INTERSECTION THEORY ON $\overline{M}_{1,4}$ AND ELLIPTIC GROMOV-WITTEN INVARIANTS

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1. Introduction

In this paper, we find a new relation among codimension 2 cycles in $\overline{\mathcal{M}}_{1,4}$. The main application of the new relation is to the calculation of elliptic Gromov-Witten invariants. As an illustration of our results, we show that the elliptic Gromov-Witten invariants are determined by the rational ones and $\langle I^V_{1,1,\beta}\rangle$, $0 \le c_1(V) \cap \beta < d$, in the following cases: for a surface V; for a threefold V, on restriction to the even dimensional cohomology; for any smooth projective variety V, on the subalgebra of $H^\bullet(V,\mathbb{Q})$ generated by the Kähler form ω.

In [11], we will prove, using mixed Hodge theory, that the cycles $[\overline{\mathcal{M}}(G)]$, as G ranges over all stable graphs of genus 1 and valence n, span the even dimensional homology of $\overline{\mathcal{M}}_{1,n}$, and that the new relation, together with those already known in genus 0, generate all relations among these cycles. This result is the analogue, in genus 1, of a theorem of Keel [15] in genus 0.

Our new relation is closely related to a relation in $A_2(\overline{\mathcal{M}}_3) \otimes \mathbb{Q}$ discovered by Faber (Lemma 4.4 of [6]); the image of his relation in $H_4(\overline{\mathcal{M}}_3,\mathbb{Q})$ under the cycle map is the same as the push-forward of our relation under the map $\overline{\mathcal{M}}_{1,4} \to \overline{\mathcal{M}}_3$ obtained by contracting the 4 tails pairwise. This suggests that our new relation should actually be a rational equivalence.

Let us illustrate our results with the case of the projective plane. The genus 1 potential of $\mathbb{CP}^2$ equals

$$F_1(\mathbb{CP}^2) = -\frac{t_1}{8} + \sum_{n=1}^{\infty} N_n^{(1)} q^n e^{nt_1} \frac{t_2^{3n}}{(3n)!},$$

where t_1 and t_2 are formal variables, of degree 0 and 2 respectively, dual to the classes $\omega \in H^2(\mathbb{CP}^2,\mathbb{Q})$ and $\omega^2 \in H^4(\mathbb{CP}^2,\mathbb{Q})$ respectively, and $N_n^{(1)}$ is the number of elliptic plane curves of degree n which meet $3n$ generic points. In Section 3, we prove that the coefficients $N_n^{(1)}$ satisfy the recursion

$$(1.1)\qquad \begin{aligned} 6N_n^{(1)} &= \sum_{n=i+j+k} \binom{3n-2}{3j-1,3k-1} ij^3k^3(2i-j-k) N_i^{(1)} N_j^{(0)} N_k^{(0)} \\ &+ 2\sum_{n=i+j} \left(\binom{3n-2}{3i} ij^2(8i-j) - \binom{3n-2}{3i-1} 2(i+j)j^3 \right) N_i^{(1)} N_j^{(0)} \\ &- \frac{1}{24}\left(\sum_{n=i+j} \binom{3n-2}{3i-1} (n^2-3n-6ij) i^3 j^3 N_i^{(0)} N_j^{(0)} + 6n^3(n-1) N_n^{(0)} \right). \end{aligned}$$

In Table 1, we list the first few of these coefficients, together with the corresponding rational Gromov-Witten invariants for comparison. We have checked that our results for $N_n^{(1)}$ agree in degrees up to 6 with those obtained by Caporaso and Harris [5].

TABLE 1. Rational and elliptic Gromov-Witten invariants of $\mathbb{CP}^2$

n	$N_n^{(0)}$	$N_n^{(1)}$
1	1	0
2	1	0
3	12	1
4	620	225
5	87 304	87 192
6	26 312 976	57 435 240
7	14 616 808 192	60 478 511 040
8	13 525 751 027 392	96 212 546 526 096
9	19 385 778 269 260 800	220 716 443 548 094 400
10	40 739 017 561 997 799 680	702 901 008 498 298 112 640
11	120 278 021 410 937 387 514 880	3 011 788 599 493 603 375 929 600
12	482 113 680 618 029 292 368 686 080	16 916 605 752 011 965 307 094 124 800

The situation for the elliptic Gromov-Witten invariants of $\mathbb{CP}^3$ is a little more complicated. The genus 0 and 1 potentials of $\mathbb{CP}^3$ have the form

$$F_0(\mathbb{CP}^3) = \frac{t_0^2 t_3}{2} + t_0 t_1 t_2 + \frac{t_1^3}{6} + \sum_{n=1}^{\infty} \sum_{4n=a+2b} N_{ab}^{(0)} q^n e^{nt_1} \frac{t_2^a t_3^b}{a!b!},$$

$$F_1(\mathbb{CP}^3) = -\frac{t_1}{4} + \sum_{n=1}^{\infty} \sum_{4n=a+2b} N_{ab}^{(g)} q^n e^{nt_1} \frac{t_2^a t_3^b}{a!b!},$$

where t_i is the formal variable, of degree $2i-2$, dual to $\omega^i \in H^{2i}(\mathbb{CP}^3, \mathbb{Q})$, and $N_{ab}^{(g)}$ is the Gromov-Witten invariant which "counts" the stable maps of genus g and degree n to $\mathbb{CP}^3$ which meet a generic lines and b generic points. The elliptic Gromov-Witten invariants are no longer positive integers: for example, $N_{02}^{(1)} = -1/12$. In [10], we use the methods of this paper to prove that the linear combination $N_{ab}^{(1)} + (2n-1)N_{ab}^{(0)}/12$ counts the number of elliptic space curves which meet a generic planes and b generic points.

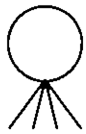
Acknowledgments. Conversations with K. Behrend, E. Looijenga, Yu. Manin and especially with C. Faber, enabled me to write this paper at all.

I am very grateful to Yu. Manin, D. Zagier and the Max-Planck-Institut für Mathematik in Bonn, where this paper was conceived, and to A. Kupiainen and the Finnish Mathematical Society for an invitation to Helsinki University, where much of it was finished.

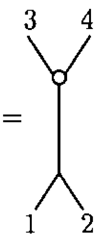
2. Intersection theory on $\overline{\mathcal{M}}_{1,4}$

In this section, we calculate the relations among certain codimension two cycles in $\overline{\mathcal{M}}_{1,4}$; one such relation was known, and we find that there is one new one.

First, we assign names to the codimension 1 strata of $\overline{\mathcal{M}}_{1,4}$. Denote by Δ_0 the boundary stratum of irreducible curves in $\overline{\mathcal{M}}_{1,4}$, associated to the stable graph

$\Delta_0 =$ 

For each subset S of $\{1,2,3,4\}$ of cardinality at least 2, let Δ_S be the boundary stratum associated to the stable graph with two vertices, of genus 0 and 1, one edge connecting them, and with those tails labelled by elements of S attached to the vertex of genus 0; there are 11 such graphs. In our pictures, we denote genus 1 vertices by a hollow dot, leaving genus 0 vertices unmarked. For example,

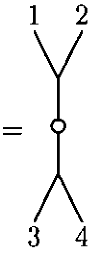
$\Delta_{\{1,2\}} =$ 

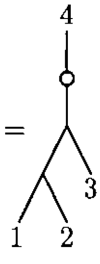
We only need the three S_4-invariant combinations of these 11 strata, which are as follows:

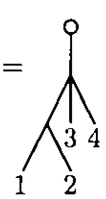
$$\begin{aligned}
\Delta_2 &= \Delta_{\{1,2\}} + \Delta_{\{1,3\}} + \Delta_{\{1,4\}} + \Delta_{\{2,3\}} + \Delta_{\{2,4\}} + \Delta_{\{3,4\}}, \\
\Delta_3 &= \Delta_{\{1,2,3\}} + \Delta_{\{1,2,4\}} + \Delta_{\{1,3,4\}} + \Delta_{\{2,3,4\}}, \\
\Delta_4 &= \Delta_{\{1,2,3,4\}}.
\end{aligned}$$

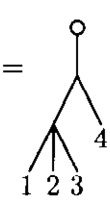
In summary, there are four invariant combinations of boundary strata: Δ_0, Δ_2, Δ_3 and Δ_4.

We now turn to enumeration of the codimension two strata. These fall into two classes, distinguished by whether they are contained in the irreducible stratum Δ_0 or not. We start by listing those which are not; each of them is the intersection of a pair of boundary strata $\Delta_S \cdot \Delta_T$. We give four examples: from these, the other strata may be obtained by the action of S_4:

$\Delta_{\{1,2\}} \cdot \Delta_{\{3,4\}} =$ 

$\Delta_{\{1,2\}} \cdot \Delta_{\{1,2,3\}} =$ 

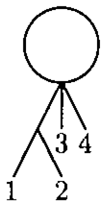
$\Delta_{\{1,2\}} \cdot \Delta_{\{1,2,3,4\}} =$ 

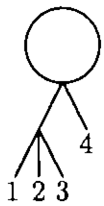
$\Delta_{\{1,2,3\}} \cdot \Delta_{\{1,2,3,4\}} =$ 

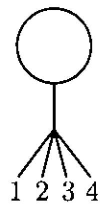
The S_4-invariant combinations of these strata are as follows:

$$
\begin{aligned}
\Delta_{2,2} &= \Delta_{\{1,2\}} \cdot \Delta_{\{3,4\}} + \Delta_{\{1,3\}} \cdot \Delta_{\{2,4\}} + \Delta_{\{1,4\}} \cdot \Delta_{\{2,3\}}, \\
\Delta_{2,3} &= \Delta_{\{1,2\}} \cdot \Delta_{\{1,2,3\}} + \Delta_{\{1,2\}} \cdot \Delta_{\{1,2,4\}} + \Delta_{\{1,3\}} \cdot \Delta_{\{1,2,3\}} + \Delta_{\{1,3\}} \cdot \Delta_{\{1,3,4\}} \\
&\quad + \Delta_{\{1,4\}} \cdot \Delta_{\{1,2,4\}} + \Delta_{\{1,4\}} \cdot \Delta_{\{1,3,4\}} + \Delta_{\{2,3\}} \cdot \Delta_{\{1,2,3\}} + \Delta_{\{2,3\}} \cdot \Delta_{\{2,3,4\}} \\
&\quad + \Delta_{\{2,4\}} \cdot \Delta_{\{1,2,4\}} + \Delta_{\{2,4\}} \cdot \Delta_{\{2,3,4\}} + \Delta_{\{3,4\}} \cdot \Delta_{\{1,3,4\}} + \Delta_{\{3,4\}} \cdot \Delta_{\{2,3,4\}}, \\
\Delta_{2,4} &= \Delta_{\{1,2\}} \cdot \Delta_{\{1,2,3,4\}} + \Delta_{\{1,3\}} \cdot \Delta_{\{1,2,3,4\}} + \Delta_{\{1,4\}} \cdot \Delta_{\{1,2,3,4\}} \\
&\quad + \Delta_{\{2,3\}} \cdot \Delta_{\{1,2,3,4\}} + \Delta_{\{2,4\}} \cdot \Delta_{\{1,2,3,4\}} + \Delta_{\{3,4\}} \cdot \Delta_{\{1,2,3,4\}}, \\
\Delta_{3,4} &= \Delta_{\{1,2,3\}} \cdot \Delta_{\{1,2,3,4\}} + \Delta_{\{1,2,4\}} \cdot \Delta_{\{1,2,3,4\}} + \Delta_{\{1,3,4\}} \cdot \Delta_{\{1,2,3,4\}} + \Delta_{\{2,3,4\}} \cdot \Delta_{\{1,2,3,4\}}.
\end{aligned}
$$

Each of the intersections $\Delta_0 \cdot \Delta_S$ is a codimension two stratum in Δ_0; for example

$\Delta_0 \cdot \Delta_{\{1,2\}} =$ 

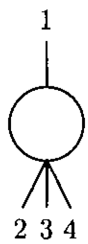
$\Delta_0 \cdot \Delta_{\{1,2,3\}} =$ 

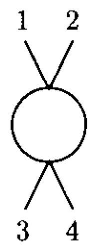
$\Delta_0 \cdot \Delta_{\{1,2,3,4\}} =$ 

From these, we may form the S_4-invariant combinations

$$
\begin{aligned}
\Delta_{0,2} &= \Delta_0 \cdot \Delta_{\{1,2\}} + \Delta_0 \cdot \Delta_{\{1,3\}} + \Delta_0 \cdot \Delta_{\{1,4\}} + \Delta_0 \cdot \Delta_{\{2,3\}} + \Delta_0 \cdot \Delta_{\{2,4\}} + \Delta_0 \cdot \Delta_{\{3,4\}}, \\
\Delta_{0,3} &= \Delta_0 \cdot \Delta_{\{1,2,3\}} + \Delta_0 \cdot \Delta_{\{1,2,4\}} + \Delta_0 \cdot \Delta_{\{1,3,4\}} + \Delta_0 \cdot \Delta_{\{2,3,4\}}, \\
\Delta_{0,4} &= \Delta_0 \cdot \Delta_{\{1,2,3,4\}}.
\end{aligned}
$$

There remain seven strata which are not expressible as intersections, which we denote by $\Delta_{\alpha,i}$, $1 \le i \le 4$, and $\Delta_{\beta,12|34}$, $\Delta_{\beta,13|24}$ and $\Delta_{\beta,14|24}$. We illustrate the stable graphs for two of these strata:

$\Delta_{\alpha,1} =$ 

$\Delta_{\beta,12|34} =$ 

Denote by Δ_α and Δ_β the S_4-invariant combinations of strata:

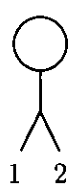
$$\Delta_\alpha = \Delta_{\alpha,1} + \Delta_{\alpha,2} + \Delta_{\alpha,3} + \Delta_{\alpha,4}, \quad \Delta_\beta = \Delta_{\beta,12|34} + \Delta_{\beta,13|24} + \Delta_{\beta,14|24}.$$

For each of these strata Δ_x, let δ_x be the corresponding cycle in $H_\bullet(\overline{\mathcal{M}}_{1,4}, \mathbb{Q})$ (in the sense of orbifolds — we divide by the order of the automorphism group of a generic point in the stratum, which in our case is always 2).

Lemma (2.1). *The following relation among cycles holds in* $H_4(\overline{\mathcal{M}}_{1,4}, \mathbb{Q})$*:*

$$\delta_{0,2} + 3\delta_{0,3} + 6\delta_{0,4} = 3\delta_\alpha + 4\delta_\beta.$$

Proof. The two strata



define the same cycle, as do any pair of codimension two strata lying in an irreducible surface. We obtain the lemma by lifting this relation by the 6 distinct projections $\overline{\mathcal{M}}_{1,4} \to \overline{\mathcal{M}}_{1,2}$ and summing the answers. □

We can now state the main result of this section.

Theorem (2.2). *The first seven rows of the intersection matrix of the nine S_4-invariant codimension two cycles in $\overline{\mathcal{M}}_{1,4}$ introduced above equals*

	$\delta_{2,2}$	$\delta_{2,3}$	$\delta_{2,4}$	$\delta_{3,4}$	$\delta_{0,2}$	$\delta_{0,3}$	$\delta_{0,4}$	δ_α	δ_β
$\delta_{2,2}$	1/8	0	0	0	−3	0	3/2	0	3/2
$\delta_{2,3}$	0	0	0	0	0	−6	6	6	0
$\delta_{2,4}$	0	0	0	−1/2	0	6	−3	0	0
$\delta_{3,4}$	0	0	−1/2	1/6	6	−2	0	0	0
$\delta_{0,2}$	−3	0	0	6	0	0	0	0	0
$\delta_{0,3}$	0	−6	6	−2	0	0	0	0	0
$\delta_{0,4}$	3/2	6	−3	0	0	0	0	0	0

Proof. The following lemma shows that many of the intersection numbers vanish. (The use of this lemma simplifies our original proof of Theorem (2.2), and was suggested to us by C. Faber.)

Lemma (2.3). *Let δ be a cycle in Δ_0. Then $\delta_0 \cdot \delta = 0$.*

Proof. Consider the projection $\pi : \overline{\mathcal{M}}_{1,n} \to \overline{\mathcal{M}}_{1,1}$ which forgets all but the first marked point, and stabilizes the marked curve which results. The divisor Δ_0 is the inverse image under π of the compactification divisor of $\overline{\mathcal{M}}_{1,1}$; thus, we may replace it in calculating intersections by any cycle of the form $\pi^{-1}(x)$, where $x \in \mathcal{M}_{1,1}$. The resulting cycle has empty intersection with δ, proving the lemma. □

This lemma shows that all intersections among the cycles $\delta_{0,2}$, $\delta_{0,3}$ and $\delta_{0,4}$ and with δ_α and δ_β vanish.

A number of other entries in the intersection matrix vanish because the associated strata do not meet: thus,

$$\delta_{2,2} \cdot \delta_{2,3} = \delta_{2,2} \cdot \delta_{3,4} = \delta_{2,2} \cdot \delta_{0,3} = \delta_{2,2} \cdot \delta_\alpha = 0,$$
$$\delta_{2,3} \cdot \delta_{2,4} = \delta_{2,3} \cdot \delta_\beta = 0,$$
$$\delta_{2,4} \cdot \delta_\alpha = \delta_{2,4} \cdot \delta_\beta = \delta_{3,4} \cdot \delta_\alpha = \delta_{3,4} \cdot \delta_\beta = 0.$$

To calculate the remaining entries of the intersection matrix, we need the excess intersection formula (Fulton [7], Section 6.3).

Proposition (2.4). *Let Y be a smooth variety, let $X \hookrightarrow Y$ be a regular immersion of codimension d, and let V be a closed subvariety of Y of dimension n. Suppose that the inclusion $W = X \cap V \hookrightarrow V$ is a regular immersion of codimension $d - e$. Then*

$$[X] \cdot [V] = c_e(E) \cap [W] \in A_{n-d}(W),$$

where $E = (N_X Y)|_W / (N_W V)$ is the excess bundle *of the intersection.*

Observe that in calculating the top four rows of our intersection matrix, at least one of the cycles which we intersect with has a regular immersion in $\overline{\mathcal{M}}_{1,4}$, since its dual graph is a tree. This makes the application of the excess intersection formula straightforward.

It remains to give a formula for the normal bundles to the strata of $\overline{\mathcal{M}}_{1,4}$.

Definition (2.5). The *tautological line bundles* are defined by

$$\omega_i = \sigma_i^* \omega_{\overline{\mathcal{M}}_{g,n+1}/\overline{\mathcal{M}}_{g,n}},$$

where $\sigma_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$, $1 \le i \le n$, are the n canonical sections of the universal stable curve $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$. Denote the Chern class $c_1(\omega_i)$ by ψ_i.

The following formula for the normal bundle of a stratum $\overline{\mathcal{M}}(G) \subset \overline{\mathcal{M}}_{g,n}$ may be found in Section 4 of Hain-Looijenga [12].

Proposition (2.6). *Let G be a stable graph of genus g and valence n, and let $\overline{\mathcal{M}}(G)$ be the closure of the associated stratum $\mathcal{M}(G)$ of $\mathcal{M}_{g,n}$. Then $\overline{\mathcal{M}}(G)$ is the quotient of the product*

$$\prod_{v \in V(G)} \overline{\mathcal{M}}_{g(v),n(v)}$$

by the automorphism group $\operatorname{Aut}(G)$ *of the graph.*

Each edge e of the graph consists of two flags $s(e)$ and $t(e)$, whose tautological line bundles may be pulled back to $\prod_v \overline{\mathcal{M}}_{g(v),n(v)}$. The vector bundle

$$\bigoplus_{e \in E(G)} \omega_{s(e)}^\vee \otimes \omega_{t(e)}^\vee$$

is invariant under the action of $\operatorname{Aut}(G)$ *on $\prod_v \overline{\mathcal{M}}_{g(v),n(v)}$, and hence descends to a vector bundle on $\overline{\mathcal{M}}(G)$, which may be identified with the normal bundle $N_{\overline{\mathcal{M}}(G)}\overline{\mathcal{M}}_{g,n}$.* □

It is now straightforward to calculate the remaining entries of the intersection matrix. We will use the integrals

$$(2.7) \qquad \int_{\overline{\mathcal{M}}_{0,4}} \psi_i = 1, \quad \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \int_{\overline{\mathcal{M}}_{1,2}} \psi_1 \cup \psi_2 = \frac{1}{24},$$

which are proved in Witten [23].

In performing the calculations, it is helpful to introduce a graphical notation for the cycle obtained from a stratum by capping with a monomial in the Chern classes $-\psi_i$: we point a small arrow along each flag i where we intersect by the class $-\psi_i$. (This notation generalizes that of Kaufmann [14], who considers the case of trees where the genus of each vertex is 0. The minus signs come from the inversion accompanying the tautological line bundles in the formula of Proposition (2.6).) One then calculates the contribution of such a graph by multiplying together factors for each vertex equal to the integral over $\overline{\mathcal{M}}_{g(v),n(v)}$ of the appropriate monomial in the classes $-\psi_i$, and dividing by the order of the automorphism group $\operatorname{Aut}(G)$: in particular, this vanishes unless there are $3(g(v)-1)+n(v)$ arrows at each vertex v.

We illustrate the sort of enumeration which arises with one of the most complicated of these calculations, that of $\delta_{2,4} \cdot \delta_{2,4}$. Two sorts of terms contribute: 6 terms of the form

$$(\delta_{\{1,2\}} \cdot \delta_{\{1,2,3,4\}})^2 = \frac{1}{24},$$

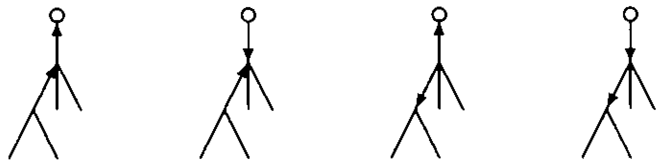
and 6 terms of the form

$$\delta_{\{1,2\}} \cdot \delta_{\{1,2,3,4\}} \cdot \delta_{\{3,4\}} \cdot \delta_{\{1,2,3,4\}} = -\frac{1}{24}.$$

Applying the excess intersection formula, we see that

$$(\delta_{\{1,2\}} \cdot \delta_{\{1,2,3,4\}})^2 = c_2(N_{\Delta_{\{1,2\}} \cap \Delta_{\{1,2,3,4\}}} \overline{\mathcal{M}}_{1,4}) \cap (\delta_{\{1,2\}} \cdot \delta_{\{1,2,3,4\}}).$$

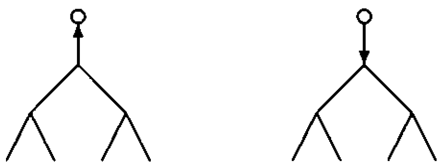
Expanding the second Chern class of the normal bundle, we see that each term contributes the sum of four graphs:



Only the first graph is nonzero, since in the other cases, the wrong number of arrows point towards the vertices. And the first graph contributes

$$\int_{\overline{\mathcal{M}}_{0,4}} (-\psi_1) \cdot \int_{\overline{\mathcal{M}}_{1,1}} (-\psi_1) = \frac{1}{24}.$$

In the case of terms of the form $\delta_{\{1,2\}} \cdot \delta_{\{1,2,3,4\}} \cdot \delta_{\{3,4\}} \cdot \delta_{\{1,2,3,4\}}$, the excess dimension e equals 1, and we must calculate the degree of the excess bundle on the stratum $\Delta_{\{1,2\}} \cap \Delta_{\{3,4\}} \cap \Delta_{\{1,2,3,4\}}$. Two graphs contribute:



Only the first of these graphs gives a nonzero value, namely

$$\int_{\overline{\mathcal{M}}_{1,1}} (-\psi_1) = -\frac{1}{24}.$$

This completes our outline of the proof of Theorem (2.2). □

The intersection matrix of Theorem (2.2) has rank 7. We now apply the results of [9], where we calculated the character of the S_n-modules $H^i(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})$: these calculations show that $\dim H^4(\overline{\mathcal{M}}_{1,4}, \mathbb{Q})^{S_4} = 7$. This shows that our 9 cycles span $H^4(\overline{\mathcal{M}}_{1,4}, \mathbb{Q})^{S_4}$, and that the nullspace of the intersection matrix gives relations among them. We already know one such relation, by Lemma (2.1). Calculating the remaining null-vector of the intersection matrix, we obtain the main theorem of this paper.

Theorem (2.8). *The following new relation among cycles holds:*

$$12\delta_{2,2} - 4\delta_{2,3} - 2\delta_{2,4} + 6\delta_{3,4} + \delta_{0,3} + \delta_{0,4} - 2\delta_\beta = 0. \quad \square$$

Using this theorem, it is easy to calculate the remaining intersections among our 9 strata:

$$\delta_\alpha \cdot \delta_\alpha = 16, \quad \delta_\alpha \cdot \delta_\beta = -12, \quad \delta_\beta \cdot \delta_\beta = 9.$$

C. Faber informs us that the direct calculation of these intersection numbers is not difficult. This would allow a different approach to the proof of Theorem (2.8), using the theorem of [11] that the strata of $\overline{\mathcal{M}}_{1,n}$ span the even-dimensional rational cohomology.

3. Gromov-Witten invariants

In the remainder of this paper, we apply the new relation to the calculation of elliptic Gromov-Witten invariants: we will do this explicitly for curves and for the projective plane $\mathbb{CP}^2$, and prove some general results in other cases.

(3.1) The Novikov ring. Let V be a smooth projective variety of dimension d. In studying the Gromov-Witten invariants, it is convenient to work with cohomology with coefficients in the Novikov ring Λ of V, which we now define.

Let $\mathrm{N}_1(V)$ be the abelian group

$$\mathrm{N}_1(V) = \mathrm{Z}_1(V)/\text{numerical equivalence},$$

and let $\mathrm{NE}_1(V)$ be its sub-semigroup

$$\mathrm{NE}_1(V) = \mathrm{ZE}_1(V)/\text{numerical equivalence},$$

where $\mathrm{Z}_1(V)$ is the abelian group of 1-cycles on V, and $\mathrm{ZE}_1(V)$ is the semigroup of effective 1-cycles. (Recall that two 1-cycles x and y are numerically equivalent $x \equiv y$ when $x \cdot Z = y \cdot Z$ for any Cartier divisor Z on V.)

The Novikov ring is

$$\begin{aligned}\Lambda &= \mathbb{Q}[\mathrm{N}_1(V)] \otimes_{\mathbb{Q}[\mathrm{NE}_1(V)]} \mathbb{Q}[\![\mathrm{NE}_1(V)]\!] \\ &= \left\{a = \textstyle\sum_{\beta \in \mathrm{N}_1(V)} a_\beta q^\beta \mid \text{there exists } \beta_0 \in \mathrm{N}_1(V) \text{ such that } \mathrm{supp}(a) \subset \beta_0 + \mathrm{NE}_1(V)\right\},\end{aligned}$$

with product $q^{\beta_1} q^{\beta_2} = q^{\beta_1+\beta_2}$ and grading $|q^\beta| = -2c_1(V) \cap \beta$. That the product is well-defined is shown by the following proposition (Kollár [18], Proposition II.4.8).

Proposition (3.2). *If V is a projective variety with Kähler form ω, the set*

$$\{\beta \in \mathrm{NE}_1(V) \mid \omega \cap \beta \leq c\}$$

is finite for each $c > 0$. □

For example, if $V = \mathbb{CP}^n$, then $\mathrm{N}_1(\mathbb{CP}^n) = \mathbb{Z} \cdot [L]$, where $[L]$ is the cycle defined by a line $L \subset \mathbb{CP}^n$, and $\Lambda \cong \mathbb{Q}(\!(q)\!)$, with grading $|q| = -2(n+1)$, since $c_1(\mathbb{CP}^n) \cap [L] = n+1$.

If $V = E$ is an elliptic curve, then $\mathrm{N}_1(E) = \mathbb{Z} \cdot [E]$, and $\Lambda \cong \mathbb{Q}(\!(q)\!)$, concentrated in degree 0.

(3.3) Stable maps. The definition of Gromov-Witten invariants is based on the study of the moduli stacks $\overline{\mathcal{M}}_{g,n}(V,\beta)$ of stable maps of Kontsevich, which have been shown by Behrend and Manin [3] to be complete Deligne-Mumford stacks (though not in general smooth).

For each $N \geq 0$, let $\pi_{n,N} : \overline{\mathcal{M}}_{g,n+N}(V,\beta) \to \overline{\mathcal{M}}_{g,n}(V,\beta)$ be the projection which forgets the last N marked points of the stable curve, and stabilizes the resulting map. In the special case $N = 1$, we obtain a fibration

$$\pi : \overline{\mathcal{M}}_{g,n+1}(V,\beta) \to \overline{\mathcal{M}}_{g,n}(V,\beta)$$

which is shown by Behrend and Manin to be the universal curve; that is, its fibre over a stable map $(f : C \to V, x_i)$ is the curve C. Denote by $f : \overline{\mathcal{M}}_{g,n+1}(V,\beta) \to V$ the universal stable map, obtained by evaluation at x_{n+1}.

(3.4) The virtual fundamental class. If $2(g-1)+n>0$, the projection $\overline{\mathcal{M}}_{g,n}(V,\beta)\to\overline{\mathcal{M}}_{g,n}$ sends the stable map $(f:C\to V,x_i)$ to the stabilization of (C,x_i), obtained by collapsing rational components of C with fewer than 3 special points to a point.

If the sheaf $R^1\pi_*f^*TV$ vanishes on $\overline{\mathcal{M}}_{g,n}(V,\beta)$, the Riemann-Roch theorem predicts that the fibres of the projection $\overline{\mathcal{M}}_{g,n}(V,\beta)\to\overline{\mathcal{M}}_{g,n}$ have dimension $d(1-g)+c_1(V)\cap\beta$, and hence that $\overline{\mathcal{M}}_{g,n}(V,\beta)$ has dimension

$$d(1-g)+c_1(V)\cap\beta+\dim\overline{\mathcal{M}}_{g,n}=(3-d)(1-g)+c_1(V)\cap\beta+n.$$

This hypothesis is only rarely true, and in any case only in genus 0. However, Behrend-Fantecchi [1, 2] and Li-Tian [20] show that there is a bivariant class

$$[\overline{\mathcal{M}}_{g,n}(V,\beta)/\overline{\mathcal{M}}_{g,n},R^\bullet\pi_*f^*TV]\in A^{d(1-g)+c_1(V)\cap\beta}(\overline{\mathcal{M}}_{g,n}(V,\beta)\to\overline{\mathcal{M}}_{g,n}),$$

the virtual relative fundamental class, which stands in for $[\overline{\mathcal{M}}_{g,n}(V,\beta)/\overline{\mathcal{M}}_{g,n}]$ in the obstructed case.

The following result is proved in [1], and sometimes permits the explicit calculation of Gromov-Witten invariants, as we will see later.

Proposition (3.5). *If the coherent sheaf $R^1\pi_*f^*TV$ on $\overline{\mathcal{M}}_{g,n}$ is locally trivial of dimension e (the* excess dimension*), then $\overline{\mathcal{M}}_{g,n}(V,\beta)$ is smooth of dimension*

$$(3-d)(1-g)+c_1(V)\cap\beta+n+e,$$

and $[\overline{\mathcal{M}}_{g,n}(V,\beta)/\overline{\mathcal{M}}_{g,n},R^\bullet\pi_*f^*TV]=c_e(R^1\pi_*f^*TV)\cap[\overline{\mathcal{M}}_{g,n}(V,\beta)/\overline{\mathcal{M}}_{g,n}]$. □

(3.6) Gromov-Witten invariants. The Gromov-Witten invariant of genus $g\geq 0$, valence $n\geq 0$ and degree $\beta\in\mathrm{NE}_1(V)$ is a cohomology operation

$$I^V_{g,n,\beta}:H^{2d(1-g)+2c_1(V)\cap\beta+\bullet}(V^n,\mathbb{Q})\to H^\bullet(\overline{\mathcal{M}}_{g,n},\mathbb{Q}),$$

defined by the formula

$$I^V_{g,n,\beta}(\alpha_1,\dots,\alpha_n)=[\overline{\mathcal{M}}_{g,n}(V,\beta)/\overline{\mathcal{M}}_{g,n},R^\bullet\pi_*f^*TV]\cap\mathrm{ev}^*(\alpha_1\boxtimes\cdots\boxtimes\alpha_n),$$

where $\mathrm{ev}:\overline{\mathcal{M}}_{g,n}(V,\beta)\to V^n$ is evaluation at the marked points:

$$\mathrm{ev}:(f:C\to V,x_i)\mapsto(f(x_1),\dots,f(x_n))\in V^n.$$

Note that $I^V_{g,n,\beta}$ is invariant under the action of the symmetric group S_n on V^n.

Capping $I^V_{g,n,\beta}$ with the fundamental class $[\overline{\mathcal{M}}_{g,n}]$, we obtain a numerical invariant

$$\langle I^V_{g,n,\beta}\rangle:H^{2(d-3)(1-g)+2c_1(V)\cap\beta+2n}(V^n,\mathbb{Q})\to\mathbb{Q}.$$

This is the n-point correlation function of two-dimensional topological gravity with the topological σ-model associated to V as a background [23]. Note that if $\beta\neq 0$, $\langle I^V_{g,n,\beta}\rangle$ may be defined even when $2(g-1)+n\leq 0$, even though $I^V_{g,n,\beta}$ does not exist.

Introducing the Novikov ring, we may define the generating function

$$I^V_{g,n}=\sum_{\beta\in\mathrm{NE}_1(V)}q^\beta I^V_{g,n,\beta}:H^*(V,\Lambda)^{\otimes n}\to H^\bullet(\overline{\mathcal{M}}_{g,n},\Lambda),$$

along with its integral over the fundamental class $[\overline{\mathcal{M}}_{g,n}]$

$$\langle I^V_{g,n}\rangle=\sum_{\beta\in\mathrm{NE}_1(V)}q^\beta\langle I^V_{g,n,\beta}\rangle:H^*(V,\Lambda)^{\otimes n}\to\Lambda,$$

In the special case of zero degree, the moduli space $\overline{\mathcal{M}}_{g,n}(V,\beta)$ is isomorphic to $\overline{\mathcal{M}}_{g,n}\times V$. This allows us to calculate the Gromov-Witten invariants $\langle I^V_{0,3,0}\rangle$ and $\langle I^V_{1,1,0}\rangle$. The former is given by the explicit formula

$$\langle I^V_{0,3,0}(\alpha_1,\alpha_2,\alpha_3)\rangle = \int_V \alpha_1 \cup \alpha_2 \cup \alpha_2.$$

This formula is very simple to prove, since the moduli space $\overline{\mathcal{M}}_{0,3}(V,0)\cong V$ is smooth, with dimension equal to its virtual dimension d, and thus the virtual fundamental class $[\overline{\mathcal{M}}_{0,3}(V,0), R^\bullet\pi_* f^*TV]$ may be identified with the fundamental class of V. A similar proof shows that $\langle I^V_{0,n,0}\rangle$ vanishes if $n>3$.

The calculation of the Gromov-Witten invariant $\langle I^V_{1,1,0}\rangle$ (see Bershadsky et al. [4]) is a good illustration of the application of Proposition (3.5).

Proposition (3.7).

$$\langle I^V_{1,1,0}(\alpha)\rangle = -\frac{1}{24}\int_V c_{d-1}(V)\cup\alpha,$$

while $\langle I^V_{1,n,0}\rangle = 0$ *if* $n>1$.

Proof. The moduli stack $\overline{\mathcal{M}}_{1,n}(V,0)$ is isomorphic to $\overline{\mathcal{M}}_{1,n}\times V$, and the obstruction bundle $R^1\pi_*f^*TV$ is isomorphic to the vector bundle $\omega^\vee \boxtimes TV$, of rank d, where $\omega = \pi_*\omega_{\overline{\mathcal{M}}_{1,n+1}/\overline{\mathcal{M}}_{1,n}}$. Hence $R^1\pi_*f^*TV$ has top Chern class

$$c_d(\omega^\vee\otimes f^*TV) = 1\boxtimes f^*c_d(V) - c_1(\omega)\boxtimes f^*c_{d-1}(V).$$

By Proposition (3.5),

$$\begin{aligned}\langle I^V_{1,n,0}(\alpha_1,\dots,\alpha_n)\rangle &= \int_{\overline{\mathcal{M}}_{1,n}\times V} c_d(\omega^\vee\otimes f^*TV)\boxtimes(\alpha_1\cup\dots\cup\alpha_n)\\ &= -\int_{\overline{\mathcal{M}}_{1,n}} c_1(\omega)\cdot\int_V c_{d-1}(V)\cup\alpha_1\cup\dots\cup\alpha_n.\end{aligned}$$

On dimensional grounds, $\langle I^V_{1,n,0}\rangle$ vanishes if $n>1$, while the formula follows when $n=1$ from $c_1(\omega)\cap[\overline{\mathcal{M}}_{1,1}] = \frac{1}{24}$. □

(3.8) The puncture axiom. One of the fundamental axioms satisfied by Gromov-Witten expressed in the relationship between virtual fundamental classes

$$[\overline{\mathcal{M}}_{g,n+1}(V,\beta)/\overline{\mathcal{M}}_{g,n+1}, R^\bullet\pi_*f^*TV] = \pi^*[\overline{\mathcal{M}}_{g,n}(V,\beta)/\overline{\mathcal{M}}_{g,n}, R^\bullet\pi_*f^*TV].$$

Here, $\pi^* : A^k(\overline{\mathcal{M}}_{g,n}(V,\beta)\to\overline{\mathcal{M}}_{g,n}) \to A^k(\overline{\mathcal{M}}_{g,n+1}(V,\beta)\to\overline{\mathcal{M}}_{g,n+1})$ is the operation of flat pullback associated to the diagram

$$\begin{array}{ccc}\overline{\mathcal{M}}_{g,n+1}(V,\beta) & \longrightarrow & \overline{\mathcal{M}}_{g,n+1}\\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi}\big\downarrow\\ \overline{\mathcal{M}}_{g,n}(V,\beta) & \longrightarrow & \overline{\mathcal{M}}_{g,n}\end{array}$$

This axiom implies that if α is a cohomology class on V of degree at most 2 and $2(g-1)+n>0$,

$$(3.9)\qquad I^V_{g,n+1,\beta}(\alpha,\alpha_1,\dots,\alpha_n) = \begin{cases}0, & |\alpha| = 0,1,\\ (\alpha\cap\beta)I^V_{g,n,\beta}(\alpha_1,\dots,\alpha_n), & |\alpha| = 2.\end{cases}$$

(3.10) Generating functions. Let $\Lambda[H]$ be the power series ring $\Lambda[\![H_{\bullet+2}(V,\mathbb{Q})]\!]$. Let $\{\gamma^a\}_{a=0}^k$ be a homogeneous basis of the graded vector space $H^\bullet(V,\mathbb{Q})$, with $\gamma^0 = 1$, and let $\{t_a\}_{a=0}^k$ be the dual basis; the (homological) degree of t_a equals the (cohomological) degree of γ^a minus 2. We may identify the ring $\Lambda[H]$ with $\Lambda[\![t_0,\dots,t_k]\!]$.

Let $F_g(V)$ be the generating function

$$F_g(V) = \sum_{n=0}^{\infty} \langle I_{g,n}^V \rangle \in \Lambda[H].$$

This is a power series of degree $2(d-3)(1-g)$. This suggests assigning to Planck's constant $\hbar$ the degree $2(d-3)(g-1)$, and forming the total generating function, homogeneous of degree 0,

$$F(V) = \sum_{g=0}^{\infty} \hbar^{g-1} F_g(V).$$

(3.11) The composition axiom. The composition axiom for Gromov-Witten invariants gives a formula for the integral of the Gromov-Witten invariant $I_{g,n}^V$ over the cycle $[\overline{\mathcal{M}}(G)]$ associated to a stable graph G which bears a strong resemblance to the Feynman rules of quantum field theory:

Let η_{ab} be the Poincaré form of V with respect to the basis $\{\gamma^a\}_{a=0}^k$ of $H^\bullet(V,\mathbb{Q})$. Then

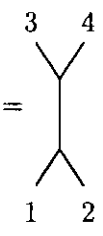
$$\int_{\overline{\mathcal{M}}(G)} I_{g,n}^V(\alpha_1,\dots,\alpha_n) = \frac{1}{\operatorname{Aut}(G)} \sum_{\substack{a(e),b(e)=0\\ e\in E(G)}}^{k} \prod_{e\in E(G)} \eta_{a(e),b(e)} \prod_{v\in V(G)} \langle I_{g(v),n(v)}^V(\dots)\rangle.$$

Here, the Gromov-Witten invariant $\langle I_{g(v),n(v)}^V(\dots)\rangle$ is evaluated on the cohomology classes α_i corresponding to the tails of G which meet the vertex v, on the $\gamma^{a(e)}$ corresponding to edges e which start at the vertex v, and on the $\gamma^{b(e)}$ corresponding to edges e which end at v. (The right-hand side is independent of the chosen orientation of the edges, by the symmetry of the Poincaré form.)

(3.12) Relations among Gromov-Witten invariants. The subvariety $\pi_{n,N}^{-1}(\overline{\mathcal{M}}(G))$ is the union of strata associated to the set of stable graphs obtained from G by adjoining N tails $\{n+1,\dots,n+N\}$ in all possible ways to the vertices of G.

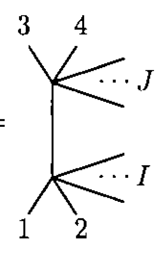
For example, consider the stratum $\Delta_{12|34} \subset \overline{\mathcal{M}}_{0,4}$, associated to the stable graph

$\Delta_{12|34} =$



The inverse image $\pi_{4,N}^{-1}(\Delta_{12|34})$ consists of the union of all strata in $\overline{\mathcal{M}}_{0,4+N}$ associated to stable graphs

$\Delta_{12I|34J} =$



where I and J form a partition of the set $\{5,\dots,N+4\}$.

If δ is a cycle in $\overline{\mathcal{M}}_{g,n}$, define the generating function

$$F(\delta, V) = \sum_{N=0}^{\infty} \int_{\pi^{-1}(\delta)} I^V_{g,n+N} : H^{\bullet+2}(V,\Lambda)^{\otimes n} \to \Lambda[\![H]\!].$$

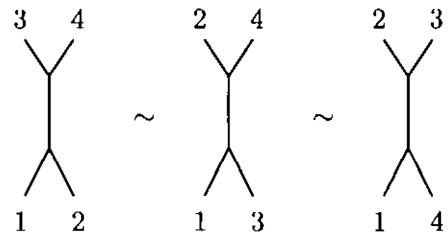
In particular, if $\delta = [\overline{\mathcal{M}}(G)]$ where G is a stable graph, we set $F(G,V) = F([\overline{\mathcal{M}}(G)], V)$. If $g > 1$, $F_g(V)$ is a special case of this construction, with $\delta = [\overline{\mathcal{M}}_{g,0}]$.

A little exercise involving Leibniz's rule shows that the composition axiom implies the following formula for these generatings functions:

$$F(G,V) = \frac{1}{\operatorname{Aut}(G)} \sum_{\substack{a(e),b(e)=0 \\ e\in E(G)}}^{k} \prod_{e\in E(G)} \eta_{a(e),b(e)} \prod_{v\in V(G)} \partial^{n(v)} F_{g(v)}(V)(\dots), \tag{3.13}$$

where as before, the multilinear form $\partial^{n(v)} F_{g(v)}(V)$ is evaluated on the cohomology classes α_i corresponding to the tails of G meeting the vertex v, on the $\gamma^{a(e)}$ corresponding to edges e which start at the vertex v, and on the $\gamma^{b(e)}$ corresponding to edges e which end at v.

The composition axiom implies that any relation among the cycles $[\overline{\mathcal{M}}(G)]$ is reflected in a relation among Gromov-Witten invariants, which, by (3.13) may be translated into a differential equation among generating functions $F_g(V)$. An example is the rational equivalence of the cycles associated to the three strata of $\overline{\mathcal{M}}_{0,4}$ of codimension 1:



The equality of the Gromov-Witten invariant $F(\delta, V)$ when evaluated on these cycles is the Witten-Dijkgraaf-Verlinde-Verlinde equation.

In order to express the relation among the Gromov-Witten invariants implied by Theorem (2.8), it is useful to introduce certain operators which act on elements of $\Lambda[H] \otimes \Lambda[\![H]\!]$ through differentiation in the first factor: the Laplacian

$$\Delta = \frac{1}{2} \sum_{a,b=0}^{k} \eta_{ab} \frac{\partial^2}{\partial t_a \partial t_b},$$

and the sequence of bilinear differential operators Γ_n by $\Gamma_0(f,g) = fg$ and

$$\Gamma_n(f,g) = \frac{1}{n}\big(\Delta\Gamma_{n-1}(f,g) - \Gamma_{n-1}(\Delta f, g) - \Gamma_{n-1}(f, \Delta g)\big).$$

(We will abbreviate $\Gamma_1(f,g)$ to $\Gamma(f,g)$.)

Proposition (3.14). *Denote the derivative $\partial^{n(v)} F_{g(v)}(V)/n(v)! \in \Lambda[H] \otimes \Lambda[\![H]\!]$ by $f_{g,n}$. (Note that $f_{g,n} = F([\overline{\mathcal{M}}_{g,n}], V)$.) Then*

$$\begin{aligned} 6\Gamma(\Gamma_1(f_{1,2}, f_{0,3}), f_{0,3}) &- 5\Gamma(f_{1,2}, \Gamma(f_{0,3}, f_{0,3})) \\ &- 2\Gamma(f_{0,3}, \Gamma(f_{1,1}, f_{0,4})) + 6\Gamma(f_{0,4}, \Gamma(f_{1,1}, f_{0,3})) \\ &+ \Gamma(f_{0,4}, \Delta f_{0,4}) + \Gamma(f_{0,5}, \Delta f_{0,3}) - \Gamma_2(f_{0,4}, f_{0,4}) = 0. \end{aligned}$$

Proof. This follows from the following table, which is obtained by application of (3.13).

δ	$F(\delta, V)$		
$\delta_{2,2}$	$\frac{1}{2}\Gamma(\Gamma(f_{1,2}, f_{0,3}), f_{0,3})$	$\delta_{0,2}$	$\Gamma(f_{0,3}, \Delta f_{0,5})$
	$-\frac{1}{4}\Gamma(f_{1,2}, \Gamma(f_{0,3}, f_{0,3}))$	$\delta_{0,3}$	$\Gamma(f_{0,4}, \Delta f_{0,4})$
$\delta_{2,3}$	$\frac{1}{2}\Gamma(f_{1,2}, \Gamma(f_{0,3}, f_{0,3}))$	$\delta_{0,4}$	$\Gamma(f_{0,5}, \Delta f_{0,3})$
$\delta_{2,4}$	$\Gamma(f_{0,3}, \Gamma(f_{1,1}, f_{0,4}))$	δ_α	$\Gamma_2(f_{0,3}, f_{0,5})$
$\delta_{3,4}$	$\Gamma(f_{0,4}, \Gamma(f_{1,1}, f_{0,3}))$	δ_β	$\frac{1}{2}\Gamma_2(f_{0,4}, f_{0,4})$

□

When we apply Proposition (3.14) with $V = \mathbb{CP}^2$ and evaluate the resulting multilinear form to $\omega^{\boxtimes 4}$, we obtain the recursion relation (1.1) for the elliptic Gromov-Witten invariants $N_n^{(1)}$ of $\mathbb{CP}^2$.

4. The symbol of the new relation

We may introduce a filtration on Gromov-Witten invariants with respect to which the leading order of our new relation takes a relatively simple form; by analogy with the case of differential operators, we call this leading order relation the symbol of the full relation. In some cases, this symbol may be used to prove that elliptic Gromov-Witten invariants are determined by rational ones.

Definition (4.1). The *symbol* of a relation $\delta = 0$ among cycles of strata in $\overline{\mathcal{M}}_{g,n}$ is the set of relations among Gromov-Witten invariants obtained by taking, for each $\beta \in \mathrm{NE}_1(V)$, the coefficient of q^β in $I^V_{g,n} \cap [\delta]$, expanding in Feynman diagrams using the composition axiom, and setting all Gromov-Witten invariants $\langle I^V_{g',n',\beta'}\rangle$ other than $\langle I^V_{g,n,\beta}\rangle$ and $\langle I^V_{0,3,0}\rangle$ to zero.

We define a total order on the symbols $\langle I^V_{g,n,\beta}\rangle$ by setting $\langle I^V_{g',n',\beta'}\rangle \prec \langle I^V_{g,n,\beta}\rangle$ if $g' < g$, or $g' = g$ and $n' < n$, or $g' = g$, $n' = n$ and $\beta = \beta' + \beta''$ where $\beta'' \in \mathrm{NE}_1(V)$ is non-zero. Thus, knowledge of the symbol determines relations among Gromov-Witten invariants such that the error in the relation on $\langle I^V_{g,n,\beta}\rangle$ involves invariants $\langle I^V_{g',n',\beta'}\rangle$ with $\langle I^V_{g',n',\beta'}\rangle \prec \langle I^V_{g,n,\beta}\rangle$. (Here, we must of course exclude $\langle I^V_{0,3,0}\rangle$.) We use the symbol $\sim$ to denote this equivalence relation.

For example, the symbol of the Witten-Dijkgraaf-Verlinde-Verlinde equation is

$$(a, b, c \cup d) + (a \cup b, c, d) \sim (-1)^{|a|(|b|+|c|)}\big((b, c, a \cup d) + (b \cup c, a, d)\big),$$

where we have abbreviated $\langle I^V_{0,n,\beta}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \dots, \alpha_n)\rangle$ to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

Next, consider the symbol of the relation

$$\pi^{-1}_{4,n-4}\big(12\delta_{2,2} - 4\delta_{2,3} - 2\delta_{2,4} + 6\delta_{3,4} + \delta_{0,3} + \delta_{0,4} - 2\beta\big) = 0$$

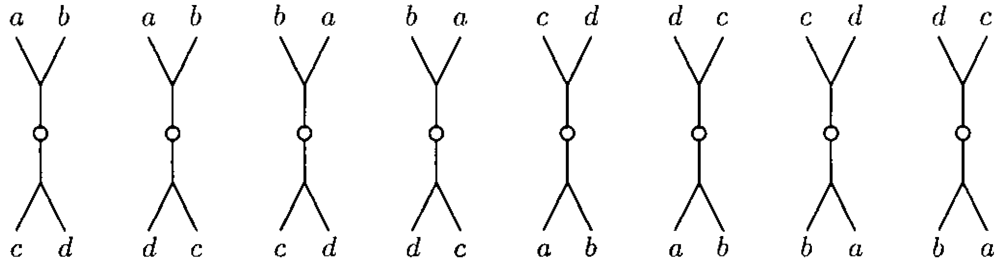
in $H_{2n-4}(\overline{\mathcal{M}}_{1,n}, \mathbb{Q})$. Only the cycles $\delta_{2,2}$ and $\delta_{2,3}$ contribute terms to the symbol. Abbreviate the Gromov-Witten class $\langle I^V_{1,n,\beta}(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n)\rangle$ to $\{\alpha_1, \alpha_2\}$. Up to a numerical factor to be determined, the cycle $\delta_{2,2}$ contributes the expression

$$\{a \cup b, c \cup d\} + (-1)^{|b|\,|c|}\{a \cup c, b \cup d\} + (-1)^{(|b|+|c|)|d|}\{a \cup d, b \cup c\}.$$

This numerical factor equals

$$\frac{1}{24} \cdot 3 \cdot 12 \cdot 8 = 12.$$

The factor 1/24 comes from symmetrization over the four inputs, the factor of 3 from the three strata making up $\delta_{2,2}$, the factor of 12 is the coefficient of the cycle in the relation, and the factor 8 is illustrated by listing all of the graphs which contribute a term $\{a \cup b, c \cup d\}$:



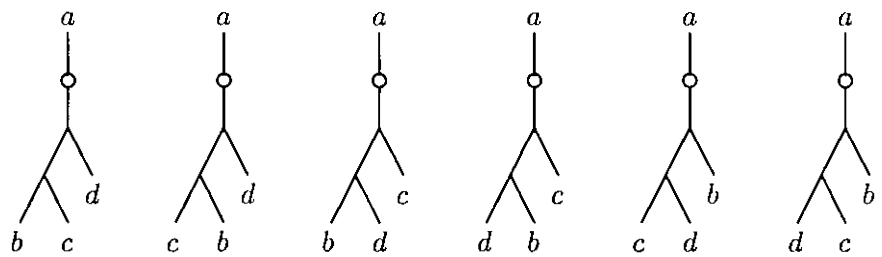
Similarly, the cycle $\delta_{2,3}$ contributes the expression

$$\{a, b{\cup}c{\cup}d\}+(-1)^{|a|\,|b|}\{b, a{\cup}c{\cup}d\}+(-1)^{(|a|+|b|)|c|}\{c, a{\cup}b{\cup}d\}+(-1)^{(|a|+|b|+|c|)|d|}\{d, a{\cup}b{\cup}c\},$$

with numerical factor

$$\frac{1}{24} \cdot 12 \cdot (-4) \cdot 6 = -12;$$

the factor 12 counts the strata making up $\delta_{2,3}$, -4 is the coefficient of the cycle in the relation, and we illustrate the factor 6 by listing all of the graphs which contribute a term $\{a, b \cup c \cup d\}$:



In conclusion, we obtain the following result.

Theorem (4.2). *Abbreviating* $\langle I^V_{1,n,\beta}(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n)\rangle$ *to* $\{\alpha_1, \alpha_2\}$, *we have*

$$\begin{aligned}\Psi(a,b,c,d) = {}& \{a \cup b, c \cup d\} + (-1)^{|b|\,|c|}\{a \cup c, b \cup d\} + (-1)^{(|b|+|c|)|d|}\{a \cup d, b \cup c\} \\ & - \{a, b \cup c \cup d\} - (-1)^{|a|\,|b|}\{b, a \cup c \cup d\} \\ & - (-1)^{(|a|+|b|)|c|}\{c, a \cup b \cup d\} - (-1)^{(|a|+|b|+|c|)|d|}\{d, a \cup b \cup c\} \sim 0.\end{aligned}$$

Note that the linear form $\Psi(a,b,c,d)$ is (graded) symmetric in its four arguments, and vanishes if any of them equals 1.

Corollary (4.3). *If* $\omega \in H^2(V, \mathbb{Q})$ *and* $a, b \in H^\bullet(V, \mathbb{Q})$, *then for* $j \geq i+3$,

$$\{\omega^i \cup a, \omega^{j-i} \cup b\} = \tbinom{i+2}{2}\{a, \omega^j \cup b\}.$$

Proof. By Theorem (4.2), we have for $i \geq 0$ and $j \geq 3$,

$$\begin{aligned}\Psi(\omega, \omega^{i+1} \cup a, \omega, \omega^{j-i-3} \cup b) &- \Psi(\omega, \omega^i \cup a, \omega, \omega^{j-i-2} \cup b) \\ \sim {}& \big(2\{\omega^{i+2} \cup a, \omega^{j-i-2} \cup b\} + \{\omega^2, \omega^{j-2} \cup a \cup b\} \\ & - \{\omega^{i+1} \cup a, \omega^{j-i-1} \cup b\} - \{\omega^{i+3} \cup a, \omega^{j-i-3} \cup b\}\big) \\ & - \big(2\{\omega^{i+1} \cup a, \omega^{j-i-1} \cup b\} + \{\omega^2, \omega^{j-2} \cup a \cup b\} \\ & - \{\omega^i \cup a, \omega^{j-i} \cup b\} - \{\omega^{i+2} \cup a, \omega^{j-i-2} \cup b\}\big) \\ \sim {}& \{\omega^i \cup a, \omega^{j-i} \cup b\} - 3\{\omega^{i+1} \cup a, \omega^{j-i-1} \cup b\} \\ & + 3\{\omega^{i+2} \cup a, \omega^{j-i-2} \cup b\} - \{\omega^{i+3} \cup a, \omega^{j-i-3} \cup b\} \sim 0.\end{aligned}$$

This implies that the function $a(i,j) = \{\omega^i \cup a, \omega^{j-i} \cup b\}$ satisfies the difference equation

$$a(i,j) - 3a(i+1,j) + 3a(i+2,j) - a(i+3,j) \sim 0$$

with solution $a(i,j) \sim \binom{i+2}{2} a(0,j)$. □

We can now prove a weak analogue for elliptic Gromov-Witten invariants of the (first) Reconstruction Theorem of Kontsevich-Manin (Theorem 3.1 of [19]). Denote the primitive cohomology

$$\operatorname{coker}(H^{j-2}(V,\mathbb{Q}) \xrightarrow{\omega\cup\cdot} H^j(V,\mathbb{Q}))$$

by $P^j(V)$, and let $H^\bullet_{[i]}(V,\mathbb{Q})$ be the graded subspace of $H^\bullet(V,\mathbb{Q})$ associated by the Lefschetz decomposition to the primitive cohomology classes of degree at most i. Let $\Pi^4(V)$ be the projection of the image of the cup product $H^2(V,\mathbb{Q}) \otimes H^2(V,\mathbb{Q}) \xrightarrow{\cup} H^4(V,\mathbb{Q})$ to $P^4(V)$.

Theorem (4.4). *(1) On restriction to $H^\bullet_{[1]}(V,\mathbb{Q})$, the elliptic Gromov-Witten invariants of V are determined by its rational Gromov-Witten invariants together with the Gromov-Witten invariants $\langle I_{1,1,\beta}(\omega^{i+1})\rangle$ for $0 \le c_1(V) \cap \beta = i < d$.*

(2) On restriction to $H^\bullet_{[2]}(V,\mathbb{Q})$, the elliptic Gromov-Witten invariants of V are determined by its rational Gromov-Witten invariants together with the Gromov-Witten invariants

- $\langle I_{1,1,\beta}(-)\rangle : H^{2i+2}_{[2]}(V,\mathbb{Q}) \to \mathbb{Q}$ *for* $0 \le c_1(V) \cap \beta = i < d$, *and*
- $\langle I_{1,2,\beta}(\omega^2, -)\rangle : \Pi^4(V) \to \mathbb{Q}$ *for* $c_1(V) \cap \beta = 2$.

Proof. We proceed by induction: by hypothesis, $\langle I^V_{g,n,\beta}\rangle$ is known for $g = 0$ or $g = 1$ and $n = 1$. Now consider the Gromov-Witten invariant $\langle I^V_{1,n,\beta}(\alpha_1, \dots, \alpha_n)\rangle$, where $n > 1$. By (3.9), we may assume that $|\alpha_i| > 2$, and under the hypotheses of the proposition, we may write it as $\omega^{p_i} \cup \gamma_i$ where $|\gamma_i| \le 2$ is a primitive cohomology class.

Step 1: If any two indices p_i and p_j satisfy $p_i + p_j > 2$, we may apply Corollary (4.3) to replace the pair $(\omega^{p_i} \cup \gamma_i, \omega^{p_j} \cup \gamma_j)$ by $(\gamma_i, \omega^{p_i+p_j} \cup \gamma_j)$. If $|\gamma_1| = 1$, the result vanishes by (3.9), while if $|\gamma_1| = 2$, we may apply (3.9) to reduce n by 1.

Step 2: We are reduced to considering $\langle I^V_{1,n,\beta}(\omega \cup \gamma_1, \dots, \omega \cup \gamma_n)\rangle$, where the classes γ_i have degree 1 or degree 2. Applying Theorem (4.2), we see that

$$\Psi(\omega, \gamma_1, \omega, \gamma_2) = 2\{\omega \cup \gamma_1, \omega \cup \gamma_2\} + \{\omega^2, \gamma_1 \cup \gamma_2\} \sim 0.$$

In particular, we may assume that $n = 2$, since otherwise, we would be able to return to Step 1. There are two cases.

Step 2a: If the classes γ_i are both of degree 1, we see that $\{\omega \cup \gamma_1, \omega \cup \gamma_2\} \sim 0$, since in that case $\gamma_1 \cup \gamma_2$ has degree 2 and we may apply (3.9). (This completes the proof of part (1) of the proposition.)

Step 2b: If the classes γ_i are both of degree 2, we see that it suffices to know the invariant $\langle I^V_{1,2,\beta}(\omega^2, \gamma_1 \cup \gamma_2)\rangle$, or rather, since we have already handled $\langle I^V_{1,2,\beta}(\omega^2, \omega \cup \gamma)\rangle$, we only need to know the invariant $\langle I^V_{1,2,\beta}(\omega^2, \alpha)\rangle$, where $\alpha \in \Pi^4(V)$ is the projection of $\gamma_1 \cup \gamma_2 \in H^4(V,\mathbb{Q})$ to $P^4(V)$. □

Three special cases of this result seem worth singling out:

i) If V is a surface, $H^\bullet_{[2]}(V,\mathbb{Q}) = H^\bullet(V,\mathbb{Q})$, and $\Pi^4(V) = 0$; thus, all elliptic Gromov-Witten invariants are determined by the rational invariants together with $\langle I_{1,1,\beta}(-)\rangle : H^{2c_1(V)\cap\beta+2}(V,\mathbb{Q}) \to \mathbb{Q}$ for $c_1(V) \cap \beta = 0, 1$.

ii) If V is a threefold, $H^\bullet_{[2]}(V,\mathbb{Q})$ contains the even degree cohomology of V, and once more $\Pi^4(V) = 0$; thus, all elliptic Gromov-Witten invariants with arguments of even degree are determined by the rational invariants together with $\langle I_{1,1,\beta}(-)\rangle$: $H^{2c_1(V)\cap\beta+2}(V,\mathbb{Q}) \to \mathbb{Q}$ for $c_1(V)\cap\beta = 0,1,2$.

iii) In all cases, the elliptic Gromov-Witten invariants with arguments in the subalgebra of $H^\bullet(V,\mathbb{Q})$ generated by ω are determined by the rational invariants together with $\langle I_{1,1,\beta}(\omega^i)\rangle$ for $1 \le i = 1 + c_1(V)\cap\beta \le d$. In particular, for $V = \mathbb{CP}^d$, the elliptic Gromov-Witten invariants are determined by the rational Gromov-Witten invariants.

5. Gromov-Witten invariants of curves

To illustrate our new relation, we start with the case where V is a curve. We will only discuss curves of genus 0 and 1, since for curves of higher genus, $I^V_{g,n,\beta} = 0$ if $\beta \neq 0$, and the new relation is identically satisfied.

(5.1) $\mathbb{CP}^1$. When $V = \mathbb{CP}^1$, the potential F_g is a power series of degree $4g-4$ in variables t_0 and t_1 (of degree -2 and 0) and the generator q of Λ, of degree $-4 = -2c_1(\mathbb{CP}^1)\cap[\mathbb{CP}^1]$. By degree counting, together with (3.9), we see that

$$F_g(\mathbb{CP}^1) = \begin{cases} t_0^2t_1/2 + qe^{t_1}, & g=0, \\ -t_1/24, & g=1, \\ 0, & g>1; \end{cases}$$

the only thing which is not immediate is the coefficient of q in $F_0(\mathbb{CP}^1)$, which is the number of maps of degree 1 from $\mathbb{CP}^1$ to itself, up to isomorphism, and clearly equals 1.

It is easy to calculate $F(\delta,\mathbb{CP}^1)$ for δ equal to one of our nine 2-cycles: all of them vanish except

$$F(\delta_{3,4},\mathbb{CP}^1) = \frac{t_1^4}{24}\otimes(-qe^{t_1}/6), \quad F(\delta_{0,4},\mathbb{CP}^1) = \frac{t_1^4}{24}\otimes qe^{t_1}, \quad F(\delta_\alpha,\mathbb{CP}^1) = \frac{t_1^4}{24}\otimes 2qe^{t_1}.$$

We see that the new relation holds among these potentials.

(5.2) Elliptic curves. Let E be an elliptic curve. Denote by ξ,η variables of degree -1 corresponding to a basis of $H_1(E,\mathbb{Z})$ such that $\langle\xi,\eta\rangle = 1$. The ring Λ has one generator q, of degree 0 (since $c_1(V) = 0$). Since there are no rational curves in E of positive degree, we have

$$F_0(E) = t_0^2t_1/2 + t_0\xi\eta.$$

It is shown in [4] that

$$F_1(E) = -\frac{t_1}{24} + \sum_{\beta=1}^{\infty}\frac{\sigma(\beta)}{\beta}q^\beta\left(e^{\beta t_1}-1\right), \tag{5.3}$$

since $\langle I^E_{1,1,\beta}(\omega)\rangle = \sigma(\beta)$ counts the number of unramified covers of degree β of the curve E up to automorphisms, which are easily enumerated. An equivalent form of (5.3) is

$$\frac{\partial F_1(E)}{\partial t_1} = G_2(qe^{t_1}),$$

where

$$G_2(q) = -\frac{1}{24} + \sum_{\beta=1}^{\infty}\sigma(\beta)q^\beta$$

is the Eisenstein series of weight 2. By degree counting, we also see that $F_g(E) = 0$ for $g > 1$.

Note that the Gromov-Witten invariants of an elliptic curve are invariant under deformation; this is true for any smooth projective variety V (Li-Tian [20]). In fact, the definition of Gromov-Witten invariants extends to any almost-Kähler manifold (a symplectic manifold with compatible almost-complex structure), and the resulting invariants are independent of the almost-complex structure (Li-Tian [21]).

It is simple to calculate the Gromov-Witten potentials $F(\delta, E)$ for our nine 2-cycles in $\overline{\mathcal{M}}_{1,4}$.

Lemma (5.4). *We have*

$$F(\delta_{2,2}, E) = \Big(\frac{5}{12}G_4(qe^{t_1}) - G_2(qe^{t_1})^2\Big)(t_0t_1 + \xi\eta)^2 = \frac{q}{2}(t_0t_1 + \xi\eta)^2 + O(q^2),$$

$F(\delta_{2,3}, E) = 3F(\delta_{2,2}, E)$, *while the remaining* 7 *potentials vanish.* □

Again, we see that the new relation holds.

6. The Gromov-Witten invariants of $\mathbb{CP}^2$

The Gromov-Witten potential $F_g(\mathbb{CP}^2)$ is a power series of degree $2g-2$ in variables t_0, t_1 and t_2, of degrees -2, 0 and 2, where t_i is dual to ω^i, and the generator q of Λ, of degree $-6 = -2c_1(\mathbb{CP}^2) \cap [L]$.

By degree counting, together with (3.9), we see that

$$F_g(\mathbb{CP}^2) = \begin{cases} \frac{1}{2}(t_0t_1^2 + t_0^2t_2) + \sum_{\beta=1}^{\infty} N_\beta^{(0)} q^\beta e^{\beta t_1} \frac{t_2^{3\beta-1}}{(3\beta-1)!}, & g = 0, \\ -\frac{t_1}{8} + \sum_{\beta=1}^{\infty} N_\beta^{(1)} q^\beta e^{\beta t_1} \frac{t_2^{3\beta}}{(3\beta)!}, & g = 1, \\ \sum_{\beta=1}^{\infty} N_\beta^{(g)} q^\beta e^{\beta t_1} \frac{t_2^{3\beta+g-1}}{(3\beta+g-1)!}, & g > 1, \end{cases}$$

where $N_\beta^{(g)}$ are certain rational coefficients.

Using the Severi theory of plane curves, we will show that $N_\beta^{(g)}$ is the answer to an enumerative problem for plane curves; in particular, it is a non-negative integer. This phenomenon is special to del Pezzo surfaces: we have already seen that the elliptic Gromov-Witten invariants of an elliptic curve are non-integral, while for $\mathbb{CP}^3$, they are not even positive.

We apply the following result, which is Proposition 2.2 of Harris [13].

Proposition (6.1). *Let S be a smooth rational surface. Let $\pi : C \to \mathcal{M}$ be a family of curves of geometric genus g with $\mathcal{M}$ irreducible, and let $f : C \to \mathcal{M}$ be a map such that on each component of a general fibre C_z of π, the restriction f_z of f to C_z is not constant and $f_z^*\omega_S$ has negative degree.*

Let W be the image of the map from $\mathcal{M}$ to the Chow variety of curves on S defined by sending $z \in \mathcal{M}$ to the curve C_z. Then $\dim(W) \le -\deg(f_z^*\omega_S) + g - 1$, and if equality holds, then f_z is birational for all $z \in \mathcal{M}$. □

Corollary (6.2). *The coefficient $N_\beta^{(g)}$ equals the number of irreducible plane curves of arithmetic genus g and degree β passing through $3\beta + g - 1$ general points in $\mathbb{CP}^2$.*

Proof. Let $\mathcal{M}$ be a component of the boundary $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^2, \beta) \setminus \mathcal{M}_{g,n}(\mathbb{CP}^2, \beta)$, and consider the family of curves $C \to \mathcal{M}$ obtained by restricting the universal curve $\overline{\mathcal{M}}_{g+1,n}(V, \beta) \to \overline{\mathcal{M}}_{g,n}(V, \beta)$ to $\mathcal{M}$ and contracting to a point all components of the fibres on which f has degree 0.

The geometric genus of the fibres of this family is bounded above by $g-1$. Applying Proposition (6.1), we see that the image of $\mathcal{M}$ in the Chow variety of plane curves has dimension at most $3\beta+g-2$.

On the other hand, if $\mathcal{M}$ is a component of $\mathcal{M}_{g,n}(\mathbb{CP}^2,\beta)$, and $\mathcal{C}\to\mathcal{M}$ is the universal family of curves $\mathcal{C}\to\mathcal{M}$, we see that the image of $\mathcal{M}$ in the Chow variety of plane curves has dimension less than $3\beta+g-1$ unless the stable maps parametrized by $\mathcal{M}$ are birational to their image.

The Gromov-Witten invariant $N_\beta^{(g)}$ equals the degree of the intersection of the image of $\overline{\mathcal{M}}_{g,3\beta+g-1}(V,\beta)$ in the Chow variety of curves in V with the cycle of curves passing through $3\beta+g-1$ general points. By Bertini's theorem for homogenous spaces [16], we see that the points of intersection are reduced and lie in the components of $\mathcal{M}_{g,n}(\mathbb{CP}^2,\beta)$ on which the map f is birational to its image, and hence an immersion. (This argument is borrowed from Section 6 of Fulton-Pandharipande [8].) The result follows. □

(6.3) Comparison with the formulas of Caporaso and Harris. Caporaso and Harris [5] have calculated the numbers $N_\beta^{(g)}$ for all $g\ge 0$, and we now turn the comparison of our results for $N_\beta^{(1)}$. We have not been able to find a proof that our answers agree, but we have verified that this is so for $\beta\le 6$.

The recursion of Caporaso and Harris for the Gromov-Witten invariants of $\mathbb{CP}^2$ is more easily applied if it is recast in terms of generating functions.

Definition (6.4). If α is a partition, denote by $\ell(\alpha)$ the number of parts of α and by $|\alpha|$ the sum $\alpha_1+\cdots+\alpha_{\ell(\alpha)}$ of the parts of α. Let $\alpha!$ be the product $\alpha!=\alpha_1!\dots\alpha_{\ell(\alpha)}!$.

Fix a line L in $\mathbb{CP}^2$. If α and β are partitions with $|\alpha|+|\beta|=d$, and Ω is a collection of $\ell(\alpha)$ general points of L, let $V^{d,\delta}(\alpha,\beta)(\Omega)=V^{d,\delta}(\alpha,\beta)$ be the generalized Severi variety: the closure of the locus of reduced plane curves of degree d not containing L, smooth except for δ double points, having order of contact α_i with L at Ω_i, and to order $\beta_1,\dots,\beta_{\ell(\beta)}$ at $\ell(\beta)$ further unassigned points of L. For example, $V^{d,\delta}(0,1^d)$ is the classical Severi variety of plane curves of degree d with δ double points, while $V^{d,\delta}(0,21^{d-1})$ is the closure of the locus of plane curves tangent to L at a smooth point.

Denote by $V_0^{d,\delta}(\alpha,\beta)$ the union of the components of $V^{d,\delta}(\alpha,\beta)$ whose general point is an irreducible curve. (By the main result of Harris [13], there is actually only one such component.) Let $N^{d,\delta}(\alpha,\beta)$ be the degree of $V^{d,\delta}(\alpha,\beta)$ and let $N_0^{d,\delta}(\alpha,\beta)$ be the degree of $V_0^{d,\delta}(\alpha,\beta)$. Form the generating functions

$$Z=\sum\frac{z^{\binom{d+1}{2}-\delta+\ell(\beta)}}{(\binom{d+1}{2}-\delta+\ell(\beta))!}\frac{p^\alpha}{\alpha!}q^\beta N^{d,\delta}(\alpha,\beta),$$
$$F=\sum\frac{z^{\binom{d+1}{2}-\delta+\ell(\beta)}}{(\binom{d+1}{2}-\delta+\ell(\beta))!}\frac{p^\alpha}{\alpha!}q^\beta N_0^{d,\delta}(\alpha,\beta).$$

The integer $\binom{d+1}{2}-\delta+\ell(\beta)$ is the dimension of the variety $V^{d,\delta}(\alpha,\beta)$. The union of curves of degree d_i, $1\le i\le n$, with δ_i double points and partitions α_i and β_i is a (reducible) curve has degree $d=d_1+\cdots+d_n$ with

$$\delta=\delta_1+\cdots+\delta_n+\sum_{i<j}\delta_i\delta_j$$

double points and partitions $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\beta=(\beta_1,\dots,\beta_n)$. This formula for δ amounts to the condition that the sum of the dimensions of the generalized Severi varieties

$V_0^{d_i,\delta_i}(\alpha_i,\beta_i)$ equals the dimension of $V^{d,\delta}(\alpha,\beta)$. The relationship of these two generating functions,

$$Z = \exp(F),$$

is an exercise in the definition of degree (see Ran [22]).

Caporaso and Harris prove a recursion which in terms of the generating function Z may be written

$$\frac{\partial Z}{\partial z} = \sum_{k=1}^{\infty} kq_k \frac{\partial Z}{\partial p_k} + \operatorname{res}_{t=0}\Big[\exp\Big(\sum_{k=1}^{\infty} t^{-k}p_k + \sum_{k=1}^{\infty} kt^k \frac{\partial}{\partial q_k}\Big)\Big] Z,$$

where $\operatorname{res}_{t=0}$ is the residue with respect to the formal variable t, in other words, the coefficient of t^{-1} when the exponential is expanded.* Dividing by Z, we obtain

$$\frac{\partial F}{\partial z} = \sum_{k=1}^{\infty} kq_k \frac{\partial F}{\partial p_k} + \operatorname{res}_{t=0}\Big[\exp\Big(\sum_{k=1}^{\infty} t^{-k}p_k + F(q_k + kt^k) - F(q_k)\Big)\Big],$$

which clearly allows the recursive calculation of the coefficients $N_0^{d,\delta}(\alpha,\beta)$.

As a special case of $Z = \exp(F)$, we have

$$1 + \sum \frac{z^{\binom{d+2}{2}-\delta-1} q^d N^{d,\delta}}{(\binom{d+2}{2}-\delta-1)!} = \exp\Big(\sum \frac{z^{\binom{d+2}{2}-\delta-1} q^d N_0^{d,\delta}}{(\binom{d+2}{2}-\delta-1)!}\Big),$$

since $\binom{d+1}{2} - \delta + d = \binom{d+2}{2} - \delta - 1$. Expanding the exponential, we obtain

$$N^{d,\delta} = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{d=d_1+\cdots+d_n} \sum_{\substack{\delta=\sum_{i<j}\delta_i\delta_j \\ +\delta_1+\cdots+\delta_n}} \frac{(\binom{d+2}{2}-\delta-1)! N_0^{d_1,\delta_1} \dots N_0^{d_n,\delta_n}}{(\binom{d_1+2}{2}-\delta_1-1)! \dots (\binom{d_i+2}{2}-\delta_i-1)!}.$$

For example, with $d = 5$, we obtain

$$N_0^{5,4} = N^{5,4} - \frac{16!}{14!2!} N_0^{4,0} N_0^{1,0} = 36975 - 120 \cdot 1 = 36855,$$
$$N_0^{5,5} = N^{5,5} - \frac{15!}{13!2!} N_0^{4,1} N_0^{1,0} = 90027 - 105 \cdot 27 = 87192,$$

while with $d = 6$ and $\delta = 9$, we obtain

$$\begin{aligned} N_0^{6,9} &= N^{6,9} - \frac{18!}{16!2!} N_0^{5,4} N_0^{1,0} - \frac{18!}{13!5!} N_0^{4,1} N_0^{2,0} - \frac{1}{2}\frac{18!}{9!9!} (N_0^{3,0})^2 - \frac{1}{2}\frac{18!}{14!2!2!} N_0^{4,0} (N_0^{1,0})^2 \\ &= 63338881 - 153 \cdot 36855 \cdot 1 + 8568 \cdot 27 \cdot 1 + \tfrac{1}{2} \cdot 48620 \cdot 1^2 + \tfrac{1}{2} \cdot 18360 \cdot 1 \cdot 1^2 \\ &= 57435240 \end{aligned}$$

in agreement with the recursion (1.1).

By Proposition (6.2), the relation between the numbers $N_0^{d,\delta}$ and the Gromov-Witten invariants is very simple: $N_d^{(g)} = N_0^{d,\delta}$ where $g = \binom{d-1}{2} - \delta$. In terms of F, the Gromov-Witten potentials $F_g(\mathbb{CP}^2)$ are given by the formula

$$\begin{aligned} \sum_{g=0}^{\infty} \hbar^{g-1} F_g(\mathbb{CP}^2) &= \frac{1}{2\hbar}(t_0^2 t_2 + t_0 t_1^2) - \frac{t_1}{8} \\ &\quad + F(p_k = 0, k \geq 1; q_1 = \hbar^{-3} q e^{t_1}; q_k = 0, k > 1; z = \hbar t_2). \end{aligned}$$

*The resemblance of the right-hand side to the Hamiltonian of the Liouville model is striking — we have no idea why operators so closely resembling vertex operators make their appearance here.

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