rat]. One can hope to find a similar structure for the action of $\mathrm{SL}_2(\mathbb{R})$ on $\Omega\mathcal{M}_g$. We conclude by reporting on recent progress on classifying invariant measures and orbit closures in $\Omega\mathcal{M}_2$.

The papers [Mc1], [Mc2] contain more details and references, and are available at `http://math.harvard.edu/~ctm/papers`.

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