THE CATEGORY OF REDUCED ORBIFOLDS

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Abstract. It is well known that reduced orbifolds and proper effective foliation groupoids are closely related. We propose a notion of maps between reduced orbifolds and a definition of a category in terms of (marked atlas) groupoids such that the arising category of orbifolds is isomorphic (not only equivalent) to this groupoid category.

Contents

1. Introduction

The purpose of this article is to propose a definition of the category of reduced (smooth) orbifolds and the definition of an isomorphic category in terms of a certain kind of Lie groupoids. In both categories, the morphisms will be explicitely given. In the orbifold category morphisms are defined via local charts and maps between them, and in the groupoid category morphisms are described as certain equivalence classes of groupoid homomorphisms. Moreover, the isomorphism between the two categories is explicitely given.

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It is well known that reduced orbifolds and proper effective foliation groupoids are intimately related. More precisely, given a reduced orbifold and an orbifold atlas representing its orbifold structure one has an explicit construction of a proper effective foliation groupoid from this data (see, e. g., [MM03]). Moreover, there is a natural notion of isomorphisms between orbifolds. Using this notion, Moerdijk and Pronk [MP97] provide a precise bijection of isomorphism classes of reduced orbifolds and Morita equivalence classes of proper effective foliation groupoids. This bijection can be used to establish an equivalence between a category of proper effective foliation groupoids and a category of reduced orbifolds (see [Moe02] for details). Originally, the morphisms in this orbifold category are defined only implicitely as the corresponding morphisms in the groupoid category. To achieve an explicit description of these orbifold morphisms (that is, in terms of local charts) one needs, as a first step, a characterization in local charts of (classical) groupoid homomorphisms. Unfortunately, the characterization in [LU04] (which to the knowledge of the author is the only attempt in the existing literature) is flawed. In this article we provide a correct characterization of groupoid homomorphisms in local charts. We use the arising maps between local charts to define a geometrically motivated notion of orbifold maps. Then we characterize orbifolds and orbifold maps in terms of groupoids and groupoid homomorphisms. This enables us to define a category in terms of groupoids (which is not the classical category of groupoids) which is isomorphic to the category formed by reduced orbifolds with orbifold maps as morphisms.

We start by recalling briefly the necessary background material on orbifolds, groupoids, pseudogroups, and the well-known construction of a groupoid from an orbifold and an orbifold atlas representing its orbifold structure. Groupoids which arise in this way will be called *atlas groupoids*. We will see that different reduced orbifolds might give rise to the same atlas groupoid. Therefore it is not possible to find a bijection between the class of all reduced orbifolds and the class of atlas groupoids (or the class of certain equivalence classes of atlas groupoids). In particular, it is not possible to establish an isomorphism between a category whose class of objects consists of all reduced orbifolds and a category whose objects are (equivalence classes of) atlas groupoids. To overcome this problem we introduce, in Sec. 3, a certain marking of atlas groupoids, which allows to recover the orbifold. In Sec. 4 we characterize homomorphisms between marked atlas groupoids in local charts. On the orbifold side, this characterization involves the choice of representatives of the orbifold structures, namely those orbifold atlases which were used to construct the marked atlas groupoids. Hence, at this point we get a notion of orbifold map with fixed representatives of orbifold structures, which we will call *charted orbifold maps*. In Sec. 5 we introduce a natural definition of composition of charted orbifold maps and a geometrically motivated definition of the identity morphism (a certain class of charted orbifold maps), which allows us to establish a natural equivalence relation on the class of charted orbifold maps. An orbifold map (which does not depend on the choice of orbifold atlases) is then an equivalence class. The leading idea for this equivalence relation is geometric: we consider charted orbifold maps as equivalent if and only if they induce the same charted orbifold map on common refinements of the orbifold atlases. Moreover, using the same idea, we define the composition of orbifold maps. In this way, we construct a category of reduced orbifolds. Finally, in Sec. 6, we characterize orbifolds as certain equivalence classes of marked atlas groupoids, and orbifold maps as equivalence classes of homomorphisms of marked atlas groupoids. These equivalence relations are natural adaptations of the classical Morita equivalence. In this way, there arises a category of marked atlas groupoids which is isomorphic to the orbifold category. An additional benefit is that the isomorphism functor is constructive.

We expect that the constructed category of marked atlas groupoids is isomorphic to a category of which the class of objects consists of equivalence classes of all marked proper effective foliation groupoids and the morphisms are given by certain equivalence classes of groupoid homomorphisms.

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Notation and conventions: The set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ denotes the set of non-negative integers. If not stated otherwise, every manifold is assumed to be real, second-countable, Hausdorff and smooth (C^∞). If M is a manifold, then $\operatorname{Diff}(M)$ denotes the group of diffeomorphisms of M. If G is a subgroup of $\operatorname{Diff}(M)$, then $G\backslash M$ denotes the space of cosets $\{gM \mid g \in G\}$ endowed with the final topology. If A_1, A_2, B are sets (manifolds) and $f_1\colon A_1 \to B$, $f_2\colon A_2 \to B$ are maps (submersions), then we denote the fibered product of f_1 and f_2 by $A_1 \,{}_{f_1}\!\times_{f_2} A_2$. As well known, $A_1 \,{}_{f_1}\!\times_{f_2} A_2$ is uniquely isomorphic to

$$\{(a_1, a_2) \in A_1 \times A_2 \mid f_1(a_1) = f_2(a_2)\}$$

for which reason we will identify $A_1 \,{}_{f_1}\!\times_{f_2} A_2$ with this set (manifold).

Finally, we say that a family $\mathcal{V} = \{V_i \mid i \in I\}$ is *indexed by* I if $I \to \mathcal{V}$, $i \mapsto V_i$, is a bijection.

2. Reduced orbifolds, groupoids, and pseudogroups

This section has a preliminary character. It recalls definitions and results concerning reduced orbifolds and groupoids.

2.1. **Reduced orbifolds.**

In the modern literature, two definitions of reduced orbifolds are used. They differ in their respective concept of orbifold atlas. In this work we prefer the definition which can be found in [BH99] and [GH06]. It is better suited to our needs than the common one used, e. g., in [MM03]. However, it is fairly easy to prove that the arising orbifold structures in both definitions are identical (cf. [MM03, Prop. 2.13]). Therefore we may use all results about orbifolds from [MM03].

Definition 2.1. Let Q be a topological space.

(i) Let $n \in \mathbb{N}_0$. A reduced *orbifold chart* of dimension n on Q is a triple (V, G, φ) where V is an open connected n-manifold, G is a finite subgroup of $\mathrm{Diff}(V)$, and $\varphi\colon V \to Q$ is a map with open image $\varphi(V)$ that induces a homeomorphism from $G\backslash V$ to $\varphi(V)$. In this case, (V, G, φ) is said to *uniformize* $\varphi(V)$.

(ii) Two reduced orbifold charts (V, G, φ), (W, H, ψ) on Q are called *compatible* if for each pair $(x, y) \in V \times W$ with $\varphi(x) = \psi(y)$ there are open connected neighborhoods $\widetilde{V}$ of x and $\widetilde{W}$ of y and a diffeomorphism $h\colon \widetilde{V} \to \widetilde{W}$ such that $\psi \circ h = \varphi|_{\widetilde{V}}$. The map h is called a *change of charts*.

(iii) A reduced *orbifold atlas* of dimension n on Q is a collection of pairwise compatible reduced orbifold charts

$$\mathcal{V} := \{(V_i, G_i, \varphi_i) \mid i \in I\}$$

of dimension n on Q such that $\bigcup_{i\in I} \varphi_i(V_i) = Q$.

(iv) Two reduced orbifold atlases are *equivalent* if their union is a reduced orbifold atlas.

(v) A reduced *orbifold structure* of dimension n on Q is a (w. r. t. inclusion) maximal reduced orbifold atlas of dimension n on Q, or equivalently, an equivalence class of reduced orbifold atlases of dimension n on Q.

(vi) A reduced *orbifold* of dimension n is a pair $(Q, \mathcal{U})$ where Q is a second-countable Hausdorff space and $\mathcal{U}$ is a reduced orbifold structure of dimension n on Q.

Let $(Q, \mathcal{U})$ be a reduced orbifold. The term "reduced" refers to the requirement that for each reduced orbifold chart (V, G, φ) in $\mathcal{U}$ the group G be a subgroup of $\mathrm{Diff}(V)$. Hence the action of G on V is effective. Orbifolds with this property are also known as "effective orbifolds". Since we are considering reduced orbifolds only, we omit the term "reduced" from now on.

Let M be a manifold and G a subgroup of $\mathrm{Diff}(M)$. A subset S of M is called *G-stable*, if it is connected and if for each $g \in G$ we either have $gS = S$ or $gS \cap S = \emptyset$.

Remark 2.2. The neighborhoods $\widetilde{V}$ and $\widetilde{W}$ and the diffeomorphism h in Def. 2.1(ii) can always be chosen in such a way that $h(x) = y$. Moreover $\widetilde{V}$ may assumed to be open G-stable. In this case, $\widetilde{W}$ is open H-stable by Prop. 2.12(i) in [MM03].

Definition 2.3. Let (V, G, φ), (W, H, ψ) be orbifold charts on the topological space Q. Then an *embedding*

$$\mu\colon (V, G, \varphi) \to (W, H, \psi)$$

between these two orbifold charts is an open embedding $\mu\colon V \to W$ between manifolds which satisfies $\psi \circ \mu = \varphi$. If μ is a diffeomorphism between V and W, then μ is called an *isomorphism* from (V, G, φ) to (W, H, ψ). Suppose that S is an open G-stable subset of V and set $G_S := \{g \in G \mid gS = S\}$, the *isotropy group* of S. Then $(S, G_S, \varphi|_S)$ is an orbifold chart on Q, the *restriction* of (V, G, φ) to S.

Remark 2.4. (i) Let (V, G, φ) be an orbifold chart on the topological space Q and that S is a G-stable subset of V. Then $(S, G_S, \varphi|_S)$ is obviously embedded into (V,G,φ) by id_S, and hence compatible with (V,G,φ). In turn, restrictions of orbifold charts in an orbifold structure $\mathcal{U}$ are themselves elements of $\mathcal{U}$.

(ii) Suppose now that (W,H,ψ) is an orbifold chart on Q which is compatible with (V,G,φ). Further suppose that $(x,y)\in V\times W$ with $\varphi(x)=\psi(y)$. Then Remark 2.2 shows that there exists a restriction $(S,G_S,\varphi|_S)$ of (V,G,φ) with $x\in S$ and an embedding $h\colon (S,G_S,\varphi|_S)\to(W,H,\psi)$ with $h(x)=y$.

(iii) Suppose that $\mu\colon (V,G,\varphi)\to(W,H,\psi)$ is an embedding. In [MM03, Prop. 2.12(i)] it is shown that $\mu(V)$ is an open H-stable subset of W, and that there is a unique group isomorphism $\overline{\mu}\colon G\to H_{\mu(V)}$ for which $\mu(gx)=\overline{\mu}(g)\mu(x)$ for $g\in G$, $x\in V$.

In the following example we construct two orbifolds with the same underlying topological space. These orbifolds are particularly simple since both orbifold structures have one-chart-representatives. Despite their simplicity they serve as motivating examples for several definitions in the following.

Example 2.5. Let $Q:=[0,1)$ be endowed with the induced topology of $\mathbb{R}$. The map

$$f\colon \begin{cases} Q & \to & Q \\ x & \mapsto & x^2 \end{cases}$$

is a homeomorphism. Further the map $\mathrm{pr}\colon(-1,1)\to[0,1)$, $x\mapsto|x|$, induces a homeomorphism $\{\pm\mathrm{id}\}\backslash(-1,1)\to Q$. Then

$$V_1 := \big((-1,1),\{\pm\mathrm{id}\},\mathrm{pr}\big)\quad\text{and}\quad V_2 := \big((-1,1),\{\pm\mathrm{id}\},f\circ\mathrm{pr}\big)$$

are two orbifold charts on Q. We claim that these two orbifold charts are not compatible. To see this assume for contradiction that they do be compatible. Since $f\circ\mathrm{pr}(0)=0=\mathrm{pr}(0)$, there exist open connected neighborhoods $\widetilde{V}_1$, $\widetilde{V}_2$ of 0 in V_1 resp. V_2 and a diffeomorphism $h\colon\widetilde{V}_2\to\widetilde{V}_1$ such that $\mathrm{pr}\circ h=f\circ\mathrm{pr}|_{\widetilde{V}_2}$. We construct all possible candidates for h. For each $x\in\widetilde{V}_2$ we have

$$|h(x)| = \mathrm{pr}(h(x)) = f(\mathrm{pr}(x)) = x^2,$$

hence $h(x)\in\{\pm x^2\}$. Now h being continuous reduces the possible candidates to the four maps

$$\begin{aligned} h_1(x) &:= x^2 \\ h_2(x) &:= -x^2 \\ h_3(x) &:= \begin{cases} x^2 & x\geq 0 \\ -x^2 & x\leq 0 \end{cases} \\ h_4(x) &:= \begin{cases} -x^2 & x\geq 0 \\ x^2 & x\leq 0, \end{cases} \end{aligned}$$

neither of which is a diffeomorphism. This gives the contradiction.

Let $\mathcal{U}_1$ be the orbifold structure on Q generated by V_1, and $\mathcal{U}_2$ be the one generated by V_2.

2.2. **Groupoids and homomorphisms.** A groupoid is a small category in which each morphism is an isomorphism. In the context of orbifolds this concept is most commonly expressed (equivalently) in terms of sets and maps. The morphisms are then called arrows.

Definition 2.6. A *groupoid* G is a tuple $G = (G_0, G_1, s, t, m, u, i)$ consisting of

(a) a set G_0, the set of *objects*, or the *base* of G,
(b) a set G_1, the set of *arrows*,
(c) a map $s\colon G_1 \to G_0$, the *source map*,
(d) a map $t\colon G_1 \to G_0$, the *target map*,
(e) a map $m\colon G_1\, {}_s\times_t\, G_1 \to G_1$, the *multiplication* or *composition*, where

$$G_1\, {}_s\times_t\, G_1 := \{(g,f) \in G_1 \times G_1 \mid s(g) = t(f)\}$$

is the fibered product of s and t,
(f) a map $u\colon G_0 \to G_1$, the *unit map*,
(g) a map $i\colon G_1 \to G_1$, the *inversion*,

which satisfy the conditions

(i) for all $(g,f) \in G_1\, {}_s\times_t\, G_1$ it holds

$$s(m(g,f)) = s(f) \quad \text{and} \quad t(m(g,f)) = t(g),$$

(ii) for all $(h,g),(g,f) \in G_1\, {}_s\times_t\, G_1$ we have

$$m(h, m(g,f)) = m(m(h,g), f).$$

Note that $(h, m(g,f)), (m(h,g), f) \in G_1\, {}_s\times_t\, G_1$ due to (i).
(iii) for all $x \in G_0$ we have

$$s(u(x)) = x = t(u(x)),$$

(iv) for all $x \in G_0$ and all $(u(x), f), (g, u(x)) \in G_1\, {}_s\times_t\, G_1$ it follows

$$m(u(x), f) = f \quad \text{and} \quad m(g, u(x)) = g,$$

(v) for all $g \in G_1$ we have

$$s(i(g)) = t(g) \quad \text{and} \quad t(i(g)) = s(g),$$

and

$$m(g, i(g)) = u(t(g)) \quad \text{and} \quad m(i(g), g) = u(s(g)).$$

The maps s, t, m, u, i are called the *structure maps* of the groupoid G. We often use the notations $m(g,f) = gf$, $u(x) = 1_x$, $i(g) = g^{-1}$, and $g\colon x \to y$ or $x \xrightarrow{g} y$ for an arrow $g \in G_1$ with $s(g) = x$, $t(g) = y$. Moreover, $G(x,y)$ denotes the set of arrows from x to y.

Definition 2.7. Let G and H be groupoids. A *homomorphism* from G to H is a functor $\varphi\colon G \to H$, i.e., it is a tuple $\varphi = (\varphi_0, \varphi_1)$ of maps $\varphi_0\colon G_0 \to H_0$ and $\varphi_1\colon G_1 \to H_1$ which commute with all structure maps.

Definition 2.8. Let G be a groupoid.

(1) The *orbit* of $x \in G_0$ is the set

$$Gx := t(s^{-1}(x)) = \left\{ y \in G_0 \;\middle|\; \exists\, g \in G_1\colon x \xrightarrow{g} y \right\}.$$

(2) Two elements $x, y \in G_0$ are called *equivalent*, $x \sim y$, if they are in the same orbit. The quotient space $G_0/_\sim$ is called the *orbit space* of G. It is denoted by $|G|$. Further, we denote the canonical quotient map $G_0 \to |G|$ by pr or pr_G, and we set $[x] := \mathrm{pr}(x)$ for $x \in G_0$.

Since we will be considering smooth orbifolds, we need groupoids with a smooth structure as well.

Definition 2.9. A *Lie groupoid* is a groupoid G for which G_0 is a smooth Hausdorff manifold, G_1 is a smooth (possibly non-Hausdorff) manifold, the structure maps $s, t\colon G_1 \to G_0$ are smooth submersions (hence $G_1\, {}_s\!\times_t G_1$, the domain of m, is a smooth manifold), and the structure maps m, u and i are smooth. A *homomorphism* between two Lie groupoids is a homomorphism $\varphi = (\varphi_0, \varphi_1)$ in sense of Def. 2.7 such that φ_0 and φ_1 are smooth maps.

2.3. **Pseudogroups and groupoids.** In this section we recall how to construct from an orbifold and a representative of its orbifold structure a Lie groupoid. This construction is well known in literature, see e.g. [MM03]. It is a two-step process in which one first assigns to the orbifold a pseudogroup, which depends on the representative of the orbifold structure. Then one constructs from the pseudogroup an étale groupoid. For purposes of generality and clarity we start with the second step.

Definition 2.10. Let M be a manifold. A *transition* on M is a diffeomorphism $f\colon U \to V$ where U, V are open subsets of M. Each of the two sets U and V is allowed to be empty. In particular, the empty map $\emptyset \to \emptyset$ is a transition on M. The *product* of two transitions $f\colon U \to V$, $g\colon U' \to V'$ is the transition

$$f \circ g\colon g^{-1}(U \cap V') \to f(U \cap V'),\ x \mapsto f(g(x)).$$

The *inverse* of f is the transition

$$f^{-1}\colon V \to U,\ f(x) \mapsto x.$$

If $f\colon U \to V$ is a transition, we denote its *domain* by $\operatorname{dom} f := U$ and its *codomain*, which here equals the image of f, by $\operatorname{cod} f$. Further, if $x \in \operatorname{dom} f$, then $\operatorname{germ}_x f$ denotes the germ of f at x, which is the set (or equivalence class) of all transitions g on M such that $x \in \operatorname{dom} g$ and that there is an open neighborhood W of x contained in $\operatorname{dom} g \cap \operatorname{dom} f$ and for which $g|_W = f|_W$.

Let $\mathcal{A}(M)$ be the set of all transitions on M. A *pseudogroup* on M is a subset P of $\mathcal{A}(M)$ which is closed under multiplication and inversion. A pseudogroup P is called *full* if $\mathrm{id}_U \in P$ for each open subset U of M. It is said to be *complete* if it is full and satisfies the following gluing property: Whenever there is a transition $f \in \mathcal{A}(M)$ and an open covering $(U_i)_{i\in I}$ of $\operatorname{dom} f$ such that $f|_{U_i} \in P$ for all $i \in I$, then $f \in P$.

A Lie groupoid is called *étale* if its source and target map are local diffeomorphisms. We now show how to construct an étale groupoid from a full pseudogroup.

Construction 2.11. Let M be a manifold and P a full pseudogroup on M. The *associated groupoid* $\Gamma := \Gamma(P)$ is given by

$$\Gamma_0 := M, \quad \Gamma_1 := \{\operatorname{germ}_x f \mid f \in P,\ x \in \operatorname{dom} f\},$$

and, in particular,

$$\Gamma(x,y) := \{\operatorname{germ}_x f \mid f \in P,\ x \in \operatorname{dom} f,\ f(x) = y\}.$$

For $f \in P$ define

$$U_f := \{\operatorname{germ}_x f \mid x \in \operatorname{dom} f\}$$

The topology and differential structure of Γ_1 is given by the germ topology and germ differential structure, that is for each $f \in P$ the bijection

$$\varphi_f \colon \begin{cases} U_f & \to \operatorname{dom} f \\ \operatorname{germ}_x f & \mapsto x \end{cases}$$

is required to be a diffeomorphism. The structure maps (s,t,m,u,i) of Γ are the obvious ones, namely

$$\begin{aligned} s(\operatorname{germ}_x f) &:= x \\ t(\operatorname{germ}_x f) &:= f(x) \\ m(\operatorname{germ}_{f(x)} g, \operatorname{germ}_x f) &:= \operatorname{germ}_x(g \circ f) \\ u(x) &:= \operatorname{germ}_x \operatorname{id}_U \quad \text{for any open neighborhood } U \text{ of } x \\ i(\operatorname{germ}_x f) &:= \operatorname{germ}_{f(x)} f^{-1}. \end{aligned}$$

All structure maps are smooth, and s, t are local diffeomorphisms (and in particular submersions). Hence $\Gamma(P)$ is an étale groupoid.

Special Case 2.12. Let $(Q,\mathcal{U})$ be an orbifold, and let

$$\mathcal{V} = \{(V_i, G_i, \pi_i) \mid i \in I\}$$

be an orbifold atlas of Q, hence a representative of $\mathcal{U}$. Suppose that $\mathcal{V}$ is indexed by I. We define

$$V := \coprod_{i\in I} V_i \quad \text{and} \quad \pi := \coprod_{i\in I} \pi_i.$$

Then

$$\Psi(\mathcal{V}) := \big\{ f \text{ transition on } V \;\big|\; \pi \circ f = \pi|_{\operatorname{dom} f} \big\}.$$

is a complete pseudogroup on V. The associated groupoid $\Gamma(\mathcal{V}) := \Gamma(\Psi(\mathcal{V}))$ is the étale groupoid we shall associate to Q and $\mathcal{V}$. Note that this groupoid depends on the choice of the representative of the orbifold structure $\mathcal{U}$ of Q. A groupoid which arises in this way we call *atlas groupoid*.

Example 2.13. Recall the orbifolds $(Q,\mathcal{U}_i)$ $(i = 1,2)$ from Example 2.5, and consider the representative $\mathcal{V}_i := \{V_i\}$ of $\mathcal{U}_i$. Prop. 2.12 in [MM03] implies that

$$\Psi(\mathcal{V}_i) = \big\{ g|_U \colon U \to g(U) \;\big|\; U \subseteq (-1,1) \text{ open},\ g \in \{\pm \operatorname{id}\} \big\}.$$

In both cases the associated groupoid $\Gamma := \Gamma(\mathcal{V}_i)$ is

$$\Gamma_0 = (-1,1)$$

$$\Gamma(x,y) = \begin{cases} \{\operatorname{germ}_0 \operatorname{id}, \operatorname{germ}_0(-\operatorname{id})\} & x = 0 = y \\ \{\operatorname{germ}_x \operatorname{id}\} & x = y \neq 0 \\ \{\operatorname{germ}_x(-\operatorname{id})\} & x = -y \neq 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

3. Marked Lie groupoids and their homomorphisms

In Example 2.13 we have seen that it may happen that the same atlas groupoid is associated to two different orbifolds. The reason for this is that in the definition of the pseudogroup which is needed for the construction of the atlas groupoid one loses information about the projection maps φ of the orbifold charts (V, G, φ). To be able to distinguish atlas groupoids constructed from different orbifolds, we mark the groupoids with a topological space and a homeomorphism. It will turn out that this marking suffices to identify the orbifold one started with from any (properly) marked atlas groupoid associated to it.

Recall that for a groupoid G we use $|G|$ to denote its orbit space, and that $[x] = \mathrm{pr}_G(x)$ denotes the image of $x \in G_0$ under the quotient map $\mathrm{pr}_G \colon G_0 \to |G|$.

Definition 3.1. A *marked Lie groupoid* is a triple (G, α, X) consisting of a Lie groupoid G, a topological space X, and a homeomorphism $\alpha\colon |G| \to X$.

Proposition 3.2. *Let $(Q, \mathcal{U})$ be an orbifold and $\mathcal{V} = \{(V_i, G_i, \pi_i) \mid i \in I\}$ an orbifold atlas of Q indexed by I. Define*

$$V := \coprod_{i\in I} V_i \quad \text{and} \quad \pi := \coprod_{i\in I} \pi_i \colon V \to Q.$$

Then

$$\alpha\colon \left\{ \begin{array}{ccc} |\Gamma(\mathcal{V})| & \to & Q \\ [x] & \mapsto & \pi(x) \end{array} \right.$$

is a homeomorphism.

Proof. To show that α is well-defined, suppose $[x_1] = [x_2]$. Then there is an arrow $x_1 \to x_2$. Hence there exists $f \in \Psi(\mathcal{V})$ such that $x_1 \in \operatorname{dom} f$ and $f(x_1) = x_2$. From this it follows that $\pi(x_1) = \pi(f(x_1)) = \pi(x_2)$.

Obviously, α is surjective. For the proof of injectivity let $\pi(x_1) = \pi(x_2)$ for some $x_1, x_2 \in V$. Then there are orbifold charts $(V_i, G_i, \pi_i) \in \mathcal{V}$ with $x_i \in V_i$ $(i = 1, 2)$. By compatibility of these orbifold charts and Remark 2.2 there is $f \in \Psi(\mathcal{V})$ such that $x_1 \in \operatorname{dom} f$ and $f(x_1) = x_2$. This means that $\mathrm{germ}_{x_1} f\colon x_1 \to x_2$ is an element of $\Gamma(\mathcal{V})_1$. Thus, $[x_1] = [x_2]$.

Consider now the commutative diagram

$$\begin{array}{ccc} V & & \\ {\scriptstyle \mathrm{pr}}\downarrow & \searrow^{\pi} & \\ |\Gamma(\mathcal{V})| & \xrightarrow{\alpha} & Q \end{array}$$

where pr is the canonical quotient map on the orbit space. One easily proves that π is continuous and open. Therefore α is continuous and open (see [Bou98, I.2.4.6, I.5.2.3]). Hence α is a homeomorphism. □

Let $(Q, \mathcal{U})$ be an orbifold. To each orbifold atlas $\mathcal{V}$ of Q we assign the marked atlas groupoid $(\Gamma(\mathcal{V}), \alpha_{\mathcal{V}}, Q)$ with $\alpha_{\mathcal{V}}$ being the homeomorphism from Prop. 3.2. We often only write $\Gamma(\mathcal{V})$ to refer to this marked groupoid.

Example 3.3. Recall from Example 2.13 the orbifolds $(Q,\mathcal{U}_i)$ $(i = 1, 2)$, their respective orbifold atlases $\mathcal{V}_i$, and the associated groupoids $\Gamma = \Gamma(\mathcal{V}_i)$. The orbit of $x \in \Gamma_0$ is $\{x, -x\}$. Hence the homeomorphism associated to $(Q,\mathcal{U}_i)$ is

$$\alpha_{\mathcal{V}_1}\colon \begin{cases} |\Gamma| & \to & Q \\ [x] & \mapsto & |x| \end{cases} \qquad \text{for } i = 1,$$

resp.

$$\alpha_{\mathcal{V}_2}\colon \begin{cases} |\Gamma| & \to & Q \\ [x] & \mapsto & x^2 \end{cases} \qquad \text{for } i = 2.$$

Thus, the associated marked groupoids $(\Gamma, \alpha_{\mathcal{V}_1}, Q)$ and $(\Gamma, \alpha_{\mathcal{V}_2}, Q)$ are different.

Proposition 3.4. *Let $(Q,\mathcal{U})$ and $(Q',\mathcal{U}')$ be orbifolds. Suppose that $\mathcal{V}$ is a representative of $\mathcal{U}$, and $\mathcal{V}'$ a representative of $\mathcal{U}'$. If the associated marked atlas groupoids $(\Gamma(\mathcal{V}), \alpha_{\mathcal{V}}, Q)$ and $(\Gamma(\mathcal{V}'), \alpha_{\mathcal{V}'}, Q')$ are equal, then the orbifolds $(Q,\mathcal{U})$ and $(Q',\mathcal{U}')$ are equal. More precisely, we even have $\mathcal{V} = \mathcal{V}'$.*

Proof. Clearly, $Q = Q'$. Suppose that

$$\mathcal{V} = \{(V_i, G_i, \pi_i) \mid i \in I\} \quad \text{and} \quad \mathcal{V}' = \{(V_j', G_j', \pi_j') \mid j \in J\},$$

where $\mathcal{V}$ is indexed by I and $\mathcal{V}'$ is indexed by J. From $\Gamma(\mathcal{V}) = \Gamma(\mathcal{V}')$ it follows that

$$\coprod_{i\in I} V_i = \Gamma(\mathcal{V})_0 = \Gamma(\mathcal{V}')_0 = \coprod_{j\in J} V_j'.$$

Since each V_i and each V_j' is connected, there is a bijection between I and J. We may assume $I = J$. Then $V_i = V_i'$ for all $i \in I$. Let $x \in V_i$. Then

$$\pi_i(x) = \alpha_{\mathcal{V}}([x]) = \alpha_{\mathcal{V}'}([x]) = \pi_i'(x).$$

Therefore $\pi_i = \pi_i'$ for all $i \in I$. In turn,

$$\coprod_{i\in I} \pi_i = \coprod_{i\in I} \pi_i',$$

and thus $\Psi(\mathcal{V}) = \Psi(\mathcal{V}')$. Let $g \in G_i$. Then $g \in \Psi(\mathcal{V}) = \Psi(\mathcal{V}')$. Hence, for each $x \in V_i$ we have $\pi_i'(g(x)) = \pi_i'(x)$. This shows that $g(x) \in G_i'x$ for each $x \in V_i$. By [MM03, Lemma 2.11] there exists a unique element $g' \in G_i'$ such that $g = g'$. Since this argument is symmetric in G_i and G_i', it follows that $G_i = G_i'$ (as acting groups). Thus, $\mathcal{V} = \mathcal{V}'$. □

A homomorphism between marked Lie groupoids is a pair consisting of a homomorphism between the Lie groupoids and a continuous map (i. e., a homomorphism in the continuous category) between the topological spaces such that these two maps are compatible.

Definition 3.5. Let (G, α, X) and (H, β, Y) be marked Lie groupoids. A *homomorphism* $(G, \alpha, X) \to (H, \beta, Y)$ is a pair (φ, ψ) consisting of a homomorphism $\varphi = (\varphi_0, \varphi_1)\colon G \to H$ of Lie groupoids and a continuous map $\psi\colon X \to Y$ such that the diagram

$$\begin{array}{ccc} G_0 & \xrightarrow{\varphi_0} & H_0 \\ {\scriptstyle \mathrm{pr}_G}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_H} \\ |G| & & |H| \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ X & \xrightarrow{\psi} & Y \end{array}$$

commutes.

In the following we show that homomorphisms of Lie groupoids naturally extend to homomorphisms of marked Lie groupoids. In turn, the category of marked Lie groupoids is an "infinite cover" of the category of Lie groupoids.

Lemma 3.6. *Let G be a Lie groupoid. Then pr_G is open.*

Proof. Let $U \subseteq G_0$ be open. Then

$$\begin{aligned} \mathrm{pr}_G^{-1}(\mathrm{pr}_G(U)) &= \left\{ y \in G_0 \;\middle|\; \exists\, x \in U \;\exists\, g \in G_1 \colon x \xrightarrow{g} y \right\} \\ &= t(s^{-1}(U)). \end{aligned}$$

Since s is continuous, $s^{-1}(U)$ is open. The map $t\colon G_1 \to G_0$ is a submersion, hence open. Therefore $t(s^{-1}(U))$ is open. This means that $\mathrm{pr}_G(U)$ is open. □

Lemma 3.7. *If $\varphi = (\varphi_0, \varphi_1)\colon G \to H$ is a homomorphism of groupoids, then φ induces a unique map $|\varphi|\colon |G| \to |H|$ such that the diagram*

$$\begin{array}{ccc} G_0 & \xrightarrow{\varphi_0} & H_0 \\ {\scriptstyle \mathrm{pr}_G}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_H} \\ |G| & \xrightarrow{|\varphi|} & |H| \end{array}$$

commutes. If φ is a homomorphism of Lie groupoids (or, more generally, if φ_0 is continuous), then $|\varphi|$ is continuous.

Proof. Let $x, y \in G_0$. If there is an arrow $g\colon x \to y$, then $\varphi_1(g)\colon \varphi_0(x) \to \varphi_0(y)$. Hence $|\varphi|\colon |G| \to |H|$, $[x] \mapsto [\varphi_0(x)]$ is a well-defined map. Now let G_0 and H_0 be topological spaces, and φ_0 be continuous. Due to the definition of the topology on $|G|$, the map $|\varphi|$ is continuous if and only if $|\varphi| \circ \mathrm{pr}_G$ is continuous. But $|\varphi| \circ \mathrm{pr}_G = \mathrm{pr}_H \circ \varphi_0$, which is continuous. □

Let $(\varphi, \psi)\colon (G, \alpha, X) \to (H, \beta, Y)$ be a homomorphism of marked Lie groupoids. Lemma 3.7 implies that ψ is completely determined by φ, α and β, namely $\psi = \beta \circ |\varphi| \circ \alpha^{-1}$. Hence we may and shall skip the map ψ from the notation of a homomorphism of marked Lie groupoids.

4. Groupoid homomorphisms in local charts

In this section we characterize homomorphisms between atlas groupoids on the orbifold side, i. e., in terms of local charts. We proceed in a two-step process. At first we define representatives of orbifold maps, each of which gives rise to exactly one homomorphism between the associated atlas groupoids. Since each groupoid homomorphism corresponds to several such representatives, we then impose an equivalence relation on the class of all representatives for fixed orbifold atlases. The equivalence classes turn out to be in bijection with the homomorphisms between the atlas groupoids. The constructions in this section are subject to a fixed choice of representatives of the orbifold structures. In the following sections we extend the constructions to be independent of the chosen orbifold atlases.

Throughout this section let $(Q,\mathcal{U})$, $(Q',\mathcal{U}')$ denote two orbifolds.

Definition 4.1. Let $f\colon Q \to Q'$ be a continuous map, and suppose that $(V,G,\pi) \in \mathcal{U}$, $(V',G',\pi') \in \mathcal{U}'$ are orbifold charts. A *local lift* of f w. r. t. (V,G,π) and (V',G',π') is a smooth map $\tilde f\colon V \to V'$ such that $\pi' \circ \tilde f = f \circ \pi$:

$$\begin{array}{ccc} V & \xrightarrow{\tilde f} & V' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ Q & \xrightarrow{f} & Q' \end{array}$$

In this case, we call $\tilde f$ a *local lift of f at q* for each $q \in \pi(V)$.

Recall the definition of the maximal pseudogroup $\mathcal{A}(M)$ from Def. 2.10.

Definition 4.2. Let M be a manifold and A a pseudogroup on M which is closed under restrictions, i. e., if $f \in A$ and $U \subseteq \operatorname{dom} f$ is open, then the map $f|_U\colon U \to f(U)$ is in A. Suppose that B is a subset of $\mathcal{A}(M)$. Then A is said to be *generated* by B if $B \subseteq A$ and for each $f \in A$ and each $x \in \operatorname{dom} f$ there exists some $g \in B$ with $x \in \operatorname{dom} g$ and an open set $U \subseteq \operatorname{dom} f \cap \operatorname{dom} g$ such that $x \in U$ and

$$f|_U = g|_U.$$

We remark that each subset B of $\mathcal{A}(M)$ generates a unique pseudogroup on M which is closed under restrictions. If we drop the latter closeness condition, then B obviously is generating for more than one pseudogroup on M.

A subset P of $\mathcal{A}(M)$ is called a *quasi-pseudogroup* on M if it satisfies the following two properties:

(i) If $f \in P$ and $x \in \operatorname{dom} f$, then there exists an open set U with $x \in U \subseteq \operatorname{dom} f$ and $g \in P$ such that there exists an open set V with $f(x) \in V \subseteq \operatorname{dom} g$ and

$$\big(f|_U\big)^{-1} = g|_V.$$

(ii) If $f, g \in P$ and $x \in f^{-1}(\operatorname{dom} g)$, then there exists $h \in P$ with $x \in \operatorname{dom} h$ such that we find an open set U with $x \in U \subseteq f^{-1}(\operatorname{dom} g) \cap \operatorname{dom} h$ and

$$g \circ f|_U = h|_U.$$

A quasi-pseudogroup is designed for work with the germs of its elements. Therefore identities (like inversion and composition) of elements in quasi-pseudogroups are only required to be satisfied locally, whereas for (ordinary) pseudogroups these identities have to be valid globally.

In the following definition of a representative of an orbifold map, the underlying continuous map f is the only entity which is stable under change of atlases or, in other words, under the choice of local lifts. The pair (P,ν) should be considered as one entity. It serves as a transport of changes of charts. We ask here for a quasi-pseudogroup P instead of working with all of $\Psi(\mathcal{V})$ for two reasons. In general, P is much smaller than $\Psi(\mathcal{V})$. Sometimes it may even be finite. In Example 4.6 below we see that for some orbifolds, P might even happen to consist of only two elements. Moreover, if the orbifold is a connected manifold, P can always be chosen as a singleton. The other reason is that it is much easier to construct some quasi-pseudogroup P and compatible map ν from a given groupoid homomorphism than a map ν defined on all of $\Psi(\mathcal{V})$.

The examples below show that in the following definition the requested objects in general are not immediate or uniquely determined.

Definition 4.3. A *representative of an orbifold map* from $(Q,\mathcal{U})$ to $(Q',\mathcal{U}')$ is a tuple

$$\hat{f} := (f, \{\tilde{f}_i\}_{i\in I}, P, \nu)$$

where

(R1) $f\colon Q \to Q'$ is a continuous map,
(R2) for each $i \in I$, the map $\tilde{f}_i$ is a local lift of f w. r. t. some orbifold charts $(V_i, G_i, \pi_i) \in \mathcal{U}$, $(V_i', G_i', \pi_i') \in \mathcal{U}'$ such that

$$\bigcup_{i\in I} \pi_i(V_i) = Q$$

and $(V_i, G_i, \pi_i) \neq (V_j, G_j, \pi_j)$ for $i, j \in I$, $i \neq j$,
(R3) P is a quasi-pseudogroup which consists of changes of charts of the orbifold atlas

$$\mathcal{V} := \{(V_i, G_i, \pi_i) \mid i \in I\}$$

and generates $\Psi(\mathcal{V})$.
(R4) Let $\psi := \coprod_{i\in I} \tilde{f}_i$. Then $\nu\colon P \to \Psi(\mathcal{U}')$ is a map which assigns to each $\lambda \in P$ an embedding

$$\nu(\lambda)\colon (W', H', \chi') \to (V', G', \varphi')$$

between some orbifold charts in $\mathcal{U}'$ such that
  (a) $\psi \circ \lambda = \nu(\lambda) \circ \psi|_{\operatorname{dom}\lambda}$,
  (b) for all $\lambda, \mu \in P$ and all $x \in \operatorname{dom}\lambda \cap \operatorname{dom}\mu$ with $\operatorname{germ}_x \lambda = \operatorname{germ}_x \mu$, we have $\operatorname{germ}_{\psi(x)} \nu(\lambda) = \operatorname{germ}_{\psi(x)} \nu(\mu)$,
  (c) for all $\lambda, \mu \in P$, for all $x \in \lambda^{-1}(\operatorname{dom}\mu)$ we have

$$\operatorname{germ}_{\psi(\lambda(x))} \nu(\mu) \cdot \operatorname{germ}_{\psi(x)} \nu(\lambda) = \operatorname{germ}_{\psi(x)} \nu(h)$$

  where h is some element of P with $x \in \operatorname{dom} h$ such that there is an open set U with $x \in U \subseteq \lambda^{-1}(\operatorname{dom}\mu) \cap \operatorname{dom} h$ and $\mu \circ \lambda|_U = h|_U$,
  (d) for all $\lambda \in P$ and all $x \in \operatorname{dom}\lambda$ such that there exists an open set U with $x \in U \subseteq \operatorname{dom}\lambda$ and $\lambda|_U = \mathrm{id}_U$ we have

$$\operatorname{germ}_{\psi(x)} \nu(\lambda) = \operatorname{germ}_{\psi(x)} \mathrm{id}_{W'}$$

  where $W' := \coprod_{i\in I} V_i'$.

The orbifold atlas $\mathcal{V}$ is called the *domain atlas* of the representative $\hat{f}$, and the set

$$\{(V_i', G_i', \pi_i') \mid i \in I\}$$

is called the *range family* of $\hat{f}$. The latter set is not neccessarily indexed by I.

Remark 4.4. (i) We show that the condition (R4c) is independent of the choice of h. Suppose that $h_1, h_2 \in P$ with $x \in \operatorname{dom} h_1 \cap \operatorname{dom} h_2$ such that there exist open sets U_1, U_2 with $x \in U_j \subseteq \lambda^{-1}(\operatorname{dom}\mu) \cap \operatorname{dom} h_j$ and $\mu \circ \lambda|_{U_j} = h_j|_{U_j}$. Then there exists an open set $V \subseteq U_1 \cap U_2$ with $x \in V$. It follows that $h_1|_V = h_2|_V$ and hence $\operatorname{germ}_x h_1 = \operatorname{germ}_x h_2$. By (R4b), $\operatorname{germ}_{\psi(x)} \nu(h_1) = \operatorname{germ}_{\psi(x)} \nu(h_2)$.

(ii) The additional condition "$(V_i, G_i, \pi_i) \neq (V_j, G_j, \pi_j)$ for $i \neq j$" in (R2) is not a restriction. By considering V_i as identified with $V_i \times \{i\}$ one can always consider two charts as being distinct. We require this property because we use I as an index set for $\mathcal{V}$ in (R3) and other places.

Example 4.5 below shows that the continuous map f in (R1) cannot be chosen arbitrarily. It is not even sufficient to require f to be a homeomorphism.

Example 4.5. Recall the orbifold $(Q, \mathcal{U}_1)$ from Example 2.5. The map $f\colon Q \to Q$, $f(x) = \sqrt{x}$, is a homeomorphism on Q. We show that f has no local lift at 0. Each orbifold chart that uniformizes a neighborhood of 0 is of the form $(I, \{\pm 1\}, \mathrm{pr})$ where $I = (-a, a)$ for some $0 < a < 1$. Seeking a contradiction assume that $\tilde{f}$ is a local lift of f at 0 with domain $I = (-a, a)$:

$$\begin{array}{ccc} (-a,a) & \xrightarrow{\tilde{f}} & (-\sqrt{a}, \sqrt{a}) \\ \downarrow{\scriptstyle \mathrm{pr}} & & {\scriptstyle \mathrm{pr}}\downarrow \\ [0,a) & \xrightarrow{f} & [0, \sqrt{a}) \end{array}$$

For each $x \in I$, necessarily $\tilde{f}(x) \in \big\{ \pm \sqrt{|x|} \big\}$. Since $\tilde{f}$ is required to be continuous, there only remain four possible candidates for $\tilde{f}$, namely

$$\begin{aligned} \tilde{f}_1(x) &= \sqrt{|x|}, \qquad \tilde{f}_2 = -\tilde{f}_1, \\ \tilde{f}_3(x) &= \begin{cases} \sqrt{x} & x \geq 0 \\ -\sqrt{-x} & x \leq 0, \end{cases} \\ \tilde{f}_4 &= -\tilde{f}_3. \end{aligned}$$

But none of these is differentiable in $x = 0$, hence there is no local lift of f at 0.

The following example shows that the pair (P, ν) is not uniquely determined by the choice of the family of local lifts.

Example 4.6. Recall the orbifold $(Q, \mathcal{U}_1)$ and the representative $\mathcal{V}_1 = \{V_1\}$ of $\mathcal{U}_1$ from Example 2.5. The map

$$f\colon \begin{cases} Q & \to & Q \\ q & \mapsto & 0 \end{cases}$$

is clearly continuous and has the local lift

$$\tilde{f}\colon \begin{cases} (-1,1) & \to & (-1,1) \\ x & \mapsto & 0 \end{cases}$$

with respect to V_1 and V_1. Consider the quasi-pseudogroup $P = \{\pm\operatorname{id}_{(-1,1)}\}$ on V_1. Prop. 2.12 in [MM03] implies that P generates $\Psi(\mathcal{V}_1)$. The tuple $(f, \tilde{f}, P)$ can be completed in the following two different ways to representatives of orbifold maps on $(Q, \mathcal{U}_1)$:

(a) $\nu_1(\pm\operatorname{id}_{(-1,1)}) := \operatorname{id}_{(-1,1)}$,
(b) $\nu_2(\operatorname{id}_{(-1,1)}) := \operatorname{id}_{(-1,1)}$, $\nu_2(-\operatorname{id}_{(-1,1)}) := -\operatorname{id}_{(-1,1)}$.

We will see in Example 4.8 below that $(f, \tilde{f}, P, \nu_1)$ and $(f, \tilde{f}, P, \nu_2)$ give rise to different groupoid homomorphisms.

Given a representative $\hat{f} = (f, \{\tilde{f}_i\}_{i\in I}, P, \nu)$ with domain atlas $\mathcal{V}$ and range family contained in $\mathcal{V}'$, the following proposition shows that $\hat{f}$ determines a homomorphism $\varphi = (\varphi_0, \varphi_1)\colon \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}')$ of marked atlas groupoids. The map φ_0 only depends on the family $\{\tilde{f}_i\}_{i\in I}$, whereas φ_1 also involves the change-of-charts-transport (P, ν). Recall the definition of $\alpha_{\mathcal{V}}$ from Prop. 3.2.

Proposition 4.7. *Let $\hat{f} = (f, \{\tilde{f}_i\}_{i\in I}, P, \nu)$ be a representative of an orbifold map from $(Q, \mathcal{U})$ to $(Q', \mathcal{U}')$. Suppose that*

$$\mathcal{V} = \{(V_i, G_i, \pi_i) \mid i \in I\},$$

is the domain atlas of $\hat{f}$, which is an orbifold atlas of $(Q, \mathcal{U})$ indexed by I. Let $\mathcal{V}'$ be an orbifold atlas of $(Q', \mathcal{U}')$ which contains the range family

$$\big\{(V_i', G_i', \pi_i') \;\big|\; i \in I\big\}.$$

Define the map $\varphi_0\colon \Gamma(\mathcal{V})_0 \to \Gamma(\mathcal{V}')_0$ by

$$\varphi_0 := \coprod_{i\in I} \tilde{f}_i.$$

Suppose that $\varphi_1\colon \Gamma(\mathcal{V})_1 \to \Gamma(\mathcal{V}')_1$ is determined by

$$\varphi_1(\operatorname{germ}_x \lambda) := \operatorname{germ}_{\varphi_0(x)} \nu(\lambda)$$

for all $\lambda \in P$, $x \in \operatorname{dom}\lambda$. Then

$$\varphi = (\varphi_0, \varphi_1)\colon \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}')$$

is a homomorphism. Moreover, $\alpha_{\mathcal{V}'} \circ |\varphi| = f \circ \alpha_{\mathcal{V}}$.

Proof. Let $\mathcal{V}'$ be indexed by J, and set

$$V := \coprod_{i\in I} V_i, \quad \pi := \coprod_{i\in I} \pi_i, \quad V' := \coprod_{j\in J} V_j' \quad \text{and} \quad \pi' := \coprod_{j\in J} \pi_j'.$$

The differential structure on V implies immediately that φ_0 is smooth.

We now show that φ_1 is a well-defined map on all of $\Gamma(\mathcal{V})_1$. To that end let $g \in \Psi(\mathcal{V})$ and $x \in \operatorname{dom} g$. Then there exists $\lambda \in P$ such that $x \in \operatorname{dom}\lambda$ and

$$g|_U = \lambda|_U$$

for some open subset $U \subseteq \operatorname{dom} g \cap \operatorname{dom} \lambda$ with $x \in U$. Hence $\operatorname{germ}_x g = \operatorname{germ}_x \lambda$. So

$$\varphi_1(\operatorname{germ}_x g) = \varphi_1(\operatorname{germ}_x \lambda) = \operatorname{germ}_{\varphi_0(x)} \nu(\lambda).$$

If there is $\mu \in P$ such that $x \in \operatorname{dom} \mu$ and $g|_W = \mu|_W$ for some open subset W of $\operatorname{dom} g \cap \operatorname{dom} \mu$ with $x \in W$, then $\operatorname{germ}_x \mu = \operatorname{germ}_x \lambda$. By (R4b), $\operatorname{germ}_{\varphi_0(x)} \nu(\mu) = \operatorname{germ}_{\varphi_0(x)} \nu(\lambda)$ and thus

$$\varphi_1(\operatorname{germ}_x \mu) = \varphi_1(\operatorname{germ}_x \lambda).$$

This shows that φ_1 is indeed well-defined on all of $\Gamma(\mathcal{V})_1$. By definition, φ commutes with the source maps. The properties (R4a), (R4c) and (R4d) yield that φ commutes with the other structure maps as well. It remains to show that φ_1 is smooth. For this, let $\operatorname{germ}_x \lambda \in \Gamma(\mathcal{V})_1$ with $\lambda \in P$. The definition of ν shows that φ_1 maps

$$U := \{\operatorname{germ}_y \lambda \mid y \in \operatorname{dom} \lambda\}$$

to

$$U' := \{\operatorname{germ}_z \nu(\lambda) \mid z \in \operatorname{dom} \nu(\lambda)\}.$$

Now the diagram

$$\begin{array}{ccc} U & \xrightarrow{\varphi_1} & U' \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ \operatorname{dom} \lambda & \xrightarrow{\varphi_0} & \operatorname{dom} \nu(\lambda) \end{array} \qquad\qquad \begin{array}{ccc} \operatorname{germ}_y \lambda & \longmapsto & \operatorname{germ}_{\varphi_0(y)} \nu(\lambda) \\ \downarrow & & \downarrow \\ y & \longmapsto & \varphi_0(y) \end{array}$$

commutes, the vertical maps (restriction of source maps) are diffeomorphisms and φ_0 is smooth, so φ_1 is smooth. Finally, suppose $x \in V_i$. Then

$$\begin{aligned} (\alpha_{\mathcal{V}'} \circ |\varphi|)\big([x]\big) &= \alpha_{\mathcal{V}'}\big([\varphi_0(x)]\big) = \alpha_{\mathcal{V}'}\big([\tilde f_i(x)]\big) = \pi'_i(\tilde f_i(x)) = f(\pi_i(x)) \\ &= (f \circ \alpha_{\mathcal{V}})\big([x]\big). \end{aligned}$$

□

Example 4.8. Recall the setting of Example 4.6 and the associated groupoid $\Gamma := \Gamma(\mathcal{V}_1)$ from Example 2.13. The homomorphism $\varphi = (\varphi_0, \varphi_1)\colon \Gamma \to \Gamma$ induced by $(f, \tilde f, P, \nu_1)$ is $\varphi_0 = \tilde f$ and

$$\varphi_1(\operatorname{germ}_x(\pm \operatorname{id}_{(-1,1)})) = \operatorname{germ}_0 \operatorname{id}_{(-1,1)}.$$

The homomorphism $\psi = (\psi_0, \psi_1)\colon \Gamma \to \Gamma$ induced by $(f, \tilde f, P, \nu_2)$ is $\psi_0 = \tilde f$ and

$$\psi_1(\operatorname{germ}_x \operatorname{id}_{(-1,1)}) = \operatorname{germ}_0 \operatorname{id}_{(-1,1)}, \quad \psi_1(\operatorname{germ}_x(-\operatorname{id}_{(-1,1)})) = \operatorname{germ}_0(-\operatorname{id}_{(-1,1)}).$$

The following proposition is the first step towards a converse of Prop. 4.7. It is complemented by Prop. 4.11 and 4.12.

Proposition 4.9. *Let $\mathcal{V}$ be an orbifold atlas of $(Q, \mathcal{U})$, and $\mathcal{V}'$ an orbifold atlas of $(Q', \mathcal{U}')$. Suppose that*

$$\varphi = (\varphi_0, \varphi_1)\colon \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}')$$

is a homomorphism. For each $f \in \Psi(\mathcal{V})$ and each $x \in \operatorname{dom}(f)$ there exist an element $g \in \Psi(\mathcal{V}')$ and an open neighborhood U of x (which may depend on g) with $U \subseteq \operatorname{dom}(f)$ such that for each $y \in U$ we have

$$\varphi_1(\operatorname{germ}_y f) = \operatorname{germ}_{\varphi_0(y)} g.$$

Proof. By definition of $\Gamma(\mathcal{V})_1$ and φ_1, there exists $g \in \Psi(\mathcal{V}')$ such that

$$\varphi_1(\operatorname{germ}_x f) = \operatorname{germ}_{\varphi_0(x)} g.$$

Since φ_1 is continuous, the preimage of the $\operatorname{germ}_{\varphi_0(x)} g$-neighborhood

$$U'_g = \{\operatorname{germ}_z g \mid z \in \operatorname{dom} g\}$$

is a neighborhood of $\operatorname{germ}_x f$. Hence there exists an open neighborhood U of x with $U \subseteq \operatorname{dom} f$ such that

$$U_{f|U} = \{\operatorname{germ}_y f \mid y \in U\} \subseteq \varphi_1^{-1}\left(U'_g\right).$$

Thus, for all $y \in U$ we have

$$\varphi_1(\operatorname{germ}_y f) = \operatorname{germ}_{\varphi_0(y)} g.$$

□

Remark 4.10. We remark that in Prop. 4.9 each two possible choices for g coincide on some neighborhood of $\varphi_0(x)$.

The proof of the following proposition is constructive. Moreover, we will use this construction to define the functor between the category of orbifolds and that of marked atlas groupoids.

Proposition 4.11. *Under the hypotheses of Prop. 4.9, there exist a family P of changes of charts of $\mathcal{V}$ and a map $\nu\colon P \to \Psi(\mathcal{V}')$ satisfying (R3) and (R4) with $\psi = \varphi_0$.*

Proof. Let $f \in \Psi(\mathcal{V})$ and $x_1, x_2 \in \operatorname{dom} f$, $x_1 \neq x_2$. For $j = 1, 2$ we choose, using Prop. 4.9, a pair (g_j, U_j) where $g_j \in \Psi(\mathcal{V}')$ is an embedding between some orbifold charts in $\mathcal{U}'$ and U_j is an open neighborhood of x_j such that $f|_{U_j}$ is a change of charts of $\mathcal{V}$. Clearly, the in respect to Prop. 4.9 additional conditions can be fulfilled. Further, to make sure that the map ν defined below is well-defined, we require that either $U_1 \neq U_2$ or $g_1 = g_2$. This condition can equally well be satisfied. Let P be the family of all changes of charts which we have chosen. By construction, P is a quasi-pseudogroup which generates $\Psi(\mathcal{V})$. We define the map $\nu\colon P \to \Psi(\mathcal{V}')$ by

$$\nu(\lambda) := g$$

where g is the unique element in $\Psi(\mathcal{V}')$ attached to $\lambda \in P$ by our choices. For $\lambda \in P$ and $x \in \operatorname{dom} \lambda$ we clearly have

$$\varphi_1(\operatorname{germ}_x \lambda) = \operatorname{germ}_{\varphi_0(x)} \nu(\lambda). \tag{1}$$

Now (R4a) is satisfied for each $\lambda \in P$ and $x \in \operatorname{dom} \lambda$ since

$$\begin{aligned}\varphi_0(\lambda(x)) &= \varphi_0(t(\operatorname{germ}_x \lambda)) = t(\varphi_1(\operatorname{germ}_x \lambda)) \\ &= t(\operatorname{germ}_{\varphi_0(x)} \nu(\lambda)) = \nu(\lambda)(\varphi_0(x)).\end{aligned}$$

For $\lambda, \mu \in P$ and $x \in \lambda^{-1}(\operatorname{dom} \mu)$, (R4c) is proven as follows. Suppose that $h \in P$ and U is an open set with $x \in U \subseteq \lambda^{-1}(\operatorname{dom} \mu) \cap \operatorname{dom} h$ such that $h|_U = \mu \circ \lambda|_U$. Then

$$\begin{aligned}\operatorname{germ}_{\varphi_0(\lambda(x))} \nu(\mu) \cdot \operatorname{germ}_{\varphi_0(x)} \nu(\lambda) &= \varphi_1(\operatorname{germ}_{\lambda(x)} \mu \cdot \operatorname{germ}_x \lambda) \\ &= \varphi_1(\operatorname{germ}_x \mu \circ \lambda) = \varphi_1(\operatorname{germ}_x h) = \operatorname{germ}_{\varphi_0(x)} \nu(h).\end{aligned}$$

Conditions (R4b) and (R4d) are proven along the same lines. □

Proposition 4.12. *Let $\mathcal{V}$ be an orbifold atlas of $(Q,\mathcal{U})$, $\mathcal{V}'$ an orbifold atlas of $(Q',\mathcal{U}')$, and*

$$\varphi = (\varphi_0, \varphi_1)\colon \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}')$$

a homomorphism. Then φ induces a representative of an orbifold map

$$(f, \{\tilde{f}_i\}_{i\in I}, P, \nu)$$

with domain atlas $\mathcal{V}$, range family contained in $\mathcal{V}'$, and

$$\tilde{f}_i = \varphi_0|_{V_i}$$

for all $i \in I$. Moreover, we have

$$f = \alpha_{\mathcal{V}'} \circ |\varphi| \circ \alpha_{\mathcal{V}}^{-1}.$$

Proof. It remains to show that the image of $\varphi_0|_{V_i}$ is contained in V'_j for some $(V'_j, G'_j, \pi'_j) \in \mathcal{V}'$. The claim then follows from the definition of $\alpha_{\mathcal{V}}$, $\alpha_{\mathcal{V}'}$ and Prop. 4.11. To that end let $x \in V_i$. Then there is a unique $(V'_j, G'_j, \pi'_j) \in \mathcal{V}'$ with $\varphi_0(x) \in V'_j$. We consider

$$A := \{y \in V_i \mid \varphi_0(y) \in V'_j\}.$$

Let $y \in A$. Since φ_0 is continuous and V'_j open, there exists an open neighborhood U of y in V_i such that $\varphi_0(U) \subseteq V'_j$. Thus $U \subseteq A$, which shows that A is open. By the same reasoning, for each $z \in V_i \setminus A$ there is a neighborhood U of z in V_i such that $U \subseteq V_i \setminus A$. Hence A is closed. By the choice of V'_j, the set A is nonempty. Thus, A is an open and closed nonempty subset of the connected set V_i. Therefore $A = V_i$. So, $\varphi_0(V_i) \subseteq V'_j$. □

Prop. 4.12 guarantees that each homomorphism

$$\varphi = (\varphi_0, \varphi_1)\colon \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}')$$

induces a representative of an orbifold map $(f, \{\tilde{f}_i\}_{i\in I}, P, \nu)$ with domain atlas $\mathcal{V}$, range family contained in $\mathcal{V}'$, $\tilde{f}_i = \varphi_0|_{V_i}$, and $f = \alpha_{\mathcal{V}'} \circ |\varphi| \circ \alpha_{\mathcal{V}}^{-1}$. For the pair (P, ν), Prop. 4.11 allows (in general) a whole bunch of choices. On the other hand, different representatives of orbifold maps may induce the same groupoid homomorphism.

The following definition and the proposition below serve to characterize these classes of representatives, and to show that the constructions are bijective on equivalence classes. In view of Prop. 4.7 and Remark 4.10, the relevant information stored by the pair (P, ν) are the germs of the elements in P and the via ν associated germs of elements in $\Psi(\mathcal{V}')$. This observation is the motivation for the equivalence relation.

Definition 4.13. Let $\hat{f} := (f, \{\tilde{f}_i\}_{i\in I}, P_1, \nu_1)$ and $\hat{g} := (g, \{\tilde{g}_i\}_{i\in I}, P_2, \nu_2)$ be two representatives of orbifold maps with the same domain atlas $\mathcal{V}$ representing the orbifold structure $\mathcal{U}$ on Q and both range families being contained in the orbifold atlas $\mathcal{V}'$ of $(Q',\mathcal{U}')$. Set $\psi := \coprod_{i\in I} \tilde{f}_i$. We say that $\hat{f}$ is *equivalent* to $\hat{g}$ if

$$f = g,$$
$$\tilde{f}_i = \tilde{g}_i \quad \text{for all } i \in I,$$

and if

$$\operatorname{germ}_{\psi(x)} \nu_1(\lambda_1) = \operatorname{germ}_{\psi(x)} \nu_2(\lambda_2)$$

for all $\lambda_1 \in P_1$, $\lambda_2 \in P_2$, $x \in \operatorname{dom} \lambda_1 \cap \operatorname{dom} \lambda_2$ with $\operatorname{germ}_x \lambda_1 = \operatorname{germ}_x \lambda_2$. Equivalence is clearly an equivalence relation. The equivalence class of $\hat{f}$ will be denoted by $[\hat{f}]$ or

$$(f, \{\tilde{f}_i\}_{i\in I}, [(P_1, \nu_1)]).$$

It is called an *orbifold map with domain atlas* $\mathcal{V}$ *and range atlas* $\mathcal{V}'$, in short *orbifold map with* $(\mathcal{V}, \mathcal{V}')$ or, if the precise atlases are not important, a *charted orbifold map*. The set of all orbifold maps with $(\mathcal{V}, \mathcal{V}')$ is denoted $\operatorname{Orb}(\mathcal{V}, \mathcal{V}')$. It will often be convenient to denote by

$$\mathcal{V} \xrightarrow{\hat{f}} \mathcal{V}'$$

an element $\hat{f} \in \operatorname{Orb}(\mathcal{V}, \mathcal{V}')$.

Proposition 4.14. *Let* $\mathcal{V}$ *be an orbifold atlas of* $(Q,\mathcal{U})$*, and* $\mathcal{V}'$ *an orbifold atlas of* $(Q',\mathcal{U}')$*. Then the set* $\operatorname{Orb}(\mathcal{V}, \mathcal{V}')$ *of all orbifold maps with* $(\mathcal{V}, \mathcal{V}')$ *and the set* $\operatorname{Hom}(\Gamma(\mathcal{V}), \Gamma(\mathcal{V}'))$ *of all homomorphisms from* $\Gamma(\mathcal{V})$ *to* $\Gamma(\mathcal{V}')$ *are in bijection. More precisely, the construction in Prop. 4.7 provides a bijection*

$$F_1 \colon \operatorname{Orb}(\mathcal{V}, \mathcal{V}') \to \operatorname{Hom}(\Gamma(\mathcal{V}), \Gamma(\mathcal{V}')),$$

and the constructions in Prop. 4.11 and 4.12 define a bijection

$$F_2 \colon \operatorname{Hom}(\Gamma(\mathcal{V}), \Gamma(\mathcal{V}')) \to \operatorname{Orb}(\mathcal{V}, \mathcal{V}'),$$

which is inverse to F_1*.*

Proof. We start by showing that F_1 is well-defined and injective. Suppose that $\hat{f}$ and $\hat{h}$ are two equivalent orbifold maps with $(\mathcal{V}, \mathcal{V}')$. The definition in Prop. 4.7 of the homomorphism from $\Gamma(\mathcal{V})$ to $\Gamma(\mathcal{V}')$ depends only on the germs of the elements in P resp. $\{\nu(\lambda) \mid \lambda \in P\}$ and not on the specific choice of P. Hence $\hat{f}$ and $\hat{h}$ induce the same homomorphism from $\Gamma(\mathcal{V})$ to $\Gamma(\mathcal{V}')$. Thus, the construction in Prop. 4.7 induces indeed a well-defined map

$$F_1 \colon \operatorname{Orb}(\mathcal{V}, \mathcal{V}') \to \operatorname{Hom}(\Gamma(\mathcal{V}), \Gamma(\mathcal{V}')).$$

Suppose now that $\hat{f} := (f, \{\tilde{f}_i\}_{i\in I}, P_1, \nu_1)$ and $\hat{g} := (g, \{\tilde{g}_i\}_{i\in I}, P_2, \nu_2)$ are two representatives of orbifold maps with $(\mathcal{V}, \mathcal{V}')$ such that $F_1([\hat{f}]) = F_1([\hat{g}])$. We set $\varphi = (\varphi_0, \varphi_1) := F_1([\hat{f})] = F_1([\hat{g}])$. Then

$$\tilde{f}_i = \varphi_0|_{V_i} = \tilde{g}_i \qquad \text{for all } i \in I.$$

Further for all $\lambda_1 \in P_1, \lambda_2 \in P_2$, $x \in \operatorname{dom} \lambda_1 \cap \operatorname{dom} \lambda_2$ with $\operatorname{germ}_x \lambda_1 = \operatorname{germ}_x \lambda_2$ we have

$$\operatorname{germ}_{\varphi_0(x)} \nu_1(\lambda_1) = \varphi_1(\operatorname{germ}_x \lambda_1) = \varphi_1(\operatorname{germ}_x \lambda_2) = \operatorname{germ}_{\varphi_0(x)} \nu_2(\lambda_2).$$

Finally φ determines f and g via

$$f = \alpha_{\mathcal{V}'} \circ |\tau| \circ \alpha_{\mathcal{V}}^{-1} = g.$$

Thus $[\hat{f}] = [\hat{g}]$, which shows that F_1 is injective.

We now prove that F_2 is well-defined. Let

$$\varphi = (\varphi_0, \varphi_1) \colon \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V}')$$

be a homomorphism, and let $(f, \{\tilde{f}_i\}_{i\in I}, P_1, \nu_1)$ and $(g, \{\tilde{g}_i\}_{i\in I}, P_2, \nu_2)$ be two representatives of orbifold maps with $(\mathcal{V}, \mathcal{V}')$ which are given by the constructions in Prop. 4.9, 4.11 and 4.12. We have to show that $\hat{f}$ and $\hat{g}$ are equivalent. By construction we have

$$f = \alpha_{\mathcal{V}'} \circ |\varphi| \circ \alpha_{\mathcal{V}}^{-1} = g,$$

and for each $i \in I$

$$\tilde{f}_i = \varphi_0|_{V_i} = \tilde{g}_i.$$

Finally, let $\lambda_1 \in P_1$, $\lambda_2 \in P_2$, $x \in \operatorname{dom}\lambda_1 \cap \operatorname{dom}\lambda_2$ with $\operatorname{germ}_x \lambda_1 = \operatorname{germ}_x \lambda_2$. From the definitions of ν_1 and ν_2 (see (1) in the proof of Prop. 4.11) it follows that

$$\operatorname{germ}_{\varphi_0(x)} \nu_1(\lambda_1) = \varphi_1(\operatorname{germ}_x \lambda_1) = \varphi_1(\operatorname{germ}_x \lambda_2) = \operatorname{germ}_{\varphi_0(x)} \nu_2(\lambda_2).$$

Thus, $\hat{f}$ and $\hat{g}$ are indeed equivalent, and hence F_2 is well-defined. The identity $F_1 \circ F_2 = \mathrm{id}$ is obvious from the definitions. This and the previous observation show that F_1 is bijective. In turn, F_2 is bijective with inverse map F_1. □

5. The category of reduced orbifolds

We are aiming at the definition of an orbifold category where the objects are orbifolds and the morphisms are equivalence classes of charted orbifold maps. To that end we have to answer the following questions:

(i) When shall two charted orbifold maps be considered as equal? In other words, what shall be the equivalence relation?
(ii) What shall be the identity morphism of an orbifold?
(iii) How does one compose $\varphi \in \operatorname{Orb}(\mathcal{V}, \mathcal{V}')$ and $\psi \in \operatorname{Orb}(\mathcal{V}', \mathcal{V}'')$?
(iv) What is the composition in the category?

The leading idea is that charted orbifold maps are equivalent if and only if they induce the same charted orbifold map on common refinements of the orbifold atlases. Therefore, we will introduce the notion of an induced charted orbifold map.

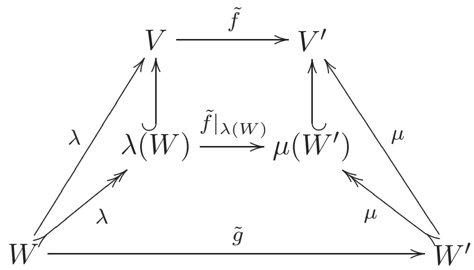
It turns out that answers to the questions (ii) and (iii) naturally extend to answers of (i) and (iv), and that the arising category has a counterpart in terms of marked atlas groupoids and homomorphisms. We start with the definition of the identity morphism of an orbifold. The definition is based on the idea that the identity morphism of $(Q,\mathcal{U})$ shall be represented by a collection of local lifts of id_Q which locally induce id_S on some orbifold charts, and that each such collection which satisfies (R2) shall be a representative.

5.1. **The identity morphism.**

Definition and Remark 5.1. Let $(Q,\mathcal{U})$ and $(Q',\mathcal{U}')$ be orbifolds and let $f\colon Q \to Q'$ be a continuous map. Suppose that $\tilde{f}$ is a local lift of f w.r.t. the orbifold charts $(V, G, \pi) \in \mathcal{U}$ and $(V', G', \pi') \in \mathcal{U}'$. Further suppose that $\lambda\colon (W, K, \chi) \to (V, G, \pi)$ and $\mu\colon (W', K', \chi') \to (V', G', \pi')$ are embeddings between orbifold charts in $\mathcal{U}$ resp. in $\mathcal{U}'$ such that $\tilde{f}(\lambda(W)) \subseteq \mu(W')$. Then the map

$$\tilde{g} := \mu^{-1} \circ \tilde{f} \circ \lambda\colon W \to W'$$

is a local lift of f w. r. t. (W, K, χ) and (W', K', χ'). We say that $\tilde{f}$ *induces the local lift* $\tilde{g}$ *w. r. t.* λ *and* μ, and we call $\tilde{g}$ the *induced lift of* f *w. r. t.* $\tilde{f}$, λ *and* μ.



Suppose that $\tilde{f}$ is a local lift of the identity id_Q for some orbifold $(Q, \mathcal{U})$. Prop. 5.3 below shows that $\tilde{f}$ induces the identity on sufficiently small orbifold charts. This means that locally $\tilde{f}$ is related to the identity itself via embeddings. In particular, $\tilde{f}$ is a local diffeomorphism. For its proof we need the following lemma.

Lemma 5.2. *Let* M *be a manifold,* G *a finite subgroup of* $\mathrm{Diff}(M)$*, and* $x \in M$*. There exist arbitrary small open* G*-stable neighborhoods* S *of* x*. Moreover, one can choose* S *so small that* $G_S = G_x$*, the isotropy group of* x*.*

Proof. Let U be a neighborhood of x, and let

$$\{x_1, \ldots, x_n\} := Gx$$

be an enumeration of the G-orbit of x, i. e., $x_i \neq x_j$ for $i \neq j$. Suppose that $x = x_1$. Since M is Hausdorff, we can choose for each $i = 1, \ldots, n$ a neighborhood U_i of x_i such that these are pairwise disjoint and $U_1 \subseteq U$. For $i = 1, \ldots, n$ define

$$G_i^1 := \{g \in G \mid gx_i = x\}$$

and set

$$U_1' := \bigcap_{i=1}^{n} \{gU_i \mid g \in G_i^1\}.$$

For $h \in G$ with $hx = x_i$ we have $h^{-1} \in G_i^1$. Then $U_1' \subseteq h^{-1}U_i$ yields $hU_1' \subseteq U_i$. If $i \neq 1$, then $U_i \cap U_1 = \emptyset$ and $U_1' \subseteq U_1$, hence

$$hU_1' \cap U_1' = \emptyset.$$

But for $i = 1$ we have $h \in G_x$ and

$$hU_1' = \bigcap_{j=1}^{n} \{hgU_j \mid g \in G_j^1\} = \bigcap_{j=1}^{n} \{gU_j \mid g \in G_j^1\} = U_1'.$$

This means that U_1' is G-stable. Now let T be the connected component of U_1' which contains x. Then M being a locally Euclidean space (hence each point has arbitrary small connected neighborhoods) shows that T is a closed neighborhood of x. Therefore $S := T^\circ$ is an open G-stable neighborhood of x with $G_S = G_x$. □

Proposition 5.3. *Let* $(Q, \mathcal{U})$ *be an orbifold and suppose that* $\tilde{f}$ *is a local lift of* id_Q *w. r. t.* (V, G, π), $(V', G', \pi') \in \mathcal{U}$*. For each* $v \in V$ *there exist a restriction* $(S, G_S, \pi|_S)$ *of* (V, G, π) *with* $v \in S$ *and a restriction* $(S', (G')_{S'}, \pi'|_{S'})$ *of*

(V', G', π') such that $\tilde{f}|_S$ is an isomorphism from $(S, G_S, \pi|_S)$ to $(S', (G')_{S'}, \pi'|_{S'})$. In particular, $\tilde{f}|_S$ induces the identity id_S w. r. t. id_S and $\left(\tilde{f}|_S\right)^{-1}$.

Proof. Let $v \in V$ and set $v' := \tilde{f}(v)$. Then $\pi(v) = \pi'(v')$. By compatibility of orbifold charts there exist a restriction (W, H, χ) of (V, G, π) with $v \in W$ and an embedding

$$\lambda\colon (W, H, \chi) \to (V', G', \pi')$$

such that $\lambda(v) = v'$. Since $M := \tilde{f}^{-1}(\lambda(W)) \cap W$ is an open neighborhood of v in V. Lemma 5.2 yields an open H-stable neighborhood S of v with $S \subseteq M$. Then $\tilde{f}(S) \subseteq \lambda(W)$. Let

$$\tilde{g} := \lambda^{-1} \circ \tilde{f}|_S \colon S \to W$$

denote the induced lift of id_Q. Since

$$\chi \circ \tilde{g} = \chi,$$

Lemma 2.11 in [MM03] shows the existence of a unique $h \in H$ such that $\tilde{g} = h|_S$. Therefore the diagram

$$\begin{array}{ccc} S & \xrightarrow{\tilde{f}} & \lambda(h(S)) \\ & {\searrow}^{h} & \uparrow{\scriptstyle\lambda} \\ & & h(S) \end{array}$$

commutes, where the non-horizontal arrows are diffeomorphisms. Hence the restriction $\tilde{f}|_S \colon S \to \lambda(h(S))$ is a diffeomorphism, and in turn

$$\tilde{f}|_S \colon (S, H_S, \chi|_S) \to (\tilde{f}(S), G'_{\tilde{f}(S)}, \pi'|_{\tilde{f}(S)})$$

is an isomorphism of orbifold charts. □

Not each local lift of the identity is a global diffeomorphism, as the following example shows.

Example 5.4. Let Q be the open annulus in $\mathbb{R}^2$ with inner radius 1 and outer radius 2 centered at the origin, i. e.,

$$Q := \{w \in \mathbb{R}^2 \mid 1 < \|w\| < 2\} = \{w \in \mathbb{C} \mid 1 < |w| < 2\}$$

where we use the Euclidean norm on $\mathbb{R}^2$. The map $\alpha\colon Q \to \mathbb{C} \times \mathbb{R}$,

$$\alpha(w) := \left(\frac{w^2}{|w|^2}, |w| - 1\right)$$

maps Q onto the cylinder

$$Z := S^1 \times (0, 1).$$

Note that $\alpha(Q)$ covers Z twice. Then the map $\beta\colon Z \to \mathbb{C}$,

$$\beta(z, s) := \frac{2}{2 - s} z$$

is the linear projection of Z from the point $(0, 2) \in \mathbb{C} \times \mathbb{R}$ to the complex plane. The composed map $\tilde{f} = \beta \circ \alpha\colon Q \to \mathbb{C}$,

$$\tilde{f}(w) := \frac{2w^2}{(3 - |w|)|w|^2}$$

is smooth and maps Q onto Q. Further it induces a homeomorphism between $Q/\{\pm\,\mathrm{id}\}$ and Q. Hence, if we endow Q with the orbifold atlas

$$\big\{\big(Q,\{\pm\,\mathrm{id}\},\tilde f\big),\big(Q,\{\mathrm{id}\},\mathrm{id}\,\big)\big\},$$

then $\tilde f$ is a local lift of id_Q w. r. t. $(Q,\{\pm\,\mathrm{id}\},\tilde f)$ and $(Q,\{\mathrm{id}\},\mathrm{id})$ which is not a global diffeomorphism.

For the proof of the following proposition recall from Remark 2.4 the notation $\overline{\mu}$ for the group isomorphism $G\to H_{\mu(V)}$ induced by the embedding $\mu\colon (V,G,\varphi)\to (W,H,\psi)$ between orbifold charts.

Proposition 5.5. *Let $(Q,\mathcal{U})$ be an orbifold and let $\{\tilde f_i\}_{i\in I}$ be a family of local lifts of id_Q which satisfies (R2). Then there exists a pair (P,ν) such that $(\mathrm{id}_Q,\{\tilde f_i\}_{i\in I},P,\nu)$ is a representative of an orbifold map on $(Q,\mathcal{U})$. The pair (P,ν) is uniquely determined up to equivalence of representatives of orbifold maps (see Def. 4.13).*

Proof. For $i\in I$ we suppose that $\tilde f_i$ is a local lift of id_Q w. r. t. the orbifold charts (V_i,G_i,π_i), $(V_i',G_i',\pi_i')\in\mathcal{U}$. We let

$$\mathcal{V}:=\{(V_i,G_i,\pi_i)\mid i\in I\}$$

be the arising representative of $\mathcal{U}$, indexed by I, and set $\psi:=\coprod_{i\in I}\tilde f_i$. We define P to be the quasi-pseudogroup of all changes of charts λ of $\mathcal{V}$ for which $\psi|_{\mathrm{dom}\,\lambda}$ and $\psi|_{\mathrm{cod}\,\lambda}$ are open embeddings. We claim that P generates $\Psi(\mathcal{V})$. To that end let $g\in\Psi(\mathcal{V})$ and $x\in\mathrm{dom}\,g$. Suppose that $x\in V_a$ and $g(x)\in V_b$. Prop. 5.3 shows that there is a restriction (W_b,H_b,χ_b) of (V_b,G_b,π_b) such that $g(x)\in W_b\subseteq\mathrm{cod}\,g$ and $\tilde f_b|_{W_b}$ is an open embedding. Then $(g^{-1}(W_b),\overline{g}^{-1}(H_b),g^{-1}\circ\chi_b)$ is a restriction of (V_a,G_a,χ_a). Invoking again Prop. 5.3 we find a restriction (W_a,H_a,χ_a) of (V_a,G_a,π_a) such that $x\in W_a\subseteq g^{-1}(W_b)$ and $\tilde f_a|_{W_a}$ is an open embedding. Then $g|_{W_a}\in P$. Therefore P satisfies (R3).

Let $\lambda\in P$ and suppose that $S:=\mathrm{dom}\,\lambda\subseteq V_a$ and $\mathrm{cod}\,\lambda\subseteq V_b$. To satisfy property (R4a) we define

$$\nu(\lambda):=\tilde f_b\circ\lambda\circ(\tilde f_a|_S)^{-1}. \tag{2}$$

Any other possibility to define $\nu(\lambda)$ must coincide with this one on $\mathrm{dom}\,\lambda$. We have to show that $\nu(\lambda)$ is an embeddig between some orbifold charts in $\mathcal{U}$. Since

$$\psi|_S=\tilde f_a|_S\colon (S,(G_a)_S,\pi_a|_S)\to(V_a',G_a',\pi_a')$$

is an open embedding, [MM03, Prop. 2.12] shows that $S':=\tilde f_a(S)$ is a G_a'-stable open set. The definition (2) yields that ν is functorial and commutes with restrictions. Hence ν satisfies (R4).

Finally let (P_1,ν_1) be any pair such that $(\mathrm{id}_Q,\{\tilde f_i\}_{i\in I},P_1,\nu_1)$ becomes a representative of an orbifold map on $(Q,\mathcal{U})$. Let $\lambda\in P_1$ and $x\in\mathrm{dom}\,\lambda$. Then we find an open neighborhood U of x such that $\lambda|_U\in P$. In particular, $\psi|_U$ is an open embedding. Therefore

$$\nu_1(\lambda)|_{\psi(U)}=\psi\circ\lambda\circ(\psi|_U)^{-1}=\nu(\lambda|_U).$$

This yields immediately that $(\mathrm{id}_Q,\{\tilde f_i\}_{i\in I},P,\nu)$ and $(\mathrm{id}_Q,\{\tilde f_i\}_{i\in I},P_1,\nu_1)$ are equivalent. ∎

The following proposition shows that whenever one has a charted orbifold map $(f, \{\tilde{f}_i\}_{i\in I}, [P,\nu])$ between orbifolds with the same underlying topological space Q such that the continuous map $f\colon Q \to Q$ is the identity id_Q and such that all elements of the family $\{\tilde{f}_i\}_{i\in I}$ are local diffeomorphisms, then the orbifolds are identical.

Proposition 5.6. *Let Q be a topological space and suppose that $\mathcal{U}$ and $\mathcal{U}'$ are orbifold structures on Q. Let*

$$\hat{f} = \big(f, \{\tilde{f}_i\}_{i\in I}, [P,\nu]\big)$$

be a charted orbifold map for which

- $f = \mathrm{id}_Q$,
- *the domain atlas $\mathcal{V}$ is a representative of $\mathcal{U}$,*
- *the range family $\mathcal{V}'$, which here is an orbifold atlas, is a representative of $\mathcal{U}'$, and*
- *for each $i \in I$, the map $\tilde{f}_i$ is a local diffeomorphism.*

Then $\mathcal{U} = \mathcal{U}'$.

Proof. Let $(V_i, G_i, \pi_i) \in \mathcal{V}$, $(V_j', G_j', \pi_j') \in \mathcal{V}'$ and $x \in V_i$, $y \in V_j'$ such that $\pi_i(x) = \pi_j'(y)$. Since $\tilde{f}_i\colon V_i \to V_i'$ is a local diffeomorphism, there are open neighborhoods U of x in V_i and U' of $\tilde{f}_i(x)$ in V_i' such that $\tilde{f}_i|_U\colon U \to U'$ is a diffeomorphism. We have

$$\pi_i'\big(\tilde{f}_i(x)\big) = \pi_i(x) = \pi_j'(y).$$

Therefore there exist open neighborhoods W of $\tilde{f}_i(x)$ in U' and W' of y in V_j' and a diffeomorphism $h\colon W \to W'$ satisfying $\pi_j' \circ h = \pi_i'$. Shrinking U shows that (V_i, G_i, π_i) and (V_j', G_j', π_j') are compatible. Thus $\mathcal{U} = \mathcal{U}'$. □

The following example shows that the requirement in Prop. 5.6 that the local lifts be local diffeomorphisms is essential.

Example 5.7. Recall the orbifolds $(Q, \mathcal{U}_i)$, $i = 1, 2$, from Example 2.5, the representatives $\mathcal{V}_1 := \{V_1\}$ and $\mathcal{V}_2 := \{V_2\}$ of $\mathcal{U}_1$ resp. $\mathcal{U}_2$, and set $g(x) := x^2$ for $x \in (-1, 1)$. Then g is a lift of id_Q w.r.t. V_2 and V_1. Further let $P := \{\pm\,\mathrm{id}_{(-1,1)}\}$ and $\nu(\pm\,\mathrm{id}_{(-1,1)}) := \mathrm{id}_{(-1,1)}$. Then $(\mathrm{id}_Q, \{g\}, [P,\nu])$ is an orbifold map with $(\mathcal{V}_2, \mathcal{V}_1)$ from $(Q, \mathcal{U}_2)$ to $(Q, \mathcal{U}_1)$, but $\mathcal{U}_1 \neq \mathcal{U}_2$.

Motivated by Prop. 5.5 and 5.6 we make the following definition.

Definition 5.8. Let $(Q, \mathcal{U})$ be an orbifold and let $\hat{f} = (f, \{\tilde{f}_i\}_{i\in I}, [P,\nu])$ be a charted orbifold map whose domain atlas is a representative of $\mathcal{U}$. If and only if $f = \mathrm{id}_Q$ and $\tilde{f}_i$ is a local diffeomorphism for each $i \in I$, we call $\hat{f}$ a *lift of the identity* $\mathrm{id}_{(Q,\mathcal{U})}$ or a *representative* of $\mathrm{id}_{(Q,\mathcal{U})}$. The set of all lifts of $\mathrm{id}_{(Q,\mathcal{U})}$ is the *identity morphism* $\mathrm{id}_{(Q,\mathcal{U})}$ of $(Q, \mathcal{U})$.

5.2. **Composition of charted orbifold maps.** We present natural definitions for the composition of two charted orbifold maps and for induced charted orbifold maps. These we use to construct the category of reduced orbifolds. The proofs and constructions in this section are quite technical due to the fact that we have to work with local charts. In Section 6 we characterize all these orbifold concepts in terms of atlas groupoids and their homomorphisms.

Construction 5.9. Let $(Q,\mathcal{U})$, $(Q',\mathcal{U}')$ and $(Q'',\mathcal{U}'')$ be orbifolds, and $\mathcal{V}$, $\mathcal{V}'$ resp. $\mathcal{V}''$ be representatives for $\mathcal{U}$, $\mathcal{U}'$ resp. $\mathcal{U}''$. Suppose that

$$\hat{f} = (f, \{\tilde{f}_i\}_{i\in I}, [P_f, \nu_f]) \in \operatorname{Orb}(\mathcal{V}, \mathcal{V}')$$

and

$$\hat{g} = (g, \{\tilde{g}_j\}_{j\in J}, [P_g, \nu_g]) \in \operatorname{Orb}(\mathcal{V}', \mathcal{V}'')$$

with $I \subseteq J$. The composition

$$\hat{g}\circ\hat{f} =: \hat{h} = (h, \{\tilde{h}_i\}_{i\in I}, [P_h, \nu_h]) \in \operatorname{Orb}(\mathcal{V}, \mathcal{V}'')$$

is given by $h := g\circ f$ and $\tilde{h}_i := \tilde{g}_i \circ \tilde{f}_i$ for all $i \in I$. To construct a representative (P_h, ν_h) of $[P_h, \nu_h]$ we fix representatives (P_f, ν_f) and (P_g, ν_g) of $[P_f, \nu_f]$ and $[P_g, \nu_g]$, resp. The leading idea to define (P_h, ν_h) is to take $P_h = P_f$ and $\nu_h = \nu_g \circ \nu_f$. But since $\nu_f(\lambda)$ is not necessarily in P_g for $\lambda \in P_f$, the composition $\nu_g \circ \nu_f$ might be ill-defined. Therefore we have to refine this idea.

Let $\mu \in P_f$ and suppose that $\operatorname{dom}\mu \subseteq V_i$ and $\operatorname{cod}\mu \subseteq V_j$ for the orbifold charts (V_i, G_i, π_i) and (V_j, G_j, π_j) in $\mathcal{V}$. By (R4a)

$$\tilde{f}_j \circ \mu = \nu_f(\mu) \circ \tilde{f}_i|_{\operatorname{dom}\mu},$$

where $\nu_f(\mu) \in \Psi(\mathcal{V}')$. For $x \in \operatorname{dom}\mu$ we set $y_x := \tilde{f}_i(x)$, which is an element of $\operatorname{dom}\nu_f(\mu)$. Hence we find (and fix a choice) $\xi_{\mu,x} \in P_g$ with $y_x \in \operatorname{dom}\xi_{\mu,x}$ and an open set $U'_{\mu,x} \subseteq \operatorname{dom}\xi_{\mu,x} \cap \operatorname{dom}\nu_f(\mu)$ such that $y_x \in U'_{\mu,x}$ and

$$\xi_{\mu,x}|_{U'_{\mu,x}} = \nu_f(\mu)|_{U'_{\mu,x}}.$$

Then we find (and fix) an open set $U_{\mu,x} \subseteq \operatorname{dom}\mu$ with $x \in U_{\mu,x}$ such that $\tilde{f}_i(U_{\mu,x}) \subseteq U'_{\mu,x}$. We may and will suppose that for $\mu_1, \mu_2 \in P_f$ and $x_1 \in \operatorname{dom}\mu_1$, $x_2 \in \operatorname{dom}\mu_2$ we either have

$$\mu_1|_{U_{\mu_1,x_1}} \neq \mu_2|_{U_{\mu_2,x_2}} \quad \text{or} \quad \xi_{\mu_1,x_1} = \xi_{\mu_2,x_2}. \tag{3}$$

Now we define

$$P_h := \left\{ \mu|_{U_{\mu,x}} \,\middle|\, \mu \in P_f,\ x \in \operatorname{dom}\mu \right\},$$

which obviously is a quasi-pseudogroup generating $\Psi(\mathcal{V})$. Further we set

$$\nu_h\big(\mu|_{U_{\mu,x}}\big) := \nu_g(\xi_{\mu,x})$$

for $\mu|_{U_{\mu,x}} \in P_h$. Property (3) yields that ν_h is a well-defined map from P_h to $\Psi(\mathcal{V}'')$. One easily sees that ν_h satisfies (R4a) - (R4d), and the equivalence class of (P_h, ν_h) does not depend on the choices we made for the construction of P_h and ν_h.

Remark 5.10. Recall the maps F_1 and F_2 from Prop. 4.14. Then the construction of the composition of two charted orbifold maps $\hat{f} \in \operatorname{Orb}(\mathcal{V}, \mathcal{V}')$ and $\hat{g} \in \operatorname{Orb}(\mathcal{V}', \mathcal{V}'')$ immediately implies that

$$F_1(\hat{g}\circ\hat{f}) = F_1(\hat{g}) \circ F_1(\hat{f}).$$

Conversely, if $\varphi \in \operatorname{Hom}(\Gamma(\mathcal{V}), \Gamma(\mathcal{V}'))$ and $\psi \in \operatorname{Hom}(\Gamma(\mathcal{V}'), \Gamma(\mathcal{V}''))$, then

$$F_2(\psi\circ\varphi) = F_2(\psi) \circ F_2(\varphi).$$

The reason for this is that the construction of (P_h, ν_h) (in the notation of Construction 5.9) only depends on germs.

The following lemma provides the definition of induced charted orbifold map and shows its relation to lifts of the identity.

Lemma and Definition 5.11. *Let $(Q,\mathcal{U})$ and $(Q',\mathcal{U}')$ be orbifolds. Further let*

$$\begin{aligned} \mathcal{V} &= \{(V_i, G_i, \pi_i) \mid i \in I\} \text{ be a representative of } \mathcal{U}, \\ \mathcal{V}' &= \{(V'_l, G'_l, \pi'_l) \mid l \in L\} \text{ be a representative of } \mathcal{U}', \text{ and} \\ \hat{f} &= \big(f, \{\tilde{f}_i\}_{i\in I}, [P_f, \nu_f]\big) \in \operatorname{Orb}(\mathcal{V}, \mathcal{V}'). \end{aligned}$$

Suppose that we have

- *a representative $\mathcal{W} = \{(W_j, H_j, \psi_j) \mid j \in J\}$ of $\mathcal{U}$, indexed by J,*
- *a subset $\{(W'_j, H'_j, \psi'_j) \mid j \in J\}$ of $\mathcal{U}'$, indexed by J (not necessarily an orbifold atlas),*
- *a map $\alpha\colon J \to I$,*
- *for each $j \in J$, an embedding*
$$\lambda_j\colon \big(W_j, H_j, \psi_j\big) \to \big(V_{\alpha(j)}, G_{\alpha(j)}, \pi_{\alpha(j)}\big),$$
and an embedding
$$\mu_j\colon \big(W'_j, H'_j, \psi'_j\big) \to \big(V'_{\alpha(j)}, G'_{\alpha(j)}, \pi'_{\alpha(j)}\big)$$
such that
$$\tilde{f}_{\alpha(j)}\big(\lambda_j(W_j)\big) \subseteq \mu_j(W'_j).$$

For each $j \in J$ set
$$\tilde{h}_j := \mu_j^{-1} \circ \tilde{f}_{\alpha(j)} \circ \lambda_j\colon W_j \to W'_j.$$
Then

(i) $\varepsilon := \big(\mathrm{id}_Q, \{\lambda_j\}_{j\in J}, [R,\sigma]\big) \in \operatorname{Orb}(\mathcal{W}, \mathcal{V})$ *(with $[R,\sigma]$ provided by Prop. 5.5) is a lift of $\mathrm{id}_{(Q,\mathcal{U})}$.*

(ii) *The set $\{(W'_j, H'_j, \psi'_j) \mid j \in J\}$ and the family $\{\mu_j\}_{j\in J}$ can be extended to a representative*
$$\mathcal{W}' = \big\{(W'_k, H'_k, \psi'_k) \;\big|\; k \in K\big\}$$
of $\mathcal{U}'$ and a family of embeddings $\{\mu_k\}_{k\in K}$ such that
$$\varepsilon' := \big(\mathrm{id}_{Q'}, \{\mu_k\}_{k\in K}, [R',\sigma']\big) \in \operatorname{Orb}(\mathcal{W}', \mathcal{V}')$$
(again with $[R',\sigma']$ provided by Prop. 5.5) is a lift of $\mathrm{id}_{(Q',\mathcal{U}')}$.

(iii) *There is a uniquely determined equivalence class $[P_h, \nu_h]$ such that*
$$\hat{h} := (f, \{\tilde{h}_j\}_{j\in J}, [P_h, \nu_h]) \in \operatorname{Orb}(\mathcal{W}, \mathcal{W}')$$
and such that the diagram
$$\begin{array}{ccccc} & \mathcal{V} & \xrightarrow{\hat{f}} & \mathcal{V}' & \\ {\scriptstyle\varepsilon}\nearrow & & & & \nwarrow{\scriptstyle\varepsilon'} \\ \mathcal{W} & & \xrightarrow{\quad\hat{h}\quad} & & \mathcal{W}' \end{array}$$
commutes.

We say that $\hat{h}$ is induced by $\hat{f}$.

Proof. (i) is clear by Prop. 5.3 and 5.5. To show that (ii) holds we construct one possible extension: Let

$$y \in Q' \setminus \bigcup_{j\in J} \psi'_j(W'_j).$$

Then there is a chart $(V', G', \pi') \in \mathcal{V}'$ such that $y \in \pi'(V')$. Extend the set $\{(W'_j, H'_j, \psi'_j) \mid j \in J\}$ with (V', G', π') and the family $\{\mu_j\}_{j\in J}$ with $\mathrm{id}_{V'}$. If this is done iteratively, one finally gets on orbifold atlas of Q' as wanted. Then Prop. 5.3 and 5.5 yields the remaining claim of (ii). The following considerations are independent of the specific choices of extensions.

Concerning (iii) we remark that each $\tilde{h}_j$ is obviously a local lift of f. Fix a representative (P_f, ν_f) of $[P_f, \nu_f]$. In the following we construct a pair (P_h, ν_h) for which $\hat{h}$ is an orbifold map and the diagram in (iii) commutes. It will be clear from the construction that the equivalence class $[P_h, \nu_h]$ is independent of the choice of (P_f, ν_f) and uniquely determined by the requirement of the commutativity of the diagram. Let $\gamma \in \Psi(\mathcal{W})$ and $x \in \operatorname{dom}\gamma$. Possibly shrinking the domain of γ, We may assume that $\operatorname{dom}\gamma \subseteq W_j$ and $\operatorname{cod}\gamma \subseteq W_k$ for some $j, k \in J$. In the following we further shrink the domain of γ to be able to define ν_h as a composition of ν_f with elements of $\{\mu_j\}_{j\in J}$. Let $y := \lambda_j(x)$. Since

$$\tilde{\gamma} := \lambda_k \circ \gamma \circ (\lambda_j|_{\operatorname{dom}\gamma})^{-1} : \lambda_j(\operatorname{dom}\gamma) \to \lambda_k(\operatorname{cod}\gamma)$$

is an element of $\Psi(\mathcal{V})$, we find $\beta_\gamma \in P_f$ such that $y \in \operatorname{dom}\beta_\gamma$ and $\operatorname{germ}_y \beta_\gamma = \operatorname{germ}_y \tilde{\gamma}$. Then

$$z := \tilde{f}_{\alpha(j)}(y) \in \operatorname{dom}\nu_f(\beta_\gamma) \cap \mu_j(W'_j).$$

Since

$$\nu_f(\beta_\gamma)(z) = \tilde{f}_{\alpha(k)}(\beta_\gamma(y)) \in \mu_k(W'_k),$$

the set

$$U' := \operatorname{dom}\nu_f(\beta_\gamma) \cap \mu_j(W'_j) \cap \nu_f(\beta_\gamma)^{-1}(\mu_k(W'_k))$$

is an open neighborhood of z. Define

$$U_1 := \{w \in \operatorname{dom}\beta_\gamma \cap \lambda_j(\operatorname{dom}\gamma) \mid \operatorname{germ}_w \beta_\gamma = \operatorname{germ}_w \tilde{\gamma}\},$$

which is an open neighborhood of y. Then also

$$U := U_1 \cap \tilde{f}_{\alpha(j)}^{-1}(U')$$

is an open neighborhood of y. We fix an open neighborhood $U_{\gamma,x} \subseteq \lambda_j^{-1}(U)$ of x. Further we suppose that for $\gamma_1, \gamma_2 \in \Psi(\mathcal{W})$, $x_1 \in \operatorname{dom}\gamma_1$, $x_2 \in \operatorname{dom}\gamma_2$, we either have

$$\gamma_1|_{U_{\gamma_1,x_1}} \neq \gamma_2|_{U_{\gamma_2,x_2}} \quad \text{or} \quad \nu_f(\beta_{\gamma_1}) = \nu_f(\beta_{\gamma_2}). \tag{4}$$

Then we define

$$P_h := \left\{\gamma|_{U_{\gamma,x}} \,\middle|\, \gamma \in \Psi(\mathcal{W}),\ x \in \operatorname{dom}\gamma\right\}$$

and set

$$\nu_h\big(\gamma|_{U_{\gamma,x}}\big) := \mu_k^{-1} \circ \nu_f(\beta_\gamma) \circ \mu_j$$

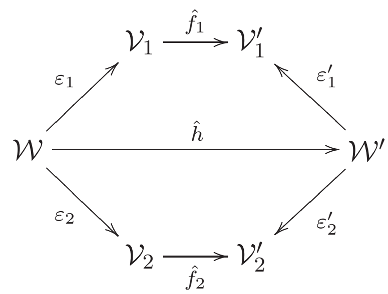
for $\gamma|_{U_{\gamma,x}} \in P_h$ with $x \in W_j$ and $\gamma(x) \in W_k$ $(j, k \in J)$. By (4), the map $\nu_h \colon P_h \to \Psi(\mathcal{W}')$ is well-defined. One easily checks that (P_h, ν_h) satisfies all requirements of (iii). □

We consider two charted orbifold maps as equivalent if they induce the same charted orbifold map on common refinements of the orbifold atlases. The following definition provides a precise specification of this idea.

Definition 5.12. Let $(Q,\mathcal{U})$ and $(Q',\mathcal{U}')$ be orbifolds. Further let $\mathcal{V}_1, \mathcal{V}_2$ be representatives of $\mathcal{U}$, and $\mathcal{V}_1', \mathcal{V}_2'$ be representatives of $\mathcal{U}'$. Suppose that $\hat{f}_1 \in \operatorname{Orb}(\mathcal{V}_1, \mathcal{V}_1')$ and $\hat{f}_2 \in \operatorname{Orb}(\mathcal{V}_2, \mathcal{V}_2')$. We call $\hat{f}_1$ and $\hat{f}_2$ *equivalent* ($\hat{f}_1 \sim \hat{f}_2$) if there are

- a representative $\mathcal{W}$ of $\mathcal{U}$,
- a representative $\mathcal{W}'$ of $\mathcal{U}'$,
- $\varepsilon_1 \in \operatorname{Orb}(\mathcal{W}, \mathcal{V}_1)$, $\varepsilon_2 \in \operatorname{Orb}(\mathcal{W}, \mathcal{V}_2)$ lifts of $\mathrm{id}_{(Q,\mathcal{U})}$,
- $\varepsilon_1' \in \operatorname{Orb}(\mathcal{W}', \mathcal{V}_1')$, $\varepsilon_2' \in \operatorname{Orb}(\mathcal{W}', \mathcal{V}_2')$ lifts of $\mathrm{id}_{(Q',\mathcal{U}')}$, and
- a map $\hat{h} \in \operatorname{Orb}(\mathcal{W}, \mathcal{W}')$

such that the diagram



commutes.

Prop. 5.15 below states that $\sim$ is indeed an equivalence relation. For its proof we need the following two lemmas. The first lemma discusses how local lifts which belong to the same charted orbifold map are related to each other, the second one shows that two charted orbifold maps which are induced from the same charted orbifold map induce the same charted orbifold map on common refinements of atlases. This means that $\sim$ satisfies the so-called diamond property.

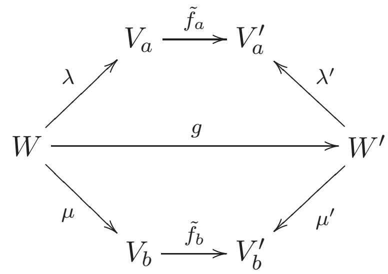
Lemma 5.13. *Let $(Q,\mathcal{U})$ and $(Q',\mathcal{U}')$ be orbifolds and let*

$$\hat{f} := (f, \{\tilde{f}_i\}_{i\in I}, [P,\nu]) \in \operatorname{Orb}(\mathcal{V}, \mathcal{V}')$$

be a charted orbifold map where $\mathcal{V}$ is a representative of $\mathcal{U}$ and $\mathcal{V}'$ one of $\mathcal{U}'$. Suppose that we have orbifold charts $(V_a, G_a, \pi_a), (V_b, G_b, \pi_b) \in \mathcal{V}$ and points $x_a \in V_a$, $x_b \in V_b$ such that $\pi_a(x_a) = \pi_b(x_b)$. Then there are orbifold charts $(W, K, \chi) \in \mathcal{U}$, $(W', K', \chi') \in \mathcal{U}'$ and embeddings

$$\begin{aligned}
\lambda &\colon (W, K, \chi) \to (V_a, G_a, \pi_a)\\
\lambda' &\colon (W', K', \chi') \to (V_a', G_a', \pi_a')\\
\mu &\colon (W, K, \chi) \to (V_b, G_b, \pi_b)\\
\mu' &\colon (W', K', \chi') \to (V_b', G_b', \pi_b')
\end{aligned}$$

with $x_a \in \lambda(W)$*,* $x_b \in \mu(W)$ *such that the induced lift* g *of* f *w.r.t.* $\tilde{f}_a, \lambda, \lambda'$ *coincides with the one induced by* $\tilde{f}_b$*,* μ*,* μ'*. In other words, the diagram*



commutes.

Proof. By compatibility of orbifold charts we find a restriction (W, K, χ) of (V_a, G_a, π_a) with $x_a \in W$ and an embedding

$$\mu\colon (W, K, \chi) \to (V_b, G_b, \pi_b)$$

such that $\mu(x_a) = x_b$ (cf. Remarks 2.2 and 2.4). Then $\mu\colon W \to \mu(W)$ is an element of $\Psi(\mathcal{V})$, hence there is a $\gamma \in P$ with $x_a \in \operatorname{dom}\gamma$ and an open neighborhood U of x_a such that $U \subseteq \operatorname{dom}\gamma \cap W$ and

$$\mu|_U = \gamma|_U.$$

W.l.o.g., $\gamma = \mu$. Property (R4a) yields that

$$\nu(\mu) \circ \tilde{f}_a|_W = \tilde{f}_b \circ \mu.$$

By shrinking the domain of $\nu(\mu)$, we can achieve that $\operatorname{cod}\nu(\mu) \subseteq V'_b$ and still $\tilde{f}_a(W) \subseteq \operatorname{dom}\nu(\mu) =: W'$. With $\mu' := \nu(\mu)$ it follows

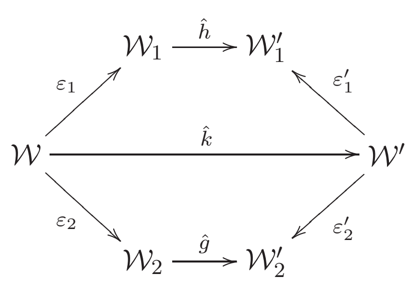
$$\tilde{f}_b(\mu(W)) = \mu'(\tilde{f}_a(W)) \subseteq \mu'(W')$$

and further

$$\tilde{f}_a|_W = (\mu')^{-1} \circ \tilde{f}_b \circ \mu.$$

This proves the claim. □

Lemma 5.14. *Let* $(Q,\mathcal{U})$ *and* $(Q',\mathcal{U}')$ *be orbifolds,* $\mathcal{V}$ *a representative of* $\mathcal{U}$*, and* $\mathcal{V}'$ *one of* $\mathcal{U}'$*. Further let* $\hat{f} \in \operatorname{Orb}(\mathcal{V}, \mathcal{V}')$*. If the charted orbifold maps* $\hat{h} \in \operatorname{Orb}(\mathcal{W}_1, \mathcal{W}'_1)$ *and* $\hat{g} \in \operatorname{Orb}(\mathcal{W}_2, \mathcal{W}'_2)$ *are induced by* $\hat{f}$*, then we find a representative* $\mathcal{W}$ *of* $\mathcal{U}$ *and two charted orbifold maps* $\varepsilon_1 \in \operatorname{Orb}(\mathcal{W}, \mathcal{W}_1)$*,* $\varepsilon_2 \in \operatorname{Orb}(\mathcal{W}, \mathcal{W}_2)$ *which are lifts of* $\operatorname{id}_{(Q,\mathcal{U})}$*, and a representative* $\mathcal{W}'$ *of* $\mathcal{U}'$ *and two charted orbifold maps* $\varepsilon'_1 \in \operatorname{Orb}(\mathcal{W}', \mathcal{W}'_1)$*,* $\varepsilon'_2 \in \operatorname{Orb}(\mathcal{W}', \mathcal{W}'_2)$ *which are lifts of* $\operatorname{id}_{(Q',\mathcal{U}')}$*, and a charted orbifold map* $\hat{k} \in \operatorname{Orb}(\mathcal{W}, \mathcal{W}')$ *such that the diagram*



commutes.

Proof. Let

$$\hat{f} = (f, \{\tilde{f}_a\}_{a\in A}, [P_f, \nu_f]),$$
$$\hat{h} = (f, \{\tilde{h}_i\}_{i\in I}, [P_h, \nu_h]),$$
$$\hat{g} = (f, \{\tilde{g}_j\}_{j\in J}, [P_g, \nu_g]),$$

and

$$\mathcal{W}_1 := \{(W_{1,i}, H_{1,i}, \psi_{1,i}) \mid i \in I\}, \text{ indexed by } I,$$
$$\mathcal{W}_1' := \{(W_{1,k}', H_{1,k}', \psi_{1,k}') \mid k \in K\}, \text{ indexed by } K,$$
$$\mathcal{W}_2 := \{(W_{2,j}, H_{2,j}, \psi_{2,j}) \mid j \in J\}, \text{ indexed by } J,$$
$$\mathcal{W}_2' := \{(W_{2,l}', H_{2,l}', \psi_{2,l}') \mid l \in L\}, \text{ indexed by } L,$$

where $I \subseteq K$ and $J \subseteq L$. Further suppose that

$$\delta_1 = (\mathrm{id}_Q, \{\lambda_{1,i}\}_{i\in I}, [R_1, \sigma_1]) \in \mathrm{Orb}(\mathcal{W}_1, \mathcal{V}),$$
$$\delta_1' = (\mathrm{id}_{Q'}, \{\mu_{1,k}\}_{k\in K}, [R_1', \sigma_1']) \in \mathrm{Orb}(\mathcal{W}_1', \mathcal{V}'),$$
$$\delta_2 = (\mathrm{id}_Q, \{\lambda_{2,j}\}_{j\in J}, [R_2, \sigma_2]) \in \mathrm{Orb}(\mathcal{W}_2, \mathcal{V}),$$
$$\delta_2' = (\mathrm{id}_{Q'}, \{\mu_{2,c}\}_{l\in L}, [R_2', \sigma_2']) \in \mathrm{Orb}(\mathcal{W}_2', \mathcal{V}')$$

are lifts of $\mathrm{id}_{(Q,\mathcal{U})}$ resp. $\mathrm{id}_{(Q',\mathcal{U}')}$ such that

$$\hat{f} \circ \delta_1 = \delta_1' \circ \hat{h} \quad \text{and} \quad \hat{f} \circ \delta_2 = \hat{g} \circ \delta_2'.$$

Further we assume that all $\lambda_{1,i}$, $\mu_{1,b}$, $\lambda_{2,j}$ and $\mu_{2,c}$ are embeddings.

We will use Lemma 5.11 to show the existence of $\hat{k}$. More precisely, we attach to each $q \in Q$ an orbifold chart $(W_q, H_q, \psi_q) \in \mathcal{U}$ with $q \in \psi_q(W_q)$ and an orbifold chart $(W_q', H_q', \psi_q') \in \mathcal{U}'$ with $f(q) \in \psi_q'(W_q')$. We may consider orbifold charts defined for distinct q to be distinct. In this way, we get an orbifold atlas

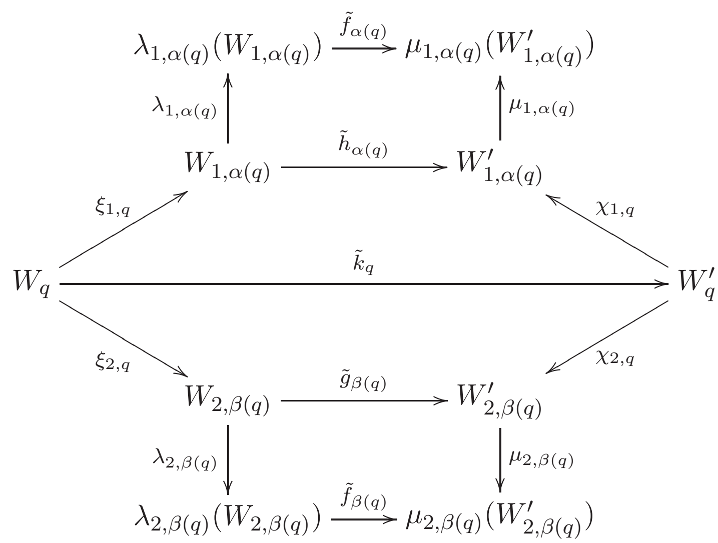
$$\mathcal{W} := \{(W_q, H_q, \psi_q) \mid q \in Q\} \tag{5}$$

of $\mathcal{U}$ which is indexed by Q, and a subset $\{(W_q', H_q', \psi_q') \mid q \in Q\}$ of $\mathcal{U}'$ indexed by Q as well. Moreover, we will find maps $\alpha\colon Q \to I$ and $\beta\colon Q \to J$ and embeddings

$$\xi_{1,q}\colon (W_q, H_q, \psi_q) \to (W_{1,\alpha(q)}, H_{1,\alpha(q)}, \psi_{1,\alpha(q)})$$
$$\xi_{2,q}\colon (W_q, H_q, \psi_q) \to (W_{2,\beta(q)}, H_{2,\beta(q)}, \psi_{2,\beta(q)})$$
$$\chi_{1,q}\colon (W_q', H_q', \psi_q') \to (W_{1,\alpha(q)}', H_{1,\alpha(q)}', \psi_{1,\alpha(q)}')$$
$$\chi_{2,q}\colon (W_q', H_q', \psi_q') \to (W_{2,\beta(q)}', H_{2,\beta(q)}', \psi_{2,\beta(q)}')$$

such that for each $q \in Q$ the local lift $\tilde{k}_q$ of f induced by $\tilde{h}_{\alpha(q)}$, $\xi_{1,q}$ and $\chi_{1,q}$ coincides with the one induced by $\tilde{g}_{\beta(q)}$, $\xi_{2,q}$ and $\chi_{2,q}$. Lemma 5.11 shows that $\hat{h}$ resp. $\hat{g}$ induce a charted orbifold map $(f, \{\tilde{k}_q\}_{q\in Q}, [P_1, \nu_1])$ resp. $(f, \{\tilde{k}_q\}_{q\in Q}, [P_2, \nu_2])$. Then it remains to show that $[P_1, \nu_1] = [P_2, \nu_2]$. For that purpose we will show that one may choose $\chi_{1,q} = \mathrm{id}$ for each $q \in Q$.

Let $q \in Q$. We fix a chart $(W_{1,i}, H_{1,i}, \psi_{1,i}) \in \mathcal{W}_1$ with $q \in \psi_{1,i}(W_{1,i})$ and we pick $w_1 \in W_{1,i}$ with $q = \psi_{1,i}(w_1)$. We set $\alpha(q) := i$. Further we fix a chart $(W_{2,j}, H_{2,j}, \psi_{2,j}) \in \mathcal{W}_2$ with $q \in \psi_{2,j}(W_{2,j})$ and and element $w_2 \in W_{2,j}$ with $q = \psi_{2,j}(w_2)$. We set $\beta(q) := j$. Lemma 5.13 shows the existence of orbifold charts $(W_q, H_q, \psi_q) \in \mathcal{U}$ with $q \in \psi_q(W_q)$, say $q = \psi_q(w_q)$, and $(W'_q, H'_q, \psi'_q) \in \mathcal{U}'$ with $f(q) \in \psi'_q(W'_q)$ and embeddings $\xi_{1,q}$, $\xi_{2,q}$, $\chi_{1,q}$, $\chi_{2,q}$ and a local lift $\tilde{k}_q$ of f such that the diagram



commutes. Now

$$\eta := \lambda_{2,\beta(q)} \circ \xi_{2,q} \circ \xi_{1,q}^{-1} \circ \lambda_{1,\alpha(q)}^{-1} \colon \lambda_{1,\alpha(q)}\big(\xi_{1,q}(W_q)\big) \to \lambda_{2,\beta(q)}\big(\xi_{2,q}(W_q)\big)$$

is an element of $\Psi(\mathcal{V})$ with $y := \lambda_{1,\alpha(q)}(\xi_{1,q}(w_q))$ in its domain. We pick a representative (P_f, ν_f) of $[P_f, \nu_f]$. Then there is $\gamma \in P_f$ with $y \in \operatorname{dom}\gamma$ and an open neighborhood U of y such that $U \subseteq \operatorname{dom}\gamma \cap \operatorname{dom}\eta$ and

$$\eta|_U = \gamma|_U.$$

By (R4a),

$$\nu_f(\gamma) \circ \tilde{f}_{\alpha(q)}|_U = \tilde{f}_{\beta(q)} \circ \gamma|_U = \tilde{f}_{\beta(q)} \circ \eta|_U.$$

The map

$$\mu := \mu_{2,\beta(q)} \circ \chi_{2,q} \circ \chi_{1,q}^{-1} \circ \mu_{1,\alpha(q)}^{-1} \colon \mu_{1,\alpha(q)}\big(\chi_{1,q}(W'_q)\big) \to \mu_{2,\beta(q)}\big(\chi_{2,q}(W'_q)\big)$$

is a diffeomorphism as well. Further there exists an open neighborhood V of y such that

$$\tilde{f}_{\beta(q)} \circ \eta|_V = \mu \circ \tilde{f}_{\alpha(q)}|_V.$$

Hence

$$\nu_f(\gamma) \circ \tilde{f}_{\alpha(q)} = \mu \circ \tilde{f}_{\alpha(q)}$$

on some neighborhood of y. Therefore, after possibly shrinking W_q, we can redefine W'_q, $\chi_{1,q}$, $\chi_{2,q}$ and $\tilde{k}_q$ such that

$$\chi_{2,q} = \mu_{2,\beta(q)}^{-1} \circ \nu_f(\gamma) \circ \mu_{1,\alpha(q)}|_{W'_q} \quad \text{and} \quad \chi_{1,q} := \mathrm{id} \tag{6}$$

and the diagram above remains commutative. We remark that this redefinition might be quite serious if $\tilde{f}_{\alpha(q)}$ and hence $\tilde{h}_{\alpha(q)}$, $\tilde{g}_{\beta(q)}$ and $\tilde{f}_{\beta(q)}$ are of low regularity. But since these maps all have the same regularity, we may perform the changes without running into problems. Let $\mathcal{W}$ be defined by (5). Lemma 5.11, more precisely its proof, shows that $\hat{h}$ resp. $\hat{g}$ induces the orbifold maps

$$\hat{k}_1 = (f, \{\tilde{k}_q\}_{q\in Q}, [P_1, \nu_1]) \quad \text{resp.} \quad \hat{k}_2 = (f, \{\tilde{k}_q\}_{q\in Q}, [P_2, \nu_2])$$

with $(\mathcal{W}, \mathcal{W}')$, where $\mathcal{W}'$ is a representative of $\mathcal{U}'$ which contains the set

$$\{(W'_q, H'_q, \psi'_q) \mid q \in Q\}$$

(the proof of Lemma 5.11 shows that we can indeed have the same $\mathcal{W}'$ for $\hat{k}_1$ and $\hat{k}_2$).

It remains to show that $[P_1, \nu_1] = [P_2, \nu_2]$. Recall from Lemma 5.11 that $[P_1, \nu_1]$ is uniquely determined by $\hat{h}$, $\{\xi_{1,q}\}_{q\in Q}$ and $\{\chi_{1,q}\}_{q\in Q}$, and analogously for $[P_2, \nu_2]$. Alternatively, we may consider $\hat{k}_1$ and $\hat{k}_2$ to be induced by $\hat{f}$. Thus, $[P_1, \nu_1]$ is uniquely determined by $\hat{f}$, $\{\lambda_{1,\alpha(q)} \circ \xi_{1,q}\}_{q\in Q}$ and $\{\mu_{1,\alpha(q)} \circ \chi_{1,q}\}_{q\in Q}$, and $[P_2, \nu_2]$ is uniquely determined by $\hat{f}$, $\{\lambda_{2,\beta(q)} \circ \xi_{2,q}\}_{q\in Q}$ and $\{\mu_{2,\beta(q)} \circ \chi_{2,q}\}_{q\in Q}$. We fix a representative (P_f, ν_f) of $[P_f, \nu_f]$. Let γ be a change of charts in $\Psi(\mathcal{W})$ and $x \in \operatorname{dom}\gamma$. Suppose $\operatorname{dom}\gamma \subseteq W_p$ and $\operatorname{cod}\gamma \subseteq W_q$. Using the same arguments and notation as in the proof of Lemma 5.11 (without discussing the necessary shrinking of domains, since we are only interested in equality in a neighborhood of x) we have

$$\begin{aligned}
\beta_h &= \lambda_{1,\alpha(q)} \circ \xi_{1,q} \circ \gamma \circ \xi_{1,p}^{-1} \circ \lambda_{1,\alpha(p)}^{-1},\\
\beta_g &= \lambda_{2,\beta(q)} \circ \xi_{2,q} \circ \gamma \circ \xi_{2,p}^{-1} \circ \lambda_{2,\beta(p)}^{-1},\\
\nu_1(\gamma) &= \chi_{1,q}^{-1} \circ \mu_{1,\alpha(q)}^{-1} \circ \nu_f(\beta_h) \circ \mu_{1,\alpha(p)} \circ \chi_{1,p},\\
\nu_2(\gamma) &= \chi_{2,q}^{-1} \circ \mu_{2,\beta(q)}^{-1} \circ \nu_f(\beta_g) \circ \mu_{2,\beta(p)} \circ \chi_{2,p}.
\end{aligned}$$

Hence

$$\beta_2 = \lambda_{2,\beta(q)} \circ \xi_{2,q} \circ \lambda_{1,\alpha(q)}^{-1} \circ \xi_{1,q}^{-1} \circ \beta_1 \circ \lambda_{1,\alpha(p)} \circ \xi_{1,p} \circ \xi_{2,p}^{-1} \circ \lambda_{2,\beta(p)}^{-1}.$$

Definition (6) shows that

$$\nu_f(\lambda_{2,\beta(q)} \circ \xi_{2,q} \circ \xi_{1,q}^{-1} \circ \lambda_{1,\alpha(q)}^{-1}) = \mu_{2,\beta(q)} \circ \chi_{2,q} \circ \mu_{1,\alpha(q)}^{-1}.$$

Then

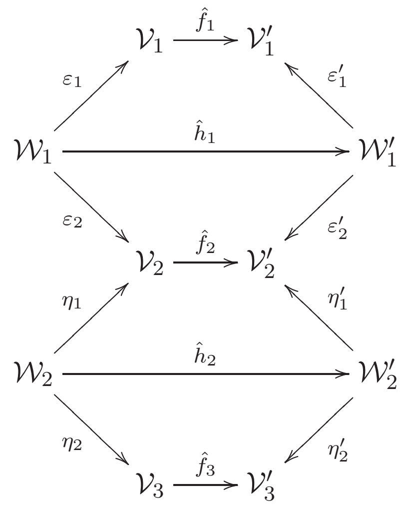
$$\nu_2(\gamma) = \mu_{1,\alpha(q)}^{-1} \circ \nu_f(\beta_h) \circ \mu_{1,\alpha(p)} = \nu_1(\gamma).$$

Hence the induced equivalence classes $[P_1, \nu_1]$ and $[P_2, \nu_2]$ indeed coincide. The lift ε_1 of $\mathrm{id}_{(Q,\mathcal{U})}$ is given by the family $\{\xi_{1,q}\}_{q\in Q}$, the lift ε_2 by $\{\xi_{2,q}\}_{q\in Q}$, the lift ε'_1 of $\mathrm{id}_{(Q',\mathcal{U}')}$ is any extension of $\{\chi_{1,q}\}_{q\in Q}$, and the lift ε'_2 is any extension of $\{\chi_{2,q}\}_{q\in Q}$. □

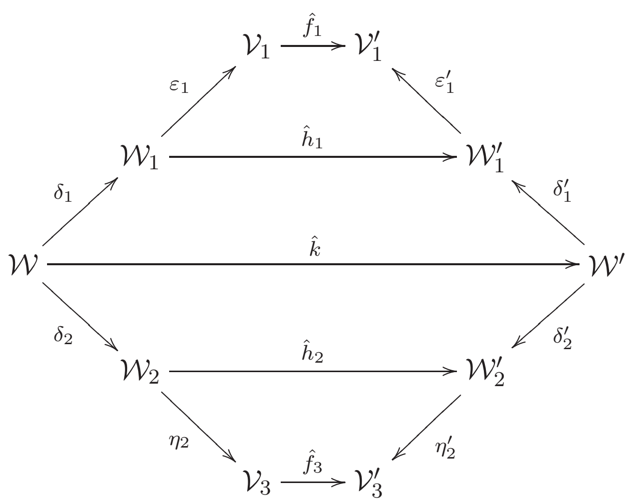
Proposition 5.15. *The relation $\sim$ from Def. 5.12 is an equivalence relation.*

Proof. Let $(Q,\mathcal{U})$ and $(Q',\mathcal{U}')$ be orbifolds. Suppose that for all $i \in \{1,2,3\}$ the orbifold atlases $\mathcal{V}_i$ are representatives of $\mathcal{U}$ and $\mathcal{V}'_i$ are representatives of $\mathcal{U}'$, and $\hat{f}_i \in \mathrm{Orb}(\mathcal{V}_i, \mathcal{V}'_i)$ are charted orbifold maps such that $\hat{f}_1 \sim \hat{f}_2$ and $\hat{f}_2 \sim \hat{f}_3$. This means that we find representatives $\mathcal{W}_1$, $\mathcal{W}_2$ of $\mathcal{U}$, representatives $\mathcal{W}'_1$, $\mathcal{W}'_2$ of $\mathcal{U}'$, charted orbifold maps $\hat{h}_1 \in \mathrm{Orb}(\mathcal{W}_1, \mathcal{W}'_1)$, $\hat{h}_2 \in \mathrm{Orb}(\mathcal{W}_2, \mathcal{W}'_2)$ and lifts of the respective identities $\varepsilon_1 \in \mathrm{Orb}(\mathcal{W}_1, \mathcal{V}_1)$, $\varepsilon_2 \in \mathrm{Orb}(\mathcal{W}_1, \mathcal{V}_2)$, $\varepsilon'_1 \in \mathrm{Orb}(\mathcal{W}'_1, \mathcal{V}'_1)$, $\varepsilon'_2 \in \mathrm{Orb}(\mathcal{W}'_1, \mathcal{V}'_2)$, $\eta_1 \in \mathrm{Orb}(\mathcal{W}_2, \mathcal{V}_2)$, $\eta_2 \in \mathrm{Orb}(\mathcal{W}_2, \mathcal{V}_3)$, $\eta'_1 \in \mathrm{Orb}(\mathcal{W}'_2, \mathcal{V}'_2)$,

$\eta_2' \in \mathrm{Orb}(\mathcal{W}_2', \mathcal{V}_3')$ such that the diagram



commutes. Since $\hat{h}_1$ and $\hat{h}_2$ are both induced by $\hat{f}_2$, Lemma 5.14 shows that there are representatives $\mathcal{W}$ of $\mathcal{U}$, $\mathcal{W}'$ of $\mathcal{U}'$, a charted orbifold map $\hat{k} \in \mathrm{Orb}(\mathcal{W}, \mathcal{W}')$ and lifts of identity $\delta_1 \in \mathrm{Orb}(\mathcal{W}, \mathcal{W}_1)$, $\delta_2 \in \mathrm{Orb}(\mathcal{W}, \mathcal{W}_2)$, $\delta_1' \in \mathrm{Orb}(\mathcal{W}', \mathcal{W}_1')$, $\delta_2' \in \mathrm{Orb}(\mathcal{W}', \mathcal{W}_2')$ such that the diagram



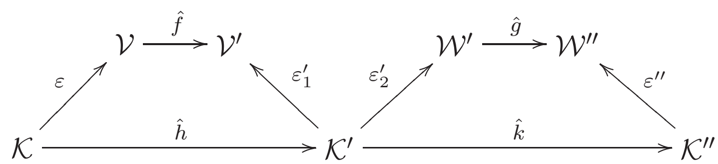
commutes. Since compositions of lifts of identity remain lifts of identity, it follows that $\hat{f}_1 \sim \hat{f}_3$. □

5.3. **The orbifold category.** Now we can define the category of reduced orbifolds.

Definition 5.16. The category Orb of reduced orbifolds is defined as follows: Its class of objects is the class of orbifolds. For two orbifolds $(Q, \mathcal{U})$ and $(Q', \mathcal{U}')$ the morphisms from $(Q, \mathcal{U})$ to $(Q', \mathcal{U}')$ are the equivalence classes $[\hat{f}]$ of charted orbifold maps $\hat{f} \in \mathrm{Orb}(\mathcal{V}, \mathcal{V}')$ where $\mathcal{V}$ is any representative of $\mathcal{U}$, and $\mathcal{V}'$ is any representative of $\mathcal{U}'$, that is

$$\begin{aligned}&\operatorname{Morph}\big((Q,\mathcal{U}),(Q',\mathcal{U}')\big) := \\ &\quad := \Big\{ [\hat f] \;\Big|\; \hat f \in \operatorname{Orb}(\mathcal{V},\mathcal{V}'),\ \mathcal{V} \text{ representative of } \mathcal{U},\ \mathcal{V}' \text{ representative of } \mathcal{U}' \Big\}.\end{aligned}$$

The composition is described in the following. Let $[\hat f] \in \operatorname{Morph}((Q,\mathcal{U}),(Q',\mathcal{U}'))$ and $[\hat g] \in \operatorname{Morph}((Q',\mathcal{U}'),(Q'',\mathcal{U}''))$. Choose representatives $\hat f \in \operatorname{Orb}(\mathcal{V},\mathcal{V}')$ of $[\hat f]$ and $\hat g \in \operatorname{Orb}(\mathcal{W}',\mathcal{W}'')$ of $[\hat g]$. Then find representatives $\mathcal{K}$, $\mathcal{K}'$, $\mathcal{K}''$ of $\mathcal{U}, \mathcal{U}', \mathcal{U}''$, resp., and lifts of identity $\varepsilon \in \operatorname{Orb}(\mathcal{K},\mathcal{V})$, $\varepsilon_1' \in \operatorname{Orb}(\mathcal{K}',\mathcal{V}')$, $\varepsilon_2' \in \operatorname{Orb}(\mathcal{K}',\mathcal{W}')$, $\varepsilon'' \in \operatorname{Orb}(\mathcal{K}'',\mathcal{W}'')$ and charted orbifold maps $\hat h \in \operatorname{Orb}(\mathcal{K},\mathcal{K}')$, $\hat k \in \operatorname{Orb}(\mathcal{K}',\mathcal{K}'')$ such that the diagram

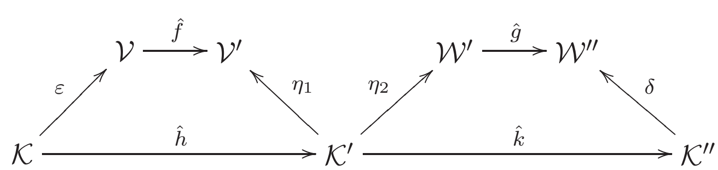


commutes. The composition of $[\hat g]$ and $[\hat f]$ is defined to be

$$[\hat g] \circ [\hat f] := [\hat k \circ \hat h].$$

We have to prove that the composition in the category of reduced orbifolds is always possible and well-defined. This means that we have to show that the induced charted orbifold maps $\hat h$ and $\hat k$ indeed exist and that the composition does not depend on the choice of the representatives $\hat f$ and $\hat g$, and neither on the choice of $\mathcal{K}$, $\mathcal{K}'$, $\mathcal{K}''$, $\hat h$ or $\hat k$. The existence is shown by the following lemma, the independence of the choices by Prop. 5.18 below.

Lemma 5.17. *Let $(Q,\mathcal{U})$, $(Q',\mathcal{U}')$ and $(Q'',\mathcal{U}'')$ be orbifolds. Further let $\mathcal{V}$ be a representative of $\mathcal{U}$, $\mathcal{V}'$ and $\mathcal{W}'$ be representatives of $\mathcal{U}'$, and $\mathcal{W}''$ a representative of $\mathcal{U}''$. Suppose that $\hat f \in \operatorname{Orb}(\mathcal{V},\mathcal{V}')$ and $\hat g \in \operatorname{Orb}(\mathcal{W}',\mathcal{W}'')$. Then there exist representatives $\mathcal{K}$ of $\mathcal{U}$, $\mathcal{K}'$ of $\mathcal{U}'$, $\mathcal{K}''$ of $\mathcal{U}''$, lifts of the respective identities $\varepsilon \in \operatorname{Orb}(\mathcal{K},\mathcal{V})$, $\eta_1 \in \operatorname{Orb}(\mathcal{K}',\mathcal{V}')$, $\eta_2 \in \operatorname{Orb}(\mathcal{K}',\mathcal{W}')$, $\delta \in \operatorname{Orb}(\mathcal{K}'',\mathcal{W}'')$, and charted orbifold maps $\hat h \in \operatorname{Orb}(\mathcal{K},\mathcal{K}')$, $\hat k \in \operatorname{Orb}(\mathcal{K}',\mathcal{K}'')$ such that the diagram*



commutes.

Proof. To fix notation suppose that

$$\begin{aligned}\hat f &= \big(f, \{\tilde f_i\}_{i\in I}, [P_f, \nu_f]\big),\\ \hat g &= \big(g, \{\tilde g_j\}_{j\in J}, [P_g, \nu_g]\big).\end{aligned}$$

Further suppose that

$$\begin{aligned}\mathcal{V} &= \{(V_i, G_i, \pi_i) \mid i \in I\}, \text{ indexed by } I,\\ \mathcal{V}' &= \{(V'_c, G'_c, \pi'_c) \mid c \in C\}, \text{ indexed by } C \text{ with } I \subseteq C,\\ \mathcal{W}' &= \{(W'_j, H'_j, \psi'_j) \mid j \in J\}, \text{ indexed by } J,\\ \mathcal{W}'' &= \{(W''_d, H''_d, \psi''_d) \mid d \in D\}, \text{ indexed by } D \text{ with } J \subseteq D.\end{aligned}$$

By Lemma 5.11 it suffices to find

- a representative $\mathcal{K} = \{(K_a, L_a, \chi_a) \mid a \in A\}$ of $\mathcal{U}$, indexed by A,
- a representative $\mathcal{K}' = \{(K'_b, L'_b, \chi'_b) \mid b \in B\}$ of $\mathcal{U}'$, indexed by B,
- a set $\{(K''_b, L''_b, \chi''_b) \mid b \in B\}$ of orbifold charts of Q'', indexed by B,
- a map $\alpha\colon A \to I$,
- an injective map $\beta\colon A \to B$,
- for each $a \in A$, an embedding
$$\lambda_a\colon (K_a, L_a, \chi_a) \to (V_{\alpha(a)}, G_{\alpha(a)}, \pi_{\alpha(a)})$$
and an embedding
$$\mu_a\colon (K'_{\beta(a)}, L'_{\beta(a)}, \chi'_{\beta(a)}) \to (V'_{\alpha(a)}, G'_{\alpha(a)}, \pi'_{\alpha(a)})$$
such that
$$\tilde{f}_{\alpha(a)}\big(\lambda_a(K_a)\big) \subseteq \mu_a(K'_{\beta(a)}),$$
- a map $\gamma\colon B \to J$,
- for each $b \in B$, an embedding
$$\varrho_b\colon (K'_b, L'_b, \chi'_b) \to (W'_{\gamma(b)}, H'_{\gamma(b)}, \psi'_{\gamma(b)})$$
and an embedding
$$\sigma_b\colon (K''_b, L''_b, \chi''_b) \to (W''_{\gamma(b)}, H''_{\gamma(b)}, \psi''_{\gamma(b)})$$
such that
$$\tilde{g}_{\gamma(b)}\big(\varrho_b(K'_b)\big) \subseteq \sigma_b(K''_b).$$

Let $q \in Q$ and set $r := f(q)$. We fix $i \in I$ and $j \in J$ such that $q \in \pi_i(V_i)$ and $r \in \psi'_j(W'_j)$. Further we choose $v' \in V'_i$ and $w' \in W'_j$ such that $\pi'_i(v') = r = \psi'_j(w')$. By compatibility of orbifold charts we find a restriction (K'_q, L'_q, χ'_q) of (V'_i, G'_i, π'_i) with $v' \in K'_q$ and an embedding

$$\varrho_q\colon (K'_q, L'_q, \chi'_q) \to (W'_j, H'_j, \psi'_j).$$

Since $\tilde{f}_i$ is continuous, there is a restriction (K_q, L_q, χ_q) of (V_i, G_i, π_i) such that $q \in \chi_q(K_q)$ and $\tilde{f}_i(K_q) \subseteq K'_q$. We set $(K''_q, L''_q, \chi''_q) := (W''_j, H''_j, \psi''_j)$. We may and shall consider charts constructed for distinct q to be distinct. Then we set

$$A := Q, \quad \alpha(q) := i, \quad \lambda_q := \mathrm{id}, \quad \mu_q := \mathrm{id},$$

$$B := Q \cup Q' \setminus f(Q) \ \text{ (disjoint union)}, \quad \beta(q) := q, \quad \gamma(q) := j, \quad \sigma_q := \mathrm{id}\,.$$

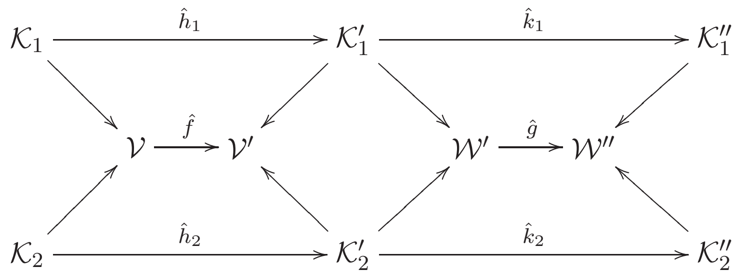
For $q' \in Q' \setminus f(Q)$ we fix $j \in J$ with $q' \in \psi'_j(W'_j)$ and set $\gamma(q') := j$. Further we set $(K'_{q'}, L'_{q'}, \chi'_{q'}) := (W'_j, H'_j, \psi'_j)$ and $(K''_{q'}, L''_{q'}, \chi''_{q'}) := (W''_j, H''_j, \psi''_j)$. Again we consider orbifold charts build for distinct q' to be distinct and define $\varrho_{q'} := \mathrm{id}$ and $\sigma_{q'} := \mathrm{id}$. Then all requirements are satisfied. □

Proposition 5.18. *The composition in* Orb *is well-defined.*

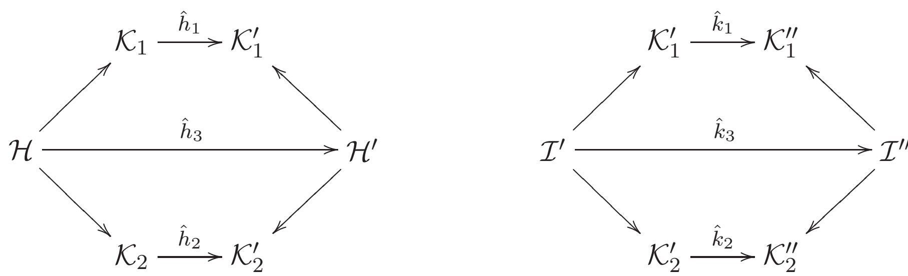
Proof. Recall the notation from the definition of the composition. We have to show that the composition of $[\hat{f}]$ and $[\hat{k}]$ does not depend on the choice

(i) of the induced orbifold maps $\hat{h}$ and $\hat{k}$,
(ii) of the representatives of $[\hat{f}]$ and $[\hat{g}]$.

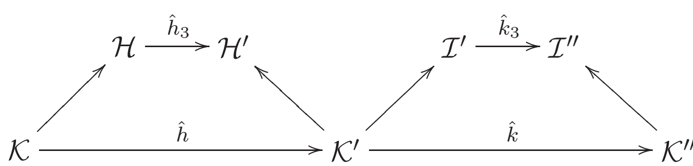
To prove (i) suppose that we have two pairs $(\hat{h}_j, \hat{k}_j)$ of induced orbifold maps $\hat{h}_j \in \mathrm{Orb}(\mathcal{K}_j, \mathcal{K}'_j)$, $\hat{k}_j \in \mathrm{Orb}(\mathcal{K}'_j, \mathcal{K}''_j)$ $(j = 1, 2)$ such that the diagram



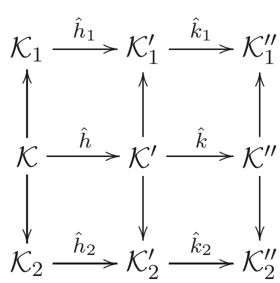
commutes. The non-horizontal maps are lifts of identity. Lemma 5.14 shows the existence of representatives $\mathcal{H}$ of $\mathcal{U}$, $\mathcal{H}', \mathcal{I}'$ of $\mathcal{U}'$, $\mathcal{I}''$ of $\mathcal{U}''$, and charted orbifold maps $\hat{h}_3 \in \mathrm{Orb}(\mathcal{H}, \mathcal{H}')$, $\hat{k}_3 \in \mathrm{Orb}(\mathcal{I}', \mathcal{I}'')$, and appropriate lifts of identity such that the diagrams



commute. By Lemma 5.17 we find representatives $\mathcal{K}, \mathcal{K}', \mathcal{K}''$ of $\mathcal{U}, \mathcal{U}', \mathcal{U}''$, resp., charted orbifold maps $\hat{h} \in \mathrm{Orb}(\mathcal{K}, \mathcal{K}')$, $\hat{k} \in \mathrm{Orb}(\mathcal{K}', \mathcal{K}'')$, and appropriate lifts of identity such that

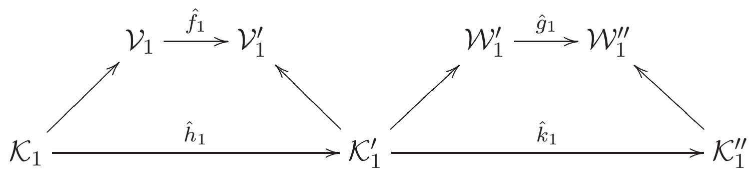


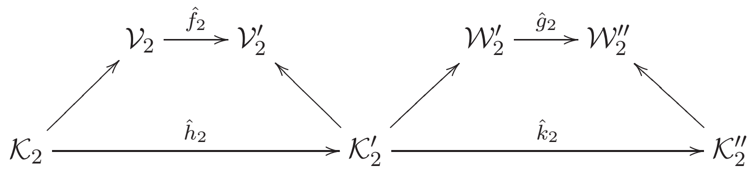
commutes. Hence, altogether we have the commutative diagram



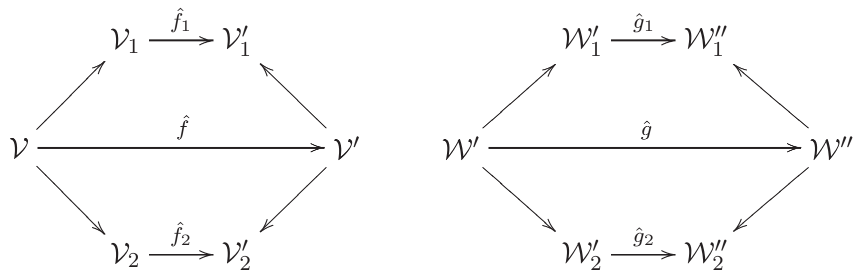
which shows that $\hat{k}_1 \circ \hat{h}_1$ and $\hat{k}_2 \circ \hat{h}_2$ are equivalent.

For the proof of (ii) let $\hat{f}_1 \in \mathrm{Orb}(\mathcal{V}_1, \mathcal{V}_1')$, $\hat{f}_2 \in \mathrm{Orb}(\mathcal{V}_2, \mathcal{V}_2')$ be representatives of $[\hat{f}]$, and $\hat{g}_1 \in \mathrm{Orb}(\mathcal{W}_1', \mathcal{W}_1'')$, $\hat{g}_2 \in \mathrm{Orb}(\mathcal{W}_2', \mathcal{W}_2'')$ be representatives of $[\hat{g}]$. Further, for $j = 1, 2$, let $\hat{h}_j \in \mathrm{Orb}(\mathcal{K}_j, \mathcal{K}_j')$ be induced by $\hat{f}_j$, and $\hat{k}_j \in \mathrm{Orb}(\mathcal{K}_j', \mathcal{K}_j'')$ be induced by $\hat{g}_j$:

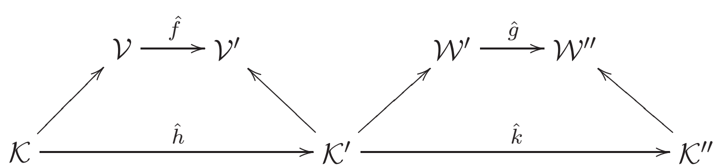




Since $\hat{f}_1$ and $\hat{f}_2$ are equivalent, we find representatives $\mathcal{V}$, $\mathcal{V}'$ of $\mathcal{U}, \mathcal{U}'$, resp., and a charted orbifold map $\hat{f} \in \mathrm{Orb}(\mathcal{V}, \mathcal{V}')$, and analogously for $\hat{g}_1$ and $\hat{g}_2$, such that the diagrams



commute. Lemma 5.17 yields the existence of $\hat{h} \in \mathrm{Orb}(\mathcal{K}, \mathcal{K}')$ and $\hat{k} \in \mathrm{Orb}(\mathcal{K}', \mathcal{K}'')$ such that



commutes. Since $\hat{h}$ is induced by $\hat{f}_1$ and by $\hat{f}_2$, and likewise, $\hat{k}$ is induced by $\hat{g}_1$ and by $\hat{g}_2$, part (i) shows that $\hat{k}_1 \circ \hat{h}_1$ and $\hat{k}_2 \circ \hat{h}_2$ are both equivalent to $\hat{k} \circ \hat{h}$. This yields that the composition map is well-defined. □

We end this section with a discussion of the equivalence class represented by a lift of identity. The following proposition shows that it is precisely the class of all lifts of identity of the considered orbifold. This justifies the notion "identity morphism" in Def. 5.8.

Proposition 5.19. *Let $(Q, \mathcal{U})$ be an orbifold and ε a lift of $\mathrm{id}_{(Q,\mathcal{U})}$. Then the equivalence class $[\varepsilon]$ of ε consists precisely of all lifts of $\mathrm{id}_{(Q,\mathcal{U})}$.*

Proof. Let $\varepsilon_1 \in \mathrm{Orb}(\mathcal{V}_1, \mathcal{W}_1)$ and $\varepsilon_2 \in \mathrm{Orb}(\mathcal{V}_2, \mathcal{W}_2)$ be two lifts of $\mathrm{id}_{(Q,\mathcal{U})}$. Prop. 5.3 (together with Prop. 5.5) implies that there is a representative $\mathcal{V}$ of $\mathcal{U}$ such that ε_1 and ε_2 both induce the orbifold map

$$\widehat{\mathrm{id}}_Q := (\mathrm{id}_Q, \{\mathrm{id}_{V_i}\}_{i\in I}, [R,\sigma])$$

on $\mathcal{V}$. Thus, each two lifts of $\mathrm{id}_{(Q,\mathcal{U})}$ are equivalent.

Let now $\hat{f}$ be a charted orbifold map which is equivalent zu ε. W.l.o.g. we may assume that $\varepsilon = \widehat{\mathrm{id}}_Q$. To fix notation let

$$\begin{aligned}
\mathcal{V} &= \{(V_i, G_i, \pi_i) \mid i \in I\}, \text{indexed by } I,\\
\mathcal{K}_1 &= \{(K_{1,a}, L_{1,a}, \chi_{1,a}) \mid a \in A\},\ \text{indexed by } A,\\
\mathcal{K}_2 &= \{(K_{2,b}, L_{2,b}, \chi_{2,b}) \mid b \in B\},\ \text{indexed by } B,\\
\mathcal{W}_1 &= \{(W_{1,j}, H_{1,j}, \psi_{1,j}) \mid j \in J\},\ \text{indexed by } J,\\
\mathcal{W}_2 &= \{(W_{2,k}, H_{2,k}, \psi_{2,k}) \mid k \in K\},\ \text{indexex by } K,
\end{aligned}$$

be representatives of $\mathcal{U}$. Further let

$$\alpha\colon A \to I, \quad \beta\colon A \to J, \quad \gamma\colon A \to B$$

and

$$\delta\colon B \to I, \quad \eta\colon B \to K, \quad \zeta\colon J \to K$$

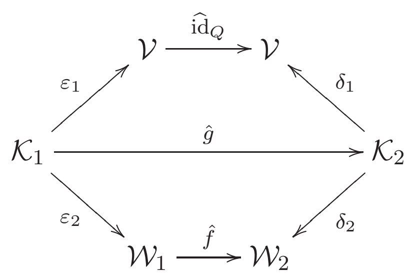
be maps, and suppose that

$$\begin{aligned}
\hat{f} &= \big(f, \{\tilde{f}_j\}_{j\in J}, [P_f, \nu_f]\big)\\
\hat{g} &= \big(g, \{\tilde{g}_a\}_{a\in A}, [P_g, \nu_g]\big)
\end{aligned}$$

are charted orbifold maps and

$$\begin{aligned}
\varepsilon_1 &= \big(\mathrm{id}_Q, \{\lambda_{1,a}\}_{a\in A}, [P_1, \nu_1]\big)\\
\varepsilon_2 &= \big(\mathrm{id}_Q, \{\lambda_{2,a}\}_{a\in A}, [P_2, \nu_2]\big)\\
\delta_1 &= \big(\mathrm{id}_Q, \{\mu_{1,b}\}_{b\in B}, [R_1, \sigma_1]\big)\\
\delta_2 &= \big(\mathrm{id}_Q, \{\mu_{2,b}\}_{b\in B}, [R_2, \sigma_2]\big)
\end{aligned}$$

are lifts of $\mathrm{id}_{(Q,\mathcal{U})}$ such that the diagram



commutes. Clearly, $g = \mathrm{id}_Q$ and hence $f = \mathrm{id}_Q$. Further for each $a \in A$, we have

$$\mathrm{id}_{V_{\alpha(a)}} \circ \lambda_{1,a} = \mu_{1,\gamma(a)} \circ \tilde{g}_a.$$

Since $\mathrm{id}_{V_{\alpha(a)}}$, $\lambda_{1,a}$ and $\mu_{1,\gamma(a)}$ are local diffeomorphisms, so is $\tilde{g}_a$. Now

$$\tilde{f}_{\beta(a)} \circ \lambda_{2,a} = \mu_{2,\gamma(a)} \circ \tilde{g}_a$$

for each $a \in A$. Hence $\tilde{f}_{\beta(a)}$ is a local diffeomorphism. Lemma 5.13 implies that $\tilde{f}_j$ is a local diffeomorphism for each $j \in J$. Therefore, $\hat{f}$ is a lift of $\mathrm{id}_{(Q,\mathcal{U})}$. □

6. The orbifold category in terms of marked atlas groupoids

Prop. 4.14 and Remark 5.10 show that charted orbifold maps and their composition correspond to homomorphisms between marked atlas groupoids and their composition. By characterizing lifts of identity and equivalence of charted orbifold maps in terms of marked atlas groupoids and their homomorphisms, we construct a category for marked atlas groupoids which is isomorphic to the one of reduced orbifolds. To that end we first show that lifts of identity correspond to unit weak equivalences, a notion we define below. Throughout this section let pr_1 denote the projection to the first component.

A homomorphism $\varphi = (\varphi_0, \varphi_1)\colon G \to H$ between Lie groupoids is called a *weak equivalence* if

(i) the map
$$t \circ \mathrm{pr}_1 \colon H\ {}_s\!\times_{\varphi_0} G_0 \to H_0$$
is a surjective submersion, and

(ii) the diagram
$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi_1} & H_1 \\ \big\downarrow{\scriptstyle (s,t)} & & \big\downarrow{\scriptstyle (s,t)} \\ G_0 \times G_0 & \xrightarrow{\varphi_0 \times \varphi_0} & H_0 \times H_0 \end{array}$$
is a fibered product.

Two Lie groupoids G, H are called *Morita equivalent* if there is a Lie groupoid K and weak equivalences
$$G \xleftarrow{\varphi} K \xrightarrow{\psi} H.$$

Definition 6.1. Let (G_1, α_1, X_1) and (G_2, α_2, X_2) be marked atlas groupoids. A homomorphism $\varphi = (\varphi_0, \varphi_1)\colon (G_1, \alpha_1, X_1) \to (G_2, \alpha_2, X_2)$ is called a *unit weak equivalence* if $\varphi\colon G_1 \to G_2$ is a weak equivalence and $\alpha_2 \circ |\varphi| \circ \alpha_1^{-1} = \mathrm{id}_{X_1}$:
$$\begin{array}{ccc} (G_1)_0 & \xrightarrow{\varphi_0} & (G_2)_0 \\ \big\downarrow{\scriptstyle \mathrm{pr}_{G_1}} & & \big\downarrow{\scriptstyle \mathrm{pr}_{G_2}} \\ |G_1| & \xrightarrow{|\varphi|} & |G_2| \\ \big\downarrow{\scriptstyle \alpha_1} & & \big\downarrow{\scriptstyle \alpha_2} \\ X_1 & = & X_2 \end{array}$$

Hence necessarily, $X_1 = X_2 =: X$. A *unit Morita equivalence* between (G_1, α_1, X) and (G_2, α_2, X) is a pair (ψ_1, ψ_2) of unit weak equivalences
$$\psi_j \colon (G, \alpha, X) \to (G_j, \alpha_j, X)$$
where (G, α, X) is some marked atlas groupoid. If such a unit Morita equivalence exists, then (G_1, α_1, X) and (G_2, α_2, X) are called *unit Morita equivalent*.

In contrast to Morita equivalence of Lie groupoids, unit Morita equivalence of marked atlas groupoids requires the third (marked) Lie groupoid to be an atlas groupoid. In Prop. 6.4 below we will show that unit Morita equivalence of marked atlas groupoids is indeed an equivalence relation.

The proof that lifts of identity correspond to unit weak equivalences needs the following lemma. Recall that a groupoid is étale if its source and target map are local diffeomorphisms.

Lemma 6.2. *Let G, H be étale groupoids and $\varphi = (\varphi_0, \varphi_1)\colon G \to H$ a weak equivalence. Then $\varphi_0\colon G_0 \to H_0$ is a local diffeomorphism.*

Proof. We show first that φ_0 is a submersion. Let $(h, x) \in H_1\, {}_s{\times}_{\varphi_0} G_0$. Choose an open neighborhood U_h of h in H_1 such that $s|_{U_h}$ and $t|_{U_h}$ are open embeddings. Further choose an open neighborhood U_x of x in G_0 such that $\varphi_0(U_x) \subseteq s(U_h)$. Then $U_h\, {}_s{\times}_{\varphi_0} U_x$ is an open subset of $H_1\, {}_s{\times}_{\varphi_0} G_0$. By definition,

$$t \circ \mathrm{pr}_1\colon H_1\, {}_s{\times}_{\varphi_0} G_0 \to H_1$$

is a submersion, hence also its restriction

$$t \circ \mathrm{pr}_1\colon U_h\, {}_s{\times}_{\varphi_0} U_x \to t(U_h).$$

Since

$$s \circ t^{-1}\colon t(U_h) \to s(U_h)$$

is by construction a diffeomorphism, the map

$$s \circ \mathrm{pr}_1 = s \circ t^{-1} \circ t \circ \mathrm{pr}_1\colon U_h\, {}_s{\times}_{\varphi_0} U_x \to s(U_h)$$

is a submersion. From the commutativity of

$$\begin{array}{ccc} U_h\, {}_s{\times}_{\varphi_0} U_x & \xrightarrow{\mathrm{pr}_2} & U_x \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \varphi_0} \\ U_h & \xrightarrow{s} & s(U_h) \end{array}$$

it follows that

$$\varphi_0 \circ \mathrm{pr}_2\colon U_h\, {}_s{\times}_{\varphi_0} U_x \to s(U_h)$$

is a submersion. For a manifold M and an element $m \in M$ let $T_m M$ denote the tangent space to M at m. Since $(1_x, x) \in H_1\, {}_s{\times}_{\varphi_0} G_0$ for each $x \in G_0$, we know that

$$\varphi_0'(x) \circ \mathrm{pr}_2'(1_x, x) = (\varphi_0 \circ \mathrm{pr}_2)'(1_x, x)\colon T_{(1_x,x)} H_1\, {}_s{\times}_{\varphi_0} G_0 \to T_{\varphi_0(x)} H_0$$

is surjective. In particular,

$$\varphi_0'(x)\colon T_x G_0 \to T_{\varphi_0(x)} H_0$$

is surjective for each $x \in G_0$. This shows that φ_0 is a submersion and

$$m := \dim G_0 \geq \dim H_0 =: n.$$

We now prove that $m = n$. Set

$$M := (G_0 \times G_0)\, {}_{\varphi_0 \times \varphi_0}{\times}_{(s,t)}\, H_1.$$

From φ being a weak equivalence, we know that the map

$$\beta\colon \begin{cases} G_1 & \to \quad M \\ g & \mapsto \quad (s(g), t(g), \varphi_1(g)) \end{cases}$$

is a diffeomorphism. Let $(x,y,h) \in M$. The submersion theorem (cf. [BG05, Thm. 1.9.11]) shows that there are (manifold) charts (U_1,ψ_1) with $x \in \psi_1(U_1)$, (U_2,ψ_2) with $y \in \psi_2(U_2)$, (V_1,χ_1) with $\varphi_0(x) \in \chi_1(V_1)$ and (V_2,χ_2) with $\varphi_0(y) \in \chi_2(V_2)$ such that

$$\eta_i := \chi_i \circ \varphi_0 \circ \psi_i^{-1} \colon (x_1,\ldots,x_n,x_{n+1},\ldots,x_m) \mapsto (x_1,\ldots,x_n)$$

for $i=1,2$. Now we choose an open neighborhood U_h of h in H_1 such that $s|_{U_h}$ and $t|_{U_h}$ are open embeddings, and such that $s(U_h) \subseteq V_1$ and $t(U_h) \subseteq V_2$. By shrinking the charts and U_h we can achieve that $s(U_h) = V_1$, $t(U_h) = V_2$ and

$$\psi_2(U_2) = V_2 = I \times J$$

with $\dim I = n$, $\dim J = m-n$. The open subset

$$(U_1 \times U_2) \;{}_{\varphi_0\times\varphi_0}\!\times_{(s,t)} U_h$$

of M reads in local coordinates as

$$\begin{aligned} N &:= (\psi_1(U_1) \times \psi_2(U_2)) \;{}_{\eta_1\times\eta_2}\!\times_{(\chi_1\circ s,\chi_2\circ t)} U_h \\ &= \{(z,w,k) \in V_1 \times V_2 \times U_h \mid (\eta_1(z),\eta_2(w)) = (\chi_2(s(k)),\chi_1(t(k))\} \\ &= \{(z,w,k) \in V_1 \times V_2 \times U_h \mid ((z_1,\ldots,z_n),(w_1,\ldots,w_n)) = (\chi_1(s(k)),\chi_2(t(k)))\} \\ &= \{ \big((z_1,\ldots,z_n,\tilde z),(\chi_2(t(s^{-1}(\chi_1^{-1}(z_1,\ldots,z_n)))),\tilde w), s^{-1}(\chi_1^{-1}(z_1,\ldots,z_n))\big) \mid \\ &\qquad\qquad \mid (z_1,\ldots,z_n,\tilde z) \in V_1, \tilde w \in J\} \end{aligned}$$

since k and $(w_1,\ldots,w_n)$ are determined by $(z_1,\ldots,z_n)$. Therefore

$$\dim M = \dim N = n + 2(m-n).$$

On the other hand,

$$\dim M = \dim G_1 = n,$$

and so $m = n$. Hence φ_0 is a submersion between manifolds of same dimension, which implies that it is a local diffeomorphism. □

Recall the maps F_1 and F_2 from Prop. 4.14.

Proposition 6.3. *Let $\mathcal{U}$ and $\mathcal{U}'$ be orbifold structures on the topological space Q. Further let*

$$\mathcal{V} = \{(V_i,G_i,\pi_i) \mid i \in I\} \quad \textit{resp.} \quad \mathcal{W}' = \{(W'_j,H'_j,\psi'_j) \mid j \in J\}$$

be a representative of $\mathcal{U}$ resp. of $\mathcal{U}'$, indexed by I resp. by J.

- (i) *Suppose that $\mathcal{U} = \mathcal{U}'$. Let $\hat f = (\mathrm{id}_Q, \{\tilde f_i\}_{i\in I}, [P,\nu]) \in \mathrm{Orb}(\mathcal{V},\mathcal{W}')$ be a lift of $\mathrm{id}_{(Q,\mathcal{U})}$. Then $F_1(\hat f)\colon (\Gamma(\mathcal{V}),\alpha_{\mathcal{V}},Q) \to (\Gamma(\mathcal{W}'),\alpha_{\mathcal{W}'},Q)$ is a unit weak equivalence.*
- (ii) *Let $\varepsilon \in \mathrm{Hom}(\Gamma(\mathcal{V}),\Gamma(\mathcal{W}'))$ be a unit weak equivalence. Then $\mathcal{U} = \mathcal{U}'$, and $F_2(\varepsilon)$ is a lift of $\mathrm{id}_{(Q,\mathcal{U})}$.*

Proof. Let $G := \Gamma(\mathcal{V})$ and $H := \Gamma(\mathcal{W}')$. We will first prove (i). By Prop. 4.7 it suffices to show that $\varepsilon = (\varepsilon_0,\varepsilon_1) := F_1(\hat f)$ is a weak equivalence. We first show that

$$t \circ \mathrm{pr}_1 \colon \begin{cases} H_1 \,{}_s\!\times_{\varepsilon_0} G_0 & \to \quad H_0 \\ \quad (h,x) & \mapsto \quad t(h) \end{cases}$$

is a submersion. Let $(h,x) \in H_1 \,{}_s\!\times_{\varepsilon_0} G_0$. Recall from Prop. 5.3 that ε_0 is a local diffeomorphism, and from Special Case 2.12 that G and H are étale groupoids.

Choose open neighborhoods U_x of x in G_0 and U_h of h in H_1 such that $\varepsilon_0|_{U_x}$ and $s|_{U_h}$ are open embeddings with $s(U_h) = \varepsilon_0(U_x)$. Then $U_h\,{}_s\!\times_{\varepsilon_0} U_x$ is open in $H_1\,{}_s\!\times_{\varepsilon_0} G_0$. Further

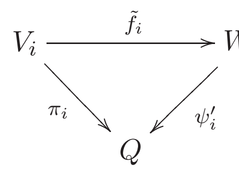
$$\begin{aligned} U_h\,{}_s\!\times_{\varepsilon_0} U_x &= \{(k,y) \in U_h \times U_x \mid s(k) = \varepsilon_0(y)\} \\ &= \left\{(k, \varepsilon_0^{-1}(s(k)) \mid k \in U_h \right\}. \end{aligned}$$

Therefore,

$$\mathrm{pr}_1 \colon U_h\,{}_s\!\times_{\varepsilon_0} U_x \to U_h$$

is a diffeomorphism. Since t is a local diffeomorphism, $t \circ \mathrm{pr}_1$ is a submersion.

Now we prove that $t \circ \mathrm{pr}_1$ is surjective. Let $y \in H_0$, say $y \in W_j'$, and set $\psi_j'(y) =: q \in Q$. Then there is an orbifold chart $(V_i, G_i, \pi_i) \in \mathcal{V}$ such that $q \in \pi_i(V_i)$, say $q = \pi_i(x)$.



Set $z := \tilde{f}_i(x)$, hence $\psi_i'(z) = q = \psi_j'(y)$. Hence, there are a restriction (S', K', χ') of (W_i', H_i', ψ_i') with $z \in S'$ and an embedding

$$\lambda \colon (S', K', \chi') \to (W_j', H_j', \psi_j')$$

such that $\lambda(z) = y$. Then $\lambda \in \Psi(\mathcal{W}')$ and $(\mathrm{germ}_z \lambda, x) \in H_1\,{}_s\!\times_{\varepsilon_0} G_0$ with $t \circ \mathrm{pr}_1(\mathrm{germ}_z \lambda, x) = t(\mathrm{germ}_z \lambda) = y$. This means that $t \circ \mathrm{pr}_1$ is surjective.

Set

$$K := (G_0 \times G_0)\,{}_{(\varepsilon_0,\varepsilon_0)}\!\times_{(s,t)} H_1.$$

It remains to show that the map

$$\beta \colon \begin{cases} G_1 & \to \quad K \\ \mathrm{germ}_x g & \mapsto \quad (x, g(x), \varepsilon_1(\mathrm{germ}_x g)) \end{cases}$$

is a diffeomorphism. Note that $\beta = (s, t, \varepsilon_1)$. Let $(x, y, \mathrm{germ}_{\varepsilon_0(x)} h) \in K$, hence $\mathrm{germ}_{\varepsilon_0(x)} h \colon \varepsilon_0(x) \to \varepsilon_0(y)$. By the definition of H_1 there are open neighborhoods U_1' of $\varepsilon_0(x)$ and U_2' of $\varepsilon_0(y)$ in $W' := \coprod_{j\in J} W_j'$ such that $h \colon U_1' \to U_2'$ is an element of $\Psi(\mathcal{W}')$. Since ε_0 is a local diffeomorphism, there are open neighborhoods U_1 of x and U_2 of y in $V := \coprod_{i \in I} V_i$ such that $\varepsilon_0|_{U_k}$ is an open embedding with $\varepsilon_0(U_k) \subseteq U_k'$ $(k = 1, 2)$. After shrinking U_k' we can assume that $\varepsilon_0(U_k) = U_k'$. Let $\gamma_k := \varepsilon_0|_{U_k}$. Then

$$g := \gamma_2^{-1} \circ h \circ \gamma_1 \colon U_1 \to U_2$$

is a diffeomorphism, hence $g \in \Psi(\mathcal{V})$. Note that $\varepsilon_1(\mathrm{germ}_x g) = \mathrm{germ}_{\varepsilon_0(x)} h$ by Prop. 5.5. Finally, we see

$$\beta(\mathrm{germ}_x g) = (x, g(x), \varepsilon_1(\mathrm{germ}_x g)) = (x, y, \mathrm{germ}_{\varepsilon_0(x)} h).$$

Therefore β is surjective. Since $\mathrm{germ}_x g$ does not depend on the choice of U_k and U_k', the map β is also injective. Finally, we will show that β is a local diffeomorphism. Since s and t are local diffeomorphisms, we only have to prove that ε_1 is one, too. Let $\mathrm{germ}_x f \in G_1$. Choose an open neighborhood U of x

such that $U \subseteq \operatorname{dom} f$ and $\varepsilon_0|_U \colon U \to \varepsilon_0(U)$ is a diffeomorphism. By the germ topology, the set

$$\tilde{U} := \{\operatorname{germ}_y f \mid y \in U\}$$

is open in G_1, and the set

$$\tilde{V} := \{\operatorname{germ}_z \nu(f) \mid z \in \varepsilon_0(U)\}$$

is open in H_1. Further the diagrams

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{\varepsilon_1} & \tilde{V} \\ \downarrow & & \downarrow \\ U & \xrightarrow[\varepsilon_0]{} & \varepsilon_0(U) \end{array} \qquad\qquad \begin{array}{ccc} \operatorname{germ}_y f & \overset{\varepsilon_1}{\longmapsto} & \operatorname{germ}_{\varepsilon_0(y)} \nu(f) \\ \downarrow & & \downarrow \\ y & \underset{\varepsilon_0}{\longmapsto} & \varepsilon_0(y) \end{array}$$

commute. The vertical arrows are diffeomorphisms by definition. Therefore $\varepsilon_1|_{\tilde{U}} \colon \tilde{U} \to \tilde{V}$ is a diffeomorphism. This completes the proof of (i).

We will now prove (ii). Prop. 3.4 shows that the orbifold atlases $\mathcal{V}$ and $\mathcal{W}'$ are determined completely by the marked atlas groupoids $\Gamma(\mathcal{V})$ and $\Gamma(\mathcal{W}')$, resp. Hence we can apply Prop. 4.12, which shows that $F_2(\varepsilon)$ is well-defined. Suppose that

$$F_2(\varepsilon) = \big(f, \{\tilde{f}_i\}_{i\in I}, [P,\nu]\big).$$

Prop. 4.12 yields $f = \mathrm{id}_Q$. By Lemma 6.2 ε_0 is a local diffeomorphism. Thus, Prop. 4.12 implies that each $\tilde{f}_i$ is a local diffeomorphism. The domain atlas of $F_2(\varepsilon)$ is $\mathcal{V}$, its range family is $\mathcal{W}'$. From Prop. 5.6 it follows that $\mathcal{U} = \mathcal{U}'$. By Def. 5.8 $F_2(\varepsilon)$ is a lift of $\mathrm{id}_{(Q,\mathcal{U})}$. □

Proposition 6.4. *Unit Morita equivalence of marked atlas groupoids is an equivalence relation.*

Proof. Since reflexivity and symmetry are easily verified, it suffices to show transitivity. Suppose that $\Gamma(\mathcal{V}_1)$, $\Gamma(\mathcal{V}_2)$, and $\Gamma(\mathcal{V}_3)$ are marked atlas groupoids such that $\Gamma(\mathcal{V}_1)$ is unit Morita equivalent to $\Gamma(\mathcal{V}_2)$, and $\Gamma(\mathcal{V}_2)$ is unit Morita equivalent to $\Gamma(\mathcal{V}_3)$. Hence there exist marked atlas groupoids $\Gamma(\mathcal{W}_1)$, $\Gamma(\mathcal{W}_2)$ and unit weak equivalences $\varepsilon_1 \colon \Gamma(\mathcal{W}_1) \to \Gamma(\mathcal{V}_1)$, $\varepsilon_2 \colon \Gamma(\mathcal{W}_1) \to \Gamma(\mathcal{V}_2)$, $\varepsilon_3 \colon \Gamma(\mathcal{W}_2) \to \Gamma(\mathcal{V}_2)$, $\varepsilon_4 \colon \Gamma(\mathcal{W}_2) \to \Gamma(\mathcal{V}_3)$:

$$\begin{array}{ccccccc} & & \Gamma(\mathcal{W}_1) & & \Gamma(\mathcal{W}_2) & & \\ & \swarrow^{\varepsilon_1} & & \searrow^{\varepsilon_2} \quad \swarrow^{\varepsilon_3} & & \searrow^{\varepsilon_4} & \\ \Gamma(\mathcal{V}_1) & & & \Gamma(\mathcal{V}_2) & & & \Gamma(\mathcal{V}_3) \end{array}$$

Prop. 6.3(ii) (in combination with Prop. 3.4) shows that the orbifold atlases $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$, $\mathcal{W}_1$, and $\mathcal{W}_2$ are all representatives of the same orbifold structure $\mathcal{U}$ on some topological space Q. Further $F_2(\varepsilon_1)$ and $F_2(\varepsilon_4)$ are lifts of $\mathrm{id}_{(Q,\mathcal{U})}$. Prop. 5.19 states that $F_2(\varepsilon_1)$ and $F_2(\varepsilon_4)$ are equivalent. Hence there exist a representative $\mathcal{W}$ of $\mathcal{U}$ and charted orbifold maps $\delta_1 \in \mathrm{Orb}(\mathcal{W}, \mathcal{W}_1)$,

$\delta_2 \in \mathrm{Orb}(\mathcal{W}, \mathcal{W}_2)$:

$$
\begin{array}{ccc}
 & & \mathcal{W}_1 \xrightarrow{F_2(\varepsilon_1)} \mathcal{V}_1 \\
 & \overset{\delta_1}{\nearrow} & \\
\mathcal{W} & & \\
 & \underset{\delta_2}{\searrow} & \\
 & & \mathcal{W}_2 \xrightarrow{F_2(\varepsilon_4)} \mathcal{V}_3
\end{array}
$$

The maps $\eta_1 := F_2(\varepsilon_1)\circ\delta_1$ and $\eta_2 := F_2(\varepsilon_4)\circ\delta_2$ are lifts of $\mathrm{id}_{(Q,\mathcal{U})}$. By Prop. 6.3(i) $F_1(\eta_1)$ and $F_1(\eta_2)$ are unit weak equivalences. Hence $\Gamma(\mathcal{V}_1)$ and $\Gamma(\mathcal{V}_3)$ are unit Morita equivalent. □

For an orbifold $(Q,\mathcal{U})$ we define

$$\Gamma(Q,\mathcal{U}) := \big\{ \big(\Gamma(\mathcal{V}), \alpha_{\mathcal{V}}, Q\big) \;\big|\; \mathcal{V} \text{ is a representative of } \mathcal{U} \big\}.$$

Proposition 6.5. *Let $(Q,\mathcal{U})$ be an orbifold and let $\mathcal{V}$ be any representative of $\mathcal{U}$. Then $\Gamma(Q,\mathcal{U})$ is the unit Morita equivalence class of the marked atlas groupoid $(\Gamma(\mathcal{V}), \alpha_{\mathcal{V}}, Q)$.*

Proof. Let $(\Gamma(\mathcal{W}), \alpha_{\mathcal{W}}, Q)$ be a marked atlas groupoid which is unit Morita equivalent to $(\Gamma(\mathcal{V}), \alpha_{\mathcal{V}}, Q)$. Then there exist a marked atlas groupoid $(\Gamma(\mathcal{K}), \alpha_{\mathcal{K}}, Q)$ and two unit weak equivalences $(\Gamma(\mathcal{K}), \alpha_{\mathcal{K}}, Q) \to (\Gamma(\mathcal{V}), \alpha_{\mathcal{V}}, Q)$ and $(\Gamma(\mathcal{K}), \alpha_{\mathcal{K}}, Q) \to (\Gamma(\mathcal{W}), \alpha_{\mathcal{W}}, Q)$. Prop. 3.4 shows that the orbifold atlases $\mathcal{W}$ and $\mathcal{K}$ on Q are determined completely by $(\Gamma(\mathcal{W}), \alpha_{\mathcal{W}}, Q)$ resp. $(\Gamma(\mathcal{K}), \alpha_{\mathcal{K}}, Q)$. Then Prop. 6.3(ii) yields that $\mathcal{W}$ and $\mathcal{K}$ are representatives of $\mathcal{U}$.

Now let $\mathcal{W}$ be a representative of $\mathcal{U}$. Consider the charted orbifold maps

$$\hat{f} = \big(\mathrm{id}_Q, \{\tilde{f}_i\}_{i\in I}, [P,\nu]\big) \in \mathrm{Orb}(\mathcal{V}, \mathcal{V}),$$
$$\hat{g} = \big(\mathrm{id}_Q, \{\tilde{g}_j\}_{j\in J}, [R,\sigma]\big) \in \mathrm{Orb}(\mathcal{W}, \mathcal{W})$$

where all $\tilde{f}_i$ and $\tilde{g}_j$ are identities (and $[P,\nu]$ and $[R,\sigma]$ are given by Prop. 5.5). Then $\hat{f}$ and $\hat{g}$ are clearly lifts of $\mathrm{id}_{(Q,\mathcal{U})}$. By Prop. 5.19 $\hat{f}$ and $\hat{g}$ are equivalent. Hence there exist a representative $\mathcal{K}$ of $\mathcal{U}$ and charted orbifold maps $\varepsilon_1 \in \mathrm{Orb}(\mathcal{K}, \mathcal{V})$, $\varepsilon_2 \in \mathrm{Orb}(\mathcal{K}, \mathcal{W})$ which are lifts of $\mathrm{id}_{(Q,\mathcal{U})}$. The charted orbifold maps $\eta_1 := \hat{f}\circ\varepsilon_1$ and $\eta_2 := \hat{g}\circ\varepsilon_2$ are lifts of $\mathrm{id}_{(Q,\mathcal{U})}$. Prop. 6.3(i) shows that

$$F_1(\eta_1)\colon (\Gamma(\mathcal{K}), \alpha_{\mathcal{K}}, Q) \to (\Gamma(\mathcal{V}), \alpha_{\mathcal{V}}, Q),$$
$$F_1(\eta_2)\colon (\Gamma(\mathcal{K}), \alpha_{\mathcal{K}}, Q) \to (\Gamma(\mathcal{W}), \alpha_{\mathcal{W}}, Q)$$

are unit weak equivalences. Thus, $(\Gamma(\mathcal{W}), \alpha_{\mathcal{W}}, Q)$ is in the unit Morita equivalence class of $(\Gamma(\mathcal{V}), \alpha_{\mathcal{V}}, Q)$. □

Equivalence of charted orbifold maps now translates to atlas groupoids as follows.

Definition 6.6. Let (G_1, α_1, X), (G_2, α_2, X), (H_1, β_1, Y), and (H_2, β_2, Y) be marked atlas groupoids. For $j = 1, 2$ let

$$\psi_j\colon (G_j, \alpha_j, X) \to (H_j, \beta_j, Y)$$

be a homomorphism of marked Lie groupoids. We call ψ_1 and ψ_2 *unit Morita equivalent* if there exist marked atlas groupoids (G, α, X) and (H, β, Y), a homomorphism $\chi\colon (G, \alpha, X) \to (H, \beta, Y)$, and unit weak equivalences $\varepsilon_j\colon (G, \alpha, X) \to (G_j, \alpha_j, X)$, $\delta_j\colon (H, \beta, Y) \to (H_j, \beta_j, Y)$ such that the diagram

$$\begin{array}{ccccc} & (G_1,\alpha_1,X) & \xrightarrow{\psi_1} & (H_1,\beta_1,Y) & \\ {}^{\varepsilon_1}\nearrow & & & & \nwarrow^{\delta_1} \\ (G,\alpha,X) & & \xrightarrow{\qquad\chi\qquad} & & (H,\beta,Y) \\ {}_{\varepsilon_2}\searrow & & & & \swarrow_{\delta_2} \\ & (G_2,\alpha_2,X) & \xrightarrow{\psi_2} & (H_2,\beta_2,Y) & \end{array}$$

commutes.

Remark 6.7. For $j \in \{1, 2\}$ let (G_j, α_j, Q) and (G'_j, α'_j, Q') be marked atlas groupoids and

$$\psi_j\colon (G_j, \alpha_j, Q) \to (G'_j, \alpha'_j, Q')$$

unit Morita equivalent homomorphisms. By definition there exist marked atlas groupoids (K, α, Q) and (K', α', Q'), a homomorphism

$$\chi\colon (K, \alpha, Q) \to (K', \alpha', Q'),$$

and unit weak equivalences

$$\delta_j\colon (K, \alpha, Q) \to (G_j, \alpha_j, Q)$$
$$\delta'_j\colon (K', \alpha', Q') \to (G'_j, \alpha'_j, Q')$$

such that the diagram

$$\begin{array}{ccccc} & (G_1,\alpha_1,Q) & \xrightarrow{\psi_1} & (G'_1,\alpha'_1,Q') & \\ {}^{\delta_1}\nearrow & & & & \nwarrow^{\delta'_1} \\ (K,\alpha,Q) & & \xrightarrow{\qquad\chi\qquad} & & (K',\alpha',Q') \\ {}_{\delta_2}\searrow & & & & \swarrow_{\delta'_2} \\ & (G_2,\alpha_2,Q) & \xrightarrow{\psi_2} & (G'_2,\alpha'_2,Q') & \end{array}$$

commutes. Let $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{W}$ be orbifold atlases on Q such that

$$(G_j, \alpha_j, Q) = (\Gamma(\mathcal{V}_j), \alpha_{\mathcal{V}_j}, Q)$$

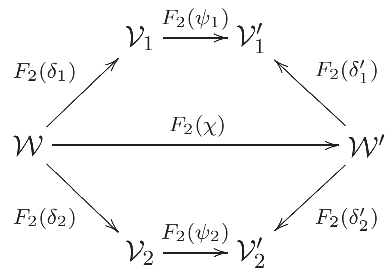
and

$$(K, \alpha, Q) = (\Gamma(\mathcal{W}), \alpha_{\mathcal{W}}, Q).$$

Likewise let $\mathcal{V}'_1$, $\mathcal{V}'_2$ and $\mathcal{W}'$ be orbifold atlases on Q' such that $(G'_j, \alpha'_j, Q') = (\Gamma(\mathcal{V}'_j), \alpha_{\mathcal{V}'_j}, Q')$ and $(K', \alpha', Q') = (\Gamma(\mathcal{W}'), \alpha_{\mathcal{W}'}, Q')$. By Prop. 3.4 all these orbifold atlases are uniquely determined. Prop. 6.3(ii) shows that $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{W}$ determine the same orbifold structure $\mathcal{U}$ on Q, and that $F_2(\delta_j)\colon \mathcal{W} \to \mathcal{V}_j$ are lifts of $\mathrm{id}_{(Q,\mathcal{U})}$. By the same reason, $\mathcal{V}'_1$, $\mathcal{V}'_2$ and $\mathcal{W}'$ determine the same orbifold structure $\mathcal{U}'$ on Q', and $F_2(\delta'_j)\colon \mathcal{W}' \to \mathcal{V}'_j$ are lifts of $\mathrm{id}_{(Q',\mathcal{U}')}$. Hence the charted orbifold maps

$$F_2(\psi_j)\colon \mathcal{V}_j \to \mathcal{V}'_j$$

are equivalent, and this equivalence is shown by the commutative diagram



An analogous argumentation allows to canonically convert equivalence of charted orbifold maps $\hat{f}$ and $\hat{g}$ to unit Morita equivalence of $F_1(\hat{f})$ and $F_1(\hat{g})$. In turn, equivalence of charted orbifold maps canonically corresponds to unit Morita equivalence of homomorphisms between marked atlas groupoids, and vice versa.

Proposition 6.8. *Unit Morita equivalence of homomorphisms between marked atlas groupoids is an equivalence relation.*

Proof. This follows immediately from Rem. 6.7 and Prop. 5.15. □

We define the category Agr of marked atlas groupoids as follows. Its class of objects consists of all $\Gamma(Q,\mathcal{U})$. The morphisms from $\Gamma(Q,\mathcal{U})$ to $\Gamma(Q',\mathcal{U}')$ are the unit Morita equivalence classes $[\varphi]$ of homomorphisms $\varphi\colon (G,\alpha,Q) \to (G',\alpha',Q')$ where (G,α,Q) is a representative of $\Gamma(Q,\mathcal{U})$ and (G',α',Q') is a representative of $\Gamma(Q',\mathcal{U}')$. More precisely,

$$\begin{aligned}&\operatorname{Morph}\big(\Gamma(Q,\mathcal{U}),\Gamma(Q',\mathcal{U}')\big) = \\ &\qquad = \big\{[\varphi] \,\big|\, \varphi \in \operatorname{Hom}\big((G,\alpha,Q),(G',\alpha',Q')\big),\ (G,\alpha,Q)\in\Gamma(Q,\mathcal{U}), \\ &\qquad\qquad (G',\alpha',Q')\in\Gamma(Q',\mathcal{U}')\big\}.\end{aligned}$$

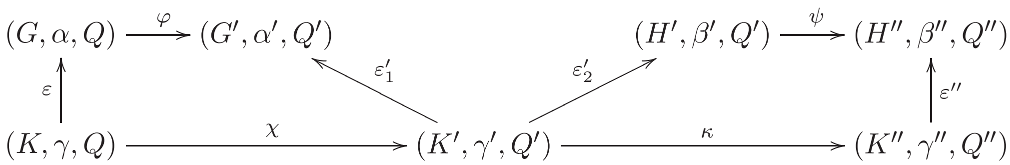
To define the composition in Agr let $[\varphi] \in \operatorname{Morph}\big(\Gamma(Q,\mathcal{U}),\Gamma(Q',\mathcal{U}')\big)$ and $[\psi] \in \operatorname{Morph}\big(\Gamma(Q',\mathcal{U}'),\Gamma(Q'',\mathcal{U}'')\big)$. Choose representatives $\varphi\colon (G,\alpha,Q) \to (G',\alpha',Q')$ of $[\varphi]$ and $\psi\colon (H',\beta',Q') \to (H'',\beta'',Q'')$ of $[\psi]$. Then find representatives (K,γ,Q), (K',γ',Q'), (K'',γ'',Q'') of $\Gamma(Q,\mathcal{U})$, $\Gamma(Q',\mathcal{U}')$, $\Gamma(Q'',\mathcal{U}'')$, resp., and unit Morita equivalences

$$\begin{aligned}\varepsilon\colon (K,\gamma,Q) &\to (G,\alpha,Q),\\ \varepsilon_1'\colon (K',\gamma',Q') &\to (G',\alpha',Q'),\\ \varepsilon_2'\colon (K',\gamma',Q') &\to (H',\beta',Q'),\\ \varepsilon''\colon (K'',\gamma'',Q'') &\to (H'',\beta'',Q''),\end{aligned}$$

and homomorphisms of marked Lie groupoids

$$\begin{aligned}\chi\colon (K,\gamma,Q) &\to (K',\gamma',Q'),\\ \kappa\colon (K',\gamma',Q') &\to (K'',\gamma'',Q'')\end{aligned}$$

such that the diagram



commutes. Then the composition of $[\varphi]$ and $[\psi]$ is defined as

$$[\psi] \circ [\varphi] := [\kappa \circ \chi].$$

Proposition 6.9. *The composition in* Agr *is well-defined.*

Proof. This follows immediately by invoking Rem. 6.7, Lemmas 5.11 and 5.17, and Prop. 5.18. ☐

Recall the orbifold category Orb from Sec. 5.3. We define an assignment F from Orb to Agr as follows. On the level of objects, F maps the orbifold $(Q,\mathcal{U})$ to $\Gamma(Q,\mathcal{U})$. Suppose that $[\hat{f}]$ is a morphism from the orbifold $(Q,\mathcal{U})$ to the orbifold $(Q',\mathcal{U}')$. Then F maps $[\hat{f}]$ to the morphism $[F_1(\hat{f})]$ from $\Gamma(Q,\mathcal{U})$ to $\Gamma(Q',\mathcal{U}')$. Now one easily deduces the following theorem.

Theorem 6.10. *The assignment F is a covariant functor from* Orb *to* Agr*. Even more, F is an isomorphism of categories.*

Remark 6.11. The functor F is constructive. With an easy extension of [MM03, Cor. 5.31], also F^{-1} is constructive.

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