STRONGLY TYPICAL REPRESENTATIONS OF THE BASIC CLASSICAL LIE SUPERALGEBRAS.

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ABSTRACT. We describe the category of representations with a strongly typical central character of a basic classical Lie superalgebra in terms of representations of its even part.

1. INTRODUCTION

1.1. In [PS1], I. Penkov and V. Serganova show that the category of representations of a basic classical Lie superalgebra $\mathfrak{g}$ of type I with a fixed typical central character is equivalent to the category of representations of the even part $\mathfrak{g}_0$ with a suitable central character. A similar result for type II was proven by I. Penkov in [P] for "generic" central characters. The aim of this paper is to understand for which central characters such an equivalence holds. We also study a simplest example when such an equivalence fails to exist, but the corresponding category of representations of a Lie superalgebra still has a good description.

1.2. The basic classical Lie superalgebras were described by V. Kac in [K1]. These Lie superalgebras are the closest to the ordinary simple Lie algebras: for instance, they can be described by their Cartan matrices. The even part of a basic classical Lie superalgebra $\mathfrak{g}$ is a reductive Lie algebra. The centre $\mathcal{Z}(\mathfrak{g})$ is described in [K2], [S1], [BZV]; it is isomorphic to a subalgebra of $\mathcal{S}(\mathfrak{h})^W$ and their fields of fractions coincide (here $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_0$ and W is the Weyl group of $\mathfrak{g}_0$).

1.3. Let $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ be a basic classical Lie superalgebra, $\widetilde{\mathcal{U}}$ be its universal enveloping superalgebra and $\mathcal{Z}(\mathfrak{g})$ be the centre of $\widetilde{\mathcal{U}}$. Let T be a special *ghost* element constructed in [G1]— see 2.3. We call a maximal ideal $\widetilde{\chi}$ of $\mathcal{Z}(\mathfrak{g})$ *strongly typical* if it does not contain T^2.

For a maximal ideal $\widetilde{\chi}$ of $\mathcal{Z}(\mathfrak{g})$ and a $\mathfrak{g}$-module $\widetilde{N}$ set $\widetilde{N}_{\widetilde{\chi}} := \{v\in\widetilde{N}|\ \widetilde{\chi}^r v=0,\ \forall r>>0\}$. For a fixed $\widetilde{\chi}\in\operatorname{Max}\mathcal{Z}(\mathfrak{g})$, denote by $\operatorname{gr}\widetilde{\mathcal{C}}_r$ the category of graded $\mathfrak{g}$-modules $\widetilde{N}$ satisfying $\widetilde{\chi}^r\widetilde{N}=0$ and by $\operatorname{gr}\widetilde{\mathcal{C}}_\infty$ the category of graded $\mathfrak{g}$-modules $\widetilde{N}$ satisfying $\widetilde{N}_{\widetilde{\chi}}=\widetilde{N}$.

Consider $\mathfrak{g}_0$ as a purely even Lie superalgebra. Denote by $\mathcal{U}$ its enveloping algebra and by $\mathcal{Z}(\mathfrak{g}_0)$ the centre of $\mathcal{U}$. For a maximal ideal χ of $\mathcal{Z}(\mathfrak{g}_0)$ and a $\mathfrak{g}_0$-module N define N_χ

The author was partially supported by TMR Grant No. FMRX-CT97-0100. Research at MSRI is supported in part by NSF grant DMS-9701755.

as above. For a fixed $\chi \in \mathcal{Z}(\mathfrak{g}_0)$, denote by $\operatorname{gr}\mathcal{C}_r$ the category of graded $\mathfrak{g}_0$-modules N satisfying $\chi^r N = 0$ and by $\operatorname{gr}\mathcal{C}_\infty$ the category of graded $\mathfrak{g}_0$-modules N satisfying $N_\chi = \widetilde{N}$. In Sect. 3 we prove the following

1.3.1. **Theorem.** *For a strongly typical $\widetilde{\chi} \in \operatorname{Max}\mathcal{Z}(\mathfrak{g})$ the category $\operatorname{gr}\widetilde{\mathcal{C}}_\infty$ is equivalent to the category $\operatorname{gr}\mathcal{C}_\infty$ where χ is a suitable maximal ideal of $\mathcal{Z}(\mathfrak{g}_0)$. The equivalence is given by the functors*

$$\widetilde{N} \mapsto \widetilde{N}_\chi, \qquad N \mapsto (\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N)_{\widetilde{\chi}}.$$

The restriction of these functors provide also the equivalence of $\operatorname{gr}\widetilde{\mathcal{C}}_r$ with $\operatorname{gr}\mathcal{C}_r$.

A maximal ideal $\chi \in \operatorname{Max}\mathcal{Z}(\mathfrak{g}_0)$ is suitable in the sense of the above theorem iff for some projective Verma $\mathfrak{g}$-module $\widetilde{M}$ satisfying $\widetilde{\chi}\widetilde{M} = 0$, the $\mathfrak{g}_0$-module $\widetilde{M}_\chi$ is Verma and, moreover, one has $\widetilde{M} = \widetilde{\mathcal{U}}\widetilde{M}_\chi$. Note that if χ is a perfect mate for $\widetilde{\chi}$ in the sense of [G2], then it fulfills these conditions. In [G2], Sect.8 we prove the existence of a perfect mate χ for each strongly typical $\widetilde{\chi} \in \operatorname{Max}\mathcal{Z}(\mathfrak{g})$. This allows one to describe the category $\operatorname{gr}\widetilde{\mathcal{C}}_\infty$ corresponding to any strongly typical $\widetilde{\chi}$ in terms of Theorem 1.3.1.

The proof is based on the theorem stating that for a strongly typical $\widetilde{\chi}$ the annihilator of a Verma $\mathfrak{g}$-module $\widetilde{M}$ satisfying $\widetilde{\chi}\widetilde{M} = 0$ is equal to $\widetilde{\mathcal{U}}\widetilde{\chi}$— see [G2].

An analogue of Theorem 1.3.1 for the basic classical Lie superalgebras of type I is proven in [PS1]; for "generic" $\widetilde{\chi}$ in type II case, it is proven in [P]. Both proofs are based on a realization of the categories $\operatorname{gr}\widetilde{\mathcal{C}}_1, \operatorname{gr}\mathcal{C}_1$ as categories of D-modules on the corresponding flag varieties.

1.4. Theorem 1.3.1 suggests a study of categories of representations of $\mathfrak{g}$ corresponding to a non strongly typical character $\widetilde{\chi}$. Call such $\widetilde{\chi}$ *weakly atypical.* If $\widetilde{\chi}$ is not typical, the category $\widetilde{\mathcal{C}}_\infty$ has a very complicated structure— for instance, it contains non-trivial extensions of non-isomorphic finite dimensional modules. An "intermediate" case, when $\widetilde{\chi}$ is typical but not strongly typical (it is possible only for the types $B(m,n), G(3)$), seems to be less complicated. We consider the case when $\mathfrak{g} = \mathfrak{osp}(1,2l)$ $(B(0,2l))$ and the maximal ideal is a "generic" weakly atypical (see the condition (10)). This means that the highest weight λ of a highest weight module annihilated by $\mathfrak{m}$ belongs to exactly one of the hyperplanes $S_\beta, \beta \in \Delta_1^+$ where $S_\beta := \{\mu \in \mathfrak{h}^* |\ (\mu + \rho, \beta) = 0\}$.

It is convenient to substitute the maximal ideal $\mathfrak{m}$ of $\mathcal{Z}(\mathfrak{g})$ by the maximal ideal $\widetilde{\chi}$ of the "ghost centre" $\widetilde{\mathcal{Z}}(\mathfrak{g})$ (see [G1] for definition) which contains $\mathfrak{m}$. For $\mathfrak{g} = \mathfrak{osp}(1,2l)$ one has $\widetilde{\mathcal{Z}}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g}) \oplus T\mathcal{Z}(\mathfrak{g})$ and the ideal $\widetilde{\chi} := \mathfrak{m} \oplus T\mathcal{Z}(\mathfrak{g})$ is a maximal ideal of $\widetilde{\mathcal{Z}}(\mathfrak{g})$. Define the categories $\operatorname{gr}\widetilde{\mathcal{C}}_r$ and $\operatorname{gr}\widetilde{\mathcal{C}}_\infty$ for such $\widetilde{\chi} \in \operatorname{Max}\widetilde{\mathcal{Z}}(\mathfrak{g})$ in the same way as it was done above for $\widetilde{\chi} \in \operatorname{Max}\mathcal{Z}(\mathfrak{g})$. For fixed χ, denote by $\mathcal{C}_r$ the category of non-graded $\mathfrak{g}_0$-modules N satisfying $\chi^r N = 0$ and by $\mathcal{C}_\infty$ the full subcategory of the category of non-graded $\mathfrak{g}_0$-modules consisting of the modules N satisfying $N_\chi = \widetilde{N}$.

Under the above assumption on $\mathfrak{m}$, we prove in Sect. 4 the following

1.4.1. **Theorem.** *The category* $\operatorname{gr}\widetilde{\mathcal{C}}_\infty$ *is equivalent to the category* $\mathcal{C}_\infty$ *where* χ *is a suitable maximal ideal of* $\mathcal{Z}(\mathfrak{g}_0)$*. The equivalence is given by the functors*

$$\widetilde{N} \mapsto \widetilde{N}_\chi \cap \widetilde{N}_0, \qquad N \mapsto (\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N)_{\widetilde{\chi}}$$

where the grading on $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N$ *is determined by the assuming* N *to be even.*

The restriction of these functors provide also the equivalence of $\operatorname{gr}\widetilde{\mathcal{C}}_r$ *with* $\mathcal{C}_r$*.*

A maximal ideal $\chi \in \operatorname{Max}\mathcal{Z}(\mathfrak{g}_0)$ is suitable in the sense of the above theorem iff for some projective Verma $\mathfrak{g}$-module $\widetilde{M}$ satisfying $\widetilde{\chi}\widetilde{M} = 0$, the $\mathfrak{g}_0$-module $M := \widetilde{M}_\chi \cap \widetilde{M}_0$ is Verma and $\widetilde{M} = \widetilde{\mathcal{U}}M$. In 4.5.3 we construct such a suitable ideal χ for each $\mathfrak{m} \in \operatorname{Max}\mathcal{Z}(\mathfrak{g})$ satisfying the above assumption.

The proof is based on the theorem stating that the annihilator of a Verma $\mathfrak{g}$-module $\widetilde{M}$ satisfying $\mathfrak{m}\widetilde{M} = 0$ is equal to $\widetilde{\mathcal{U}}\widetilde{\chi}$ (see [GL1], 6.2) and on Proposition 4.7 which describes the locally finite part $F(\widetilde{M}, \widetilde{M})$ of endomorphisms of a Verma module $\widetilde{M}$ through the image of $\widetilde{\mathcal{U}}$ in it.

1.5. **Acknowledgments.** The results of this paper were obtained when the author was a visitor at MSRI and at Max-Planck Institut für Mathematik at Bonn. I express my gratitude to these institutions for the hospitality and excellent working conditions. I wish to thank V. Serganova and I. Penkov for helpful discussions.

2. Preliminaries

Everywhere in the paper $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ denotes one (unless otherwise specified, an arbitrary one) of the basic classical complex Lie superalgebras $\mathfrak{gl}(m,n)$, $\mathfrak{sl}(m,n)$, $\mathfrak{osp}(m,n)$, $\mathfrak{psl}(n,n)$. Each of these Lie superalgebras possesses the following properties: it admits a $\mathfrak{g}$-invariant bilinear form which is non-degenerate on $[\mathfrak{g},\mathfrak{g}]$ and the even part $\mathfrak{g}_0$ is a reductive Lie algebra.

2.1. **Conventions.** In this paper the ground field is $\mathbb{C}$. We denote by $\mathbb{N}^+$ the set of positive integers. If A is an algebra, N is an A-module and X, Y are subsets of A and N respectively, we denote by XY the submodule spanned by the products xy where $x \in X, y \in Y$.

For a $\mathbb{Z}_2$-homogeneous element u of a superalgebra denote by $d(u)$ its $\mathbb{Z}_2$-degree. In all formulae where this notation is used, u is assumed to be $\mathbb{Z}_2$-homogeneous.

For a Lie superalgebra $\mathfrak{m}$ denote by $\mathcal{U}(\mathfrak{m})$ its universal enveloping algebra and by $\mathcal{S}(\mathfrak{m})$ its symmetric algebra. All modules in the text are assumed to be left modules unless otherwise specified. An $\mathfrak{m}$-module N is called locally finite if $\dim \mathcal{U}(\mathfrak{m})v < \infty$ for all $v \in N$. Set $\widetilde{\mathcal{U}} := \mathcal{U}(\mathfrak{g})$ and $\mathcal{U} := \mathcal{U}(\mathfrak{g}_0)$.

The symbol $\widetilde{V}$ (resp., V) is always used for a simple $\mathfrak{g}$ (resp., $\mathfrak{g}_0$) module and the symbol $\widetilde{M}$ (resp., M) for a Verma $\mathfrak{g}$ (resp., $\mathfrak{g}_0$) module.

2.2. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$— see [PS2] for definition. Denote by Δ the set of all non-zero roots of $\mathfrak{g}$ and by Δ_0^+ (resp., Δ_1^+) the set of non-zero positive even (resp., odd) roots of $\mathfrak{g}$. Set $\overline{\Delta}_1^+ := \{\beta \in \Delta_1^+ |\ 2\beta \notin \Delta_0^+\}$. and

$$\rho := \frac{1}{2}\Big(\sum_{\alpha\in\Delta_0^+} \alpha - \sum_{\beta\in\Delta_1^+} \beta\Big).$$

Let W be the Weyl group of $\mathfrak{g}_0$. For $w \in W, \mu \in \mathfrak{h}^*$ set

$$w.\mu = w(\mu + \rho) - \rho.$$

Denote by $(-,-)$ a $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$ which is non-degenerate on $[\mathfrak{g},\mathfrak{g}]$ and also the induced W-invariant bilinear form on $\mathfrak{h}^*$. One has $\overline{\Delta}_1^+ = \{\beta \in \Delta_1^+ |\ (\beta,\beta) = 0\}$.

2.3. Define the adjoint action of $\mathfrak{g}$ on $\widetilde{\mathcal{U}}$ by setting

$$(\operatorname{ad} g)u = gu - (-1)^{d(g)d(u)}ug,\ \forall g \in \mathfrak{g}, u \in \widetilde{\mathcal{U}}$$

By default, the action of $\mathfrak{g}$ on $\widetilde{\mathcal{U}}$ is assumed to be the adjoint action. The centre $\mathcal{Z}(\mathfrak{g})$ of $\widetilde{\mathcal{U}}$ is equal to $\widetilde{\mathcal{U}}^{\operatorname{ad}\mathfrak{g}}$.

Define the twisted adjoint action of $\mathfrak{g}$ on $\widetilde{\mathcal{U}}$ by setting

$$(\operatorname{ad}' g)u = gu - (-1)^{d(g)(d(u)+1)}ug,\ \forall g \in \mathfrak{g}, u \in \widetilde{\mathcal{U}}.$$

The anticentre $\mathcal{A}(\mathfrak{g}) := \widetilde{\mathcal{U}}^{\operatorname{ad}'\mathfrak{g}}$ contains an element T defined in [G1]. This is a unique element of $\mathcal{A}(\mathfrak{g})$ satisfying $\mathcal{P}(T)(\lambda) = \prod_{\beta\in\Delta_1^+}(\beta, \lambda+\rho)$ for any $\lambda \in \mathfrak{h}^*$; here $\mathcal{P} : \widetilde{\mathcal{U}} \to \mathcal{S}(\mathfrak{h})$ is the Harish-Chandra projection. The element T is even. Since T belongs to $\mathcal{A}(\mathfrak{g})$, it commutes with the even elements of $\widetilde{\mathcal{U}}$ and anticommutes with the odd ones; in particular, $T^2 \in \mathcal{Z}(\mathfrak{g})$.

The Harish-Chandra projection $\mathcal{P}$ provides a monomorphism $\iota : \mathcal{Z}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{h})^{W.}$. The image of ι is described in [K2], [S1], [BZV]; ι is bijective iff $\mathfrak{g} = \mathfrak{osp}(1, 2l)$. The centre $\mathcal{Z}(\mathfrak{g})$ contains an element Q such that

$$\mathcal{P}(Q)(\lambda) = \prod_{\beta\in\overline{\Delta}_1^+} (\beta, \lambda + \rho).$$

The localized algebra $\mathcal{Z}(\mathfrak{g})[T^{-2}]$ is isomorphic to a localization of a polynomial algebra $\mathcal{S}(\mathfrak{h})^W$ by $\mathcal{P}(Q)$—see [K3]. The localized algebra $\mathcal{Z}(\mathfrak{g})[T^{-2}]$ is isomorphic to a localization of a polynomial algebra $\mathcal{S}(\mathfrak{h})^W$.

2.4. **The category $\tilde{\mathcal{O}}$ and Verma modules.** Denote by $\mathcal{O}$ the full subcategory of the category of $\mathfrak{g}_0$-modules consisting of finitely generated $\mathfrak{h}$-diagonalizable $\mathfrak{g}_0$-modules which are $\mathfrak{n}_0^+$-locally finite. Denote by $\tilde{\mathcal{O}}$ the similarly defined category of $\mathfrak{g}$-modules. Since $\mathcal{U}(\mathfrak{n}^+)$ is finite over $\mathcal{U}(\mathfrak{n}_0^+)$, a $\mathfrak{g}$-module N belongs $\tilde{\mathcal{O}}$ iff as a $\mathfrak{g}_0$-module N belongs to $\mathcal{O}$. In particular, any module of category $\tilde{\mathcal{O}}$ has a finite length.

2.4.1. For $\lambda \in \mathfrak{h}^*$ denote by $\mathbb{C}_\lambda$ a one-dimensional $\mathfrak{b}$-module such that $\mathfrak{n}^+ v = 0$ and $hv = \lambda(h)v$ for any $h \in \mathfrak{h}, v \in \mathbb{C}_\lambda$. Define a Verma module $\widetilde{M}(\lambda)$ by setting

$$\widetilde{M}(\lambda) := \tilde{\mathcal{U}} \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda.$$

A Verma $\mathfrak{g}_0$-module $M(\lambda)$ is defined similarly.

2.4.2. A maximal ideal $\tilde{\chi} \in \operatorname{Max} \mathcal{Z}(\mathfrak{g})$ is called *typical* if it does not contain Q defined in 2.3. If $\tilde{\chi} \in \operatorname{Max} \mathcal{Z}(\mathfrak{g})$ is typical then the set

$$W(\tilde{\chi}) := \{\lambda \in \mathfrak{h}^* |\ \tilde{\chi}\widetilde{M}(\lambda) = 0\}$$

forms a single W.-orbit. For $\mathfrak{g} = \mathfrak{osp}(1, 2l)$ all $\tilde{\chi} \in \operatorname{Max} \mathcal{Z}(\mathfrak{g})$ are typical.

Call a $\mathfrak{g}$-module N typical if $\operatorname{Ann}_{\mathcal{Z}(\mathfrak{g})} N$ is a typical maximal ideal of $\mathcal{Z}(\mathfrak{g})$. Define the partial order on $\mathfrak{h}^*$ by setting $\mu \geq \nu \iff (\mu - \nu) \in \sum_{\alpha \in \Delta^+} \mathbb{N}\alpha$. A typical Verma module $\widetilde{M}(\lambda)$ is projective (resp., simple) in $\tilde{\mathcal{O}}$ if λ is maximal (resp., minimal) in $W.\lambda$—see, for instance, [G2], 2.5.3. In particular, for a typical $\tilde{\chi} \in \operatorname{Max} \mathcal{Z}(\mathfrak{g})$ there exists a projective Verma module $\widetilde{M}$ satisfying $\tilde{\chi}\widetilde{M} = 0$.

A maximal ideal $\tilde{\chi} \in \operatorname{Max} \mathcal{Z}(\mathfrak{g})$ is called *strongly typical* if $T^2 \not\in \tilde{\chi}$. A strongly typical central character is typical. Call a $\mathfrak{g}$-module N strongly typical if $\operatorname{Ann}_{\mathcal{Z}(\mathfrak{g})} N$ is a strongly typical maximal ideal of $\mathcal{Z}(\mathfrak{g})$.

If $\mathfrak{g}$ is not of types $B(m, n), G(3)$ then $\Delta_1^+ = \overline{\Delta}_1^+$ and so the notion of typical and strongly typical module coincide.

2.5. Throughout the paper we shall write "$\operatorname{ad}\mathfrak{g}$-module" instead "$\mathfrak{g}$-module with respect to the adjoint action". For any $\mathfrak{g}$-modules N_1, N_2 view $\operatorname{Hom}(N_1, N_2) := \operatorname{Hom}_{\mathbb{C}}(N_1, N_2)$ as a $\mathfrak{g}$-module with respect to the adjoint action and denote by $F(N_1, N_2)$ the locally finite part of the $\operatorname{ad}\mathfrak{g}$-module $\operatorname{Hom}(N_1, N_2)$. Similarly for any $\mathfrak{g}_0$-modules N_1, N_2 denote by $F(N_1, N_2)$ the locally finite part of the $\operatorname{ad}\mathfrak{g}_0$-module $\operatorname{Hom}(N_1, N_2)$. Notice that for any $\mathfrak{g}$-module N its locally finite part coincides with its $\operatorname{ad}\mathfrak{g}_0$-locally finite part, since $\tilde{\mathcal{U}}$ is a finite extension of $\mathcal{U}$.

2.5.1. Let N be a $\mathfrak{g}_0$-module and E be a finite dimensional $\mathfrak{g}_0$-module. It is easy to check that $A := F(N, N), B := F(N \otimes E, N \otimes E)$ are subalgebras of $\operatorname{Hom}(N, N)$ and $\operatorname{Hom}(N \otimes E, N \otimes E)$ respectively. The algebra A acts on $F(N, N \otimes E)$ from the right and the algebra B acts on $F(N, N \otimes E)$ from the left; these actions commute. We claim that

$F(N, N\otimes E)$ is a free right A-module whose rank is equal to the dimension of E and, moreover, $\mathrm{End}_A(F(N,N\otimes E)) = B$.

Indeed, consider the map $\iota : F(N,N)\otimes E \to \mathrm{Hom}(N, N\otimes E)$ given by $\iota(f\otimes v)(n) = f(n)\otimes v$ for any $f \in F(N,N), v\in E$ and the map $\iota' : F(N, N\otimes E) \to \mathrm{Hom}(N,N)\otimes E$ given by $\psi \mapsto \sum_i p_i\circ\psi\otimes e_i$ where $\{e_i\}$ is a basis of E and $p_i : N\otimes E \to N$ are given by $p_i(n\otimes e_j) := \delta_{i,j} n\otimes e_j$. One can easily sees that $\mathrm{Im}\,\iota \subseteq F(N,N\otimes E), \mathrm{Im}\,\iota' \subseteq F(N,N)\otimes E$ and that $\iota'\circ\iota = \mathrm{id},\ \iota\circ\iota' = \mathrm{id}$. Thus $F(N,N\otimes E) \cong F(N,N)\otimes E$ is a free right A-module whose rank is equal to the dimension of E. Similarly, the map $\iota'' : A\otimes \mathrm{End}_{\mathbb{C}}(E) \to B$ given by $\iota''(f\otimes\phi)(n\otimes v) := f(n)\otimes \phi(v)$ for any $f\in A = F(N,N), n\in N, v\in E, \phi\in \mathrm{End}_{\mathbb{C}}(E)$ is bijective. For any $f, f'\in A, v\in E, \phi\in \mathrm{End}_{\mathbb{C}}(E)$ one has $\iota''(f\otimes\phi)(\iota(f'\otimes v)) = \iota(ff'\otimes\phi(v))$. This implies $\mathrm{End}_A(F(N,N\otimes E)) = B$ since the A-module $F(N,N\otimes E)$ is freely generated by the elements $\iota(1\otimes e_i)$ (here 1 is the unit of A).

2.6. Let M be a Verma $\mathfrak{g}$-module. By Duflo's theorem $\mathrm{Ann}\, M = \mathcal{U}\,\mathrm{Ann}_{\mathcal{Z}(\mathfrak{g}_0)} M$—see [D]. By [J1], 6.4 the natural map $\mathcal{U}/(\mathrm{Ann}\, M) \to F(M,M)$ is bijective.

Let $\widetilde{M}$ be a strongly typical Verma $\mathfrak{g}$-module. Then $\mathrm{Ann}\,\widetilde{M} = \widetilde{\mathcal{U}}\,\mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})}\widetilde{M}$ and the natural map $\widetilde{\mathcal{U}}/(\mathrm{Ann}\,\widetilde{M}) \to F(\widetilde{M},\widetilde{M})$ is bijective (see [G2], 9.4,9.5).

2.7. For a $\mathfrak{g}$-module $\widetilde{N}$ and a maximal ideal $\widetilde{\chi}\in\mathcal{Z}(\mathfrak{g})$ set

$$\widetilde{N}_{\widetilde{\chi}} := \{v\in\widetilde{N} |\ \widetilde{\chi}^r v = 0,\ r >> 0\}.$$

We say that a $\widetilde{\mathcal{U}}$-module $\widetilde{N}$ has *a finite support* $\mathrm{supp}_{\mathcal{Z}(\mathfrak{g})}\widetilde{N} = \{\widetilde{\chi}_1,\dots,\widetilde{\chi}_k\}$ if for any $v\in\widetilde{N}$ there exist $r_1,\dots,r_k\in\mathbb{N}^+$ such that $\prod_i \widetilde{\chi}_i^{r_i} v = 0$. In this case,

$$\widetilde{N} = \oplus_i \widetilde{N}_{\widetilde{\chi}_i}$$

and each $\widetilde{N}_{\widetilde{\chi}_i}$ is canonically isomorphic to the localization of the module $\widetilde{N}$ at $\widetilde{\chi}_i$. If $\widetilde{N}$ has a finite support and $0\to\widetilde{N}'\to\widetilde{N}\to\widetilde{N}''\to 0$ is an exact sequence then, for any $\widetilde{\chi}'\in\mathrm{Max}\,\mathcal{Z}(\mathfrak{g})$, the sequence $0\to\widetilde{N}'_{\widetilde{\chi}'}\to\widetilde{N}_{\widetilde{\chi}'}\to\widetilde{N}''_{\widetilde{\chi}'}\to 0$ is also exact.

We adopt the similar notation for $\mathcal{Z}(\mathfrak{g}_0)$ and $\mathcal{U}$-modules. For a graded $\mathfrak{g}$-module $\widetilde{N}$ and a maximal ideal χ of $\mathcal{Z}(\mathfrak{g}_0)$ we set

$$\widetilde{N}_{\chi;i} := \widetilde{N}_{\chi}\cap\widetilde{N}_i$$

for $i = 0,1$.

2.7.1. Let $\widetilde{\chi}$ be a strongly typical central character, $\widetilde{M}$ be such that $\widetilde{\chi}\widetilde{M} = 0$. Then

$$\widetilde{\mathcal{U}}\widetilde{\chi}\cap\mathcal{Z}(\mathfrak{g}_0) = \mathrm{Ann}\,\widetilde{M}\cap\mathcal{Z}(\mathfrak{g}_0) = \prod_{\chi\in\mathrm{supp}_{\mathcal{Z}(\mathfrak{g}_0)}\widetilde{M}} \chi^{r(\chi)}$$

where $r(\chi)$ is the minimal r such that $\chi^r\widetilde{M}_{\chi} = 0$. In particular, if $\chi\in\mathrm{Max}\,\mathcal{Z}(\mathfrak{g}_0)$ is such that $\widetilde{M}_{\chi}$ is a Verma $\mathfrak{g}_0$-module then $r(\chi) = 1$. Moreover, any $\mathfrak{g}$-module $\widetilde{N}$ satisfying

$\tilde{\chi}\widetilde{N} = 0$ has a finite support in $\mathcal{Z}(\mathfrak{g})$ (which is a subset of $\operatorname{supp}_{\mathcal{Z}(\mathfrak{g}_0)} \widetilde{M}$) and one has $\chi\widetilde{N}_\chi = 0$ if $\widetilde{M}_\chi$ is a Verma $\mathfrak{g}_0$-module.

2.7.2. For a strongly typical $\tilde{\chi} \in \operatorname{Max} \mathcal{Z}(\mathfrak{g})$ call $\chi \in \operatorname{Max} \mathcal{Z}(\mathfrak{g}_0)$ *a perfect mate* if

(i) For any Verma $\mathfrak{g}$-module $\widetilde{M}$ annihilated by $\tilde{\chi}$, the $\mathfrak{g}_0$-module $\widetilde{M}_\chi$ is Verma.

(ii) For any non-trivial $\mathfrak{g}$-module $\widetilde{N}$ annihilated by $\tilde{\chi}$, the $\mathfrak{g}_0$-module $\widetilde{N}_\chi$ is non-trivial.

In [G2], Sect.8 we describe a perfect mate for each strongly typical $\tilde{\chi} \in \operatorname{Max} \mathcal{Z}(\mathfrak{g})$.

2.8. For a graded $\mathfrak{g}_0$-module L denote by $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ the vector space $\widetilde{\mathcal{U}} \otimes_{\mathcal{U}} L$ (here $\widetilde{\mathcal{U}}$ is considered as a right $\mathcal{U}$-module and a left $\widetilde{\mathcal{U}}$-module through the multiplication) equipped with the natural structure of a left graded $\widetilde{\mathcal{U}}$-module. Denote by $\operatorname{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ the vector space $\operatorname{Hom}_{\mathcal{U}}(\widetilde{\mathcal{U}}, L)$ (here $\widetilde{\mathcal{U}}$ is considered as a left $\mathcal{U}$-module) equipped with the following structure of a left graded $\widetilde{\mathcal{U}}$-module: $(uf)(u') := f(u'u)$ for any $f \in \operatorname{Hom}_{\mathcal{U}}(\widetilde{\mathcal{U}}, L),\ u, u' \in \widetilde{\mathcal{U}}$. For a graded $\mathfrak{g}$-module $\widetilde{N}$ and a graded $\mathfrak{g}_0$-module L one has the canonical bijections

$$\begin{array}{c} \operatorname{Hom}_{\mathfrak{g}_0}(\widetilde{N}, L) \xrightarrow{\sim} \operatorname{Hom}_{\mathfrak{g}}(\widetilde{N}, \operatorname{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} L), \\ \operatorname{Hom}_{\mathfrak{g}_0}(L, \widetilde{N}) \xrightarrow{\sim} \operatorname{Hom}_{\mathfrak{g}}(\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L, \widetilde{N}). \end{array} \tag{1}$$

By [BF], $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L \cong \operatorname{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ for any graded $\mathfrak{g}_0$-module L.

The same formulae define non-graded versions of $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ and $\operatorname{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$. The same canonical bijections (1) take place.

3. Equivalence of Categories for a strongly typical central character

In this section we prove that the category of $\widetilde{\mathcal{U}}$-modules $\widetilde{N}$ satisfying $\widetilde{N} = \widetilde{N}_{\tilde{\chi}}$ is equivalent to the category of $\mathcal{U}$-modules N satisfying $N = N_\chi$ provided $\tilde{\chi} \in \operatorname{Max} \mathcal{Z}(\mathfrak{g})$ is strongly typical and $\chi \in \operatorname{Max} \mathcal{Z}(\mathfrak{g}_0)$ is its perfect mate—see 2.7.2. The corresponding quasi-inverse functors are given by $\widetilde{N} \mapsto \widetilde{N}_\chi$ and by $N \mapsto (\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N)_{\tilde{\chi}}$. There are two versions of this result: graded and non-graded. We consider first a non-graded version. A graded version is easily deduced from the non-graded one.

3.1. **Notation.** Take a strongly typical $\tilde{\chi} \in \operatorname{Max} \mathcal{Z}(\mathfrak{g})$ and let $\chi \in \operatorname{Max} \mathcal{Z}(\mathfrak{g}_0)$. Denote by $\widetilde{\mathcal{C}}_r$ ($r \in \mathbb{N}^+$) the category of non-graded $\widetilde{\mathcal{U}}$-modules $\widetilde{N}$ satisfying $\tilde{\chi}^r\widetilde{N} = 0$ and by $\widetilde{\mathcal{C}}_\infty$ the category of non-graded $\widetilde{\mathcal{U}}$-modules $\widetilde{N}$ satisfying $\widetilde{N} = \widetilde{N}_{\tilde{\chi}}$. Similarly, let $\mathcal{C}_r$ be the category of $\mathcal{U}$-modules N satisfying $\chi^r N = 0$ and $\mathcal{C}_\infty$ be the category of $\mathcal{U}$-modules N satisfying $N = N_{\tilde{\chi}}$. Evidently $\widetilde{\mathcal{C}}_r$ is a full subcategory of $\widetilde{\mathcal{C}}_{r+1}$ and any module in $\widetilde{\mathcal{C}}_\infty$ is a direct limit of modules belonging to $\widetilde{\mathcal{C}}_r$ for $r \to \infty$.

3.1.1. Let M be a Verma $\mathfrak{g}_0$-module such that $\chi M = 0$. We shall use the following equality

$$\mathrm{Ann}_{\tilde{\mathcal{U}}}(\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M) = \tilde{\mathcal{U}}\chi$$

which follows from the fact that $\tilde{\mathcal{U}}$ is free over $\mathcal{U}$.

Let us show that

$$\mathrm{supp}_{\mathcal{Z}(\mathfrak{g})} \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N \subseteq \mathrm{supp}_{\mathcal{Z}(\mathfrak{g})} \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M \tag{2}$$

for any $N \in \mathcal{C}_\infty$. Indeed, for any $N \in \mathcal{C}_1$

$$\mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})}(\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N) \supseteq \tilde{\mathcal{U}}\chi \cap \mathcal{Z}(\mathfrak{g}) = \mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})}(\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M)$$

that implies the inclusion (2) (for $N \in \mathcal{C}_1$). Any $N \in \mathcal{C}_r$ admits a finite filtration with the factors belonging to $\mathcal{C}_1$ and so the inclusion (2) holds for such N. To deduce (2) for any $N \in \mathcal{C}_\infty$, observe that for any $v \in \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N$ there exists a finitely generated submodule N' of N such that $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N'$ contains v. Since N' is finitely generated, it lies in $\mathcal{C}_r$ for a suitable $r \in \mathbb{N}^+$. This implies the inclusion (2).

Hence for any $N \in \mathcal{C}_\infty$

$$\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N = \oplus_{\tilde{\chi}' \in \mathrm{supp}_{\mathcal{Z}(\mathfrak{g})} \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M} (\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N)_{\tilde{\chi}'} \tag{3}$$

Similarly, choose $\widetilde{M}$ such that $\tilde{\mathcal{U}}\tilde{\chi} = \mathrm{Ann}\, \widetilde{M}$. Then for any $\widetilde{N} \in \tilde{\mathcal{C}}_\infty$

$$\widetilde{N} = \oplus_{\chi' \in \mathrm{supp}_{\mathcal{Z}(\mathfrak{g}_0)} \widetilde{M}} \widetilde{N}_{\chi'}. \tag{4}$$

3.1.2. Consider the map $f : \tilde{\mathcal{U}}/(\tilde{\mathcal{U}}\chi) \to F(M, \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M)$ induced by the natural map $\tilde{\mathcal{U}} \to F(M, \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M)$ given by $u \mapsto (m \mapsto u \otimes m)$. We claim f is an isomorphism of left $\tilde{\mathcal{U}}$ and right $\mathcal{U}$-modules. Since $\tilde{\mathcal{U}}$ is free over $\mathcal{U}$ and $\mathrm{Ann}_{\mathcal{U}} M = \mathcal{U}\chi$, the map f is injective. Kostant's Separation Theorem (see [Ko]) states the existence of an $\mathrm{ad}\,\mathfrak{g}_0$-submodule H' of $\mathcal{S}(\mathfrak{g}_0)$ such that the multiplication map provides an isomorphism $H' \otimes \mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0} \xrightarrow{\sim} \mathcal{S}(\mathfrak{g}_0)$. Then

$$\mathcal{S}(\mathfrak{g}) = \Lambda\mathfrak{g}_1 \otimes \mathcal{S}(\mathfrak{g}_0) = (\Lambda\mathfrak{g}_1 \otimes H') \otimes \mathcal{S}(\mathfrak{g}_0)^{\mathfrak{g}_0}.$$

Using [BL], 5.4, it is easy to deduce the existence of an $\mathrm{ad}\,\mathfrak{g}_0$-submodule H of $\tilde{\mathcal{U}}$ such that the multiplication map provides the isomorphism $H \otimes \mathcal{Z}(\mathfrak{g}_0) \to \tilde{\mathcal{U}}$. Moreover, $H \cong \Lambda\mathfrak{g}_1 \otimes H'$ as an $\mathrm{ad}\,\mathfrak{g}_0$-module. By 2.6, $H' \cong F(M, M)$ as $\mathrm{ad}\,\mathfrak{g}_0$-modules. The following chain of $\mathrm{ad}\,\mathfrak{g}_0$-isomorphisms

$$\tilde{\mathcal{U}}/(\tilde{\mathcal{U}}\chi) \cong \Lambda\mathfrak{g}_1 \otimes H' \cong \Lambda\mathfrak{g}_1 \otimes F(M, M) \cong F(M, \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M)$$

implies the surjectivity of f.

The bijectivity of f gives the following useful formula

$$\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N = F(M, \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M) \otimes_{\mathcal{U}} N. \tag{5}$$

for any $N \in \mathcal{C}_1$.

3.2. Fix any $\chi \in \operatorname{Max} \mathcal{Z}(\mathfrak{g}_0), \tilde{\chi} \in \operatorname{Max} \mathcal{Z}(\mathfrak{g})$. Define the functors $\Psi : \tilde{\mathcal{C}}_\infty \to \mathcal{C}_\infty$ and $\Phi : \mathcal{C}_\infty \to \tilde{\mathcal{C}}_\infty$ by the formulae

$$\Psi(\widetilde{N}) = \widetilde{N}_\chi, \quad \Phi(N) = (\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N)_{\tilde{\chi}}.$$

3.2.1. **Lemma.** *The functor Ψ is left and right adjoint to Φ.*

Proof. Using (3), one obtains for any $N \in \mathcal{C}_\infty, \widetilde{N} \in \tilde{\mathcal{C}}_\infty$

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{g}}(\Phi(N), \widetilde{N}) = \operatorname{Hom}_{\mathfrak{g}}((\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N)_{\tilde{\chi}}, \widetilde{N}) &= \operatorname{Hom}_{\mathfrak{g}}(\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N, \widetilde{N}) = \operatorname{Hom}_{\mathfrak{g}_0}(N, \widetilde{N}) \\ &= \operatorname{Hom}_{\mathfrak{g}_0}(N, \widetilde{N}_\chi) = \operatorname{Hom}_{\mathfrak{g}_0}(N, \Psi(\widetilde{N})) \end{aligned}$$

and also, using (4),

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{g}}(\widetilde{N}, \Phi(N)) &= \operatorname{Hom}_{\mathfrak{g}}(\widetilde{N}, (\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N)_{\tilde{\chi}}) = \operatorname{Hom}_{\mathfrak{g}}(\widetilde{N}, \operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N) \xrightarrow{\sim} \operatorname{Hom}_{\mathfrak{g}}(\widetilde{N}, \operatorname{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} N) \\ &= \operatorname{Hom}_{\mathfrak{g}_0}(\widetilde{N}, N) = \operatorname{Hom}_{\mathfrak{g}_0}(\widetilde{N}_\chi, N) = \operatorname{Hom}_{\mathfrak{g}_0}(\Psi(\widetilde{N}), N) \end{aligned}$$

where $=$ stands for the natural isomorphisms and $\xrightarrow{\sim}$ is induced by an isomorphism $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N \cong \operatorname{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} N$ (see 2.8). □

3.3. **Conventions.** Fix a strongly typical $\tilde{\chi} \in \operatorname{Max} \mathcal{Z}(\mathfrak{g})$ and a Verma $\mathfrak{g}$-module $\widetilde{M}$ which is projective in $\tilde{\mathcal{O}}$ and such that $\tilde{\chi}\widetilde{M} = 0$. Take $\chi \in \operatorname{Max} \mathcal{Z}(\mathfrak{g}_0)$ such that

$$\begin{array}{ll} (a) & M := \widetilde{M}_\chi \text{ is a Verma } \mathfrak{g}_0\text{-module} \\ (b) & \widetilde{M} = \tilde{\mathcal{U}} M. \end{array} \tag{6}$$

For instance, one can choose χ to be a perfect mate for $\tilde{\chi}$ (see 2.7.2).

Till the end of this section $\tilde{\chi}$, $\widetilde{M}$ and χ chosen as above are assumed to be fixed.

3.3.1. **Theorem.** *The functors*

$$\begin{array}{ll} \Psi : \tilde{\mathcal{C}}_\infty \to \mathcal{C}_\infty & \widetilde{N} \mapsto \widetilde{N}_\chi, \\ \Phi : \mathcal{C}_\infty \to \tilde{\mathcal{C}}_\infty & N \mapsto (\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N)_{\tilde{\chi}} \end{array}$$

are mutually quasi-inverse. Moreover, their restrictions provide the equivalence of the categories $\tilde{\mathcal{C}}_r$ and $\mathcal{C}_r$ for any $r \in \mathbb{N}^+$.

Outline of the proof. We know by 3.2.1 that the functors Ψ, Φ are adjoint. In 3.4.1 we reduce the required assertion to the "case $r = 1$" that is to the statement that the restriction Ψ_1, Φ_1 of the functors Ψ, Φ to the categories $\tilde{\mathcal{C}}_1, \mathcal{C}_1$ provide an equivalence of the categories. Not that 3.4.1 does not use the condition (6).

Observe that the inclusion $\Psi_1(\tilde{\mathcal{C}}_1) \subseteq \mathcal{C}_1$ immediately follows from 2.7.1, but it is not clear apriori that $\Phi_1(\mathcal{C}_1) \subseteq \tilde{\mathcal{C}}_1$. To prove that Ψ_1, Φ_1 provide an equivalence of the categories we show, using Proposition 3.4.3, that the functor $\Phi' : N \mapsto F(M, \widetilde{M}) \otimes_{\mathcal{U}} N$ provides an equivalence of the categories $\mathcal{C}_1 \to \tilde{\mathcal{C}}_1$. It is easy to show that Ψ_1 is the left quasi-inverse to Φ' and so it provides an equivalence of the categories $\tilde{\mathcal{C}}_1 \to \mathcal{C}_1$. Using this fact, we show that $\Phi(M) \cong \widetilde{M}$ and deduce that Φ_1 is isomorphic to Φ'. This will complete the proof.

3.4. **Proof of Theorem 3.3.1.**

3.4.1. *Reduction to the case $r = 1$.* The formula (3) implies that $\Phi(N)$ is isomorphic to the localization of $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N$, considered as a $\mathcal{Z}(\mathfrak{g})$-module, at the maximal ideal $\widetilde{\chi}$ (that is by the set $\mathcal{Z}(\mathfrak{g}) \setminus \widetilde{\chi}$).

Similarly, the formula (4) implies that $\Psi(\widetilde{N})$ is isomorphic to the localization of $\widetilde{N}$, considered as a $\mathcal{Z}(\mathfrak{g}_0)$-module, at the maximal ideal χ.

Taking into account that the induction and the localization functors are exact and commute with direct limits, one concludes that the assertion of Theorem 3.3.1 is equivalent to the statement that the restrictions Ψ_1, Φ_1 of the functors Ψ, Φ to the categories $\widetilde{\mathcal{C}}_1, \mathcal{C}_1$ provide an equivalence of the categories.

3.4.2. *Case $r = 1$.* It remains to show that the restriction Ψ_1 of Ψ to the subcategory $\widetilde{\mathcal{C}}_1$ and the restriction Φ_1 of Φ to the subcategory $\mathcal{C}_1$ provide an equivalence of the categories $\widetilde{\mathcal{C}}_1$ and $\mathcal{C}_1$.

We start with the following technical proposition.

3.4.3. **Lemma.** *Assume that $\widetilde{N}$ is a $\mathfrak{g}$-module, N is a $\mathfrak{g}_0$-direct summand of $\widetilde{N}$ and $\widetilde{N}$ is a $\mathfrak{g}$-direct summand of $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N$. Then the $F(\widetilde{N}, \widetilde{N})$-$F(N, N)$ bimodule $F(N, \widetilde{N})$ provides a Morita equivalence between the algebras $F(N, N)$ and $F(\widetilde{N}, \widetilde{N})$.*

Proof. Denote the algebra $F(N, N)$ by A. For any $\mathfrak{g}$-module X endow the vector space $F(N, X)$ with the natural right A-module structure. Since N is a $\mathfrak{g}_0$-direct summand of $\widetilde{N}$, A as a right module over itself is a direct summand of $F(N, \widetilde{N})$. To obtain the statement one has to check that $F(N, \widetilde{N})$ is a finitely generated projective right A-module and that $\mathrm{End}_A(F(N, \widetilde{N})) = F(\widetilde{N}, \widetilde{N})$.

Denote $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N$ by I. Recall that as a $\mathfrak{g}_0$-module $I = \Lambda\mathfrak{g}_1 \otimes N$. By 2.5.1, $F(N, I)$ is a free A-module whose rank is equal to the dimension of $\Lambda\mathfrak{g}_1$. Since $\widetilde{N}$ is a direct summand of I, the A-module $F(N, \widetilde{N})$ is a direct summand of $F(N, I)$. Hence $F(N, \widetilde{N})$ is a finitely generated projective right A-module. One has $\mathrm{End}_A(F(N, \widetilde{N})) = p\,\mathrm{End}_A(F(N, I))p$ where $p \in \mathrm{End}_A(F(N, I))$ is the idempotent with the image $F(N, \widetilde{N})$ corresponding to the decomposition $F(N, I) = F(N, \widetilde{N}) \oplus F(N, G)$. By 2.5.1, the left action of $F(I, I)$ on $F(N, I)$ induces an isomorphism $F(I, I) \xrightarrow{\sim} \mathrm{End}_A(F(N, I))$. Thus the left action of $F(\widetilde{N}, \widetilde{N})$ on $F(N, \widetilde{N})$ induces an isomorphism $F(\widetilde{N}, \widetilde{N}) \xrightarrow{\sim} \mathrm{End}_A(F(N, \widetilde{N}))$. The assertion follows. $\square$

3.4.4. Retain the notation of 3.3. Consider a canonical map $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M \to \widetilde{M}$. By the property (b) of (6), this map is surjective. Recall $\widetilde{M}$ is projective in $\widetilde{\mathcal{O}}$. Consequently, $\widetilde{M}$ is a $\mathfrak{g}$-direct summand of $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M$ and so the pair $(\widetilde{M}, M)$ satisfies the assumptions of Lemma 3.4.3. From 2.6 it follows that $F(\widetilde{M}, \widetilde{M}) \cong \widetilde{\mathcal{U}}_{\widetilde{\chi}}$ and $F(M, M) \cong \mathcal{U}_\chi$ as algebras. Taking into account Lemma 3.4.3 one concludes that the functor $\Phi' : \mathcal{C}_1 \to \widetilde{\mathcal{C}}_1$ defined by

$$\Phi'(N) := F(M, \widetilde{M}) \otimes_{\mathcal{U}_\chi} N = F(M, \widetilde{M}) \otimes_{\mathcal{U}} N$$

provides an equivalence of the categories.

3.4.5. One has

$$\Psi_1 \circ \Phi'(N) = F(M, \widetilde{M}_\chi) \otimes_{\mathcal{U}} N = F(M, M) \otimes_{\mathcal{U}} N = N$$

since $F(M, M) \cong \mathcal{U}_\chi$. Hence Ψ_1 is quasi-inverse to Φ' and it provides an equivalence of the categories $\widetilde{\mathcal{C}}_1 \to \mathcal{C}_1$.

Let us verify that the functors Φ_1 and Φ' are isomorphic. Formula (5) implies that

$$\Phi_1(N) = F(M, (\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M)_{\widetilde{\chi}}) \otimes_{\mathcal{U}} N = F(M, \Phi_1(M)) \otimes_{\mathcal{U}} N \tag{7}$$

for any $N \in \mathcal{C}_1$. Thus to show that $\Phi_1(N) \cong \Phi'(N)$ for all $N \in \mathcal{C}_1$ it is enough to verify only that

$$(\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M)_{\widetilde{\chi}} \cong \widetilde{M}. \tag{8}$$

As we have shown above, $\widetilde{M}$ is a $\mathfrak{g}$-direct summand of $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M$. Thus it is enough to check that

$$\mathrm{Hom}_{\mathfrak{g}}(\widetilde{N}, \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M) = 0$$

for all simple $\widetilde{N} \in \widetilde{\mathcal{C}}_1$ such that $\widetilde{N} \not\cong \mathrm{Soc}\, \widetilde{M}$ and that

$$\dim \mathrm{Hom}_{\mathfrak{g}}(\mathrm{Soc}\, \widetilde{M}, \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M) = 1.$$

Recall that $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M \cong \mathrm{Coind}_{\mathfrak{g}_0}^{\mathfrak{g}} M$ (see 2.8) and so

$$\mathrm{Hom}_{\mathfrak{g}}(\widetilde{N}, \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M) = \mathrm{Hom}_{\mathfrak{g}_0}(\widetilde{N}, M) = \mathrm{Hom}_{\mathfrak{g}_0}(\widetilde{N}_\chi, M).$$

If $\widetilde{N} \in \widetilde{\mathcal{C}}_1$ is simple, $\widetilde{N}_\chi = \Psi_1(\widetilde{N}) \in \mathcal{C}_1$ is also simple. Taking into account that the $\mathfrak{g}_0$-socle of M is a simple Verma $\mathfrak{g}_0$-module, one concludes that $\mathrm{Hom}_{\mathfrak{g}_0}(\widetilde{N}_\chi, M) = 0$ if $\widetilde{N}_\chi \not\cong \mathrm{Soc}\, M$ and $\dim \mathrm{Hom}_{\mathfrak{g}_0}(\widetilde{N}_\chi, M) = 1$ otherwise. If $\Psi_1(\widetilde{N}) = \widetilde{N}_\chi \cong \mathrm{Soc}\, M$ then $\widetilde{N} \cong \mathrm{Soc}\, \widetilde{M}$ since $\Psi_1 : \widetilde{\mathcal{C}}_1 \to \mathcal{C}_1$ provides an equivalence of the categories and $\Psi_1(\widetilde{M}) = M$. This proves (8).

Hence the functors Φ_1 and Φ' are isomorphic. This completes the proof of Theorem 3.3.1.

3.5. **Remarks.**

3.5.1. Recall that any $\widetilde{N} \in \widetilde{\mathcal{C}}_\infty$ is generated by $\widetilde{N}_\chi$ and so is a homomorphic image of $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0} \widetilde{N}_\chi$. Consequently, the restrictions of the functors Ψ, Φ provide an equivalence between $\widetilde{\mathcal{C}}_\infty \cap \widetilde{\mathcal{O}}$ and $\mathcal{C}_\infty \cap \mathcal{O}$. The same is true for the categories of weight modules. Also, one can easily see that the Verma modules of $\widetilde{\mathcal{C}}_\infty \cap \widetilde{\mathcal{O}}$ correspond to the Verma modules of $\mathcal{C}_\infty \cap \mathcal{O}$.

3.5.2. Formula (8) which says that $\Phi(M) \cong \widetilde{M}$ is crucial for the whole proof. If one had an independent proof of this formula, one could deduce from it a straightforward proof of Theorem 3.3.1.

3.6. **Graded case.** Denote by $\mathrm{gr}\,\widetilde{\mathcal{C}}_\infty$ (resp., $\mathrm{gr}\,\widetilde{\mathcal{C}}_r$) the full category of $\mathbb{Z}_2$-graded $\widetilde{\mathcal{U}}$-modules $\widetilde{N}$ satisfying $\widetilde{N}_{\widetilde{\chi}} = \widetilde{N}$ (resp., $\widetilde{\chi}^r \widetilde{N} = 0$). Consider $\mathcal{U}$ as a purely even superalgebra and define similarly $\mathrm{gr}\,\mathcal{C}_\infty, \mathrm{gr}\,\mathcal{C}_r$. Denote by $\#$ the forgetful functors $\mathrm{gr}\,\widetilde{\mathcal{C}}_\infty \to \widetilde{\mathcal{C}}_\infty$ and $\mathrm{gr}\,\mathcal{C}_\infty \to \mathcal{C}_\infty$.

3.6.1. Evidently any $N' \in \mathcal{C}_\infty$ is isomorphic to $N^\#$ for some $N \in \mathrm{gr}\,\mathcal{C}_\infty$— for instance, one can consider N' as a purely even (odd) module. It turns out that the similar assertion holds for $\widetilde{\mathcal{C}}_\infty$. Indeed, since $\widetilde{\chi}$ is strongly typical, it contains an element $(T^2 - c^2)$ for some non-zero $c \in \mathbb{C}$. Denote by $\widetilde{\mathcal{U}}_0$ the even part of $\widetilde{\mathcal{U}}$; recall that T lies in the centre of $\widetilde{\mathcal{U}}_0$. Any $\widetilde{N} \in \widetilde{\mathcal{C}}_\infty$ is a direct sum of the $\widetilde{\mathcal{U}}_0$-modules N_+, N_- where

$$N_+ := \{v \in \widetilde{M} |\ (T-c)^r v = 0,\ \forall r >> 0\}, \quad N_- := \{v \in \widetilde{M} |\ (T+c)^r v = 0,\ \forall r >> 0\}.$$

One can define a $\mathbb{Z}_2$-grading on $\widetilde{N}$ by putting $\widetilde{N}_0 := N_+$, $\widetilde{N}_1 := N_-$. In such a way, one obtains a functor $\widetilde{\mathcal{C}}_\infty \to \mathrm{gr}\,\widetilde{\mathcal{C}}_\infty$ which is left quasi-inverse to the functor $\#$. This implies, in particular, that for any irreducible graded $\widetilde{\mathcal{U}}_{\widetilde{\chi}}$-module $\widetilde{N}$, its image $\widetilde{N}^\#$ remains irreducible (non-graded) module.

3.6.2. **Theorem.** *The functors*

$$\begin{aligned} \Psi^{\mathrm{gr}} &: \mathrm{gr}\,\widetilde{\mathcal{C}}_\infty \to \mathrm{gr}\,\mathcal{C}_\infty, \quad \widetilde{N} \mapsto \widetilde{N}_\chi, \\ \Phi^{\mathrm{gr}} &: \mathrm{gr}\,\mathcal{C}_\infty \to \mathrm{gr}\,\widetilde{\mathcal{C}}_\infty, \quad N \mapsto (\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0} N)_{\widetilde{\chi}} \end{aligned}$$

are mutually quasi-inverse. Their restriction provides an equivalence of the categories $\mathrm{gr}\,\widetilde{\mathcal{C}}_r$ *and* $\mathrm{gr}\,\mathcal{C}_r$ *for any* $r \in \mathbb{N}^+$.

Proof. Repeating the arguments of Lemma 3.2.1, one shows that the functor Ψ^{gr} is left and right adjoint to the functor Φ^{gr}.

The functors Ψ, Φ are quasi-inverse and so the canonical homomorphism $\alpha_N : N \to \Psi \circ \Phi(N)$ is an isomorphism for any $N \in \mathcal{C}_\infty$. For any $N \in \mathrm{gr}\,\mathcal{C}_\infty$ denote by α_N the canonical homomorphism $N \to \Psi^{\mathrm{gr}} \circ \Phi^{\mathrm{gr}}(N)$. One has $\# \circ \Psi^{\mathrm{gr}} = \Psi \circ \#$ and $\# \circ \Phi^{\mathrm{gr}} = \Phi \circ \#$. Therefore $\#(\alpha_N) = \alpha_{N\#}$. Since $\alpha_{N\#}$ is an isomorphism, α_N is also an isomorphism.

Similarly, the canonical homomorphism $\beta_{\widetilde{N}}:\Phi^{\mathrm{gr}}\circ\Psi^{\mathrm{gr}}(\widetilde{N})\to\widetilde{N}$ is an isomorphism for any $\widetilde{N}\in\operatorname{gr}\widetilde{\mathcal{C}}_\infty$. Hence $\operatorname{gr}\Psi$ and $\operatorname{gr}\Phi$ are quasi-inverse.

The equality $\Psi^{\mathrm{gr}}(\operatorname{gr}\widetilde{\mathcal{C}}_r)=\operatorname{gr}\mathcal{C}_r$ follows from the equality $\Psi(\operatorname{gr}\widetilde{\mathcal{C}}_r)=\mathcal{C}_r$. This implies the second claim. □

4. An Example

Say that a $\widetilde{\mathcal{U}}$-central character is weakly atypical if it is not strongly typical. In this section we consider a "generic" weakly atypical central characters for $\mathfrak{g}:=\mathfrak{osp}(1,2l)$. We show that, for such a character $\mathfrak{m}\in\operatorname{Max}\mathcal{Z}(\mathfrak{g})$, the category $\operatorname{gr}\widetilde{\mathcal{C}}_\infty$ of graded $\mathfrak{g}$-modules $\widetilde{N}$ satisfying $\widetilde{N}_{\mathfrak{m}}=\widetilde{N}$ is equivalent to the category $\mathcal{C}_\infty$ of $\mathfrak{g}_0$-modules N satisfying $N_\chi=N$ for an appropriate χ. We see that in this case the category $\operatorname{gr}\widetilde{\mathcal{C}}_\infty$ is "twice smaller" than one could expect in the strongly typical case.

Throughout this section all $\mathfrak{g}$-modules are assumed to be graded.

4.1. The superalgebra $\mathfrak{osp}(1,2l)$ does not have isotropic roots: $(\beta,\beta)\neq 0$ for any $\beta\in\Delta_1$. As a result, $\mathfrak{osp}(1,2l)$ has many features of the simple Lie algebras. For instance, $\widetilde{\mathcal{U}}$ is a domain, its centre $\mathcal{Z}(\mathfrak{g})$ is a polynomial algebra and all finite dimensional representations are completely reducible. In [M1], Musson proved the existence of a harmonic space H in $\widetilde{\mathcal{U}}$: this is an $\operatorname{ad}\mathfrak{g}$-submodule of $\widetilde{\mathcal{U}}$ such that the multiplication map provides the isomorphism $H\otimes\mathcal{Z}(\mathfrak{g})\xrightarrow{\sim}\widetilde{\mathcal{U}}$.

If $\widetilde{M}$ is strongly typical its annihilator is a centrally generated ideal.

Suppose that $\widetilde{M}$ is not strongly typical; this means that $\mathfrak{m}:=\operatorname{Ann}_{\mathcal{Z}(\mathfrak{g})}\widetilde{M}$ contains T^2. Then, by [GL1], 6.2,

$$\operatorname{Ann}\widetilde{M}=\widetilde{\mathcal{U}}(T\mathcal{Z}(\mathfrak{g})+\mathfrak{m})$$

and $\widetilde{\chi}:=(T\mathcal{Z}(\mathfrak{g})+\mathfrak{m})$ is the maximal ideal of the "ghost centre" $\widetilde{\mathcal{Z}}(\mathfrak{g})=\mathcal{Z}(\mathfrak{g})+T\mathcal{Z}(\mathfrak{g})$. The ideal $\widetilde{\mathcal{U}}\widetilde{\chi}$ is primitive since a Verma module $\widetilde{M}(\mu)$ is simple if $\mu\in W(\mathfrak{m})$ is minimal.

Moreover, by [GL1], 5.3, for any simple $\mathfrak{g}_0$-module V one has

$$\dim\operatorname{Hom}_{\mathfrak{g}_0}(V,\widetilde{\mathcal{U}}/(\operatorname{Ann}\widetilde{M}))=\frac{1}{2}\dim\operatorname{Hom}_{\mathfrak{g}_0}(V,H)\tag{9}$$

where H is the harmonic space mentioned above.

4.2. Retain the notation of 2.4.2. Recall that $\mathfrak{m}\in\operatorname{Max}\mathcal{Z}(\mathfrak{g})$ is strongly typical iff the elements $\lambda\in W(\mathfrak{m})$ do not belong to the hyperplanes

$$S_\beta:=\{\mu\in\mathfrak{h}^*|\ (\mu+\rho,\beta)=0\}$$

for $\beta \in \Delta_1^+$. In this section we consider $\mathfrak{m} \in \operatorname{Max} \mathcal{Z}(\mathfrak{g})$ such that any $\lambda \in W(\mathfrak{m})$ belongs to exactly one hyperplane S_β that is

$$\exists! \beta \in \Delta_1^+ : \quad (\lambda + \rho, \beta) = 0. \tag{10}$$

In particular, $\mathfrak{m}$ is not strongly typical and so $T^2 \in \mathfrak{m}$. Set

$$\tilde{\chi} := T\mathcal{Z}(\mathfrak{g}) + \mathfrak{m}.$$

For any $\mathfrak{g}$-module $\widetilde{N}$ the vector space

$$\widetilde{N}_{\tilde{\chi}} := \{v \in \widetilde{N} |\ \tilde{\chi}^r v = 0, \quad r >> 0\}$$

is a $\mathfrak{g}$-submodule of $\widetilde{N}$. Let $\operatorname{gr} \widetilde{\mathcal{C}}_r$ (resp., $\operatorname{gr} \widetilde{\mathcal{C}}_\infty$) be the category of graded $\mathfrak{g}$-modules $\widetilde{N}$ satisfying $\tilde{\chi}^r \widetilde{N} = 0$ (resp., $\widetilde{N}_{\tilde{\chi}} = \widetilde{N}$). Observe that $\widetilde{\mathcal{C}}_1 \subseteq \widetilde{\mathcal{C}}_1' \subseteq \widetilde{\mathcal{C}}_2$ where $\widetilde{\mathcal{C}}_1'$ is the category of graded $\mathfrak{g}$-modules $\widetilde{N}$ satisfying $\mathfrak{m}\widetilde{N} = 0$.

Fix a projective Verma module $\widetilde{M}$ annihilated by $\tilde{\chi}$. Fix $\chi \in \operatorname{Max} \mathcal{Z}(\mathfrak{g}_0)$ such that

$$\begin{array}{ll} (a) & M := \widetilde{M}_{\chi;0} \text{ is a Verma } \mathfrak{g}_0\text{-module} \\ (b) & \widetilde{M} = \widetilde{\mathcal{U}} M. \end{array} \tag{11}$$

(the existence of χ satisfying (11) will be shown in 4.5). Let $\mathcal{C}_\infty$ (resp., $\mathcal{C}_r$) be the category of $\mathfrak{g}_0$-modules N satisfying $N_\chi = N$ (resp., $\chi^r N = 0$).

4.3. **Theorem.** *The functors*

$$\begin{array}{ll} \Psi : \operatorname{gr} \widetilde{\mathcal{C}}_\infty \to \mathcal{C}_\infty & \widetilde{N} \mapsto (\widetilde{N}_0)_\chi, \\ \Phi : \mathcal{C}_\infty \to \operatorname{gr} \widetilde{\mathcal{C}}_\infty & N \mapsto (\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N)_{\tilde{\chi}} \end{array}$$

where the grading on $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N = \widetilde{\mathcal{U}} \otimes_{\mathcal{U}} N$ *is determined by assuming* N *to be even, are quasi-inverse. Moreover their restrictions to* $\operatorname{gr} \widetilde{\mathcal{C}}_r$ *and* $\mathcal{C}_r$ *respectively provide an equivalence of these categories.*

Proof. First steps of the proof are the same as those in the proof of Theorem 3.3.1. Considering $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N$ as the graded module with respect to the grading defined above one obtains

$$\operatorname{Hom}_{\mathfrak{g}}(\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N, \widetilde{N}) = \operatorname{Hom}_{\mathfrak{g}_0}(N, \widetilde{N}_0)$$

for any graded $\mathfrak{g}$-module $\widetilde{N}$. Repeating 3.2.1, one concludes that the functors Ψ, Φ are adjoint. Repeating 3.4.1, one reduces the required assertion to "the case $r = 1$" that is to the statement that the restrictions Ψ_1, Φ_1 of the functors Ψ, Φ to the categories $\operatorname{gr} \widetilde{\mathcal{C}}_1, \mathcal{C}_1$ provide an equivalence of the categories.

Evidently, M is a $\mathfrak{g}_0$-direct summand of $\widetilde{M}$. The embedding $M \to \widetilde{M}$ gives rise to a non-zero $\mathfrak{g}$-map $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M \to \widetilde{M}$. Using $\widetilde{M} = \widetilde{\mathcal{U}} M$ and the projectivity of $\widetilde{M}$, one concludes that $\widetilde{M}$ is a $\mathfrak{g}$-direct summand of $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} M$. Denote by B the algebra $F(\widetilde{M}, \widetilde{M})$ and by

B-Mod the category of B-modules. Taking into account Lemma 3.4.3, one concludes that the functor $\Phi'' : \mathcal{C}_1 \to B$-Mod defined by

$$\Phi''(N) := F(M, \widetilde{M}) \otimes_{\mathcal{U}} N$$

provides an equivalence of the categories.

The category $\operatorname{gr}\widetilde{\mathcal{C}}_1$ is the category of graded modules over the superalgebra $A := \widetilde{\mathcal{U}}/(\widetilde{\mathcal{U}}_{\widetilde{\chi}})$. In Proposition 4.7 below we prove that $B = A \oplus A\theta$ where $\theta \in F(\widetilde{M}, \widetilde{M})$ is given by

$$\theta(v) = (-1)^{d(v)} v.$$

In particular, $\theta^2 = 1$ and $\theta a = (-1)^{d(a)} a\theta$ for any $a \in A$. Define the functor $\mathrm{Gr} : B$-Mod $\to \operatorname{gr}\widetilde{\mathcal{C}}_1$ as follows: for any $X \in B$-Mod the A-module structure on $\mathrm{Gr}(X)$ is given by "the restriction of scalars" and the grading is given by

$$\mathrm{Gr}(X)_0 := \{v \in X |\ \theta v = v\}, \quad \mathrm{Gr}(X)_1 := \{v \in X |\ \theta v = -v\}.$$

It is easy to see that Gr provides an equivalence of the categories B-Mod and $\operatorname{gr}\widetilde{\mathcal{C}}_1$. Hence $\Phi' := \mathrm{Gr} \circ \Phi''$ provides an equivalence of the categories $\mathcal{C}_1$ and $\operatorname{gr}\widetilde{\mathcal{C}}_1$.

The functor $\Phi' : \mathcal{C}_1 \to \operatorname{gr}\widetilde{\mathcal{C}}_1$ is given by the formula $N \mapsto F(M, \widetilde{M}) \otimes_{\mathcal{U}} N$ where the grading on $F(M, \widetilde{M})$ is defined by $F(M, \widetilde{M})_i := F(M, \widetilde{M}_i)$ $(i = 0, 1)$.

The last step of the proof repeats 3.4.5. Indeed,

$$\Psi_1 \circ \Phi'(N) = F(M, (\widetilde{M}_{\chi;0}) \otimes_{\mathcal{U}} N = F(M, M) \otimes_{\mathcal{U}} N = N$$

since $F(M, M) \cong \mathcal{U}_\chi$. Hence Ψ_1 is quasi-inverse to Φ' and it provides an equivalence of the categories $\widetilde{\mathcal{C}}_1 \to \mathcal{C}_1$. Now, in order to verify that the functors Φ_1 and Φ' are isomorphic, one can simply repeat 3.4.5 for the graded modules. $\square$

4.4. The restrictions of the functors Ψ, Φ provide an equivalence between $\widetilde{\mathcal{C}}_\infty \cap \widetilde{\mathcal{O}}$ and $\mathcal{C}_\infty \cap \mathcal{O}$ and between the corresponding categories of weight modules. The Verma modules of $\widetilde{\mathcal{C}}_\infty \cap \widetilde{\mathcal{O}}$ correspond to the Verma modules of $\mathcal{C}_\infty \cap \mathcal{O}$.

4.4.1. The category $\operatorname{gr}\widetilde{\mathcal{C}}_\infty$ has a canonical involution Π given by

$$(\Pi\widetilde{N})_0 = \widetilde{N}_1, \quad (\Pi\widetilde{N})_1 = \widetilde{N}_0.$$

This leads to an interesting involution $\Pi' = \Psi \circ \Pi \circ \Phi$ on the category $\mathcal{C}_\infty$. One has

$$\begin{aligned} N &= ((\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N)_{\widetilde{\chi}})_{\chi;0}, \\ \Pi'(N) &= ((\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N)_{\widetilde{\chi}})_{\chi;1} \end{aligned}$$

Theorem 4.3 implies that $M' := \widetilde{M}_{\chi;1} = \Psi(\Pi(\widetilde{M}))$ is a Verma $\mathfrak{g}_0$-module. For $N \in \mathcal{C}_1$ one has $\Phi(N) \cong \Phi'(N)$ and so

$$\Pi'(N) = F(M, M') \otimes_{\mathcal{U}} N.$$

4.5. It remains to prove Proposition 4.7 and to find $\chi \in \operatorname{Max} \mathcal{Z}(\mathfrak{g}_0)$ satisfying (11). To achieve these goals we proceed in a manner similar to [G2], Sect.8.

Call $\chi \in \operatorname{Max} \mathcal{Z}(\mathfrak{g}_0)$ *a mate* for $\tilde{\chi}$ if for a Verma module $\widetilde{M}$ annihilated by $\tilde{\chi}$ the $\mathfrak{g}_0$-modules $\widetilde{M}_{\chi;0}, \widetilde{M}_{\chi;1}$ are Verma modules. Call $\chi \in \operatorname{Max} \mathcal{Z}(\mathfrak{g}_0)$ *a perfect mate* for $\tilde{\chi}$ if it is a mate and for a simple highest weight module $\widetilde{V}(\lambda)$ annihilated by $\tilde{\chi}$ the $\mathfrak{g}_0$-module $\widetilde{V}(\lambda)_{\chi;0}$ is non-zero.

In this subsection we construct a perfect mate χ for $\tilde{\chi}$ satisfying (10).

4.5.1. The root system of $\mathfrak{g}$ takes form

$$\Delta_1^+ = \{\sigma_i\}_{1\leq i\leq l}, \qquad \Delta_0^+ = \{\sigma_i \pm \sigma_j; 2\sigma_i\}_{1\leq i<j\leq l}$$

and $(\sigma_i, \sigma_j) = \delta_{i,j}$. The Weyl group W acts on $\{\sigma_i\}_1^l$ by the signed permutations.

Set

$$\begin{aligned}
\Gamma : \ &= \left\{ \sum_1^l r_i\sigma_i |\ r_i \in \{0,1\} \right\}, \\
\Gamma_0 : \ &= \left\{ \sum_1^l r_i\sigma_i |\ r_i \in \{0,1\}, \sum_1^l r_i \text{ is even} \right\}, \\
\Gamma_1 : \ &= \left\{ \sum_1^l r_i\sigma_i |\ r_i \in \{0,1\}, \sum_1^l r_i \text{ is odd} \right\}.
\end{aligned}$$

Define the action of the Weyl group W on Γ by setting

$$w_*\gamma = w(\gamma - \rho_1) + \rho_1.$$

Take an arbitrary $\lambda \in \mathfrak{h}^*$ and fix a $\mathbb{Z}_2$-grading on a Verma module $\widetilde{M}(\lambda)$ in such a way that a highest weight vector becomes even. As a $\mathfrak{g}_0$-module, $\widetilde{M} = \widetilde{M}_0 \oplus \widetilde{M}_1$; the module $\widetilde{M}(\lambda)_i$ has a filtration such that the set of factors coincides with the set $\{M(\lambda-\gamma) : \gamma \in \Gamma_i\}$— see [M1], 3.2.

It is easy to check that for any $w \in W, \gamma \in \Gamma$

$$w.\lambda - w_*\gamma + \rho_0 = w(\lambda - \gamma + \rho_0). \tag{12}$$

Therefore the $\mathfrak{g}_0$-central characters of $M(w.\lambda - w_*\gamma)$ and $M(\lambda-\gamma)$ coincide. Thus the multiset of $\mathfrak{g}_0$-central characters of $\{M(w.\lambda-\gamma) : \gamma \in \Gamma\}$ does not depend on the choice of $w \in W$. Recall that the set

$$W(\tilde{\chi}) := \{\mu \in \mathfrak{h}^* | \tilde{\chi}\widetilde{M}(\mu) = 0\}$$

forms a single W.-orbit.

4.5.2. Say that $\mu, \mu' \in \sum_i \mathbb{Z}\sigma_i$ have the same parity if $(\mu - \mu') \in \mathbb{Z}\Delta_0^+$. Suppose that $\gamma, \gamma' \in \Gamma$ both lie either in Γ_0 or in Γ_1. This is equivalent to the condition that the elements γ, γ' have the same parity. For any $w \in W$ one has $w_*\gamma - w_*\gamma' = w(\gamma - \gamma')$ and so the elements $w_*\gamma, w_*\gamma'$ also have the same parity. Hence both $w_*\gamma, w_*\gamma'$ lie either in Γ_0 or in Γ_1. This has the following consequence: the multiset of $\mathfrak{g}_0$-central characters of $\{M(w.\lambda - \gamma) : \gamma \in \Gamma_0\}$ coincides either with the multiset of $\mathfrak{g}_0$-central characters of $\{M(\lambda - \gamma) : \gamma \in \Gamma_0\}$ or with the multiset of $\mathfrak{g}_0$-central characters of $\{M(\lambda - \gamma) : \gamma \in \Gamma_1\}$. In particular, if $\chi \in \operatorname{Max} \mathcal{Z}(\mathfrak{g}_0)$ is such that for some $\lambda \in W(\tilde{\chi})$ one has $\widetilde{M}(\lambda)_\chi = M(\lambda - \gamma_0) \oplus M(\lambda - \gamma_1)$ for certain $\gamma_i \in \Gamma_i (i = 0, 1)$ then for any $\lambda' \in W(\tilde{\chi})$ the $\mathfrak{g}_0$-modules $\widetilde{M}(\lambda')_{\chi;i}$ are Verma modules.

4.5.3. Define a lexicographic order on $\mathbb{C}$ by setting $c_1 > c_2$ if $\operatorname{Re} c_1 > \operatorname{Re} c_2$ or $\operatorname{Re} c_1 = \operatorname{Re} c_2$ and $\operatorname{Im} c_1 > \operatorname{Im} c_2$.

The condition (10) implies the existence of $\lambda \in W(\tilde{\chi})$ such that

$$\lambda + \rho = \sum_1^l k_i\sigma_i, \quad k_1, \dots, k_{l-1} > k_l = 0.$$

Set $\widetilde{M} := \widetilde{M}(\lambda)$ and

$$\chi = \operatorname{Ann}_{\mathcal{Z}(\mathfrak{g}_0)} M(\lambda).$$

One has $\rho_1 = \frac{1}{2}\sum_1^l \sigma_i$ and so

$$\lambda + \rho_0 = \sum_1^l (k_i + \frac{1}{2})\sigma_i.$$

One easily sees that

$$\lambda + \rho_0 - \gamma \in W(\lambda + \rho_0)$$

only for $\gamma = \sigma_l, 0$. Therefore $\chi M(\lambda - \gamma) = 0$ iff $\gamma = \sigma_l, 0$. This implies $\widetilde{M}_\chi = M(\lambda) \oplus M(\lambda - \sigma_l)$. By 4.5.2, χ is a mate for $\tilde{\chi}$.

4.5.4. Let us show that χ is a perfect mate. Suppose this is not true. Then $\tilde{V}(w.\lambda)_{\chi;i} = 0$ for some $w \in W$ and $i \in \{0, 1\}$. The equality (12) implies that for any $y \in W$

$$\widetilde{M}(y.\lambda)_\chi = M(y.\lambda - y_*0) \oplus M(\lambda - y_*\sigma_l)$$

and so $\tilde{V}(y.\lambda)_\chi$ is a homomorphic image of $M(y.\lambda - y_*0) \oplus M(\lambda - y_*\sigma_l)$. The module $\tilde{V}(w.\lambda)$ is a homomorphic image of the module $\widetilde{M}(w.\lambda)$; denote the kernel of this homomorphism by $\widetilde{N}$. The module $\widetilde{N}$ has finite length and the factors of its Jordan-Gölder series have form $\tilde{V}(\mu)$ for some $\mu \in W.\lambda$ satisfying $\mu < w.\lambda$. Since $0 = \tilde{V}(w.\lambda)_{\chi;i} = (\widetilde{M}(w.\lambda)/\widetilde{N})_{\chi;i}$, one concludes that the $\mathfrak{g}_0$-module $\widetilde{N}_{\chi;i} = \widetilde{M}(w.\lambda)_{\chi;i}$ has a finite filtration whose factors are quotients of either $M(y.\lambda - y_*0)$ or $M(y.\lambda - y_*\sigma_l)$ for some $y \in W$ satisfying $y.\lambda < w.\lambda$. Hence the highest weight of $\widetilde{M}(w.\lambda)_{\chi;i}$ belongs to the set

$$X := \{y.\lambda - y_*0,\ y.\lambda - y_*\sigma_l\}_{y \in W \ s.t.\ y.\lambda < w.\lambda}.$$

The module $\widetilde{M}(w.\lambda)_{\chi;i}$ is isomorphic either to $M(w.\lambda - w_*0)$ or to $M(w.\lambda - w_*\sigma_l)$. Therefore either $w.\lambda - w_*0$ or $w.\lambda - w_*\sigma_l$ belongs to the set X.

If $w.\lambda - w_*0 = y.\lambda - y_*0$ then, by (12), $w(\lambda+\rho_0) = y(\lambda+\rho_0)$, that is $w^{-1}y \in \mathrm{Stab}_W(\lambda+\rho_0)$. One has

$$\mathrm{Stab}_W(\lambda+\rho_0) = \mathrm{Stab}_W(\sum_1^l (k_j+\frac{1}{2})\sigma_j) \subseteq \mathrm{Stab}_W(\sum_1^l k_j\sigma_j) = \mathrm{Stab}_W(\lambda+\rho)$$

since $k_1,\dots,k_{l-1} > k_l = 0$. Thus $w.\lambda - w_*0 = y.\lambda - y_*0$ implies $w.\lambda = y.\lambda$. Hence $(w.\lambda - w_*0) \notin \{y.\lambda - y_*0\}_{y\in W, y.\lambda < w.\lambda}$.

If $w.\lambda - w_*0 = y.\lambda - y_*\sigma_l$ then, by (12), $w(\lambda+\rho_0) = y(\lambda+\rho_0-\sigma_l)$ or, equivalently, $w(\lambda+\rho_0) = ys_{\beta_l}(\lambda+\rho_0)$ where $s_{\beta_l} \in W$ is the reflection with respect to the root β_l. Therefore $w^{-1}ys_{\beta_l} \in \mathrm{Stab}_W(\lambda+\rho_0)$. As we already saw $\mathrm{Stab}_W(\lambda+\rho_0) \subseteq \mathrm{Stab}_W(\lambda+\rho)$ and so $w^{-1}ys_{\beta_l} \in \mathrm{Stab}_W(\lambda+\rho)$. Then $w^{-1}y \in \mathrm{Stab}_W(\lambda+\rho)$ since $s_{\beta_l} \in \mathrm{Stab}_W(\lambda+\rho)$. Hence $w.\lambda - w_*0 = y.\lambda - y_*\sigma_l$ forces $w^{-1}y \in \mathrm{Stab}_W(\lambda+\rho)$. We conclude that $(w.\lambda - w_*0) \notin X$.

Similarly, if $w.\lambda - w_*\sigma_l = y.\lambda - y_*0$ then, by (12), $w(\lambda+\rho_0-\sigma_l) = y(\lambda+\rho_0)$. As we have shown above, this implies $w^{-1}y \in \mathrm{Stab}_W(\lambda+\rho)$. Hence $(w.\lambda - w_*\sigma_l) \notin \{y.\lambda - y_*0\}_{y\in W, y.\lambda < w.\lambda}$.

Finally, if $w.\lambda - w_*\sigma_l = y.\lambda - y_*\sigma_l$ then, by (12), $w^{-1}y \in \mathrm{Stab}_W(\lambda+\rho_0-\sigma_l)$. One can easily deduce from the equality

$$\lambda+\rho_0-\sigma_l = \sum_1^{l-1}(k_j+\frac{1}{2})\sigma_j - \frac{1}{2}\sigma_l$$

that $\mathrm{Stab}_W(\lambda+\rho_0-\sigma_l) \subseteq \mathrm{Stab}_W(\lambda+\rho)$. Thus $(w.\lambda - w_*\sigma_l) \notin X$ as required.

Hence $\{w.\lambda - w_*0; w.\lambda - w_*\sigma_l\} \cap X = \emptyset$. This proves that $\widetilde{V}(w.\lambda)_{\chi;i} \neq 0$ for $i = 0,1$.

4.5.5. **Corollary.** *The ideal $\chi \in \mathrm{Max}\,\mathcal{Z}(\mathfrak{g}_0)$ described in 4.5.3 is a perfect mate for $\widetilde{\chi}$.*

4.6. Suppose that $\chi \in \mathrm{Max}\,\mathcal{Z}(\mathfrak{g}_0)$ is a perfect mate for $\widetilde{\chi}$.

The ideal $\widetilde{\mathcal{U}}\widetilde{\chi}$ is equal to the annihilator of a Verma module $\widetilde{M}$ which has a finite support in $\mathcal{Z}(\mathfrak{g}_0)$. Therefore any $\mathfrak{g}$-module annihilated by $\widetilde{\mathcal{U}}\widetilde{\chi}$ has a finite support in $\mathcal{Z}(\mathfrak{g}_0)$. Arguing as in [G2], 8.3.2, 8.3.3, one can deduce from the definition of perfect mate that for a graded $\mathfrak{g}$-module $\widetilde{N}$ annihilated by $\widetilde{\mathcal{U}}\widetilde{\chi}$ one has $\widetilde{N} = \widetilde{\mathcal{U}}\widetilde{N}_{\chi;0}$. In particular, $\widetilde{N}_{\chi;0} \neq 0$ if $\widetilde{N} \neq 0$.

For a graded $\mathfrak{g}$-bimodule L set

$${}_{\chi}L_{\chi;0} := \{f \in L_0 |\ \chi^r f = f\chi^r = 0,\ \forall r >> 0\}.$$

Arguing as in [G2], 8.4 one concludes ${}_{\chi}L_{\chi;0} \neq 0$ provided $L \neq 0$ and $\widetilde{\chi}L = L\widetilde{\chi} = 0$.

4.7. **Proposition.** *Let $\widetilde{M}$ be a Verma module annihilated by $\widetilde{\chi}$ and let A be the image of $\widetilde{\mathcal{U}}$ in $F(\widetilde{M},\widetilde{M})$ under the natural map. Then*

$$F(\widetilde{M},\widetilde{M}) = A \oplus A\theta$$

where $\theta \in F(\widetilde{M},\widetilde{M})$ is given by

$$\theta(v) = (-1)^{d(v)}v.$$

Proof. Suppose that $\widetilde{M}$ is simple. By [G2], 11.1.5, for any simple $\mathfrak{g}$-module $\widetilde{V}$ one has $\dim \mathrm{Hom}_{\mathfrak{g}}(\widetilde{V}, F(\widetilde{M},\widetilde{M})) = \dim \mathrm{Hom}_{\mathfrak{g}}(\widetilde{V}, H)$ where H is a harmonic space —see 4.1. The complete reducibility of both H and $F(\widetilde{M},\widetilde{M})$ forces $H \cong F(\widetilde{M},\widetilde{M})$. The element θ is $\mathrm{ad}\,\mathfrak{g}_0$-invariant and so $A \cong A\theta$ as $\mathrm{ad}\,\mathfrak{g}_0$-modules. In the light of (9), as $\mathrm{ad}\,\mathfrak{g}_0$-modules $H \cong A \oplus A$. Since the multiplicity of any finite dimensional $\mathfrak{g}_0$-module in H is finite, in order to prove the equality $F(\widetilde{M},\widetilde{M}) = A \oplus A\theta$ it is enough to show that $F(\widetilde{M},\widetilde{M}) = A + A\theta$. Since $\theta u = (-1)^{d(u)}u\theta$ for any $u \in \widetilde{\mathcal{U}}$, $(A + A\theta)$ is a graded $\widetilde{\mathcal{U}}$-subbimodule of $F(\widetilde{M},\widetilde{M})$. Let $\chi \in \mathcal{Z}(\mathfrak{g}_0)$ be a perfect mate for $\widetilde{\chi}$. Then both $M := \widetilde{M}_{\chi;0}$ and $M' := \widetilde{M}_{\chi;1}$ are Verma $\mathfrak{g}_0$-modules. The restriction of endomorphisms of $\widetilde{M}$ to $M \oplus M'$ induces an isomorphism

$${}_{\chi}F(\widetilde{M},\widetilde{M})_{\chi;0} \xrightarrow{\sim} F(M,M) \oplus F(M',M').$$

Recall that the natural maps $\mathcal{U}/(\mathcal{U}\chi) \to F(M,M)$ and $\mathcal{U}/(\mathcal{U}\chi) \to F(M',M')$ are bijective; identify ${}_{\chi}F(\widetilde{M},\widetilde{M})_{\chi;0}$ with $\mathcal{U}/(\mathcal{U}\chi) \oplus \mathcal{U}/(\mathcal{U}\chi)$ through these maps. Write

$$\mathrm{Ann}_{\mathcal{Z}(\mathfrak{g}_0)}\widetilde{M} = \chi \prod_1^s \chi_j^{r_j}$$

where $\mathrm{supp}_{\mathcal{Z}(\mathfrak{g}_0)} = \{\chi, \chi_1, \ldots, \chi_s\}$ and $r_1, \ldots, r_s \in \mathbb{N}^+$. Take $a \in \prod_1^s \chi_j^{r_j}$ such that $a = 1$ modulo χ. The image of $\mathcal{U}a$ in $F(\widetilde{M},\widetilde{M})$ lies in ${}_{\chi}F(\widetilde{M},\widetilde{M})_{\chi;0}$. Since for any $u \in \mathcal{U}$ the element ua acts on both M and M' by the multiplication by u, the image J of $\mathcal{U}a$ in ${}_{\chi}F(\widetilde{M},\widetilde{M})_{\chi;0}$ is equal to the diagonal copy of $\mathcal{U}/(\mathcal{U}\chi)$ inside $\mathcal{U}/(\mathcal{U}\chi) \oplus \mathcal{U}/(\mathcal{U}\chi)$:

$$J = \{(u,u) |\ u \in \mathcal{U}/(\mathcal{U}\chi)\}.$$

Then $J\theta = \{(u,-u) |\ u \in \mathcal{U}/(\mathcal{U}\chi)\}$ and thus

$$J + J\theta = \mathcal{U}/(\mathcal{U}\chi) \oplus \mathcal{U}/(\mathcal{U}\chi) =_{\chi} F(\widetilde{M},\widetilde{M})_{\chi;0}.$$

Hence

$${}_{\chi}F(\widetilde{M},\widetilde{M})_{\chi;0} =_{\chi} (A + A\theta)_{\chi;0}.$$

By 4.6, this forces $F(\widetilde{M},\widetilde{M}) = A + A\theta$.

We have shown that $F(\widetilde{M},\widetilde{M}) = \mathrm{Im}\,\widetilde{\mathcal{U}} \oplus \theta\, \mathrm{Im}\,\widetilde{\mathcal{U}}$ provided $\widetilde{M}$ is simple. Take an arbitrary Verma module $\widetilde{M}$ satisfying $\widetilde{\chi}\widetilde{M} = 0$. The module $\widetilde{M}$ contains a simple Verma submodule $\widetilde{M}'$ since $\widetilde{\mathcal{U}}$ is a domain. Let E be an $\mathrm{ad}\,\mathfrak{g}$-submodule of $\widetilde{\mathcal{U}}$ such that the restriction of the natural map $f' : \widetilde{\mathcal{U}} \to F(\widetilde{M}',\widetilde{M}')$ to E provides a bijection $E \to \mathrm{Im}\, f'$. Then

$F(\widetilde{M}',\widetilde{M}') = f'(E)\oplus f'(E)\theta$. The restriction of the natural map $f:\widetilde{\mathcal{U}}\to F(\widetilde{M},\widetilde{M})$ to E provides a bijection $E\to \operatorname{Im} f$ since $\ker f=\ker f'=\widetilde{\mathcal{U}}\widetilde{\chi}$. Suppose that $f(u_1)=f(u_2)\theta$ for some $u_1,u_2\in E$; then $(u_1-u_2)\widetilde{M}_0=0$ and so $f'(u_1)=f'(u_2)\theta$ that implies $u_1=u_2=0$ since $f'(E)\cap f'(E)\theta=0$. Therefore $f(E)\cap f(E)\theta=0$. Thus $F(\widetilde{M},\widetilde{M})$ contains $f(E)\oplus f(E)\theta$. A Joseph's reasoning based on the use of GK-dimension, shows that the map $F(\widetilde{M}',\widetilde{M}')\to F(\widetilde{M}',\widetilde{M})$ is bijective and the map $F(\widetilde{M},\widetilde{M})\to F(\widetilde{M}',\widetilde{M})$ is injective (both maps are induced by the embedding $\widetilde{M}'$ to $\widetilde{M}$)— see [J2], 8.3.9 or [G2], 9.2. In particular, $F(\widetilde{M},\widetilde{M})$ is isomorphic to an $\operatorname{ad}\mathfrak{g}$-submodule of $F(\widetilde{M}',\widetilde{M}')$. On the other hand, $F(\widetilde{M},\widetilde{M})$ contains $f(E)\oplus f(E)\theta$ which is isomorphic, as an $\operatorname{ad}\mathfrak{g}_0$-module, to $F(\widetilde{M}',\widetilde{M}')$. Since the multiplicity of each finite dimensional $\mathfrak{g}_0$-module in $F(\widetilde{M}',\widetilde{M}')$ is finite, one concludes that $F(\widetilde{M},\widetilde{M})=f(E)\oplus f(E)\theta=\operatorname{Im} f\oplus\theta\operatorname{Im} f$ as required. $\square$

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