Jacobian Conjecture and Nilpotent Mappings

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Abstract

We prove the equivalence of the Jacobian Conjecture (JC(n)) and the Conjecture on the cardinality of the set of fixed points of a polynomial nilpotent mapping (JN(n)) and prove a series of assertions confirming JN(n).

Introduction

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a morphism of complex affine spaces. If we choose coordinates in the image and preimage, then F is defined by n polynomials $Y_i = F_i(X_1, ..., X_n) \in \mathbb{C}[X] = \mathbb{C}[X_1, ..., X_n]$ in n variables. Put

$$F'(X) = \det JF(X),$$

where

$$JF = \left(\frac{\partial F_i}{\partial X_j}\right)_{1\leq i,j\leq n}$$

is the Jacobi martix of F.

It is well-known that F is locally invertible in a neighborhood of some point $x \in \mathbb{C}^n$ (as an analytic mapping) if and only if $F'(x) \neq 0$. Therefore, "$F'(x) \neq 0$ for all $x \in \mathbb{C}^n$" is a necessary condition for the invertibility of F.

The famous *Jacobian Conjecture* ($JC(n)$) claims:

Conjecture $JC(n)$. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a morphism such that $F'(x) \neq 0$ for all $x \in \mathbb{C}^n$ (or, equivalently, $F'(X) \in \mathbb{C}^*$), then F is an isomorphism, i.e. F possesses an inverse mapping which is also given by polynomials.*

Without loss of generality we can assume that $F'(X) \equiv 1$ if $F'(X) \in \mathbb{C}^*$. The following reformulations of conjecture $JC(n)$ are well-known

Theorem 1 *(cf. [2]). Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a morphism with $F'(X) \in \mathbb{C}^*$. The Jacobian conjecture $JC(n)$ is equivalent to either of the following assertions:*

(i) F is injective;

(ii) The degree $\deg F$ of F is equal to 1 (i.e. $\mathbb{C}(X_1, ..., X_n)) = \mathbb{C}(F_1(X), ..., F_n(X))$);

(iii) The ring $\mathbb{C}[X_1, ..., X_n]$ is a finitely generated $\mathbb{C}[F_1(X), ..., F_n(X)]$-module;

(iv) The ring $\mathbb{C}[X_1, ..., X_n]$ is a projective $\mathbb{C}[F_1(X), ..., F_n(X)]$-module;

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(v) The ring $\mathbb{C}[F_1(X), ..., F_n(X)]$ is integrally closed in $\mathbb{C}[X_1, ..., X_n]$;
(vi) The field $C(X_1, ..., X_n)$ is a Galois extension over $\mathbb{C}(F_1(X), ..., F_n(X))$;
(vii) F is proper.

The Jacobian Conjecture have been formulated by O.H.Keller in 1939 [7] in the case $n = 2$ for the polynomials with integer coefficients. For $n = 1$ conjecture $JC(n)$ is trivial since a polynomial $F(X_1) \in \mathbb{C}[X_1]$ with $F'(X_1) \in \mathbb{C}^*$ is of degree one and, therefore, defines an isomorphism $F : \mathbb{C}^1 \to \mathbb{C}^1$. The Jacobian Conjecture $JC(n)$ remains an open problem even in the case $n = 2$ in spite of attempts of many mathematicians to prove it or to find a counterexample (a survey of the results on the Jacobian Conjecture one can find in [2], [4], [5]). These attempts allow to obtain many partial results connected with the Jacobian Conjecture and even several false proofs were published (for the critique of these false proofs we refer to [12] and [2]).

The aim of this short note is to attract attention of the reader to a connection of $JC(n)$ and one of conjectures on the nilpotent mappings.

Definition 1 *A morphism $N = (N_1, ..., N_n) : \mathbb{C}^n \to \mathbb{C}^n$ is called nilpotent if $JN(X)$ is a nilpotent matrix, i.e.*

$$(JN(X))^n = \left(\frac{\partial N_i}{\partial X_i}\right)^n \equiv 0$$

in the ring of matrices $\mathcal{M}(n, \mathbb{C}[X])$ of n-th order with coefficients in $\mathbb{C}[X]$.

We consider nilpotent mappings as endomorphisms of $\mathbb{C}^n$. A point $x \in \mathbb{C}^n$ is called a *fixed point* of a nilpotent mapping N if $N(x) = x$. It is well-known that if N is linear and nilpotent, then $N^n(\mathbb{C}^n)$ consists of a unique point. In particular, a linear nilpotent mapping N possesses only one fixed point.
Conjecture $JN(n)$. *A nilpotent mapping $N : \mathbb{C}^n \to \mathbb{C}^n$ possesses at most one fixed point.*
The following theorem describes a connection between the conjectures $JC(n)$ and $JN(n)$.

Theorem 2 *The conjectures $JC(n)$ and $JN(n)$ are equivalent, i.e. $JC(n)$ is true for all n if and only if $JN(n)$ is true for all n.*

In section 1, we recall some well-known results relating to the Jacobian Conjecture. In section 2, we prove Theorem 2 and a series of assertions confirming $JN(n)$.

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1 Stable equivalence of mappings and the Jacobian Conjecture

1.1. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ and $G : \mathbb{C}^n \to \mathbb{C}^n$ be morphisms given by polynomials $F_1(X), ..., F_n(X)$ and $G_1(X), ..., G_n(X)$ respectively. We shall say that F and G are *equivalent* if there exist automorphisms $H, R : \mathbb{C}^n \to \mathbb{C}^n$ such that the following diagram is commutative

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "H"] \arrow[d, "G"'] & \mathbb{C}^n \arrow[d, "F"] \\ \mathbb{C}^n \arrow[r, "R"'] & \mathbb{C}^n. \end{tikzcd}$$

In other words, F and G are equivalent if

$$R_i(G_1(X), ..., G_n(X)) = F_i(H_1(X), ..., H_n(X))$$

for all i, where $H_1(X), ..., H_n(X)$ and $R_1(X), ..., R_n(X)$ are polynomials in $\mathbb{C}[X_1, ..., X_n]$ defining the automorphisms H and R.

Definition 2 *Morphisms* $F : \mathbb{C}^n \to \mathbb{C}^n$ *and* $G : \mathbb{C}^m \to \mathbb{C}^m$, $n \leq m$, *are called stably equivalent if* $F \times Id : \mathbb{C}^m = \mathbb{C}^n \times \mathbb{C}^{m-n} \to \mathbb{C}^n \times \mathbb{C}^{m-n} = \mathbb{C}^m$ *and* G *are equivalent.*

The following assertion is a consequence of the definition.

Claim 1 *If* F *and* G *are stably equivalent, then*

$$\deg F = \deg G.$$

In particular, if $F'(X) \in \mathbb{C}^*$ *for* F *and* F *and* G *are stably equivalent, then* F *and* G *are invertible (or not invertible) simultaneously.*

1.2. To avoid confusion with the degree of a morphism, we shall call *p-degree* or *polynomial degree* of a morphism $F = (F_1, ..., F_n) : \mathbb{C}^n \to \mathbb{C}^n$ the maximum of the degrees of $F_i(X)$ defining F,

$$\deg_p F = \max \deg F_i(X).$$

Of course, p-degree depends on a choice of affine coordinates in the image and preimage.

A usual construction to simplify the form of polynomials defining a morphism is the following: one replaces a mapping by a stably equivalent one and performs linear and triangular automorphisms of the image and the preimage.

Definition 3 *An automorphism* $H : \mathbb{C}^n \to \mathbb{C}^n$ *is called triangular if, in some coordinate system, it can be given by polynomials* $H_i(X_1, ..., X_n)$, $i = 1, ..., n$, *of the form*

$$H_i(X_1, ..., X_n) = X_i + H_i(X_1, ..., X_{i-1}),$$

where $H_i(X_1, ..., X_{i-1}) \in \mathbb{C}[X_1, ..., X_{i-1}]$ *is a polynomial in variables* $X_1, ..., X_{i-1}$.

The following theorem is well-known.

Theorem 3 *([2], [15]). For any morphism* $F : \mathbb{C}^n \to \mathbb{C}^n$ *there exists a stably equivalent morphism* $G : \mathbb{C}^m \to \mathbb{C}^m$ *such that* $\deg_p G \leq 3$.

1.3. Let $F : \mathbb{C}^n \to \mathbb{C}^n$, given by polynomials $F_i(X)$, map the origin to the origin, i.e. $F(0) = 0$. A morphism $F_t : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ given by the polynomials

$$\begin{aligned} \widetilde{F}_i(X_1, ..., X_n, T) &= \sum_{j=1}^{\deg F_i} T^{j-1} F_{(j)i}(X_1, ..., X_n), \quad i = 1, ..., n, \\ \widetilde{F}_{n+1}(X_1, ..., X_n, T) &= T, \end{aligned}$$

is called *a blow up* of F, where $F_{(j)i}(X)$ is the homogeneous component of degree j of the polynomial $F_i(X)$. It is easy to check the following assertion.

Claim 2 *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a morphism such that $F(0) = 0$, then*
(i) $F_t'(X_1, ..., X_n, T) = F'(TX_1, ..., TX_n)$;
(ii) F, with $F'(X) \in \mathbb{C}^*$, is invertible if and only if F_t is invertible.

Geometrically, to obtain F_t from F, one needs to perform σ-processes with centers at the origins of the image and the preimage of the morphism $F \times Id : \mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C}^1 \to \mathbb{C}^n \times \mathbb{C}^1 = \mathbb{C}^{n+1}$ and to restrict the obtained morphism onto coordinates neighborhoods corresponding the last coordinate T.

In [3], theorem 3 was made more precise.

Theorem 4 *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a morphism such that $F(0) = 0$ and $F'(0) \neq 0$. Then the blow up F_t of F is stably equivalent to $G : \mathbb{C}^m \to \mathbb{C}^m$ defined by polynomials $G_i(X)$ of the form*

$$G_i(X_1, ..., X_m) = X_i + G_{(3)i}(X_1, ..., X_m), \tag{1}$$

where $G_{(3)i}(X_1, ..., X_m)$ are homogeneous forms of degree 3.

Since the space of the homogeneous polynomials of degree k is generated by k-th powers of the linear polynomials, the following theorem is a simple consequence of Theorem 4.

Theorem 5 *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be given by polynomials*

$$F_i(X_1, ..., X_n) = X_i + F_{(3)i}(X_1, ..., X_n).$$

Then F is stably equivalent to $G : \mathbb{C}^m \to \mathbb{C}^m$ given by polynomials $G_i(X)$ of the form

$$G_i(X_1, ..., X_m) = X_i + (\sum a_{i,j} X_j)^3. \tag{2}$$

1.4. The Jacobian conjecture was proved in [14] and [6] for the morphisms of the form (1) with $m \leq 4$ and in [6] for the morphisms of the form (2) with $m \leq 7$. Besides, in [13] the following theorem was proved

Theorem 6 *The conjecture $JC(n)$ is true for $F : \mathbb{C}^n \to \mathbb{C}^n$ of polynomial degree* $\deg_p F \leq 2$.

This theorem will be obtained in the next section as a consequence of more general result (see Proposition 1).

Note also that the conjecture $JC(n)$ was proved in [8] in the following two cases: when F is given by polynomials $F_i(X) \in X_i\mathbb{C}[X]$ and when $F_i(X) = X_i + \lambda_i M_i(X)$, where $M_i(X)$ are monomials.

1.5. In the case $F : \mathbb{C}^2 \to \mathbb{C}^2$ the Jacobian conjecture holds for all F of polynomial degree $\deg_p F \leq 100$ [10]. In [9] (see also [5]), it was proved that if the degrees of polynomials defining F with $F'(X) \in \mathbb{C}^*$ are coprime, then F is invertible and moreover in this case one of polynomials must be linear. In particular, the Jacobian conjecture holds if one of the polynomials defining F is of prime degree. In [1], this result was generalized to the case when the degree of one of the polynomials defining F is a product of two different prime integers.

In addition, in [11], it was shown that if $F : \mathbb{C}^2 \to \mathbb{C}^2$ with $F'(X) \equiv 1$ is not invertible, then $\deg F \geq 4$.

1.6. A morphism $F : \mathbb{C}^n \to \mathbb{C}^n$ induces a polynomial mapping $\widetilde{F} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by

$$\widetilde{F} = (Re\, F_1, Im\, F_1, ..., Re\, F_n, Im\, F_n),$$

where F_i are polynomials defining F. It is easy to see that

$$\det J\widetilde{F} = \mid \det JF \mid^2 .$$

Hence, $\det JF \in \mathbb{C}^*$ if and only if $\det J\widetilde{F} \in \mathbb{R}^*$.

Conjecture $RJC(n)$. *A polynomial mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ with $\det JF \in \mathbb{R}^*$ is injective.*

It is easy to show that if $RJC(n)$ is true, then $F : \mathbb{R}^n \to \mathbb{R}^n$ with $\det JF \in \mathbb{R}^*$ is invertible and the following proposition holds.

Proposition 1 *The conjecture $JC(n)$ is true for all n if and only if the conjecture $RJC(n)$ is true for all n.*

Consider $F : \mathbb{R}^n \to \mathbb{R}^n$ with $\det JF \in \mathbb{R}^*$. Without loss of generality, we can assume that $F(0) = 0$ and $JF(0) = E$, where E is the identity matrix. Write F in the form

$$F = Id + F_{(2)} + ... + F_{(m)}, \tag{3}$$

where $F_{(i)}$ are morphisms defined by homogeneous polynomials of degree i.

Definition 4 *A polynomial mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ of the form (3) is called positive (resp. negative) if all non-zero coefficients of the $F_{(i)}$ are positive (resp. negative).*

In [16], the following theorem has been proved.

Theorem 7 *(i) The conjecture $RJC(n)$ holds for all n and all negative polynomial mappings $F : \mathbb{R}^n \to \mathbb{R}^n$.*

(ii) It suffices to prove $RJC(n)$ for all positive polynomial mappings $F : \mathbb{R}^n \to \mathbb{R}^n$.

It follows from Theorem 7 (i) (passing to the blow up of F) that

The conjecture $RJC(n)$ holds for all n and for all positive polynomials mappings $F : \mathbb{R}^n \to \mathbb{R}^n$ of the form $F = Id + F_{(2)} + F_{(4)} + ... + F_{(2m)}$, and also for all polynomial mappings F of the form $F = Id + F_{(2)} - F_{(3)} + ... + (-1)^m F_{(m)}$, where $F_{(i)}$ are morphisms defined by homogeneous polynomials of degree i with non-negative coefficients.

2 Nilpotent Mappings

2.1. It is well-known that a matrix $A \in \mathcal{M}(n, \mathbb{C})$ is nilpotent if and only if $\det(tE - A) \equiv t^n$, where E is the identity matrix. As a consequence we obtain the following Lemma.

Lemma 1 *A morphism $N : \mathbb{C}^n \to \mathbb{C}^n$ given by polynomials $N_i(X)$ is nilpotent if and only if $\widetilde{F} = (Id - T \cdot N) \times Id : \mathbb{C}^n \times \mathbb{C}^1 \to \mathbb{C}^n \times \mathbb{C}^1$, given by the polynomials*

$$\begin{aligned} \widetilde{N}_i(X_1, ..., X_n, T) &= X_i - TN_i(X_1, ..., X_n) \qquad \textit{for } i = 1, \dots, n, \\ \widetilde{F}_{n+1}(X_1, ..., X_n, T) &= T, \end{aligned}$$

has $\widetilde{F}'(X, T) \equiv 1$.

If we apply lemma 1 to the restriction of $\widetilde{F} = Id - T \cdot N$ to the hypersurface $\mathbb{C}^n = \{T = 1\}$ we obtain that for the nilpotent mapping N the corresponding morphism $F = Id - N : \mathbb{C}^n \to \mathbb{C}^n$ has $F'(X) \equiv 1$. Besides, if there exist d distinct fixed points of N, then for $F = Id - N$ the preimage of the origin consists of d distinct points, and hence degree of $F = Id - N$ is not less than d. In particular, if a nilpotent mapping $N : \mathbb{C}^n \to \mathbb{C}^n$ possesses $d > 1$ fixed points, then $F = Id - N : \mathbb{C}^n \to \mathbb{C}^n$ is not invertible.

2.2. Theorem 2 is a consequence of the following theorem.

Theorem 8 *Any morphism $F : \mathbb{C}^n \to \mathbb{C}^n$ of degree $\deg F = d$ with $F'(X) \equiv 1$ is stably equivalent to $\overline{F}$ of the form*

$$\overline{F} = Id - N,$$

where N is a nilpotent morphism possessing d fixed points.

Conversely, if a nilpotent mapping $N : \mathbb{C}^n \to \mathbb{C}^n$ possesses d distinct fixed points, then $\overline{F} = Id - N$ is of degree not less than d.

Proof. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a morphism of degree $\deg F = d$ with $F'(X) \equiv 1$. Performing a translation and linear transformations, if necessary, we can assume that $F(0) = 0$, the number of the preimages of the origin $\#\{F^{-1}(0)\} = \deg F$, and F is given by polynomials

$$F_i(X_1, ..., X_n) = X_i - F_{(\geq 2)i}(X_1, ..., X_n), \tag{4}$$

where $F_{(\geq 2)i}(X_1, ..., X_n)$ are polynomials containing no monomials of degree zero and one.

Consider the blow up $F_t : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ of F. The morphism F_t is defined by polynomials

$$\begin{aligned} Y_i &= \widetilde{F}_i(X_1, ..., X_n, T) &&= X_i - \textstyle\sum_{j=2}^{\deg F_i} T^{j-1} F_{(j)i}(X_1, ..., X_n), \quad i = 1, ..., n, \\ \widetilde{T} &= \widetilde{F}_{n+1}(X_1, ..., X_n, T) &&= T. \end{aligned}$$

Let us show that F_t is stably equivalent to $\widetilde{G} = (Id - T \cdot G) \times Id : \mathbb{C}^m \times \mathbb{C}^1 \to \mathbb{C}^m \times \mathbb{C}^1$ given by $\widetilde{Y}_i = \widetilde{G}_i(X_1, ..., X_m, T) = X_i - TG_i(X_1, ..., X_m)$ for $i = 1, ..., m$, and $\widetilde{G}_{m+1}(X_1, ..., X_n, T) = T$, and of degree $\deg \widetilde{G} = \deg F$, and the number of the preimage of the point with coordinates $(0, ..., 0, 1)$ is $\#\{\widetilde{G}^{-1}((0, ..., 0, 1))\} = \deg F$. For this we must get rid of all monomials of the form

$T^kX_1^{r_1}\cdots X_n^{r_n}$ with $k>1$ and then apply Lemma 1 to $G:\mathbb{C}^m\to\mathbb{C}^m$ defined by $G_i(X_1,...,X_m)$, $i=1,...,m$. By Lemma 1, G is nilpotent and the number of the fixed points of G is equal to $\deg F$.

Removing the monomial $T^kX_1^{r_1}\cdots X_n^{r_n}$ with $k>1$ can be performed by adding new variables (passing to stably equivalent morphisms) and performing linear and triangular transformations. In fact, let for some i a polynomial $\widetilde{Y}_i$ be of the form

$$\widetilde{Y}_i=\widetilde{X}_i-\sum_{j=1}^{k_i}T^jF_{(j)i}(\widetilde{X}_1,...,\widetilde{X}_n)$$

(here $F_{(j)i}(\widetilde{X}_1,...,\widetilde{X}_n)$ is not necessary homogeneous). We add a new variable $\widetilde{Y}_{n+1}=\widetilde{X}_{n+1}$ and perform a triangular transformation in the preimage:

$$\begin{array}{lcll} X_i & = & \widetilde{X}_i, & i=1,...,n \\ X_{n+1} & = & \widetilde{X}_{n+1}+TF_{(k_i)i}(\widetilde{X}_1,...,\widetilde{X}_n). & \end{array}$$

Next we perform a triangular transformation in the image:

$$\begin{array}{rll} Y_l= & \widetilde{Y}_l, & l=1,...,i-1,i+1,...,n+1 \\ Y_i= & \widetilde{Y}_i-T^{k_i-1}\widetilde{Y}_{n+1}. & \end{array}$$

It is easy to see that in the coordinates $X_1,...,X_{n+1},T$ and $Y_1,...,Y_{n+1},T$ the morphism $F_t\times Id$ is given by equations:

$$\begin{array}{lcll} Y_s & = & X_s-\sum_{j=1}^{k_s}T^jF_{(j)s}(X_1,...,X_n), & s=1,...,i-1,i+1,...,n \\ Y_i & = & X_i-\sum_{j=1}^{k_i-1}T^jF_{(j)l}(X_1,...,X_n)-T^{k_i-1}X_{n+1}, & \\ Y_{n+1} & = & X_{n+1}-TF_{(k_i)i}(X_1,...,X_n). & \end{array}$$

Thus, we removed the summand of $\widetilde{F}_i(X_1,...,X_n,T)$ containing the variable T is contained with the maximal multiplicity k_i. Obviously, after a finite number of similar steps we obtain a morphism stably equivalent to F_t and satisfying the required conditions.

2.3. Let $F:\mathbb{C}^n\to\mathbb{C}^n$ be of the form $F=Id-F_{(k)}$, i.e. F is defined by

$$F_i(X)=X_i-F_{(k)i}(X_1,...,X_n),\qquad i=1,...,n,$$

where $F_{(k)i}(X_1,...,X_n)$ are homogeneous forms of degree k. If $F'(X)\equiv 1$, then $F_{(k)}$ is nilpotent. In fact, by Theorem 2, the Jacobian $F_t'(X,T)$ of the blow up F_t is equal to

$$F_t'(X,T)=F'(TX_1,...,TX_n)=\det(E-T^{k-1}JF_{(k)})\equiv 1.$$

Therefore, the Jacobian of $\widehat{F}_t$ which is defined by polynomials

$$\begin{array}{lcll} Y_i & = & X_i-TF_{(k)i}(X_1,...,X_n), & i=1,...,n, \\ T & = & T, & \end{array}$$

is also equal to 1. Hence, by Lemma 1, $F_{(k)}$ is nilpotent.

Proposition 2 *The conjecture $JN(n)$ holds for nilpotent morphisms $N_{(k)} : \mathbb{C}^n \to \mathbb{C}^n$ defined by homogeneous forms of degree k.*

Proof. The origin, i.e. the point $o = (0, ..., 0)$, is a fixed point of $N = N_{(k)}$. Let $x = (x_1, ..., x_n) \neq o$ be another fixed point of $N_{(k)}$. This contradicts the nilpotency of N, since the vector $\overline{x} = (x_1, ..., x_n)$ is an eigenvector (corresponding to a non-zero eigenvalue (equals to k)) of the linear mapping $JN_{(k)}(X)$ at the point $X = x$. In fact, by Euler's formula

$$\begin{aligned} JN_{(k)}(x)(\overline{x}) &= (\textstyle\sum_{j=1}^n x_j \frac{\partial N_1}{\partial X_j}(x), ..., \sum_{j=1}^n x_j \frac{\partial N_n}{\partial X_j}(x)) \\ &= (kN_1(x), ..., kN_n(x)) = \\ &= (kx_1, ..., kx_n) = k \cdot \overline{x}, \end{aligned}$$

where $N_i(X)$ are polynomials defining $N = N_{(k)}$.

Theorem 6 is an easy consequence of this theorem. For let F be of polynomial degree $\deg_p F = 2$. After a linear changes of variables, we can assume that F is of the form $F = Id - F_{(2)}$ and the preimage $F^{-1}(0)$ consists of $\deg F$ distinct points. Hence, the nilpotent morphism $F_{(2)}$ possesses $\deg F$ distinct fixed points. Therefore, $\deg F = 1$, i.e. F is invertible.

As a consequence of above considerations, Lemma 1, Theorem 6, and the poof of Theorem 8 we obtain the following theorem.

Theorem 9 *The conjecture $JN(n)$ holds for $N : \mathbb{C}^n \to \mathbb{C}^n$ of polynomial degree $\deg_p N \leq 2$.*

Proposition 2 may be generalized as follows.

Proposition 3 *The conjecture $JN(n)$ holds for $N : \mathbb{C}^n \to \mathbb{C}^n$ of the form $N = N_{(k_1)} + N_{(k_2)}$, where $N_{(k_i)}$ are morphisms defined by homogeneous forms of degree $k_i > 0$.*

Proof. The origin is a fixed point of N. Let $x = (x_1, ..., x_n) \neq o$ be another fixed point of N. As in the proof of Proposition 2 this contradicts the nilpotency of N, since the vector $\overline{x} = (x_1, ..., x_n)$ is an eigenvector (corresponding to a non-zero eigenvalue) of the linear mapping $JN(X)$ at the point $X = \lambda x = (\lambda x_1, ..., \lambda x_n)$, where

$$\lambda = \left(\frac{k_1}{k_2}\right)^{\frac{1}{k_2 - k_1}}.$$

In fact,

$$\begin{aligned} JN(\lambda x)(\overline{x}) &= (\textstyle\sum_{j=1}^n x_j \frac{\partial N_{(k_1)1}}{\partial X_j}(\lambda x), ..., \sum_{j=1}^n x_j \frac{\partial N_{(k_1)n}}{\partial X_j}(\lambda x)) + \\ &+ (\textstyle\sum_{j=1}^n x_j \frac{\partial N_{(k_2)1}}{\partial X_j}(\lambda x), ..., \sum_{j=1}^n x_j \frac{\partial N_{(k_2)n}}{\partial X_j}(\lambda x)) = \\ &= (k_1 \lambda^{k_1 - 1} N_{(k_1)1}(x), ..., k_1 \lambda^{k_1 - 1} N_{(k_1)n}(x)) + \\ &+ (k_2 \lambda^{k_2 - 1} N_{(k_2)1}(x), ..., k_2 \lambda^{k_2 - 1} N_{(k_2)n}(x)) = \\ &= k_1 \left(\tfrac{k_1}{k_2}\right)^{\frac{k_1 - 1}{k_2 - k_1}} \cdot N_{(k_1)}(x) + k_2 \left(\tfrac{k_1}{k_2}\right)^{\frac{k_2 - 1}{k_2 - k_1}} \cdot N_{(k_2)}(x) = \end{aligned}$$

$$
\begin{aligned}
&= \frac{k_1^{\frac{k_2-1}{k_2-k_1}}}{k_2^{\frac{k_1-1}{k_2-k_1}}} \cdot N_{(k_1)}(x) + \frac{k_1^{\frac{k_2-1}{k_2-k_1}}}{k_2^{\frac{k_1-1}{k_2-k_1}}} \cdot N_{(k_2)}(x) \\
&= \frac{k_1^{\frac{k_2-1}{k_2-k_1}}}{k_2^{\frac{k_1-1}{k_2-k_1}}} \cdot (N_{(k_1)}(x) + N_{(k_2)}(x)) \\
&= \frac{k_1^{\frac{k_2-1}{k_2-k_1}}}{k_2^{\frac{k_1-1}{k_2-k_1}}} \cdot \overline{x}.
\end{aligned}
$$

2.4. The following result is a consequence of Theorem 3 and the proof of Theorem 8

Theorem 10 *If each nilpotent morphism $N : \mathbb{C}^n \to \mathbb{C}^n$ of polynomial degree* $\deg_p N = 3$ *has a unique fixed point, then $JN(n)$ holds.*

In the case $\deg_p N = 3$ the morphism N is of the form

$$N = N_{(1)} + N_{(2)} + N_{(3)},$$

where $N_{(i)}$ is a homogeneous form of degree i, and Proposition 3 can be reformulated as follows:

The conjecture $JN(n)$ holds for $N : \mathbb{C}^n \to \mathbb{C}^n$ of the form $N = N_{(1)} + N_{(3)}$ and $N = N_{(2)} + N_{(3)}$.

It follows from [6] that:

The conjecture $JN(3)$ holds for $N : \mathbb{C}^3 \to \mathbb{C}^3$ of polynomial degree $\deg_p N \leq 3$.

2.5. The following proposition is a consequence of Theorem 7.

Proposition 4 *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial mapping of the form $F = Id + N$, where N is a positive nilpotent morphism. Then F is invertible.*

Proof. Without loss of generality, we can assume that $N(0) = 0$. Consider a morphism $\overline{F} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ defined by $\overline{F}_i = X_i - TN_i(X)$, $i = 1, ..., n$, and $\overline{F}_{n+1} = T$. By Theorem 7, $\overline{F}$ is invertible. The restriction of $\overline{F}$ to the hyperplane $T = -1$ coincides with F. Therefore, F is invertible.

In particular, it follows from this that:

The conjecture $JN(n)$ holds for $N : \mathbb{R}^n \to \mathbb{R}^n$ if N is either a positive or a negative nilpotent morphism.

It follows from Theorem 7 (if we use a blow up) that:

The conjecture $JN(n)$ holds for the nilpotent morphisms $N : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$N = \pm(N_{(1)} - N_{(2)} + ... + (-1)^{m+1} N_{(m)}),$$

where $N_{(i)}$ are defined by positive homogeneous forms of degree i.

2.6. Let $r = rk\, JN(X)$ be the rank of a nilpotent morphism $N : \mathbb{C}^n \to \mathbb{C}^n$. In [2] it was shown that if $r = 1$, then $F = Id - N$ is invertible:

The conjecture $JN(n)$ holds for nilpotent morphisms $N : \mathbb{C}^n \to \mathbb{C}^n$ of rank $r = 1$. In particular, the conjecture $JN(2)$ is true.

References

[1] *H. Appelgate, H. Onishi,* The Jacobian Conjecture in two variables // J.P.A.A., **37** (1985), 215-227.

[2] *H. Bass, E. Connell, D. Wright,* The Jacobian Conjecture: Reduction of degree and formal expansion of the inverse // Bulletin of the AMS., **7**, No. 2 (1982), 287-330.

[3] *L. Drużkowski,* An affective approach to Keller's Jacobian Conjecture // Math. Ann. **264** (1983), 303-313.

[4] *A. van den Essen,* Seven lectures on polynomial automorphisms // Univ. of Nijmegen, Report No. 9442 (October 1994).

[5] *A. van den Essen,* Polynomial automorphisms and the Jacobian Conjecture // Univ. of Nijmegen, Report No. 9540 (October 1995).

[6] *E. Hubberts,* The Jacobian Conjecture: Cubic homogeneous maps in dimension four // Master's thesis, Univ. of Nijmegen, Febr. 1994.

[7] *O. Keller,* Ganze Cremona-Transformationen // Monatsh. Math. Phys. **47** (1939), 299-306.

[8] *J. Lang, S.Maslamani,* Some results on the Jacobian Conjecture in higher dimension // J. of Pure and Applied Algebra **94** (1994), 327-330.

[9] *A. Magnus,* On polynomial solutions of a differential equation // Math. Scand. **3** (1955), 255-260.

[10] *T.-T. Moh,* On the global Jacobian Conjecture and the configuration of roots // J. Angew. Math. **340** (1983), 140-212.

[11] *S.Yu. Orevkov,* On three-sheeted polynomial mappings of $\mathbb{C}^2$ // Math. of the USSR, Izvestiya, vol. **29** (1987), 587-596.

[12] *A.G. Vitushkin,* On polynomial transformation of $\mathbb{C}^n$ manifolds (Tokyo, 1973) // Tokyo Univ. Press, Tokyo, 1975, 415-417.

[13] *S. Wang,* A Jacobian criterion for separability // J. of Algebra **65** (1980), 453-494.

[14] *D. Wright,* The Jacobian Conjecture: linear triangulation for cubics in dimension three // Linear and Multilinear Algebra **34** (1993), 85-97.

[15] *A. Yagzhev,* On Keller's problem // Siberian Math. J. 21 (1980), 747-754.

[16] *J. Yu,* On the Jacobian Conjecture: Reduction of coefficients // J. of Algebra **171** (1995), 515-523.

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