DEFORMATIONS OF ALGEBROID STACKS

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Abstract. In this paper we consider deformations of an algebroid stack on an etale groupoid. We construct a differential graded Lie algebra (DGLA) which controls this deformation theory. In the case when the algebroid is a twisted form of functions we show that this DGLA is quasiisomorphic to the twist of the DGLA of Hochschild cochains on the algebra of functions on the groupoid by the characteristic class of the corresponding gerbe.

Contents

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1. Introduction

The two main results of the paper are

(1) classification of deformations of an algebroid stack on an etale groupoid by Maurer-Cartan elements of a differential graded Lie algebra (DGLA) canonically associated to the algebroid stack, see Theorem 5.22;
(2) in the particular case, when the algebroid is a twisted form of the structure sheaf (i.e. is associated to a gerbe on the groupoid) a construction of a quasiisomorphism of the DGLA alluded to above with a twist (by the class of the gerbe) of the DGLA of Hochschild cochains on the algebra of functions on the groupoid, see Theorems 5.25 and 5.26

In the case when the étale groupoid is a manifold these results were established in [6]. The principal example motivating the present work is as follows.

Let Σ be a coisotropic submanifold of a symplectic manifold M. The holonomy groupoid of the characteristic foliation on Σ is an étale groupoid with a symplectic structure. The canonical deformation quantization of this étale groupoid is an algebroid stack. Similarly, for a complex manifold with a (holomorphic) symplectic structure the canonical deformation quantization is an algebroid stack. The first obstruction for this algebroid stack to be (equivalent to) a sheaf of algebras is the first Rozansky-Witten invariant.

The canonical deformation quantization of the symplectic étale groupoid associated to a coisotropic submanifold naturally arises in the study of the following question motivated by problems in microlocal analysis. For a coisotropic submanifold Σ of a symplectic manifold M consider the graph Λ of the characteristic foliation which is a Lagrangian submanifold of the product $M \times M^{op}$. When $M = T^*X$ for a manifold X, and Σ is conic, the Lagrangian Λ is conic as well, hence determines a class of Fourier integral operators given by kernels whose wave fronts are contained in Λ. These operators form an algebra under composition since the composition $\Lambda \circ \Lambda$ coincides with Λ. An asymptotic version of the operator product gives rise to a deformation of the foliation algebra which we will discuss in a subsequent work. The canonical deformation quantization of the holonomy groupoid is, in a suitable sense, Morita equivalent to this algebra.

The DGLA whose Maurer-Cartan elements classify deformations of an algebroid stack is constructed as follows. From an algebroid stack on Hausdorff etale groupoid one passes to a cosimplicial matrix algebra

on the nerve of the groupoid. If the groupoid is non-Hausdorff one has to replace the groupoid by its embedding category, cf. [25].

For a cosimplicial matrix algebra we form the Hochschild cochain complex which happens to be a cosimplicial sheaf of DGLAs not on the nerve itself, but on its first subdivision. From this we pass to a cosimplicial DGLA and to its totalization which is an ordinary DGLA.

For a gerbe on an etale groupoid this DGLA can be replaced by another of a more familiar geometric nature, leading to the Theorems 5.25 and 5.26.

This paper is organized as follows. In the section 2 we give overview of the preliminaries from the category theory and the theory of (co)simplicial spaces. We also describe I. Moerdijk construction of embedding category and stacks on etale categories.

In the section 3 we review relation between the deformation and differential graded Lie algebras as well as discuss in this context deformations of matrix Azumaya algebras.

In the section 4 we introduce the notion of cosimplicial matrix algebra and construct a DGLA governing deformations of matrix algebras (cf. Theorem 4.5). We then specialize to the case of cosimplicial matrix Azumaya algebras. In this case using the differential geometry of the infinite jet bundle we are able to show that the deformation theory of a cosimplicial matrix Azumaya algebra $\mathcal{A}$ is controlled by DGLA of jets of Hochschild cochains twisted by the cohomology class of the gerbe associated with $\mathcal{A}$.

Finally in the section 5 we apply the results of the previous section to study the deformation theory of stacks on etale groupoids. We also use these methods to study the deformation theory of a twisted convolution algebra of etale groupoid.

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2. Preliminaries

2.1. **Categorical preliminaries.**

2.1.1. **The simplicial category.** For $n = 0, 1, 2, \ldots$ we denote by $[n]$ the category with objects $0, \ldots, n$ freely generated by the graph

$$0 \to 1 \to \cdots \to n \ .$$

For $0 \leq i \leq j \leq n$ we denote by (ij) the unique arrow $i \to j$ in $[n]$. We denote by Δ the full subcategory of **Cat** with objects the categories $[n]$ for $n = 0, 1, 2, \ldots$.

For a category $\mathcal{C}$ we refer to a functor $\lambda\colon [n] \to \mathcal{C}$ as *a(n n-)simplex in* $\mathcal{C}$. For a morphism $f\colon [m] \to [n]$ in Δ and a simplex $\lambda\colon [n] \to \mathcal{C}$ we denote by $f^*(\lambda)$ the simplex $\lambda \circ f$.

Suppose that $f\colon [m] \to [n]$ is a morphism in Δ and $\lambda\colon [n] \to \mathcal{C}$ is a simplex. Let $\mu = f^*(\lambda)$ for short. The morphism $(0n)$ in $[n]$ factors uniquely into the composition $0 \to f(0) \to f(m) \to n$ which, under λ, gives the factorization of $\lambda(0n)\colon \lambda(0) \to \lambda(n)$ in $\mathcal{C}$ into the composition

$$\lambda(0) \xrightarrow{a} \mu(0) \xrightarrow{b} \mu(m) \xrightarrow{c} \lambda(n)\ , \tag{2.1.1}$$

where $b = \mu((0m))$.

For $0 \leq i \leq n+1$ we denote by $\partial_i = \partial_i^n\colon [n] \to [n+1]$ the i^{th} face map, i.e. the unique injective map whose image does not contain the object $i \in [n+1]$.

For $0 \leq i \leq n-1$ we denote by $s_i = s_i^n\colon [n] \to [n-1]$ the i^{th} degeneracy map, i.e. the unique surjective map such that $s_i(i) = s_i(i+1)$.

2.1.2. **Subdivision.** For $\lambda\colon [n] \to \Delta$ define λ_k by $\lambda(k) = [\lambda_k]$, $k = 0, 1, \ldots, n$. Let $sd^\lambda\colon [n] \to \lambda(n)$ be a morphism in Δ defined by

$$sd^\lambda(k) = \lambda(kn)(\lambda_k). \tag{2.1.2}$$

For a morphism $f\colon [m] \to [n]$ in Δ let $sd(f)^\lambda = c$ in the notations of (2.1.1). Then, the diagram

$$\begin{array}{ccc} [m] & \xrightarrow{f} & [n] \\ {\scriptstyle sd^{f^*(\lambda)}}\downarrow & & \downarrow{\scriptstyle sd^\lambda} \\ f^*(\lambda)(m) & \xrightarrow{sd(f)^\lambda} & \lambda(n) \end{array}$$

is commutative.

2.1.3. **(Co)simplicial objets.** A simplicial object in a category $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$. For a simplicial object X we denote $X([n])$ by X_n.

A cosimplicial object in a category $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$. For a cosimplicial object Y we denote $Y([n])$ by Y^n.

Simplicial (respectively, cosimplicial) objects in $\mathcal{C}$ form a category in the standard way.

2.1.4. **Nerve.** For $n = 0, 1, 2, \ldots$ let $N_n\mathcal{C} := \mathrm{Hom}([n], \mathcal{C})$. The assignment $n \mapsto N_n\mathcal{C}$, $([m] \xrightarrow{f} [n]) \mapsto N\mathcal{C}(f) := (\lambda \mapsto \lambda \circ f)$ defines the simplicial set $N\mathcal{C}$ called *the nerve of* $\mathcal{C}$.

The effect of the face map ∂_i^n (respectively, the degeneracy map s_i^n) will be denoted by $d^i = d_n^i$ (respectively, $\varsigma_i = \varsigma_i^n$).

2.1.5. **Subdivision of (co)simplicial objects.** Assume that coproducts in $\mathcal{C}$ are represented. Let $X \in \mathcal{C}^{\Delta^{op}}$.

For $\lambda\colon [n] \to \Delta$ let $|X|_\lambda := X_{\lambda(n)}$, $|X|_n := \coprod_{\lambda\colon [n]\to\Delta} |X|_\lambda$.

For a morphism $f\colon [m] \to [n]$ in Δ let $|X|(f)\colon |X|_n \to |X|_m$ denote the map whose restriction to $|X|_\lambda$ is the map $X(sd(f)^\lambda)$.

The assignment $[n] \mapsto |X|_n$, $f \mapsto |X|(f)$ defines the simplicial object $|X|$ called *the subdivision* of X.

Let $sd(X)_n\colon |X|_n \to X_n$ denote the map whose restriction to $|X|_\lambda$ is the map $X(sd^\lambda)$. The assignment $[n] \mapsto sd(X)_n$ defines the canonical morphism of simplicial objects

$$sd(X)\colon |X| \to X.$$

Suppose that $\mathcal{C}$ has products. For $V \in \mathcal{C}^\Delta$, $\lambda\colon [n] \to \Delta$ let $|V|^\lambda = V^{\lambda(n)}$, $|V|^n = \prod_{[n]\xrightarrow{\lambda}\Delta} |V|^\lambda$. For a morphism $f\colon [m] \to [n]$ in Δ let $|V|(f)\colon |V|_m \to |V|_n$ denote the map such that $\mathtt{pr}_\lambda \circ |V|(f) = V(sd(f)^\lambda) \circ \mathtt{pr}_{f^*(\lambda)}$. The assignment $[n] \mapsto |V|_n$, $f \mapsto |V|(f)$ defines the cosimplicial object $|V|$ called *the subdivision* of V.

Let $sd(V)^n\colon V^n \to |V|^n$ denote the map such that $\mathtt{pr}_\lambda \circ sd(V)^n = V(sd^\lambda)$. The assignment $[n] \mapsto sd(V)_n$ defines the canonical morphism of simplicial objects

$$sd(V)\colon V \to |V|.$$

2.1.6. **Totalization of cosimplicial vector spaces.** Next, we recall the definition of the functor $\mathtt{Tot}$ which assigns to a cosimplicial vector space a complex of vector spaces.

For $n = 0, 1, 2, \ldots$ let Ω_n denote the polynomial de Rham complex of the n-dimensional simplex. In other words, Ω_n is the DGCA generated by $t_0, \ldots, t_n$ of degree zero and $dt_0, \ldots, dt_n$ of degree one subject to the relations $t_0 + \cdots + t_n = 1$ and $dt_0 + \cdots + dt_n = 0$; the differential d_Δ is determined by the Leibniz rule and $t_i \mapsto dt_i$. The assignment $\Delta \ni [n] \mapsto \Omega_n$ extends in a natural way to a simplicial DGCA.

Suppose that V is a cosimplicial vector space. For each morphism $f\colon [p] \to [q]$ in Δ we have the morphisms $V(f)\colon V^p \to V^q$ and $\Omega(f)\colon \Omega_q \to \Omega_p$. Let $\mathtt{Tot}(V)^k \subset \prod_n \Omega_n^k \otimes V^n$ denote the subspace which consists of those $a = (a_n)$ which satisfy the conditions $(\mathtt{Id} \otimes V(f))(a_p) = (\Omega(f) \otimes \mathtt{Id})(a_q)$ for all $f\colon [p] \to [q]$. The de Rham differential $d_\Delta\colon \Omega^k \to \Omega^{k+1}$ induces the differential in $\mathtt{Tot}(V)$. It is clear that the assignment

$V \mapsto \mathtt{Tot}(V)$ is a functor on the category of cosimplicial vectors spaces with values in the category of complexes of vector spaces.

2.1.7. **Cohomology of cosimplicial vector spaces.** For a cosimplicial vector space V we denote by $C(V)$ (respectively, $N(V)$), the associated complex (respectively, normalized complex). These are given by $C^i(V) = N^i(V) = 0$ for $i < 0$ and, otherwise, by

$$C^n(V) = V^n, \qquad N^n(V) = \bigcap_i \ker(s_i \colon V^n \to V^{n-1})$$

In either case the differential $\partial = \partial_V$ is given by $\partial^n = \sum_i (-1)^i \partial_i^n$. The natural inclusion $N(V) \hookrightarrow C(V)$ is a quasiisomorphism (see e.g. [13]). Let $H^\bullet(V) \colon = H^\bullet(N(V)) = H^\bullet(C(V))$.

We will also need the following result, see e.g.[29]:

Lemma 2.1. *Suppose that V is a cosimplicial vector space. The map $H^\bullet(sd(V)) \colon H^\bullet(V) \to H^\bullet(|V|)$ is an isomorphism.*

2.1.8. There is a natural quasiisomorphism

$$\int_V : \mathtt{Tot}(V) \to N(V)$$

(see e.g. [12]) such that the composition

$$H^0(V) \xrightarrow{1 \otimes \mathtt{Id}} \mathtt{Tot}(V) \xrightarrow{\int_V} N(V)$$

coincides with the inclusion $H^0(V) \hookrightarrow N(V)$.

Suppose that (V, d) is a cosimplicial complex of vector spaces. Then, for $i \in \mathbb{Z}$, we have the cosimplicial vector space $V^{\bullet,i} \colon [n] \mapsto (V^n)^i$. Applying $\mathtt{Tot}(\ )$, $N(\ )$ and $C(\ )$ componentwise we obtain double complexes whose associated total complexes will be denoted, respectively, by $\mathtt{Tot}(V)$, $N(V)$ and $C(V)$. The maps

$$H^0(V^{\bullet,i}) \to \mathtt{Tot}(V^{\bullet,i})$$

give rise to the map of complexes

$$\ker(V^0 \rightrightarrows V^1) \to \mathtt{Tot}(V) \tag{2.1.3}$$

Lemma 2.2. *Suppose that V is a cosimplicial complex of vector spaces such that*

1. *there exists $M \in \mathbb{Z}$ such that, for all $i < M$, $n = 0, 1, \ldots$, $V^{n,i} = 0$ (i.e. the complexes V^n are bounded below uniformly in n)*
2. *for all $i \in \mathbb{Z}$ and $j \neq 0$, $H^j(V^{\bullet,i}) = 0$.*

Then, the map (2.1.3) *is a quasiisomorphism.*

Proof. It suffices to show that the composition

$$\ker(V^0 \rightrightarrows V^1) \to \mathtt{Tot}(V) \xrightarrow{\int_V} N(V)$$

is a quasiisomorphism. Note that, for $i \in \mathbb{Z}$, the acyclicity assumption implies that the composition

$$\ker(V^{0,i} \rightrightarrows V^{1,i}) = H^0(V^{\bullet,i}) \to \mathtt{Tot}(V^{\bullet,i}) \to N(V^{\bullet,i}))$$

is a quasiisomorphism. Since the second map is a quasiisomorphism so is the first one. Since, by assumption, V is uniformly bounded below the claim follows. □

2.2. Cosimplicial sheaves and stacks. We will denote by $\mathrm{Sh}_R(X)$ the category of sheaves of R-modules on X.

2.2.1. Cosimplicial sheaves. Suppose that X is a simplicial space.

A cosimplicial sheaf of vector spaces (or, more generally, a cosimplicial complex of sheaves) $\mathcal{F}$ on X is given by the following data:

(1) for each $p = 0, 1, 2, \ldots$ a sheaf $\mathcal{F}^p$ on X_p and
(2) for each morphism $f\colon [p] \to [q]$ in Δ a morphism $f_*\colon X(f)^{-1}\mathcal{F}^p \to \mathcal{F}^q$.

These are subject to the condition: for each pair of composable arrows $[p] \xrightarrow{f} [g] \xrightarrow{g} [r]$ the diagram map $(g\circ f)_*\colon X(g\circ f)^{-1}\mathcal{F}^p \to \mathcal{F}^r$ is equal to the composition $X(g \circ f)^{-1}\mathcal{F}^p \cong X(g)^{-1}X(f)^{-1} \xrightarrow{X(g)^{-1}f_*} X(g)^{-1}\mathcal{F}^q \xrightarrow{g_*} \mathcal{F}^r$.

Definition 2.3. A cosimplicial sheaf $\mathcal{F}$ is *special* if the structure morphisms $f_*\colon X(f)^{-1}\mathcal{F}^p \to \mathcal{F}^q$ are isomorphisms for all f.

2.2.2. Cohomology of cosimplicial sheaves. For a cosimplicial sheaf of vector spaces $\mathcal{F}$ on X let $\Gamma(X;\mathcal{F})^n = \Gamma(X_n;\mathcal{F}^n)$. For a morphism $f\colon [p] \to [q]$ in Δ let $f_* = \Gamma(X;\mathcal{F})(f)\colon \Gamma(X;\mathcal{F})^p \to \Gamma(X;\mathcal{F})^q$ denote the composition

$$\Gamma(X_p;\mathcal{F}^p) \xrightarrow{X(f)^*} \Gamma(X_q;X(f)^{-1}\mathcal{F}^p) \xrightarrow{\Gamma(X_q;f_*)} \Gamma(X_q;\mathcal{F}^q)$$

The assignments $[p] \mapsto \Gamma(X;\mathcal{F})^p$, $f \mapsto f_*$ define a cosimplicial vector space denoted $\Gamma(X;\mathcal{F})$.

The functor $\mathcal{F} \mapsto H^0(\Gamma(X;\mathcal{F}))$ from the (abelian) category of cosimplicial sheaves of vector spaces on X to the category of vector spaces is left exact. Let $\mathbf{R}\Gamma(X;\mathcal{F})) = \mathbf{R}H^0(\Gamma(X;\mathcal{F}),\ H^i(X;\mathcal{F}) = \mathbf{R}^i\Gamma(X;\mathcal{F}))$.

Assume that $\mathcal{F}$ satisfies $H^i(X_j;\mathcal{F}^j) = 0$ for $i \neq 0$. For complexes of sheaves the assumption on $\mathcal{F}^j$ is that the canonical morphism in the

derived category $\Gamma(X_j;\mathcal{F}^j) \to \mathbf{R}\Gamma(X_j;\mathcal{F}^j)$ is an isomorphism. Under the acyclicity assumption on $\mathcal{F}$ we have

$$H^j(X;\mathcal{F}) = H^j(\Gamma(X;\mathcal{F}))$$

2.2.3. Sheaves on the subdivision. For a cosimplicial sheaf $\mathcal{F}$ on X let

$$|\mathcal{F}| := sd(X)^{-1}\mathcal{F}\ .$$

Thus, $|\mathcal{F}|$ is a cosimplicial sheaf on $|X|$ and

$$\Gamma(|X|;|\mathcal{F}|)^n = \Gamma(|X|_n;|\mathcal{F}|_n) = \prod_{\lambda\colon [n]\to\Delta} \Gamma(X_{\lambda(n)};\mathcal{F}^{\lambda(n)}) = |\Gamma(X;\mathcal{F})|_n$$

i.e. the canonical isomorphism of cosimplicial vector spaces

$$\Gamma(|X|;|\mathcal{F}|) = |\Gamma(X;\mathcal{F})|$$

Lemma 2.4. *The map $sd(X)^*\colon H^\bullet(X;\mathcal{F}) \to H^\bullet(|X|;|\mathcal{F}|)$ is an isomorphism.*

Proof. Follows from Lemma 2.1 □

2.2.4. We proceed with the notations introduced above. Suppose that $\mathcal{F}$ is a special cosimplicial sheaf on X. Then, $|\mathcal{F}|$ admits an equivalent description better suited for our purposes.

Let $|\mathcal{F}|'^n$ denote the sheaf on $|X|_n$ whose restriction to $|X|_\lambda$ is given by $|\mathcal{F}|'_\lambda := X(\lambda(0n))^{-1}\mathcal{F}^{\lambda(0)}$. For a morphism of simplices $f\colon \mu \to \lambda$ the corresponding (component of the) structure map f'_* is defined as the unique map making the diagram

$$\begin{array}{ccc} X(c)^{-1}X(\mu((0m)))^{-1}\mathcal{F}^{\mu(0)} & \xrightarrow{f'_*} & X(\lambda(0n))^{-1}\mathcal{F}^{\lambda(0)} \\ {\scriptstyle X(c)^{-1}\mu((0m))_*}\downarrow & & \downarrow{\scriptstyle \lambda(0n)_*} \\ X(c)^{-1}\mathcal{F}^{\mu(m)} & \xrightarrow{c_*} & \mathcal{F}^{\lambda(n)} \end{array}$$

commutative. Note that f'_* exists and is unique since the vertical maps are isomorphisms as $\mathcal{F}$ is special.

It is clear that $|F|'$ is a cosimplicial sheaf on $|X|$; moreover there is a canonical isomorphism $|F|' \to |F|$ whose restriction to $|X|_\lambda$ is given by the structure map $\lambda(0n)_*$.

2.2.5. Stacks. We refer the reader to [1] and [31] for basic definitions. We will use the notion of fibered category interchangeably with that of a pseudo-functor. A *prestack* $\mathcal{C}$ on a space Y is a category fibered over the category of open subsets of Y, equivalently, a pseudo-functor $U \mapsto \mathcal{C}(U)$, satisfying the following additional requirement. For an

open subset U of Y and two objects $A, B \in \mathcal{C}(U)$ we have the presheaf $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,B)$ on U defined by $U \supseteq V \mapsto \mathrm{Hom}_{\mathcal{C}(V)}(A|_V, B|_B)$. The fibered category $\mathcal{C}$ is a prestack if for any U, $A, B \in \mathcal{C}(U)$, the presheaf $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,B)$ is a sheaf. A prestack is a *stack* if, in addition, it satisfies the condition of effective descent for objects. For a prestack $\mathcal{C}$ we denote the associated stack by $\widetilde{\mathcal{C}}$.

A stack in groupoids $\mathcal{G}$ is a gerbe if it is locally nonempty and locally connected, i.e.it satisfies

(1) any point $y \in Y$ has a neighborhood U such that $\mathcal{G}(U)$ is nonempty;
(2) for any $U \subseteq Y$, $y \in U$, $A, B \in \mathcal{G}(U)$ there exists a neighborhood $V \subseteq U$ of y and an isomorphism $A|_V \cong B|_V$.

2.2.6. Cosimplicial stacks. Suppose that X is an étale simplicial space.

A cosimplicial stack $\mathcal{C}$ on X is given by the following data:

(1) for each $[p] \in \Delta$ as stack $\mathcal{C}^p$ on X_p ;
(2) for each morphism $f\colon [p] \to [q]$ in Δ a 1-morphism of stacks $\mathcal{C}_f\colon X(f)^{-1}\mathcal{C}^p \to \mathcal{C}^q$;
(3) for any pair of morphisms $[p] \xrightarrow{f} [q] \xrightarrow{g} [r]$ a 2-morphism $\mathcal{C}_{f,g}\colon \mathcal{C}_g \circ X(g)^*(\mathcal{C}_f) \to \mathcal{C}_{g\circ f}$

These are subject to the associativity condition: for a triple of composable arrows $[p] \xrightarrow{f} [q] \xrightarrow{g} [r] \xrightarrow{h} [s]$ the equality of 2-morphisms

$$\mathcal{C}_{g\circ f,h} \circ (X(h)^{-1}\mathcal{C}_{f,g} \otimes \mathtt{Id}_{\mathcal{C}_h}) = \mathcal{C}_{f,h\circ g} \circ (\mathtt{Id}_{X(h\circ g)^{-1}\mathcal{C}_f} \otimes \mathcal{C}_{g,h})$$

holds. Here and below we use $\otimes$ to denote the horizontal composition of 2-morphisms.

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are cosimplicial stacks on X. A 1-morphism $\phi\colon \mathcal{C} \to \mathcal{D}$ is given by the following data:

(1) for each $[p] \in \Delta$ a 1-morphism $\phi^p\colon \mathcal{C}^p \to \mathcal{D}^p$
(2) for each morphism $f\colon [p] \to [q]$ in Δ a 2-isomorphism $\phi_f\colon \phi^q \circ \mathcal{C}_f \to \mathcal{D}_f \circ X(f)^*(\phi^p)$

which, for every pair of morphisms $[p] \xrightarrow{f} [q] \xrightarrow{g} [r]$ satisfy

$$(\mathcal{D}_{f,g} \otimes \mathtt{Id}_{X(gf)^*(\phi^p)}) \circ (X(g)^*(\phi_f) \otimes \mathtt{Id}_{\mathcal{D}_g}) \circ (\mathtt{Id}_{X(g)^*(\mathcal{C}_f)} \otimes \phi_g) = \phi_{g\circ f} \circ (\mathtt{Id}_{\phi^r} \otimes \mathcal{C}_{f,g})$$

Suppose that ϕ and ψ are 1-morphisms of cosimplicial stacks $\mathcal{C} \to \mathcal{D}$. A 2-morphism $b\colon \phi \to \psi$ is given by a collection of 2-morphisms

$b^p\colon \phi^p \to \psi^p$, $p = 0, 1, 2, \ldots$, which satisfy

$$\psi_f \circ (b^q \otimes \mathtt{Id}_{\mathcal{C}_f}) = (\mathtt{Id}_{\mathcal{D}_f} \otimes X(f)^*(b^p)) \circ \phi_f$$

for all $f\colon [p] \to [q]$ in Δ.

2.2.7. Cosimplicial gerbes. Suppose that A is an abelian cosimplicial sheaf on X. A cosimplicial A-gerbe $\mathcal{G}$ on X is a cosimplicial stack on X such that

(1) for each $[p] \in \Delta$, $\mathcal{G}^p$ is a A^p-gerbe on X_p.
(2) For each morphism $f\colon [p] \to [q]$ in Δ the 1-morphism

$$\mathcal{G}_f\colon X(f)^{-1}\mathcal{G}^p \to \mathcal{G}^q$$

of stacks compatible with the map $A_f\colon X(f)^{-1}A^p \to A^q$.

2.3. Sheaves and stacks on étale categories.

2.3.1. Étale categories. In what follows (C^∞-)manifolds are not assumed to be Hausdorff. Note, however, that, by the very definition a manifold is a locally Hausdorff space. An *étale* map of manifolds is a local diffeomorphism.

An *étale category* $\mathcal{E}$ is a category in manifolds with the manifold of objects $N_0\mathcal{E}$, the manifold of morphisms $N_1\mathcal{E}$ and étale structure maps. Forgetting the manifold structures on the sets of objects and morphisms one obtains the underlying category.

If the underlying category of an étale category G is a groupoid and the inversion map $N_1G \to N_1G$ is C^∞ one says that G is an *étale groupoid*.

We will identify a manifold X with the category with the space of objects X and only the identity morphisms. In particular, for any étale category G there is a canonical embedding $N_0G \to G$ which is identity on objects.

2.3.2. Notation. We will make extensive use of the following notational scheme. Suppose that X is a simplicial space (such as the nerve of a topological category) and $f\colon [p] \to [q]$ is a morphism in Δ. The latter induced the map $X(f)\colon X_q \to X_p$ of spaces. Note that f is determined by the number q and the sequence $\vec{f} = (f(0), \ldots, f(p-1))$. For an object A associated to (or, rather, "on") X_p (such as a function, a sheaf, a stack) for which the inverse image under $X(f)$ is define we will denote the resulting object on X_q by $A^{(q)}_{\vec{f}}$.

2.3.3. Sheaves on étale categories. Suppose that $\mathcal{E}$ is an étale category. A sheaf (of sets) on $\mathcal{E}$ is a pair $\underline{F} = (F, F_{01})$, where

- F is a sheaf on the space of objects $N_0\mathcal{E}$ and
- $F_{01}\colon F_1^{(1)} \to F_0^{(1)}$ is an isomorphisms of sheaves on the space of morphisms $N_1 G$

which is multiplicative, i.e. satisfies the "cocycle" condition

$$F_{02}^{(2)} = F_{01}^{(2)} \circ F_{12}^{(2)},$$

(on $N_2 G$) and unit preserving, i.e.

$$F_{00}^{(0)} = \mathtt{Id}_F.$$

We denote by $\mathrm{Sh}(G)$ the category of sheaves on G.

A morphism $f\colon \underline{F} \to \underline{F'}$ of sheaves on $\mathcal{E}$ is a morphism of sheaves $f\colon F \to F'$ on $N_0\mathcal{E}$ which satisfies the "equivariance" condition

$$f_0^{(1)} \circ F_{01} = F_{01} \circ f_\cdot^{(1)}$$

A morphism of étale categories $\phi\colon G \to G'$ induces the functor of inverse image (or pull-back)

$$\phi^{-1}\colon \mathrm{Sh}(G') \to \mathrm{Sh}(G).$$

For $\underline{F} = (F, F_{01}) \in \mathrm{Sh}(G')$, the sheaf on $N_0 G$ underlying $\phi^{-1}\underline{F}$ is given by $(N_0\phi)^{-1}F$. The image of F_{01} under the map

$$\begin{aligned}(N_1\phi)^*\colon \mathrm{Hom}(F_1^{(1)}, F_0^{(1)}) &\\ \to \mathrm{Hom}((N_1\phi)^{-1}F_1^{(1)}, (N_1\phi)^{-1}F_0^{(1)}) &\\ \cong \mathrm{Hom}(((N_0\phi)^{-1}F)_1^{(1)}, ((N_0\phi)^{-1}F)_0^{(1)}) &\end{aligned}$$

The category $\mathrm{Sh}(G)$ has a final object which we will denote by $*$. For $\underline{F} \in \mathrm{Sh}(G)$ let $\Gamma(G;\underline{F})\colon = \mathrm{Hom}_{\mathrm{Sh}(G)}(*, \underline{F})$. The set $\Gamma(G;\underline{F})$ is easily identified with the subset of G-invariant sections of $\Gamma(N_0 G; F)$.

A sheaf $\underline{F}$ on G gives rise to a cosimplicial sheaf on NG which we denote $\underline{F}_\Delta$. The latter is defined as follows. For $n = 0, 1, 2, \ldots$ let $\underline{F}_\Delta^n = F_0^{(n)} \in \mathrm{Sh}(N_n G)$. For $f\colon [p] \to [q]$ in Δ the corresponding structure map f_* is defined as

$$NG(f)^{-1}\underline{F}_\Delta^p = F_{f(0)}^{(q)} \xrightarrow{F_{0f(0)}^{(q)}} F_0^{(q)} = \underline{F}_\Delta^q.$$

2.3.4. Stacks on étale categories.

Definition 2.5. A stack on G is a triple $\underline{\mathcal{C}} = (\mathcal{C}, \mathcal{C}_{01}, \mathcal{C}_{012})$ which consists of

(1) a stack $\mathcal{C}$ on N_0G
(2) an *equivalence* $\mathcal{C}_{01}\colon \mathcal{C}_1^{(1)} \to \mathcal{C}_0^{(1)}$
(3) an *isomorphism* $\mathcal{C}_{012}\colon \mathcal{C}_{01}^{(2)} \circ \mathcal{C}_{12}^{(2)} \to \mathcal{C}_{02}^{(2)}$ of 1-morphisms $\mathcal{C}_2^{(2)} \to \mathcal{C}_0^{(2)}$.

which satisfy

- $\mathcal{C}_{023}^{(3)} \circ (\mathcal{C}_{012}^{(3)} \otimes \mathtt{Id}) = \mathcal{C}_{013}^{(3)} \circ (\mathtt{Id} \otimes \mathcal{C}_{123}^{(3)})$
- $\mathcal{C}_{00}^{(0)} = \mathtt{Id}$

Suppose that $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ are stacks on G.

Definition 2.6. A 1-morphism $\underline{\phi}\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a pair $\underline{\phi} = (\phi_0, \phi_{01})$ which consists of

(1) a 1-morphism $\phi_0\colon \mathcal{C} \to \mathcal{D}$ of stacks on N_0G
(2) a 2-isomorphism $\phi_{01}\colon \phi_0^{(1)} \circ \mathcal{C}_{01} \to \mathcal{D}_{01} \circ \phi_1^{(1)}$ of 1-morphisms $\mathcal{C}_1^{(1)} \to \mathcal{D}_0^{(1)}$

which satisfy

- $(\mathcal{D}_{012} \otimes \mathtt{Id}_{\phi_2^{(2)}}) \circ (\mathtt{Id}_{\mathcal{D}_{01}^{(2)}} \otimes \phi_{12}^{(2)}) \circ (\phi_{01}^{(2)} \otimes \mathtt{Id}_{\mathcal{C}_{12}^{(2)}}) = \phi_{02}^{(2)} \circ (\mathtt{Id}_{\phi_0^{(2)}} \otimes \mathcal{C}_{012})$
- $\phi_{00}^{(0)} = \mathtt{Id}$

Suppose that $\underline{\phi}$ and $\underline{\psi}$ are 1-morphisms $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$.

Definition 2.7. A 2-morphism $b\colon \underline{\phi} \to \underline{\psi}$ is a 2-morphism $b\colon \phi_0 \to \psi_0$ which satisfies $\psi_{01} \circ (b_0^{(1)} \otimes \mathtt{Id}_{\mathcal{C}_{01}}) = (\mathtt{Id}_{\mathcal{D}_{01}} \otimes b_1^{(1)}) \circ \phi_{01}$

Suppose that $\phi; G \to G'$ is a morphism of étale categories and $\underline{\mathcal{C}} = (\mathcal{C}, \mathcal{C}_{01}, \mathcal{C}_{012})$ is a stack on G'. The inverse image $\phi^{-1}\underline{\mathcal{C}}$ is the stack on G given by the triple $(\mathcal{D}, \mathcal{D}_{01}, \mathcal{D}_{012})$ with $\mathcal{D} = (N_0\phi)^{-1}\mathcal{C}$, $\mathcal{D}_{01}$ equal to the image of $\mathcal{C}_{01}$ under the map

$$\begin{aligned}(N_1\phi)^*\colon \operatorname{Hom}(\mathcal{C}_1^{(1)}, \mathcal{C}_0^{(1)}) &\\ \to \operatorname{Hom}((N_1\phi)^{-1}\mathcal{C}_1^{(1)}, (N_1\phi)^{-1}\mathcal{C}_0^{(1)}) &\\ \cong \operatorname{Hom}(\mathcal{D}_1^{(1)}, \mathcal{D}_0^{(1)})&\end{aligned}$$

and $\mathcal{D}_{012}$ induced by $\mathcal{C}_{012}$ in a similar fashion.

A stack $\underline{\mathcal{C}}$ on G gives rise to a cosimplicial stack on NG which we denote $\underline{\mathcal{C}}_\Delta$. The latter is defined as follows. For $n = 0, 1, 2, \ldots$ let $\underline{\mathcal{C}}_\Delta^n =$

$\mathcal{C}_0^{(n)}$. For $f\colon [p] \to [q]$ in Δ the corresponding structure 1-morphism $\underline{\mathcal{C}}_{\Delta f}$ is defined as

$$NG(f)^{-1}\underline{\mathcal{C}}^p_\Delta = \mathcal{C}^{(p)}_{f(0)} \xrightarrow{\mathcal{C}^{(q)}_{0f(0)}} \mathcal{C}^{(q)}_0 = \underline{\mathcal{C}}^q_\Delta.$$

For a pair of morphisms $[p] \xrightarrow{f} [q] \xrightarrow{g} [r]$ let $\underline{\mathcal{C}}_{\Delta f,g} = \mathcal{C}_{0g(0)g(f(0))}$.

2.3.5. Gerbes on étale categories. Suppose that G is an étale category, $\underline{A}$ is an abelian sheaf on G.

An $\underline{A}$-gerbe $\underline{\mathcal{G}}$ on G is a stack on G such that

(1) $\mathcal{G}$ is an A^0-gerbe on N_0G ;
(2) the 1-morphism $\mathcal{C}_{01}$ is compatible with the morphism A_{01}.

If $\underline{\mathcal{G}}$ is an $\underline{A}$-gerbe G, then $\underline{\mathcal{G}}_\Delta$ is a cosimplicial $\underline{A}_\Delta$-gerbe on NG.

2.3.6. The category of embeddings. Below we recall the basics of the "category of G-embeddings" associated with an étale category G and a basis of the topology on the space of objects of G, which was introduced by I. Moerdijk in [25].

For an étale category G and a basis $\mathbb{B}$ for the topology on N_0G we denote by $\mathcal{E}_\mathbb{B}(G)$ or, simply by $\mathcal{E}$, the following category.

The space of objects is given by $N_0\mathcal{E} = \coprod_{U\in\mathbb{B}} U$, the disjoint union of the elements of $\mathbb{B}$. Thus, the space of morphism decomposes as

$$N_1\mathcal{E} = \coprod_{(U,V)\in\mathbb{B}\times\mathbb{B}} (N_1\mathcal{E})_{(U,V)},$$

where $(N_1\mathcal{E})_{(U,V)}$ is defined by the pull-back square

$$\begin{array}{ccc} (N_1\mathcal{E})_{(U,V)} & \longrightarrow & N_1\mathcal{E} \\ \downarrow & & \downarrow{\scriptstyle (d_0^1,d_0^0)} \\ U\times V & \longrightarrow & N_0\mathcal{E}\times N_0\mathcal{E}. \end{array}$$

(the bottom arrow being the inclusion). Now, $(N_1\mathcal{E})_{(U,V)} \subset (N_1G)_{(U,V)}$ (the latter defined in the same manner as the former replacing $\mathcal{E}_\mathbb{B}(G)$ by G) is given by $(N_1\mathcal{E})_{(U,V)} = \coprod_\sigma \sigma(U)$, where $\sigma\colon U \to (N_1G)_{(U,V)}$ is a section of the "source" projection $d_0^1\colon (N_1G)_{(U,V)} \to U$ such that the composition $U \xrightarrow{\sigma} \sigma(U) \xrightarrow{d_0^0} V$ is an embedding.

With the structure (source, target, composition, "identity") maps induced from those of G, $\mathcal{E}_\mathbb{B}(G)$ is an étale category equipped with the canonical functor

$$\lambda_\mathbb{B}(G)\colon \mathcal{E}_\mathbb{B}(G) \to G.$$

Note that the maps $N_i\lambda_\mathbb{B}(G)\colon N_i\mathcal{E}_\mathbb{B}(G) \to N_iG$ are étale surjections.

The canonical map $i\colon N_0G \to G$ induces the map of the respective embedding categories $\mathcal{E}_{\mathbb{B}}(i)\colon \mathcal{E}_{\mathbb{B}}(N_0G) \to \mathcal{E}_{\mathbb{B}}(G)$ and the diagram

$$\begin{array}{ccc} \mathcal{E}_{\mathbb{B}}(N_0G) & \xrightarrow{\mathcal{E}_{\mathbb{B}}(i)} & \mathcal{E}_{\mathbb{B}}(G) \\ {\scriptstyle \lambda_{\mathbb{B}}(N_0G)}\downarrow & & \downarrow{\scriptstyle \lambda_{\mathbb{B}}(G)} \\ N_0G & \xrightarrow{i} & G \end{array} \tag{2.3.1}$$

is commutative.

2.3.7. First consider the particular case when $G = X$ is a space and $\mathbb{B}$ is a basis for the topology on X. For an open subset $V \subseteq X$ let $\mathbb{B}\cap V \subseteq \mathbb{B} = \{U \in \mathbb{B} | U \subseteq V\}$. There is an obvious embedding $\mathcal{E}_{\mathbb{B}\cap V}(V) \to \mathcal{E}_{\mathbb{B}}(X)$.

For $\underline{F} = (F, F_{01}) \in \mathrm{Sh}(\mathcal{E}_{\mathbb{B}}(X))$, $V \subseteq X$ let $\lambda_!\underline{F}$ denote the presheaf on X defined by

$$V \mapsto \Gamma(\mathcal{E}_{\mathbb{B}\cap V}(V); \underline{F}) = \varprojlim_{U \in \mathbb{B}\cap V} F(U),$$

where $\mathbb{B} \cap V$ is partially ordered by inclusion. The presheaf $\lambda_!\underline{F}$ is in fact a sheaf. It is characterized by the following property: $\lambda_!\underline{F}|_U = F|_U$ for any $U \in \mathbb{B}$.

Let $\lambda^{-1}\lambda_!\underline{F} = (H, H_{01})$. For V an open subset of $N_0\mathcal{E}_{\mathbb{B}}(X)$ we have

$$H(V) = (\lambda_!\underline{F})(V) = \varprojlim_{U \in \mathbb{B}\cap V} F(U) = F(V)$$

naturally in V. We leave it to the reader to check that this extends to an isomorphism $\lambda^{-1}\lambda_!\underline{F} = \underline{F}$ natural in $\underline{F}$, i.e. to an isomorphism of functors $\lambda^{-1}\lambda_! = \mathtt{Id}$.

On the other hand, for $H \in \mathrm{Sh}(X)$, V an open subset of X, put $\lambda^{-1}H = (F, F_{01})$; we have

$$(\lambda_!\lambda^{-1}H)(V) = \varprojlim_{U \in \mathbb{B}\cap V} F(U) = \varprojlim_{U \in \mathbb{B}\cap V} H(U) = H(V)$$

naturally in V. We leave it to the reader to check that this extends to an isomorphism $H = \lambda_!\lambda^{-1}H$ natural in H, i.e. to an isomorphism of functors $\mathtt{Id} = \lambda_!\lambda^{-1}$.

2.3.8. We now consider the general case. To simplify notations we put $\mathcal{E} := \mathcal{E}_{\mathbb{B}}(G)$, $\mathcal{E}' := \mathcal{E}_{\mathbb{B}}(N_0G)$, $\lambda := \lambda_{\mathbb{B}}(G)$, $\lambda' := \lambda_{\mathbb{B}}(N_0G)$. Let $\underline{F} = (F, F_{01}) \in \mathrm{Sh}(\mathcal{E})$ and let $\underline{F}' = (F', F'_{01}) := \mathcal{E}_{\mathbb{B}}(i)^{-1}\underline{F}$.

Applying the construction of 2.3.7 to $\underline{F}'$ we obtain the sheaf $\lambda'_!\underline{F}'$ on N_0G. Note that $(N_0\lambda)^{-1}\lambda'_!\underline{F}' = F$. The properties of the map $N_1\lambda$

imply that the pull-back map

$$\begin{aligned}(N_1\lambda)^*\colon \operatorname{Hom}&((\lambda'_!\underline{F}')^{(1)}_1,(\lambda'_!\underline{F}')^{(1)}_0)\\ &= \Gamma(N_1G;\underline{\operatorname{Hom}}((\lambda'_!\underline{F}')^{(1)}_1,(\lambda'_!\underline{F}')^{(1)}_0)\\ &\qquad\to \Gamma(N_1\mathcal{E};(N_1\lambda)^{-1}\underline{\operatorname{Hom}}((\lambda'_!\underline{F}')^{(1)}_1,(\lambda'_!\underline{F}')^{(1)}_0)\end{aligned}$$

is injective. Combining the latter with the canonical isomorphisms

$$\begin{aligned}(N_1\lambda)^{-1}\underline{\operatorname{Hom}}((\lambda'_!\underline{F}')^{(1)}_1,(\lambda'_!\underline{F}')^{(1)}_0) &= \underline{\operatorname{Hom}}((N_1\lambda)^{-1}(\lambda'_!\underline{F}')^{(1)}_1,(N_1\lambda)^{-1}(\lambda'_!\underline{F}')^{(1)}_0)\\ &= \underline{\operatorname{Hom}}(((N_0\lambda)^{-1}\lambda'_!\underline{F}')^{(1)}_1,((N_0\lambda)^{-1}\lambda'_!\underline{F}')^{(1)}_0)\\ &= \underline{\operatorname{Hom}}(F_1^{(1)},F_0^{(1)})\end{aligned}$$

we obtain the injective map

$$\operatorname{Hom}((\lambda'_!\underline{F}')^{(1)}_1,(\lambda'_!\underline{F}')^{(1)}_0)\to\operatorname{Hom}(F_1^{(1)},F_0^{(1)}). \tag{2.3.2}$$

We leave it to the reader to verify that the map F_{01} lies in the image of (2.3.2); let $(\lambda'_!\underline{F}')_{01}$ denote the corresponding element of $\operatorname{Hom}(t^{-1}\lambda'_!\underline{F}',s^{-1}\lambda'_!\underline{F}'))$. The pair $(\lambda'_!\underline{F}',(\lambda'_!\underline{F}')_{01})$ is easily seen to determine a sheaf on G henceforth denoted $\lambda_!\underline{F}$. The assignment $\underline{F}\mapsto\lambda_!\underline{F}$ extends to a functor, denoted

$$\lambda_!\colon \operatorname{Sh}(\mathcal{E})\to\operatorname{Sh}(G).$$

quasi-inverse to the inverse image functor λ^{-1}. Hence, $\lambda^{-1}(*)=*$ and, for $\underline{F}\in\operatorname{Sh}(G)$ the map $\Gamma(G;\underline{F})\to\Gamma(\mathcal{E};\lambda^{-1}\underline{F})$ is an isomorphism. Similarly, $\lambda_!(*)=*$ and, for $\underline{H}\in\operatorname{Sh}(\mathcal{E})$ the map $\Gamma(\mathcal{E};\underline{H})\to\Gamma(G;\lambda_!\underline{H})$ is an isomorphism.

2.3.9. The functors λ^{-1} and $\lambda_!$ restrict to mutually quasi-inverse exact equivalences of abelian categories

$$\lambda^{-1}\colon \operatorname{ShAb}(G)\rightleftarrows\operatorname{ShAb}(\mathcal{E})\colon\lambda_!$$

The morphism $\lambda^*\colon \mathbf{R}\Gamma(G;\underline{F})\to\mathbf{R}\Gamma(\mathcal{E};\lambda^{-1}\underline{F})$ is an isomorphism in the derived category.

Suppose that $\underline{F}\in\operatorname{ShAb}(G)$ is $\mathbb{B}$-acyclic, i.e., for any $U\in\mathbb{B}$, $i\neq 0$, $H^i(U;F)=0$. Then, the composition $C(\Gamma(N\mathcal{E};\lambda^{-1}\underline{F}))\to\mathbf{R}\Gamma(\mathcal{E};\lambda^{-1}\underline{F})\cong\mathbf{R}\Gamma(G;\underline{F})$ is an isomorphism in the derived category.

2.3.10. Suppose given G, $\mathbb{B}$ as in 2.3.6; we proceed in the notations introduced in 2.3.8. The functor of inverse image under λ establishes an equivalence of (2-)categories of stack on G and those on $\mathcal{E}$. Below we sketch the construction of the quasi-inverse along the lines of 2.3.7 and 2.3.8.

First consider the case $G = X$ a space. Let $\mathcal{C}$ be a stack on $\mathcal{E}$. For an open subset $V \subseteq X$ let

$$\lambda_!\mathcal{C}(V) \colon = \varprojlim_{U \in \mathbb{B} \cap V} \mathcal{C}(U),$$

where the latter is described in [18], Definition 19.1.6. Briefly, an object of $\lambda_!\mathcal{C}(V)$ is a pair (A, φ) which consists of a function $A \colon \mathbb{B} \cap V \ni U \mapsto A_U \in \mathcal{C}(U)$ and a function $\varphi \colon (U \subseteq U') \mapsto (\varphi_{UU'} \colon A'_U|_U \xrightarrow{\sim} A_U)$; the latter is required to satisfy a kind of a cocycle condition with respect to compositions of inclusions of basic open sets. For (A, φ), (A', φ') as above the assignment $\mathbb{B} \cap V \ni U \mapsto \mathrm{Hom}_{\mathcal{C}}(A_U, A'_U)$ extends to a presheaf on $\mathbb{B}$. By definition,

$$\mathrm{Hom}_{\lambda_!\mathcal{C}(V)}((A, \varphi), (A', \varphi')) = \varprojlim_{U \in \mathbb{B} \cap V} \mathrm{Hom}_{\mathcal{C}}(A_U, A'_U).$$

The assignment $V \mapsto \lambda_!\mathcal{C}(V)$ extends to a stack on X denoted $\lambda_!\mathcal{C}$. We have natural equivalences

$$\lambda^{-1}\lambda_!\underline{\mathcal{D}} \cong \underline{\mathcal{D}}, \quad \lambda_!\lambda^{-1}\mathcal{C} \cong \mathcal{C}.$$

Continuing with the general case, let $\underline{\mathcal{C}}$ be a stack on $\mathcal{E}$. The stack $\lambda_!\underline{\mathcal{C}}$ on G is given by the triple $(\mathcal{D}, \mathcal{D}_{01}, \mathcal{D}_{012})$ with $\mathcal{D} = \lambda'_!\mathcal{E}_{\mathbb{B}}(i)^{-1}\mathcal{C}$. The morphisms $\mathcal{D}_{01}$ and $\mathcal{D}_{012}$ are induced, respectively, by $\mathcal{C}_{01}$ and $\mathcal{C}_{012}$. We omit the details.

2.4. Jet bundle. Let $\mathtt{pr}_i \colon X \times X \to X$, $i = 1, 2$, denote the projection on the i^{th} factor. The restriction of the canonical map

$$\mathtt{pr}_i^* \colon \mathtt{pr}_i^{-1}\mathcal{O}_X \to \mathcal{O}_{X \times X}$$

to the subsheaf $\mathcal{O}_X$ takes values in the subsheaf $\mathcal{O}_{X \times X/}$ hence induces the map

$$\mathtt{pr}_i^* \colon \mathcal{O}_X \to (\mathtt{pr}_i)_*\mathcal{O}_{X \times X}.$$

Let $\Delta_X \colon X \to X \times X$ denote the diagonal embedding. It follows from the Leibniz rule that the restriction of the canonical map

$$\Delta_X^* \colon \mathcal{O}_{X \times X} \to (\Delta_X)_*\mathcal{O}_X$$

to the subsheaf $\mathcal{O}_{X \times X}$ takes values in the subsheaf $(\Delta_X)_*\mathcal{O}_X$. Let

$$\mathcal{I}_X \colon = \ker(\Delta_X^*) \cap \mathcal{O}_{X \times X}.$$

The sheaf $\mathcal{I}_X$ plays the role of the defining ideal of the "diagonal embedding $X \to X \times X$": there is a short exact sequence of sheaves on $X \times X$

$$0 \to \mathcal{I}_X \to \mathcal{O}_{X \times X} \to (\Delta_X)_*\mathcal{O}_X \to 0$$

For a locally-free $\mathcal{O}_X$-module of finite rank $\mathcal{E}$ let

$$\begin{aligned}\mathcal{J}^k_X(\mathcal{E}) &:= (\mathtt{pr}_1)_*\left(\mathcal{O}_{X\times X}/\mathcal{I}^{k+1}_X \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_X} \mathtt{pr}_2^{-1}\mathcal{E}\right),\\ \mathcal{J}^k_X &:= \mathcal{J}^k_X(\mathcal{O}_X).\end{aligned}$$

It is clear from the above definition that $\mathcal{J}^k_X$ is, in a natural way, a commutative algebra and $\mathcal{J}^k_X(\mathcal{E})$ is a $\mathcal{J}^k_X$-module.

Let

$$\mathbf{1}^{(k)}\colon \mathcal{O}_X \to \mathcal{J}^k_X$$

denote the composition

$$\mathcal{O}_X \xrightarrow{\mathtt{pr}_1^*} (\mathtt{pr}_1)_*\mathcal{O}_{X\times X} \to \mathcal{J}^k_X$$

In what follows, unless stated explicitly otherwise, we regard $\mathcal{J}^k_X(\mathcal{E})$ as a $\mathcal{O}_X$-module via the map $\mathbf{1}^{(k)}$.

Let

$$j^k\colon \mathcal{E} \to \mathcal{J}^k_X(\mathcal{E})$$

denote the composition

$$\mathcal{E} \xrightarrow{e\mapsto 1\otimes e} (\mathtt{pr}_1)_*\mathcal{O}_{X\times X} \otimes_{\mathbb{C}} \mathcal{E} \to \mathcal{J}^k_X(\mathcal{E})$$

The map j^k is not $\mathcal{O}_X$-linear unless $k = 0$.

For $0 \le k \le l$ the inclusion $\mathcal{I}^{l+1}_X \to \mathcal{I}^{k+1}_X$ induces the surjective map $\pi_{l,k}\colon \mathcal{J}^l_X(\mathcal{E}) \to \mathcal{J}^k_X(\mathcal{E})$. The sheaves $\mathcal{J}^k_X(\mathcal{E})$, $k = 0, 1, \ldots$ together with the maps $\pi_{l,k}$, $k \le l$ form an inverse system. Let $\mathcal{J}_X(\mathcal{E}) = \mathcal{J}^\infty_X(\mathcal{E}) := \varprojlim \mathcal{J}^k_X(\mathcal{E})$. Thus, $\mathcal{J}_X(\mathcal{E})$ carries a natural topology.

The maps $\mathbf{1}^{(k)}$ (respectively, j^k), $k = 0, 1, 2, \ldots$ are compatible with the projections $\pi_{l,k}$, i.e. $\pi_{l,k} \circ \mathbf{1}^{(l)} = \mathbf{1}^{(k)}$ (respectively, $\pi_{l,k} \circ j^l = j^k$). Let $\mathbf{1} := \varprojlim \mathbf{1}^{(k)}$, $j^\infty := \varprojlim j^k$.

Let

$$\begin{aligned}d_1\colon \mathcal{O}_{X\times X} \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_X} \mathtt{pr}_2^{-1}\mathcal{E} \to\\ \to \mathtt{pr}_1^{-1}\Omega^1_X \otimes_{\mathtt{pr}_1^{-1}\mathcal{O}_X} \mathcal{O}_{X\times X} \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_X} \mathtt{pr}_2^{-1}\mathcal{E}\end{aligned}$$

denote the exterior derivative along the first factor. It satisfies

$$\begin{aligned}d_1(\mathcal{I}^{k+1}_X \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_X} \mathtt{pr}_2^{-1}\mathcal{E}) \subset\\ \mathtt{pr}_1^{-1}\Omega^1_X \otimes_{\mathtt{pr}_1^{-1}\mathcal{O}_X} \mathcal{I}^k_X \otimes_{\mathtt{pr}_2^{-1}\mathcal{O}_X} \mathtt{pr}_2^{-1}\mathcal{E}\end{aligned}$$

for each k and, therefore, induces the map

$$d_1^{(k)}\colon \mathcal{J}^k(\mathcal{E}) \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{J}^{k-1}(\mathcal{E})$$

The maps $d_1^{(k)}$ for different values of k are compatible with the maps $\pi_{l,k}$ giving rise to the *canonical flat connection*

$$\nabla^{can}_{\mathcal{E}}\colon \mathcal{J}_X(\mathcal{E}) \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{J}_X(\mathcal{E})\ .$$

Let

$$\begin{aligned}
\mathcal{J}_X(\mathcal{E}) &:= \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{J}_X(\mathcal{E}) \\
\mathcal{J}_X &:= \mathcal{J}_X(\mathcal{O}_X) \\
\overline{\mathcal{J}}_X &:= \mathcal{J}_X/\mathbf{1}(\mathcal{O}_X) \\
\mathcal{J}_{X,0} &:= \ker(\mathcal{J}_X(\mathcal{O}_X) \to \mathcal{O}_X)
\end{aligned}$$

Here and below by abuse of notation we write $(.) \otimes_{\mathcal{O}_X} \mathcal{J}_X(\mathcal{E})$ for $\varprojlim (.) \otimes_{\mathcal{O}_X} \mathcal{J}^k(\mathcal{E})$.

The canonical flat connection extends to the flat connection

$$\nabla^{can}_{\mathcal{E}}\colon \mathcal{J}_X(\mathcal{E}) \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{J}_X(\mathcal{E}).$$

2.4.1. De Rham complexes. Suppose that $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\nabla\colon \mathcal{F} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{F}$ is a flat connection. The flat connection ∇ extends uniquely to a differential ∇ on $\Omega^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{F}$ subject to the Leibniz rule with respect to the $\Omega^\bullet_X$-module structure. We will make use of the following notation :

$$(\Omega^i_X \otimes_{\mathcal{O}_X} \mathcal{F})^{cl} := \ker(\Omega^i_X \otimes_{\mathcal{O}_X} \mathcal{F} \xrightarrow{\nabla} \Omega^{i+1}_X \otimes_{\mathcal{O}_X} \mathcal{F})$$

Suppose that $(\mathcal{F}^\bullet, d)$ is a complex of $\mathcal{O}_X$-modules with a flat connection $\nabla = (\nabla^i)_{i\in\mathbb{Z}}$, i.e. for each $i \in \mathbb{Z}$, ∇^i is a flat connection on $\mathcal{F}^i$ and $[d, \nabla] = 0$. Then, $(\Omega^\bullet_X \otimes_{\mathcal{O}_X} \mathcal{F}^\bullet, \nabla, \mathtt{Id} \otimes d)$ is a double complex. We denote by $\mathtt{DR}(\mathcal{F})$ the associated simple complex.

2.5. Characteristic classes of cosimplicial $\mathcal{O}^\times$-gerbes. In this section we consider an étale simplicial manifold X, i.e. a simplicial manifold $X\colon [p] \mapsto X_p$ such that

(2.5.1) for each morphism $f\colon [p] \to [q]$ in Δ,
the induced map $X(f)\colon X_q \to X_p$ is *étale*

As a consequence, the collection of sheaves $\mathcal{O}_{X_p}$ (respectively, $\mathcal{J}_{X_p}$, etc.) form a *special* (see Definition 2.3) cosimplicial sheaf on X which will be denoted $\mathcal{O}_X$ (respectively, $\mathcal{J}_X$, etc.)

The goal of this section is to associate to a cosimplicial $\mathcal{O}^\times$-gerbe $\mathcal{S}$ on X a cohomology class $[\mathcal{S}] \in H^2(|X|; \mathtt{DR}(\overline{\mathcal{J}}_{|X|}))$, where $|X|$ is the subdivision of X (see 2.1.5).

2.5.1. Gerbes on manifolds. Suppose that Y is a manifold. We begin by sketching a construction which associated to an $\mathcal{O}_Y^\times$-gerbe $\mathcal{S}$ on Y a characteristic class in $H^2(Y;\mathtt{DR}(\mathcal{J}_Y/\mathcal{O}_Y))$ which lifts the more familiar de Rham characteristic class $[\mathcal{S}]_{dR}\in H^3_{dR}(Y)$.

The map $\mathbf{1}\colon \mathcal{O}_Y\to\mathcal{J}_Y$ of sheaves of rings induces the map of sheaves of abelian groups $\mathbf{1}\colon \mathcal{O}_Y^\times\to\mathcal{J}_Y^\times$. Let $\overline{(\,.\,)}\colon \mathcal{J}_Y^{(\times)}\to\mathcal{J}_Y^{(\times)}/\mathcal{O}_Y^{(\times)}$ denote the projection.

Suppose that $\mathcal{S}$ is an $\mathcal{O}_Y^\times$-gerbe. The composition

$$\mathcal{O}_Y^\times \xrightarrow{\mathbf{1}} \mathcal{J}_Y^\times \xrightarrow{\nabla^{can}\log} (\Omega^1_Y\otimes\mathcal{J}_Y)^{cl}$$

gives rise to the $\Omega^1_Y\otimes\mathcal{J}_Y$-gerbe $\nabla^{can}\log\mathbf{1}\mathcal{S}$. Let

$$\eth\colon (\Omega^1_Y\otimes\mathcal{J}_Y)^{cl}[1]\to\nabla^{can}\log\mathbf{1}\mathcal{S}$$

be a trivialization of the latter.

Since $\nabla^{can}\circ\nabla^{can}=0$, the $(\Omega^2_Y\otimes\mathcal{J}_Y)^{cl}$-gerbe $\nabla^{can}\nabla^{can}\log\mathbf{1}\mathcal{S}$ is canonically trivialized. Therefore, the trivialization $\nabla^{can}\eth$ of $\nabla^{can}\log\mathbf{1}\mathcal{S}$ induced by $\eth$ may (and will) be considered as a $(\Omega^2_Y\otimes\mathcal{J}_Y)^{cl}$-torsor. Let

$$B\colon (\Omega^2_Y\otimes\mathcal{J}_Y)^{cl}\to\nabla^{can}\eth$$

be a trivialization of the $\nabla^{can}\eth$.

Since $\nabla^{can}\log\circ\mathbf{1}=\mathbf{1}\circ d\log$, it follows that the $(\Omega^1_Y\otimes\overline{\mathcal{J}}_Y)^{cl}$-gerbe $\overline{\nabla^{can}\log\mathbf{1}\mathcal{S}}$ is canonically trivialized. Therefore, its trivialization $\overline{\eth}$ may (and will) be considered as a $(\Omega^1_Y\otimes\overline{\mathcal{J}}_Y)^{cl}$-torsor. Moreover, since $\nabla^{can}\nabla^{can}=0$ the $(\Omega^2_Y\otimes\overline{\mathcal{J}}_Y)^{cl}$-torsor $\nabla^{can}\overline{\eth}\cong\overline{\nabla^{can}\eth}$ is canonically trivialized, the trivialization induced by B is a section $\overline{B}$ of $(\Omega^2_Y\otimes\overline{\mathcal{J}}_Y)^{cl}$, i.e. $\overline{B}$ is a cocycle in $\Gamma(X;\mathtt{DR}(\overline{\mathcal{J}}_Y))$.

One can show that the class of $\overline{B}$ in $H^2(X;\mathtt{DR}(\overline{\mathcal{J}}_Y))$

(1) depends only on $\mathcal{S}$ and not on the auxiliary choices of $\eth$ and B made
(2) coincides with the image of the class of $\mathcal{S}$ under the map $H^2(Y;\mathcal{O}_Y^\times)\to H^2(X;\mathtt{DR}(\overline{\mathcal{J}}_Y))$ induced by the composition

$$\mathcal{O}_Y^\times\xrightarrow{\overline{(\cdot)}}\mathcal{O}_Y^\times/\mathbb{C}^\times\xrightarrow{\log}\mathcal{O}_Y/\mathbb{C}\xrightarrow{j^\infty}\mathtt{DR}(\overline{\mathcal{J}}_Y)$$

On the other hand, $H=\nabla^{can}B$ is a trivialization of the canonically trivialized (by $\nabla^{can}\circ\nabla^{can}=0$) $(\Omega^3_Y\otimes\mathcal{J}_Y)^{cl}$-torsor $\nabla^{can}\nabla^{can}\eth$, hence a section of $(\Omega^3_Y\otimes\mathcal{J}_Y)^{cl}$ which, clearly, satisfies $\overline{H}=0$, i.e. is a closed 3-form. Moreover, as is clear from the construction, the class of H in $H^3_{dR}(Y)$ is the image of the class of $\overline{B}$ under the boundary map $H^2(X;\mathtt{DR}(\overline{\mathcal{J}}_Y))\to H^3_{dR}(Y)$.

Below we present a generalization of the above construction to étale simplicial manifolds.

2.5.2. Suppose that $\mathcal{G}$ is a cosimplicial $(\Omega^1_X \otimes \mathcal{J}_X)^{cl}$-gerbe on X. An example of such is $\nabla^{can} \log \mathbf{1}(\mathcal{S})$, where $\mathcal{S}$ is an $\mathcal{O}_X^\times$-gerbe.

Consider the $(\Omega^2_X \otimes \mathcal{J}_X)^{cl}$-gerbe $\nabla^{can}\mathcal{G}$. Since the composition

$$(\Omega^1_{X_p} \otimes \mathcal{J}_{X_p})^{cl} \hookrightarrow \Omega^1_{X_p} \otimes \mathcal{J}_{X_p} \xrightarrow{\nabla^{can}} (\Omega^2_{X_p} \otimes \mathcal{J}_{X_p})^{cl}$$

is equal to zero, $\nabla^{can}\mathcal{G}^p$ is canonically trivialized for all $p = 0, 1, 2, \ldots$. Therefore, for a morphism $f\colon [p] \to [q]$, $\nabla^{can}\mathcal{G}_f$, being a morphism of trivialized gerbes, may (and will) be regarded as a $(\Omega^2_{X_q} \otimes \mathcal{J}_{X_q})^{cl}$-torsor.

Assume given, for each $p = 0, 1, 2, \ldots$

(i) a choice of trivialization

$$\eth^p\colon (\Omega^1_{X_p} \otimes \mathcal{J}_{X_p})^{cl} \to \mathcal{G}^p \ ;$$

it induces the trivialization

$$\nabla^{can}\eth^p\colon (\Omega^2_{X_p} \otimes \mathcal{J}_{X_p})^{cl}[1] \to \nabla^{can}\mathcal{G}^p \ .$$

Since $\nabla^{can}\mathcal{G}^p$ is canonically trivialized, $\nabla^{can}\eth^p$ may (and will) be regarded as a $(\Omega^2_{X_p} \otimes \mathcal{J}_{X_p})^{cl}$-torsor.

(ii) a choice of trivialization

$$B^p\colon (\Omega^2_{X_p} \otimes \mathcal{J}_{X_p})^{cl} \to \nabla^{can}\eth^p \ .$$

For a morphism $f\colon [p] \to [q]$ in Δ the trivialization $\eth^p$ induces the trivialization

$$X(f)^{-1}\eth^p\colon (\Omega^1_{X_q} \otimes \mathcal{J}_{X_q})^{cl} = X(f)^{-1}(\Omega^1_{X_p} \otimes \mathcal{J}_{X_p})^{cl} \to X(f)^{-1}\mathcal{G}^p \ ;$$

thus, $\mathcal{G}_f$ is a morphism of trivialized gerbes, i.e. a $(\Omega^1_{X_q} \otimes \mathcal{J}_{X_q})^{cl}$-torsor.

Assume given for each morphism $f\colon [p] \to [q]$

(iii) a choice of trivialization $\beta_f\colon (\Omega^1_{X_q} \otimes \mathcal{J}_{X_q})^{cl} \to \mathcal{G}_f$.

For a pair of composable arrows $[p] \xrightarrow{f} [q] \xrightarrow{g} [r]$ the discrepancy

$$\mathcal{G}_{f,g}\colon = (\beta_g + X(g)^{-1}\beta_f) - \beta_{g\circ f}$$

is global section of $(\Omega^1_{X_r} \otimes \mathcal{J}_{X_r})^{cl}$. Since the map $\mathcal{J}_{X_r,0} \xrightarrow{\nabla^{can}} (\Omega^1_{X_r} \otimes \mathcal{J}_{X_r})^{cl}$ is an isomorphism there is a unique section $\beta_{f,g}$ of $\mathcal{J}_{X_r,0}$ such that

$$\nabla^{can}\beta_{f,g} = \mathcal{G}_{f,g} \ . \tag{2.5.2}$$

Lemma 2.8. *For any triple of composable arrows $[p] \xrightarrow{f} [q] \xrightarrow{g} [r] \xrightarrow{h} [s]$ the relation $h_*\beta_{f,g} = \beta_{g,h} - \beta_{gf,h} + \beta_{f,hg}$ holds.*

Proof. A direct calculation shows that

$$h_*\mathcal{G}_{f,g} = \mathcal{G}_{g,h} - \mathcal{G}_{gf,h} + \mathcal{G}_{f,hg} \ .$$

Therefore ((2.5.2)),

$$\nabla^{can}(h_*\beta_{f,g} - (\beta_{g,h} - \beta_{gf,h} + \beta_{f,hg})) = 0 \ .$$

The map $\mathcal{J}_{X_r,0} \xrightarrow{\nabla^{can}} (\Omega^1_{X_r} \otimes \mathcal{J}_{X_r})^{cl}$ is an isomorphism, hence the claim follows. □

2.5.3. We proceed in the notations introduced above. Let

$$\mathtt{pr}_X \colon \Delta^q \times X_p \to X_p$$

denote the projection. Let

$$\mathtt{pr}_X^\dagger \colon \mathtt{pr}_X^{-1}(\Omega^1_{X_p} \otimes_{\mathcal{O}_{X_p}} \mathcal{J}_{X_p}) \to \Omega^1_{X_p\times\Delta^q} \otimes_{\mathtt{pr}_X^{-1}\mathcal{O}_{X_p}} \mathtt{pr}_X^{-1}\mathcal{J}_{X_p} \tag{2.5.3}$$

denote the canonical map. For a $(\Omega^1_{X_p} \otimes_{\mathcal{O}_{X_p}} \mathcal{J}_{X_p})$-gerbe $\mathcal{G}$, let

$$\mathtt{pr}_X^*\mathcal{G} = (\mathtt{pr}_X^\dagger)_*(\mathtt{pr}_X^{-1}\mathcal{G})$$

Let

$$\widetilde{\nabla}^{can} = \mathtt{pr}_X^*\nabla^{can} = d_\Delta \otimes \mathtt{Id} + \mathtt{Id} \otimes \nabla^{can} \ .$$

Since $\sum_{i=0}^q t_i = 1$ the composition

$$\Omega^1_{X_p\times\Delta^q} \otimes_{\mathtt{pr}_X^{-1}\mathcal{O}_{X_p}} \mathtt{pr}_X^{-1}\mathcal{J}_{X_p} \to (\Omega^1_{X_p\times\Delta^q} \otimes_{\mathtt{pr}_X^{-1}\mathcal{O}_{X_p}} \mathtt{pr}_X^{-1}\mathcal{J}_{X_p})^{\times(p+1)} \to \Omega^1_{X_p\times\Delta^q} \otimes_{\mathtt{pr}_X^{-1}\mathcal{O}_{X_p}} \mathtt{pr}_X^{-1}\mathcal{J}_{X_p}$$

of the diagonal map with $(\alpha_0, \ldots, \alpha_p) \mapsto \sum_{i=0}^q t_i \cdot \alpha_i$ is the identity map, it follows that the composition

$$\mathtt{pr}_X^{-1}(\Omega^1_{X_p} \otimes_{\mathcal{O}_{X_p}} \mathcal{J}_{X_p}) \xrightarrow{\mathtt{pr}_X^\dagger} \Omega^1_{X_p\times\Delta^q} \otimes_{\mathtt{pr}_X^{-1}\mathcal{O}_{X_p}} \mathtt{pr}_X^{-1}\mathcal{J}_{X_p} \to (\Omega^1_{X_p\times\Delta^q} \otimes_{\mathtt{pr}_X^{-1}\mathcal{O}_{X_p}} \mathtt{pr}_X^{-1}\mathcal{J}_{X_p})^{\times(p+1)} \to \Omega^1_{X_p\times\Delta^q} \otimes_{\mathtt{pr}_X^{-1}\mathcal{O}_{X_p}} \mathtt{pr}_X^{-1}\mathcal{J}_{X_p} \tag{2.5.4}$$

is equal to the map $\mathtt{pr}_X^\dagger$. Since, by definition, the $(\Omega^1_{X_p\times\Delta^q} \otimes_{\mathtt{pr}_X^{-1}\mathcal{O}_{X_p}} \mathtt{pr}_X^{-1}\mathcal{J}_{X_p})$-gerbe $\sum_{i=0}^p t_i \cdot \mathtt{pr}_X^*\mathcal{G}^p$ is obtained from $\mathtt{pr}_X^{-1}\mathcal{G}^p$ via the "change of structure group" along the composition (2.5.4), it is canonically equivalent to $\mathtt{pr}_X^*\mathcal{G}^p$.

Consider a simplex $\lambda \colon [n] \to \Delta$. Let

$$\eth^\lambda \colon = \sum_{i=0}^n t_i \cdot \mathtt{pr}_X^*\mathcal{G}_{\lambda(in)}(X(\lambda(in))^{-1}\eth^{\lambda(i)});$$

thus $\eth^\lambda$ is a trivialization of $\mathtt{pr}_X^*\mathcal{G}^{\lambda(n)}$. Since $\lambda(in)_*\eth^{\lambda(i)} = \eth^{\lambda(n)} - \mathcal{G}_{\lambda(in)}$

$$\eth^\lambda = \mathtt{pr}_X^*\eth^{\lambda(n)} - \sum_{i=0}^n t_i \cdot \mathtt{pr}_X^*\mathcal{G}_{\lambda(in)}$$

Therefore,

$$\widetilde{\nabla}^{can}\eth^\lambda \quad = \quad \sum_{i=0}^n t_i \ \cdot \ \mathtt{pr}_X^*\lambda(in)_*\nabla^{can}\eth^{\lambda(i)} \ - \ \sum_{i=0}^n dt_i \ \wedge \ \mathtt{pr}_X^*\mathcal{G}_{\lambda(in)}.$$

Let B^λ denote the trivialization of $\widetilde{\nabla}^{can}\eth^\lambda$ given by

$$B^\lambda = \sum_{i=0}^n t_i \cdot \mathtt{pr}_X^*\lambda(in)_*B^{\lambda(i)} - \sum_{i=0}^n dt_i \wedge \mathtt{pr}_X^*\beta_{\lambda(in)} - \widetilde{\nabla}^{can}\left(\sum_{0\le i<j\le n}(t_i dt_j - t_j dt_i)\wedge \mathtt{pr}_X^*\beta_{\lambda;ij}\right) \tag{2.5.5}$$

where $\beta_{\lambda;ij} = \beta_{\lambda((ij)),\lambda((jn))}$.

2.5.4. Suppose that $\mu\colon [m] \to \Delta$ is another simplex and a morphism $\phi\colon [m] \to [n]$ such that $\mu = \lambda \circ \phi$, i.e. ϕ is a morphism of simplices $\mu \to \lambda$. Let $f\colon \mu(m) \to \lambda(n)$ denote the map $\lambda(\phi(m) \to n)$. The map f induces the maps

$$\Delta^m \times X_{\mu(m)} \xleftarrow{\mathtt{Id}\times X(f)} \Delta^m \times X_{\lambda(n)} \xrightarrow{f\times \mathtt{Id}} \Delta^n \times X_{\lambda(n)}$$

Proposition 2.9. *In the notations introduced above*

(1) $(\mathtt{Id} \times X(f))^*\eth^\mu = (\phi \times \mathtt{Id})^*\eth^\lambda$
(2) $(\mathtt{Id} \times X(f))^*B^\mu = (\phi \times \mathtt{Id})^*B^\lambda$

Proof. For $\phi\colon [m] \to [n]$ the induced map $\phi^*\colon \Omega^\bullet_{\Delta^n} \to \Omega^\bullet_{\Delta^m}$ is given by $\phi^*(t_j) = \sum_{\phi(i)=j} t_i$.

$$(\mathtt{Id} \times X(f))^*\eth^\mu =$$

$$(\mathtt{Id} \times X(f))^* \sum_{i=0}^m t_i \cdot \mathtt{pr}^*_{\mu(m)}\mu((im))_*\eth^{\mu(i)} =$$

$$\sum_{i=0}^m t_i \cdot \mathtt{pr}^*_{\lambda(n)}f_*\mu((im))_*\eth^{\mu(i)} =$$

$$\sum_{j=0}^n \sum_{\phi(i)=j} t_i \cdot \mathtt{pr}^*_{\lambda(n)}\lambda((jn))_*\eth^{\lambda(j)} = (\phi \times \mathtt{Id})^*\eth^\lambda$$

Therefore, $(\mathtt{Id}\times X(f))^*\widetilde{\nabla}^{can}\widetilde{\partial}^{\mu} = (\phi\times\mathtt{Id})^*\widetilde{\nabla}^{can}\widetilde{\partial}^{\lambda}$.

By Lemma 2.10, it suffices to verify the second claim for $\phi\colon [1]\to[n]$. Let $k=\lambda(\phi(0))$, $l=\lambda(\phi(1))$, $p=\lambda(n)$. With these notations $f=(lp)$ and the left hand side reduces to

$$t_0\cdot \mathtt{pr}_p^*(kp)_*B^k + t_1\cdot \mathtt{pr}_p^*(lp)_*B^l + dt_0\wedge \mathtt{pr}_p^*(lp)_*\beta_{(kl)}$$

while the right hand side reads

$$\begin{aligned} t_0\cdot \mathtt{pr}_p^*(kp)_*B^k + t_1\cdot \mathtt{pr}_p^*(lp)_*B^l + \\ dt_0\wedge \mathtt{pr}_p^*\beta_{(kp)} + dt_1\wedge \mathtt{pr}_p^*\beta_{(lp)} + \\ (t_1dt_0 - t_0dt_1)\wedge \mathtt{pr}_p^*\alpha_{(kl),(lp)} \end{aligned}$$

Using $t_1dt_0 - t_0dt_1 = dt_0 = -dt_1$ and the definition of $\alpha_{(kl),(lp)}$ one sees that the two expressions are indeed equal. □

Lemma 2.10. *The map*

$$\Omega_n^{\leq 1}\to \prod_{\mathrm{Hom}_\Delta([1],[n])}\Omega_1$$

induced by the maps $\Delta((ij))^*\colon \Omega_n^{\leq 1}\to\Omega_1$, $i\leq j$, is injective on the subspace of form with coefficients of degree at most one.

Proof. Left to the reader. □

2.5.5. Suppose that $\mathcal{F}$ is a cosimplicial $\Omega_X^{1,cl}$-gerbe on X.

The composition

$$\mathcal{O}_X \xrightarrow{\mathbf{1}} \mathcal{J}_X \xrightarrow{\nabla^{can}} \Omega_X^1\otimes\mathcal{J}_X$$

is a derivation, hence factors canonically through the map denoted

$$\mathbf{1}\colon \Omega_X^1\to\Omega_X^1\otimes\mathcal{J}_X$$

which maps closed forms to closed forms.

Put $\mathcal{G}=\mathbf{1}(\mathcal{F})$ in 2.5.2. Since the composition

$$\Omega_{X_p}^{1,cl}\to(\Omega_{X_p}^1\otimes\mathcal{J}_{X_p})^{cl}\to(\Omega_{X_p}^1\otimes\overline{\mathcal{J}}_{X_p})^{cl}$$

is equal to the zero map, the $(\Omega_{X_p}^1\otimes\overline{\mathcal{J}}_{X_p})^{cl}$-gerbe $\overline{\mathcal{G}^p}$ is canonically trivialized for each p. Therefore, $\overline{\widetilde{\partial}^p}$ may (and will) be regarded as a $(\Omega_{X_p}^1\otimes\overline{\mathcal{J}}_{X_p})^{cl}$-torsor. The $(\Omega_{X_p}^2\otimes\overline{\mathcal{J}}_{X_p})^{cl}$-torsor $\nabla^{can}\overline{\widetilde{\partial}^p}$ is canonically trivialized, therefore, $\overline{B^p}$ may (and will) be regarded as a section of $(\Omega_{X_p}^2\otimes\overline{\mathcal{J}}_{X_p})^{cl}$.

Since, for a morphism $f\colon[p]\to[q]$, $\mathcal{G}_f=\mathbf{1}(\mathcal{F}_f)$, it follows that $\overline{\mathcal{G}_f}$ is canonically trivialized, hence $\overline{\beta_f}$ may (and will) be regarded as a section of $(\Omega_{X_p}^1\otimes\overline{\mathcal{J}}_{X_p})^{cl}$.

It follows that, for a simplex $\lambda\colon [n] \to \Delta$, the formula (2.5.5) gives rise to a section $\overline{B^\lambda}$ of $\Omega^2_{\Delta^n \times X_{\lambda(n)}} \otimes \mathtt{pr}_X^* \overline{\mathcal{J}}_{X_{\lambda(n)}}$ which clearly satisfies $\widetilde{\nabla}^{can} \overline{B^\lambda} = 0$.

Proposition 2.11. *In the notations of 2.5.5*

(1) *the assignment* $\overline{B}\colon \lambda \mapsto \overline{B}^\lambda$ *defines a cycle in the complex* $\mathtt{Tot}(\Gamma(|X|; \mathtt{DR}(\overline{\mathcal{J}}_{|X|}))$

(2) *The class of* $\overline{B}$ *in* $H^2(\mathtt{Tot}(\Gamma(|X|; \mathtt{DR}(\overline{\mathcal{J}}_{|X|}))))$ *coincides with the image of the class* $[\mathcal{F}] \in H^2(X; \Omega_X^{1,cl})$ *of the gerbe* $\mathcal{F}$ *under the composition*

$$H^2(X; \Omega_X^{1,cl}) \to H^2(|X|; |\Omega_X^{1,cl}|) \to H^2(\mathtt{Tot}(\Gamma(|X|; \mathtt{DR}(\overline{\mathcal{J}}_{|X|}))))$$

where the first map is the canonical isomorphism (see 2.2.3) and the second map is induced by the map $\Omega_X^{1,cl} \cong \mathcal{O}_X/\mathbb{C} \xrightarrow{j^\infty} \overline{\mathcal{J}}_X$. *In particular, the class of* $\overline{B}$ *does not depend on the choices made.*

2.5.6. In the rest of the section we will assume that $\mathcal{S}$ is a cosimplicial $\mathcal{O}_X^\times$-gerbe on X such that the gerbes $\mathcal{S}^p$ are trivial for all p, i.e. $\mathcal{S}^p = \mathcal{O}_X^\times[1]$. Our present goal is to obtain simplified expressions for B and $\overline{B}$ in this case.

Since a morphism of trivial $\mathcal{O}^\times$-gerbes is an $\mathcal{O}^\times$-torsor (equivalently, a locally free $\mathcal{O}$-module of rank one), a cosimplicial gerbe $\mathcal{S}$ as above is given by the following collection of data:

(1) for each morphism $f\colon [p] \to [q]$ in Δ a line bundle $\mathcal{S}_f$ on X_q,

(2) for each pair of morphisms $[p] \xrightarrow{f} [q] \xrightarrow{g} [r]$ an isomorphism $\mathcal{S}_{f,g}\colon \mathcal{S}_g \otimes X(g)^*(\mathcal{S}_f) \to \mathcal{S}_{g\circ f}$ of line bundles on X_r

These are subject to the associativity condition: for a triple of composable arrows $[p] \xrightarrow{f} [q] \xrightarrow{g} [r] \xrightarrow{h} [s]$

$$\mathcal{S}_{g\circ f,h} \otimes X(h)^{-1}\mathcal{S}_{f,g} = \mathcal{S}_{f,h\circ g} \otimes \mathcal{S}_{g,h}$$

In order to calculate the characteristic class of $\mathcal{S}$ we will follow the method (and notations) of 2.5.2 and 2.5.5 with $\mathcal{F} = d\log(\mathcal{S})$ and the following choices:

(1) $\tilde{\partial}^p$ is the canonical isomorphism $\Omega^1_{X_p} \otimes \mathcal{J}_{X_p}[1] = \nabla^{can} \log \mathbf{1}(\mathcal{O}^\times_{X_p})[1]$, i.e. is given by the trivial torsor $\Omega^1_{X_p} \otimes \mathcal{J}_{X_p} = \nabla^{can} \log \mathbf{1}(\mathcal{O}^\times_{X_p})$;

(2) B^p is the canonical isomorphism $\Omega^2_{X_p} \otimes \mathcal{J}_{X_p} = \nabla^{can}(\Omega^1_{X_p} \otimes \mathcal{J}_{X_p})$.

Then, $\mathcal{G}_f = \nabla^{can} \log \mathbf{1}(\mathcal{S}_f) = \mathbf{1} d\log(\mathcal{S}_f)$ and B_f is equal to the canonical trivialization of $\nabla^{can}\mathcal{G}_f = \nabla^{can}\nabla^{can} \log \mathbf{1}(\mathcal{S}_f)$ (stemming from $\nabla^{can} \circ \nabla^{can} = 0$).

For each $f\colon [p]\to[q]$ we choose

(3) a $\mathcal{J}_{X_q}$-linear isomorphism $\sigma_f\colon \mathcal{S}_f\otimes\mathcal{J}_{X_q}\to\mathcal{J}_{X_q}(\mathcal{S}_f)$ which reduced to the identity modulo $\mathcal{J}_{X_q,0}$

(4) a trivialization of $d\log(\mathcal{S}_f)$, i.e. a connection ∇_f on the line bundle $\mathcal{S}_f$

Let $\beta_f=\sigma_f^{-1}\circ\nabla^{can}_{\mathcal{S}_f}\circ\sigma_f$; thus, β_f is a trivialization of $\mathcal{G}_f$. The choice of ∇_f determines another trivialization of $\mathcal{G}_f$, namely $\mathbf{1}(\nabla_f)=\nabla_f\otimes\mathtt{Id}+\mathtt{Id}\otimes\nabla^{can}_{\mathcal{O}}$.

Let $F_f\colon =\beta_f-\mathbf{1}(\nabla_f)\in\Gamma(X_q;\Omega^1_{X_q}\otimes\mathcal{J}_{X_q})$. Flatness of $\nabla^{can}_{\mathcal{S}_f}$ implies that F_f satisfies $\nabla^{can}_{\mathcal{O}}F_f+\mathbf{1}(c(\nabla_f))=0$ which implies that $\nabla^{can}\overline{F}_f=0$.

For a pair of composable arrows $[p]\xrightarrow{f}[q]\xrightarrow{g}[r]$ we have

$$\nabla^{can}\overline{\beta}_{f,g}=\overline{F}_g+X(g)^*\overline{F}_f-\overline{F}_{g\circ f}\ . \tag{2.5.6}$$

With these notations, for $\lambda\colon[n]\to\Delta$, we have:

$$B^\lambda=-\sum_i dt_i\wedge(\mathbf{1}(\nabla_{\lambda(in)})+F_{\lambda(in)})-\widetilde{\nabla}^{can}\left(\sum_{0\le i<j\le n}(t_idt_j-t_jdt_i)\wedge\mathtt{pr}_X^*\beta_{\lambda;ij}\right)$$

and

$$\overline{B}^\lambda=-\sum_i dt_i\wedge\overline{F}_{\lambda(in)}-\widetilde{\nabla}^{can}\left(\sum_{0\le i<j\le n}(t_idt_j-t_jdt_i)\wedge\mathtt{pr}_X^*\overline{\beta}_{\lambda;ij}\right)\ . \tag{2.5.7}$$

Proposition 2.12. *In the notations of 2.5.6*

(1) *the assignment $\overline{B}\colon\lambda\mapsto\overline{B}^\lambda$ defines a cycle in the complex $\mathtt{Tot}(\Gamma(|X|;\mathtt{DR}(\overline{\mathcal{J}}_{|X|}))$*

(2) *The class of $\overline{B}$ in $H^2(\mathtt{Tot}(\Gamma(|X|;\mathtt{DR}(\overline{\mathcal{J}}_{|X|}))))$ coincides with the image of the class $[\mathcal{S}]\in H^2(X;\mathcal{O}_X^\times)$ of the gerbe $\mathcal{S}$ under the composition*

$$H^2(X;\mathcal{O}_X^\times)\to H^2(|X|;|\mathcal{O}_X^\times|)\to H^2(\mathtt{Tot}(\Gamma(|X|;\mathtt{DR}(\overline{\mathcal{J}}_{|X|}))))$$

where the first map is the canonical isomorphism (see 2.2.3) and the second map is induced by the map $\mathcal{O}_X^\times\to\mathcal{O}_X^\times/\mathbb{C}^\times\xrightarrow{\log}\mathcal{O}_X/\mathbb{C}\xrightarrow{j^\infty}\overline{\mathcal{J}}_X$. In particular, the class of $\overline{B}$ does not depend on the choices made.

3. Deformations and DGLA

3.1. Deligne 2-groupoid.

3.1.1. Deligne 2-groupoid. In this subsection we review the construction of Deligne 2-groupoid of a nilpotent differential graded algebra (DGLA). We follow [16, 15] and the references therein.

Suppose that $\mathfrak{g}$ is a nilpotent DGLA such that $\mathfrak{g}^i = 0$ for $i < -1$.

A Maurer-Cartan element of $\mathfrak{g}$ is an element $\gamma \in \mathfrak{g}^1$ satisfying

$$d\gamma + \frac{1}{2}[\gamma, \gamma] = 0. \tag{3.1.1}$$

We denote by $\mathrm{MC}^2(\mathfrak{g})_0$ the set of Maurer-Cartan elements of $\mathfrak{g}$.

The unipotent group $\exp \mathfrak{g}^0$ acts on the set of Maurer-Cartan elements of $\mathfrak{g}$ by gauge equivalences. This action is given by the formula

$$(\exp X) \cdot \gamma = \gamma - \sum_{i=0}^{\infty} \frac{(\mathtt{ad}\, X)^i}{(i+1)!}(dX + [\gamma, X])$$

If $\exp X$ is a gauge equivalence between two Maurer-Cartan elements γ_1 and $\gamma_2 = (\exp X) \cdot \gamma_1$ then

$$d + \mathtt{ad}\, \gamma_2 = \mathrm{Ad} \exp X\, (d + \mathtt{ad}\, \gamma_1). \tag{3.1.2}$$

We denote by $\mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$ the set of gauge equivalences between γ_1, γ_2. The composition

$$\mathrm{MC}^2(\mathfrak{g})_1(\gamma_2, \gamma_3) \times \mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2) \to \mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_3)$$

is given by the product in the group $\exp \mathfrak{g}^0$.

If $\gamma \in \mathrm{MC}^2(\mathfrak{g})_0$ we can define a Lie bracket $[\cdot, \cdot]_\gamma$ on $\mathfrak{g}^{-1}$ by

$$[a,\, b]_\gamma = [a,\, db + [\gamma,\, b]]. \tag{3.1.3}$$

With this bracket $\mathfrak{g}^{-1}$ becomes a nilpotent Lie algebra. We denote by $\exp_\gamma \mathfrak{g}^{-1}$ the corresponding unipotent group, and by $\exp_\gamma$ the corresponding exponential map $\mathfrak{g}^{-1} \to \exp_\gamma \mathfrak{g}^{-1}$. If γ_1, γ_2 are two Maurer-Cartan elements, then the group $\exp_\gamma \mathfrak{g}^{-1}$ acts on $\mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$. Let $\exp_\gamma t \in \exp_\gamma \mathfrak{g}^{-1}$ and let $\exp X \in \mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$. Then

$$(\exp_\gamma t) \cdot (\exp X) = \exp(dt + [\gamma, t]) \exp X \in \exp \mathfrak{g}^0$$

Such an element $\exp_\gamma t$ is called a 2-morphism between $\exp X$ and $(\exp t) \cdot (\exp X)$. We denote by $\mathrm{MC}^2(\mathfrak{g})_2(\exp X, \exp Y)$ the set of 2-morphisms between $\exp X$ and $\exp Y$. This set is endowed with a vertical composition given by the product in the group $\exp_\gamma \mathfrak{g}^{-1}$.

Let $\gamma_1, \gamma_2, \gamma_3 \in \mathrm{MC}^2(\mathfrak{g})_0$. Let $\exp X_{12}, \exp Y_{12} \in \mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$ and $\exp X_{23}, \exp Y_{23} \in \mathrm{MC}^2(\mathfrak{g})_1(\gamma_2, \gamma_3)$. Then one defines the horizontal composition

$$\otimes\colon \mathrm{MC}^2(\mathfrak{g})_2(\exp X_{23}, \exp Y_{23}) \times \mathrm{MC}^2(\mathfrak{g})_2(\exp X_{12}, \exp Y_{12}) \to \mathrm{MC}^2(\mathfrak{g})_2(\exp X_{23} \exp X_{12}, \exp X_{23} \exp Y_{12})$$

as follows. Let $\exp_{\gamma_2} t_{12} \in \mathrm{MC}^2(\mathfrak{g})_2(\exp X_{12}, \exp Y_{12})$ and let $\exp_{\gamma_3} t_{23} \in \mathrm{MC}^2(\mathfrak{g})_2(\exp X_{23}, \exp Y_{23})$. Then

$$\exp_{\gamma_3} t_{23} \otimes \exp_{\gamma_2} t_{12} = \exp_{\gamma_3} t_{23} \exp_{\gamma_3}(e^{\mathtt{ad}\, X_{23}}(t_{12}))$$

To summarize, the data described above forms a 2-groupoid which we denote by $\mathrm{MC}^2(\mathfrak{g})$ as follows:

(1) the set of objects is $\mathrm{MC}^2(\mathfrak{g})_0$
(2) the groupoid of morphisms $\mathrm{MC}^2(\mathfrak{g})(\gamma_1, \gamma_2)$, $\gamma_i \in \mathrm{MC}^2(\mathfrak{g})_0$ consists of:
   - objects i.e. 1-morphisms in $\mathrm{MC}^2(\mathfrak{g})$ are given by $\mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$ – the gauge transformations between γ_1 and γ_2.
   - morphisms between $\exp X$, $\exp Y \in \mathrm{MC}^2(\mathfrak{g})_1(\gamma_1, \gamma_2)$ are given by $\mathrm{MC}^2(\mathfrak{g})_2(\exp X, \exp Y)$.

A morphism of nilpotent DGLA $\phi\colon \mathfrak{g} \to \mathfrak{h}$ induces a functor $\phi\colon \mathrm{MC}^2(\mathfrak{g}) \to \mathrm{MC}^2(\mathfrak{g})$.

We have the following important result ([17], [16] and references therein).

Theorem 3.1. *Suppose that $\phi\colon \mathfrak{g} \to \mathfrak{h}$ is a quasi-isomorphism of DGLA and let $\mathfrak{m}$ be a nilpotent commutative ring. Then the induced map $\phi\colon \mathrm{MC}^2(\mathfrak{g} \otimes \mathfrak{m}) \to \mathrm{MC}^2(\mathfrak{h} \otimes \mathfrak{m})$ is an equivalence of 2-groupoids.*

Suppose now that $\mathfrak{G}\colon [n] \to \mathfrak{G}^n$ is a cosimplicial DGLA. We assume that each $\mathfrak{G}^n$ is a nilpotent DGLA. We denote its component of degree i by $\mathfrak{G}^{n,i}$ and assume that $\mathfrak{G}^{n,i} = 0$ for $i < -1$. Then the morphism of complexes

$$\ker(\mathfrak{G}^0 \rightrightarrows \mathfrak{G}^1) \to \mathtt{Tot}(\mathfrak{G}) \tag{3.1.4}$$

(cf. (2.1.3)) is a morphism of DGLA.

Proposition 3.2. *Assume that $\mathfrak{G}$ satisfies the following condition:*

$$\text{for all } i \in \mathbb{Z},\ H^p(\mathfrak{G}^{\bullet,i}) = 0 \text{ for } p \neq 0. \tag{3.1.5}$$

Then the morphism of 2-groupoids

$$\mathrm{MC}^2(\ker(\mathfrak{G}^0 \rightrightarrows \mathfrak{G}^1)) \to \mathrm{MC}^2(\mathtt{Tot}(\mathfrak{G}))$$

induced by (3.1.4) *is an equivalence.*

Proof. Follows from Lemma 2.2 and Theorem 3.1 □

3.2. Deformations and the Deligne 2-groupoid.

3.2.1. Pseudo-tensor categories. In order to give a unified treatment of the deformation complex we will employ the formalism of pseudo-tensor categories. We refer the reader to [3] for details.

Let X be a topological space. The category $\mathrm{Sh}_k(X)$ has a canonical structure of a pseudo-tensor $\mathrm{Sh}_k(X)$-category. In particular, for any finite set I, an I-family of $\{L_i\}_{i\in I}$ of sheaves and a sheaf L we have the *sheaf*

$$P_I^{\mathrm{Sh}_k(X)}(\{L_i\}, L) = \underline{\mathrm{Hom}}_k(\otimes_{i\in I} L_i, L)$$

of I-operations.

In what follows we consider not necessarily full pseudo-tensor $\mathrm{Sh}_k(X)$-subcategories Ψ of $\mathrm{Sh}_k(X)$. Given such a category Ψ, with the notations as above, we have the subsheaf $P_I^{\Psi}(\{L_i\}, L)$ of $P_I^{\mathrm{Sh}_k(X)}(\{L_i\}, L)$.

We shall always assume that the pseudo-tensor category Ψ under consideration satisfies the following additional assumptions:

(1) For any object L of Ψ and any finite dimensional k-vector space V the sheaf $L \otimes_k V$ is in Ψ;
(2) for any k-linear map of finite dimensional vector spaces $f\colon V \to W$ the map $\mathtt{Id} \otimes f$ belongs to $\Gamma(X; P_{\mathbf{1}}^{\Psi}((L \otimes_k V)^{\mathbf{1}}, L \otimes_k W))$.

3.2.2. Examples of pseudo-tensor categories.

`DIFF` Suppose that X is a manifold. Let `DIFF` denote the following pseudo-tensor category. The objects of `DIFF` are locally free modules of finite rank over $\mathcal{O}_X$. In the notations introduced above, $P_I^{\mathtt{DIFF}}(\{L_i\}, L)$ is defined to be the sheaf of multidifferential operators $\bigotimes_{i\in I} L_i \to L$ (the tensor product is over $\mathbb{C}$).

`JET` For X as in the previous example, let `JET` denote the pseudo-tensor category whose objects are locally free modules of finite rank over $\mathcal{J}_X$. In the notations introduced above, $P_I^{\mathtt{JET}}(\{L_i\}, L)$ is defined to be the sheaf of continuous $\mathcal{O}_X$-(multi)linear maps $\bigotimes_{i\in I} L_i \to L$ (the tensor product is over $\mathcal{O}_X$).

`DEF` For Ψ as in 3.2.1 and an Artin k-algebra R let $\widetilde{\Psi(R)}$ denote the following pseudo-tensor category. An object of $\widetilde{\Psi(R)}$ is an R-module in Ψ (i.e. an object M of Ψ together with a morphism of k-algebras $R \to \Gamma(X; P_{\mathbf{1}}^{\Psi}(M^{\mathbf{1}}, M))$) which *locally on* X is R-linearly isomorphic in Ψ to an object of the form $L \otimes_k R$, where L is an object in Ψ. In the notations introduced above, the sheaf

$P_I^{\widetilde{\Psi(R)}}(\{M_i\}, M)$ of I-operations is defined as the subsheaf of R-multilinear maps in $P_I^{\Psi}(\{M_i\}, M)$.

Let $\Psi(R)$ denote the full subcategory of $\widetilde{\Psi(R)}$ whose objects are isomorphic to objects of the form $L \otimes_k R$, L in Ψ.

Note that a morphism $R \to S$ of Artin k-algebras induces the functor $(\,.\,) \otimes_R S \colon \widetilde{\Psi(R)} \to \widetilde{\Psi(S)}$ which restricts to the functor $(\,.\,) \otimes_R S \colon \Psi(R) \to \Psi(S)$. It is clear that the assignment $R \mapsto \widetilde{\Psi(R)}$ (respectively, $R \mapsto \Psi(R)$) defines a functor on ArtAlg_k and the inclusion $\Psi(R)(\,.\,) \to \widetilde{\Psi(\,.\,)}$ is a morphism thereof.

3.2.3. Hochschild cochains. For $n = 1, 2, \ldots$ we denote by $\mathbf{n}$ the set $\{1, 2, \ldots, n\}$. For an object $\mathcal{A}$ of Ψ we denote by $\mathcal{A}^{\mathbf{n}}$ the $\mathbf{n}$-collection of objects of Ψ each of which is equal to $\mathcal{A}$ and set

$$C^n(\mathcal{A}) = P_{\mathbf{n}}^{\Psi}(\mathcal{A}^{\mathbf{n}}, \mathcal{A}) \ ,$$

and $C^0(\mathcal{A}) \colon = \mathcal{A}$. The sheaf $C^n(\mathcal{A})$ is called the sheaf of Hochschild cochains on $\mathcal{A}$ of degree n.

The graded sheaf of vector spaces $\mathfrak{g}(\mathcal{A}) \colon = C^\bullet(\mathcal{A})[1]$ has a canonical structure of a graded Lie algebra under the Gerstenhaber bracket denoted by $[\ ,\ ]$ below. Namely, $C^\bullet(\mathcal{A})[1]$ is canonically isomorphic to the (graded) Lie algebra of derivations of the free associative co-algebra generated by $\mathcal{A}[1]$.

For an operation $m \in \Gamma(X; P_{\mathbf{2}}^{\Psi}(\mathcal{A}^{\mathbf{2}}, \mathcal{A})) = \Gamma(X; C^2(\mathcal{A})) = \Gamma(X; \mathfrak{g}(\mathcal{A}))$ the associativity of m is equivalent to the condition $[m, m] = 0$.

Suppose that $\mathcal{A}$ is an associative algebra with the product m as above. Let $\delta = [m, .]$. Thus, δ is a derivation of the graded Lie algebra $\mathfrak{g}(\mathcal{A})$ of degree one. The associativity of m implies that $\delta \circ \delta = 0$, i.e. δ defines a differential on $\mathfrak{g}(\mathcal{A})$ called the Hochschild differential.

The algebra is called *unital* if it is endowed with a global section $1 \in \Gamma(X; \mathcal{A})$ with the usual properties with respect to the product m. For a unital algebra the subsheaf of *normalized* cochains (of degree n) $\overline{C}^n(\mathcal{A})$ of $C^n(\mathcal{A})$ is defined as the subsheaf of Hochschild cochains which vanish whenever evaluated on 1 as one the arguments for $n > 0$; by definition, $\overline{C}^0(\mathcal{A}) = C^0(\mathcal{A})$.

The graded subsheaf $\overline{C}^\bullet(\mathcal{A})[1]$ is closed under the Gerstenhaber bracket and the action of the Hochschild differential, and the inclusion

$$\overline{C}^\bullet(\mathcal{A})[1] \hookrightarrow C^\bullet(\mathcal{A})[1]$$

is a quasi-isomorphism of DGLA.

Suppose in addition that R is a commutative Artin k-algebra with the (nilpotent) maximal ideal $\mathfrak{m}_R$. Then, $\Gamma(X;\mathfrak{g}(A)\otimes_k\mathfrak{m}_R)$ is a nilpotent DGLA concentrated in degree greater than or equal to -1. Therefore, the Deligne 2-groupoid $\mathrm{MC}^2(\mathfrak{g}(A)\otimes_k\mathfrak{m}_R)$ is defined. Moreover, it is clear that the assignment

$$R\mapsto \mathrm{MC}^2(\Gamma(X;\mathfrak{g}(\mathcal{A})\otimes_k\mathfrak{m}_R)$$

extends to a functor on the category of commutative Artin algebras.

3.2.4. DIFF and JET. Unless otherwise stated, from now on a locally free module over $\mathcal{O}_X$ (respectively, $\mathcal{J}_X$) of finite rank is understood as an object of the pseudo-tensor category DIFF (respectively, JET) defined in 3.2.2.

Suppose that $\mathcal{A}$ is an $\mathcal{O}_X$-Azumaya algebra, i.e. a sheaf of (central) $\mathcal{O}_X$-algebras locally isomorphic to $\mathtt{Mat}_n(\mathcal{O}_X)$. The canonical flat connection $\nabla^{can}_{\mathcal{A}}$ on $\mathcal{J}_X(\mathcal{A})$ induces the flat connection, still denoted $\nabla^{can}_{\mathcal{A}}$, on $C^n(\mathcal{J}_X(\mathcal{A}))$.

The flat connection $\nabla^{can}_{\mathcal{A}}$ acts by derivation of the Gerstenhaber bracket which commute with the Hochschild differential δ. Hence, we have the DGLA $\Omega^\bullet_X\otimes_{\mathcal{O}_X}C^\bullet(\mathcal{J}_X(\mathcal{A}))[1]$ with the differential $\nabla^{can}_{\mathcal{A}}+\delta$.

Lemma 3.3. *The de Rham complex* $\mathtt{DR}(C^n(\mathcal{J}_X(\mathcal{A}))) := (\Omega^\bullet_X\otimes_{\mathcal{O}_X}C^n(\mathcal{J}_X(\mathcal{A})),\nabla^{can}_{\mathcal{A}})$ *satisfies*

1. $H^i\mathtt{DR}(C^n(\mathcal{J}_X(\mathcal{A})))=0$ *for* $i\neq 0$
2. *The map* $j^\infty\colon C^n(\mathcal{A})\to C^n(\mathcal{J}_X(\mathcal{A}))$ *is an isomorphism onto* $H^0\mathtt{DR}(C^n(\mathcal{J}_X(\mathcal{A})))$.

Corollary 3.4. *The map*

$$j^\infty\colon C^\bullet(\mathcal{A})[1]\to\Omega^\bullet_X\otimes_{\mathcal{O}_X}C^\bullet(\mathcal{J}_X(\mathcal{A}))[1]$$

is a quasi-isomorphism of DGLA.

3.2.5. Star products. Suppose that $\mathcal{A}$ is an object of Ψ with an associative multiplication m and the unit 1 as above.

We denote by ArtAlg_k the category of (finitely generated, commutative, unital) Artin k-algebras. Recall that an Artin k-algebra is a local k-algebra R with the maximal ideal $\mathfrak{m}_R$ which is nilpotent, i.e. there exists a positive integer N such that $\mathfrak{m}_R^N=0$; in particular, an Artin k-algebra is a finite dimensional k-vector space. There is a canonical isomorphism $R/\mathfrak{m}_R\cong k$. A morphism of Artin algebras $\phi\colon R\to S$ is a k-algebra homomorphism which satisfies $\phi(\mathfrak{m}_R)\subseteq\mathfrak{m}_S$.

Definition 3.5. For $R\in\mathrm{ArtAlg}_k$ an *R-star product* on $\mathcal{A}$ is an R-bilinear operation $m'\in\Gamma(X;P^\Psi_{\mathbf{2}}(\mathcal{A}^{\mathbf{2}},\mathcal{A}\otimes_k R))$ which is associative and

whose image in $\Gamma(X; P_{\mathbf{2}}^{\Psi}(\mathcal{A}^{\mathbf{2}}, \mathcal{A}))$ (under composition with the canonical map $\mathcal{A} \otimes_k R \to \mathcal{A}$) coincides with m.

An R-star product m' as in 3.5 determines an R-bilinear operation $m' \in P_{\mathbf{2}}^{\Psi}((\mathcal{A} \otimes_k R)^{\mathbf{2}}, \mathcal{A} \otimes_k R)$ which endows $\mathcal{A} \otimes_k R$ with a structure of a unital associative R-algebra.

The 2-category of R-star products on $\mathcal{A}$, denoted $\mathrm{Def}(\mathcal{A})(R)$, is defined as follows:

- objects are the R-star products on $\mathcal{A}$,
- a 1-morphism $\phi\colon m_1 \to m_2$ between the R-star products m_i is an operation $\phi \in P_{\mathbf{1}}^{\Psi}(\mathcal{A}^{\mathbf{1}}, \mathcal{A} \otimes_k R)$ whose image in $\phi \in P_{\mathbf{1}}^{\Psi}(\mathcal{A}^{\mathbf{1}}, \mathcal{A})$ (under the composition with the canonical map $\mathcal{A} \otimes_k R \to \mathcal{A}$) is equal to the identity, and whose R-linear extension $\phi \in P_{\mathbf{1}}^{\Psi}(\mathcal{A}^{\mathbf{1}}, \mathcal{A} \otimes_k R)$ is a morphism of R-algebras.
- a 2-morphism $b\colon \phi \to \psi$, where $\phi, \psi\colon m_1 \to m_2$ are two 1-morphisms, are elements $b \in 1 \otimes 1 + \Gamma(X; \mathcal{A} \otimes_k \mathfrak{m}_R) \subset \Gamma(X; \mathcal{A} \otimes_k R)$ such that $m_2(\phi(a), b) = m_2(b, \psi(a))$ for all $a \in \mathcal{A} \otimes_k R$.

It follows easily from the above definition and the nilpotency of $\mathfrak{m}_R$ that $\mathrm{Def}(\mathcal{A})(R)$ is a 2-groupoid.

Note that $\mathrm{Def}(\mathcal{A})(R)$ is non-empty: it contains the trivial deformation, i.e. the star product, still denoted m, which is the R-bilinear extension of the product on $\mathcal{A}$.

It is clear that the assignment $R \mapsto \mathrm{Def}(\mathcal{A})(R)$ extends to a functor on ArtAlg_k.

3.2.6. Star products and the Deligne 2-groupoid. We continue in notations introduced above. In particular, we are considering a sheaf of associative k-algebras $\mathcal{A}$ with the product $m \in \Gamma(X; C^2(\mathcal{A}))$. The product m determines the element, still denoted m in $\Gamma(X; \mathfrak{g}^1(\mathcal{A}) \otimes_k R)$ for any commutative Artin k-algebra R, hence, the Hochschild differential $\delta\colon = [m, \ ]$ in $\mathfrak{g}(\mathcal{A}) \otimes_k R$.

Suppose that m' is an R-star product on $\mathcal{A}$. Since $\mu(m')\colon = m' - m = 0 \mod \mathfrak{m}_R$ we have $\mu(m') \in \mathfrak{g}^1(\mathcal{A}) \otimes_k \mathfrak{m}_R$. Moreover, the associativity of m' implies that $\mu(m')$ satisfies the Maurer-Cartan equation, i.e. $\mu(m') \in \mathrm{MC}^2(\Gamma(X; \mathfrak{g}(\mathcal{A}) \otimes_k \mathfrak{m}_R))_0$.

It is easy to see that the assignment $m' \mapsto \mu(m')$ extends to a functor

$$\mathrm{Def}(\mathcal{A})(R) \to \mathrm{MC}^2(\Gamma(X; \mathfrak{g}(\mathcal{A}) \otimes_k \mathfrak{m}_R)) \ . \tag{3.2.1}$$

The following proposition is well-known (cf. [14, 16, 15]).

Proposition 3.6. *The functor* (3.2.1) *is an isomorphism of* 2*-groupoids.*

3.3. Matrix Azumaya algebras.

3.3.1. Azumaya algebras. Suppose that K is a sheaf of commutative $\mathbb{C}$-algebras.

An *Azumaya K-algebra on X* is a sheaf of central K-algebras which is a twisted form of (i.e. locally isomorphic to) $\mathtt{Mat}_n(K)$ for suitable n. In our applications the algebra K will be either $\mathcal{O}_X$ of $\mathcal{J}_X$.

There is a central extension of Lie algebras

$$0 \to K \to \mathcal{A} \xrightarrow{\delta} \mathtt{Der}_K(\mathcal{A}) \to 0 \tag{3.3.1}$$

where the map δ is given by $\delta(a)\colon b \mapsto [a,b]$.

For an $\mathcal{O}_X$-Azumaya algebra we denote by $\mathcal{C}(\mathcal{A})$ the $\Omega^1_X \otimes_{\mathcal{O}_X} \mathtt{Der}_{\mathcal{O}_X}(\mathcal{A})$-torsor of (locally defined) connections ∇ on $\mathcal{A}$ which satisfy the Leibniz rule $\nabla(ab) = \nabla(a)b + a\nabla(b)$.

For the sake of brevity we will refer to Azumaya $\mathcal{O}_X$-algebras simply as Azumaya algebras (on X).

3.3.2. Splittings. Suppose that $\mathcal{A}$ is an Azumaya algebra.

Definition 3.7. A *splitting* of $\mathcal{A}$ is a pair $(\mathcal{E}, \phi)$ consisting of a vector bundle $\mathcal{E}$ and an isomorphism $\phi\colon \mathcal{A} \to \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E})$.

A morphism $f\colon (\mathcal{E}_1, \phi_1) \to (\mathcal{E}_2, \phi_2)$ of splittings is an isomorphism $f\colon \mathcal{E}_1 \to \mathcal{E}_2$ such that $\mathrm{Ad}(f) \circ \phi_2 = \phi_1$.

Let $S(\mathcal{A})$ denote the stack such that, for $U \subseteq X$, $S(\mathcal{A})(U)$ is the category of splittings of $\mathcal{A}|_U$.

The sheaf of automorphisms of any object is canonically isomorphic $\mathcal{O}_X^\times$ (the subgroup of central units). As is easy to see, $S(\mathcal{A})$ is an $\mathcal{O}_X^\times$-gerbe.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Azumaya algebras on X and F is an $\mathcal{O}_X$-linear equivalence of respective categories of modules.

Lemma 3.8. *If $(\mathcal{E}, \phi)$ is a splitting of $\mathcal{A}$, then the $\mathcal{B}$-module $F(\mathcal{E}, \phi)$ is a splitting of $\mathcal{B}$.*

Corollary 3.9. *The induced functor $S(F)\colon S(\mathcal{A}) \to S(\mathcal{B})$ is an equivalence.*

In fact, it is clear that $S(.)$ extends to functor from the 2-category of Azumaya algebras on X to the 2-category of $\mathcal{O}_X^\times$-gerbes.

3.3.3. Matrix algebras. Until further notice we work in a fixed pseudo-tensor subcategory Ψ of $\mathrm{Sh}_k(X)$ as in 3.2.1. In particular, all algebraic structures are understood to be given by operations in Ψ.

Suppose that K is a sheaf of commutative algebras on X.

Definition 3.10. A *matrix K-algebra* is a sheaf of associative K-algebras $\mathcal{A}$ (on X) together with a decomposition $\mathcal{A} = \sum_{i,j=0}^{p} \mathcal{A}_{ij}$ into a *direct* sum of K-submodules which satisfies

(1) for $0 \leq i, j, k \leq p$ the product on $\mathcal{A}$ restricts to a map $\mathcal{A}_{ij} \otimes_K \mathcal{A}_{jk} \to \mathcal{A}_{ik}$ in $\Gamma(X; P^{\Psi}(\{\mathcal{A}_{ij}, \mathcal{A}_{jk}\}, \mathcal{A}_{ik}))$
(2) for $0 \leq i, j \leq p$ the left (respectively, right) action of $\mathcal{A}_{ii}$ (respectively, $\mathcal{A}_{jj}$) on $\mathcal{A}_{ij}$ given by the restriction of the product is unital.

Note that, for $0 \leq i \leq p$, the composition $K \xrightarrow{1} \mathcal{A} \to \mathcal{A}_{ii}$ (together with the restriction of the product) endows the sheaf $\mathcal{A}_{ii}$ with a structure of an associative algebra. The second condition in Definition 3.10 says that $\mathcal{A}_{ij}$ is a unital $\mathcal{A}_{ii}$-module (respectively, $\mathcal{A}_{jj}^{op}$-module).

For a matrix algebra as above we denote by $\mathfrak{d}(\mathcal{A})$ the subalgebra of "diagonal" matrices, i.e. $\mathfrak{d}(\mathcal{A}) = \sum_{i=0}^{p} \mathcal{A}_{ii}$.

Suppose that $\mathcal{A} = \sum_{i,j=0}^{p} \mathcal{A}_{ij}$ and $\mathcal{B} = \sum_{i,j=0}^{p} \mathcal{B}_{ij}$ are two matrix K-algebras (of the same size).

Definition 3.11. A 1-morphism $F\colon \mathcal{A} \to \mathcal{B}$ of matrix algebras is a morphism of sheaves of K-algebras in $\Gamma(X; P^{\Psi}(\{\mathcal{A}\}, \mathcal{B}))$ such that $F(\mathcal{A}_{ij}) \subseteq \mathcal{B}_{ij}$.

Suppose that $F_1,\ F_2\colon \mathcal{A} \to \mathcal{B}$ are 1-morphisms of matrix algebras.

Definition 3.12. A 2-morphism $b\colon F_1 \to F_2$ is a section $b \in \Gamma(X; \mathfrak{d}(B))$ such that $b \cdot F_1 = F_2 \cdot b$.

In what follows we will assume that a matrix algebra satisfies the following condition:

for $0 \leq i, j \leq p$ the sheaf $\mathcal{A}_{ij}$ is a locally free module of rank one over $\mathcal{A}_{ii}$ and over $\mathcal{A}_{jj}^{op}$.

3.3.4. Combinatorial restriction for matrix algebras. Suppose that $\mathcal{A} = \sum_{i,j=0}^{q} \mathcal{A}_{ij}$ is a matrix K-algebra and $f\colon [p] \to [q]$ is a morphism in Δ.

Definition 3.13. The *the combinatorial restriction $f^\sharp\mathcal{A}$ (of $\mathcal{A}$ along f)* is the matrix algebra with the underlying sheaf

$$f^\sharp\mathcal{A} = \bigoplus_{i,j=0}^{p} (f^\sharp\mathcal{A})_{ij}, \quad (f^\sharp\mathcal{A})_{ij} = \mathcal{A}_{f(i)f(j)}\ .$$

The product on $f^\sharp\mathcal{A}$ is induced by the product on $\mathcal{A}$.

Suppose that $F\colon \mathcal{A} \to \mathcal{B}$ is a 1-morphism of matrix algebras. The combinatorial restriction $f^\sharp F\colon f^\sharp\mathcal{A} \to f^\sharp\mathcal{B}$ is defined in an obvious

manner. For a 2-morphism $b\colon F_1 \to F_2$ the combinatorial restriction $f^\sharp b\colon f^\sharp F_1 \to f^\sharp F_2$ is given by $(f^\sharp b)_{ii} = b_{f(i)f(i)}$.

For $0 \leq i \leq q$, $0 \leq j \leq p$ let $(\mathcal{A}_f)_{ij} = \mathcal{A}_{if(j)}$. Let

$$\mathcal{A}_f = \bigoplus_{i=0}^{q} \bigoplus_{j=0}^{p} (\mathcal{A}_f)_{ij}$$

The sheaf $\mathcal{A}_f$ is endowed with a structure of a $\mathcal{A} \otimes_K (f^\sharp \mathcal{A})^{op}$-module given by

$$(abc)_{il} = \sum_{k=0}^{q} \sum_{j=0}^{p} a_{ij} b_{jk} c_{kl} \tag{3.3.2}$$

where $a = \sum_{i,j=0}^{q} a_{ij} \in \mathcal{A}$, $b = \sum_{i=0}^{q} \sum_{j=0}^{p} b_{ij} \in \mathcal{A}_f$, $c = \sum_{i,j=0}^{p} c_{ij} \in f^\sharp \mathcal{A}$ and $abc \in \mathcal{A}_f$.

The $f^\sharp \mathcal{A} \otimes_K \mathcal{A}^{op}$-module $\mathcal{A}_f^{-1}$ is defined in a similar fashion, with $(\mathcal{A}_f^{-1})_{ij} = \mathcal{A}_{f(i)j}$, $0 \leq i \leq p$, $0 \leq j \leq q$.

Let $\overrightarrow{\alpha}_f\colon \mathcal{A}_f \otimes_{f^\sharp \mathcal{A}} \mathcal{A}_f^{-1} \to \mathcal{A}$ be defined by

$$\alpha(b \otimes c)_{ij} = \sum_{k=0}^{p} b_{ik} c_{kj} \tag{3.3.3}$$

Where $b = \sum_{i=0}^{q} \sum_{j=0}^{p} b_{ij} \in \mathcal{A}_f$, $c = \sum_{i=0}^{p} \sum_{j=0}^{q} c_{ij} \in \mathcal{A}_f^{-1}$. Similarly one constructs an isomorphism $\overleftarrow{\alpha}_f\colon \mathcal{A}_f^{-1} \otimes_{\mathcal{A}} \mathcal{A}_f \to f^\sharp \mathcal{A}$ of $f^\sharp \mathcal{A}$ bimodules.

Lemma 3.14. *The bimodules $\mathcal{A}_f$ and $\mathcal{A}_f^{-1}$ together with the maps $\overrightarrow{\alpha}_f$ and $\overleftarrow{\alpha}_f$ implement a Morita equivalence between $f^\sharp \mathcal{A}$ and $\mathcal{A}$.*

3.3.5. Matrix Azumaya algebras.

Definition 3.15. A *matrix Azumaya K-algebra on X* is a matrix K-algebra $\mathcal{A} = \sum_{i,j} \mathcal{A}_{ij}$ which satisfies the additional condition $\mathcal{A}_{ii} = K$.

A matrix Azumaya K-algebra is, in particular, an Azumaya K-algebra. Let $\mathtt{Der}_K(\mathcal{A})^{loc}$ denote the sheaf of K-linear derivations which preserve the decomposition. Note that $\mathtt{Der}_K(\mathcal{A})^{loc}$ is a sheaf of *abelian* Lie algebras. The short exact sequence (3.3.1) restricts to the short exact sequence

$$0 \to K \to \mathfrak{d}(\mathcal{A}) \xrightarrow{\delta} \mathtt{Der}_K(\mathcal{A})^{loc} \to 0$$

For an $\mathcal{O}_X$ matrix Azumaya algebra we denote by $\mathcal{C}(\mathcal{A})^{loc}$ the subsheaf of $\mathcal{C}(\mathcal{A})$ whose sections are the connections which preserve the decomposition. The sheaf $\mathcal{C}(\mathcal{A})^{loc}$ is an $\Omega^1_X \otimes_{\mathcal{O}_X} \mathtt{Der}_{\mathcal{O}_X}(\mathcal{A})^{loc}$-torsor.

If $\mathcal{A}$ is a matrix Azumaya algebra, then both $\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{J}_X$ and $\mathcal{J}_X(\mathcal{A})$ are matrix Azumaya $\mathcal{J}_X$-algebras. Let $\underline{\mathrm{Isom}}_0(\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{J}_X, \mathcal{J}_X(\mathcal{A}))^{loc}$ denote the sheaf of $\mathcal{J}_X$-matrix algebra isomorphisms $\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{J}_X \to \mathcal{J}_X(\mathcal{A})$ making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{J}_X & \longrightarrow & \mathcal{J}_X(\mathcal{A}) \\ {\scriptstyle \mathtt{Id}\otimes p}\downarrow & & \downarrow{\scriptstyle p_{\mathcal{A}}} \\ \mathcal{A} & \xrightarrow{\mathtt{Id}} & \mathcal{A} \end{array}$$

where $p_{\mathcal{A}}$ is the canonical projection and $p\colon = p_{\mathcal{O}_X}$.

Let $\underline{\mathrm{Aut}}_0(\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{J}_X)^{loc}$ denote the sheaf of $\mathcal{J}_X$-matrix algebra automorphisms of $\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{J}_X$ making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{J}_X & \longrightarrow & \mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{J}_X \\ {\scriptstyle \mathtt{Id}\otimes p}\downarrow & & \downarrow{\scriptstyle \mathtt{Id}\otimes p} \\ \mathcal{A} & \xrightarrow{\mathtt{Id}} & \mathcal{A} \end{array}$$

The sheaf $\underline{\mathrm{Isom}}_0(\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{J}_X, \mathcal{J}_X(\mathcal{A}))^{loc}$ is a torsor under $\underline{\mathrm{Aut}}_0(\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{J}_X)^{loc}$ and the latter is soft.

Note that $\underline{\mathrm{Aut}}_0(\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{J}_X)^{loc}$ is a sheaf of pro-unipotent Abelian groups and the map

$$\exp\colon \mathtt{Der}_{\mathcal{O}_X}(\mathcal{A})^{loc}\otimes_{\mathcal{O}_X}\mathcal{J}_{X,0} \to \underline{\mathrm{Aut}}_0(\mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{J}_X)^{loc} \tag{3.3.4}$$

is an isomoprhism of sheaves of groups.

3.4. DGLA of local cochains.

3.4.1. Local cochains on matrix algebras. Suppose that $\mathcal{B} = \bigoplus_{i,j=0}^{p}\mathcal{B}_{ij}$ is a sheaf of matrix k-algebras. Under these circumstances one can associate to $\mathcal{B}$ a DGLA of *local cochains* defined as follows.

Let $C^0(\mathcal{B})^{loc} = \mathfrak{d}(\mathcal{B})$. For $n \geq 1$ let $C^n(\mathcal{B})^{loc}$ denote the subsheaf of $C^n(\mathcal{B})$ of multilinear maps D such that for any collection of $s_{i_kj_k} \in \mathcal{B}_{i_kj_k}$

(1) $D(s_{i_1j_1}\otimes\cdots\otimes s_{i_nj_n}) = 0$ unless $j_k = i_{k+1}$ for all $k = 1,\ldots,n-1$
(2) $D(s_{i_0i_1}\otimes s_{i_1i_2}\otimes\cdots\otimes s_{i_{n-1}i_n}) \in \mathcal{B}_{i_0i_n}$

For $I = (i_0,\ldots,i_n) \in [p]^{\times n+1}$ let

$$C^I(\mathcal{B})^{loc}\colon = C^n(\mathcal{B})^{loc}\bigcap \underline{\mathrm{Hom}}_k(\otimes_{j=0}^{n-1}\mathcal{B}_{i_ji_{j+1}}, \mathcal{B}_{i_0i_n})\ .$$

The restriction maps along the embeddings $\otimes_{j=0}^{n-1}\mathcal{B}_{i_ji_{j+1}} \hookrightarrow \mathcal{B}^{\otimes n}$ induce an isomorphism $C^n(\mathcal{B})^{loc} \to \oplus_{I\in[p]^{\times n+1}}C^I(\mathcal{B})^{loc}$.

The sheaf $C^\bullet(\mathcal{B})^{loc}[1]$ is a subDGLA of $C^\bullet(\mathcal{B})[1]$ and the inclusion map is a quasi-isomorphism.

3.4.2. Combinatorial restriction of local cochains. As in 3.4.1, $\mathcal{B} = \bigoplus_{i,j=0}^{q} \mathcal{B}_{ij}$ is a sheaf of matrix K-algebras.

The DGLA $C^\bullet(\mathcal{B})^{loc}[1]$ has additional variance not exhibited by $C^\bullet(\mathcal{B})[1]$. Namely, for $f\colon [p] \to [q]$ there is a natural map of DGLA

$$f^\sharp\colon C^\bullet(\mathcal{B})^{loc}[1] \to C^\bullet(f^\sharp\mathcal{B})^{loc}[1] \tag{3.4.1}$$

defined as follows. Let $f_\sharp^{ij}\colon (f^\sharp\mathcal{B})_{ij} \to \mathcal{B}_{f(i)f(j)}$ denote the tautological isomorphism. For each collection $I = (i_0, \dots, i_n) \in [p]^{\times(n+1)}$ let

$$f_\sharp^I\colon = \otimes_{j=0}^{n-1} f_\sharp^{i_j i_{j+1}}\colon \otimes_{j=0}^{n-1} (f^\sharp\mathcal{B})_{i_j i_{j+1}} \to \otimes_{i=0}^{n-1} \mathcal{B}_{f(i_j)f(i_{j+1})} \ .$$

Let $f_\sharp^n\colon = \oplus_{I\in[p]^{\times(n+1)}} f_\sharp^I$. The map (3.4.1) is defined as restriction along $f_\sharp^n$.

Lemma 3.16. *The map* (3.4.1) *is a morphism of DGLA*

$$f^\sharp\colon C^\bullet(\mathcal{B})^{loc}[1] \to C^\bullet(f^\sharp\mathcal{B})^{loc}[1] \ .$$

3.4.3. Deformations of matrix algebras. For a matrix algebra $\mathcal{B}$ on X we denote by $\mathrm{Def}(\mathcal{B})^{loc}(R)$ the subgroupoid of $\mathrm{Def}(\mathcal{B})(R)$ with objects R-star products which respect the decomposition given by $(\mathcal{B}\otimes_k R)_{ij} = \mathcal{B}_{ij} \otimes_k R$ and 1- and 2-morphisms defined accordingly. The composition

$$\mathrm{Def}(\mathcal{B})^{loc}(R) \to \mathrm{Def}(\mathcal{B})(R) \to \mathrm{MC}^2(\Gamma(X; C^\bullet(\mathcal{B})[1]) \otimes_k \mathfrak{m}_R)$$

takes values in $\mathrm{MC}^2(\Gamma(X; C^\bullet(\mathcal{B})^{loc}[1]) \otimes_k \mathfrak{m}_R)$ and establishes an isomorphism of 2-groupoids $\mathrm{Def}(\mathcal{B})^{loc}(R) \cong \mathrm{MC}^2(\Gamma(X; C^\bullet(\mathcal{B})^{loc}[1]) \otimes_k \mathfrak{m}_R)$.

4. Deformations of cosimplicial matrix Azumaya algebras

4.1. Cosimplicial matrix algebras. Suppose that X is a simplicial space. We assume given a cosimplicial pseudo-tensor category $\Psi\colon [p] \mapsto \Psi^p$, $p = 0, 1, 2, \dots$, where Ψ^p is a pseudo-tensor subcategory $\mathrm{Sh}_k(X_p)$ (see 3.2.1) so that for any morphism $f\colon [p] \to [q]$ in Δ the corresponding functor $X(f)^{-1}\colon \mathrm{Sh}_k(X_p) \to \mathrm{Sh}_k(X_q)$ restricts to a functor $X(f)^{-1}\colon \Psi^p \to \Psi^q$. If X is an étale simplicial manifold, then both `DIFF` and `JET` are examples of such. In what follows all algebraic structures are understood to be given by operations in Ψ.

Suppose that K is a special cosimplicial sheaf (see Definition 2.3) of commutative algebras on X.

Definition 4.1. A *cosimplicial matrix K-algebra $\mathcal{A}$ on X* is given by the following data:

(1) for each $p = 0, 1, 2, \dots$ a matrix K^p-algebra $\mathcal{A}^p = \sum_{i,j=0}^{p} \mathcal{A}^p_{ij}$

(2) for each morphism $f\colon [p] \to [q]$ in Δ an isomorphism of matrix K^q-algebras $f_*\colon X(f)^{-1}\mathcal{A}^p \to f^\sharp \mathcal{A}^q$.

These are subject to the associativity condition: for any pair of composable arrows $[p] \xrightarrow{f} [q] \xrightarrow{g} [r]$

$$(g \circ f)_* = f^\sharp(g_*) \circ X(g)^{-1}(f_*)$$

Remark 4.2. As is clear from the above definition, a cosimplicial matrix algebra is *not* a cosimplicial sheaf of algebras in the usual sense.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two cosimplicial matrix algebras on X. A 1-*morphism of cosimplicial matrix algebras* $F\colon \mathcal{A} \to \mathcal{B}$ is given by the collection of morphisms

$$F^p\colon \mathcal{A}^p \to \mathcal{B}^p$$

of matrix K^p-algebras subject to the compatibility condition: for any morphism $f\colon [p] \to [q]$ in Δ the diagram

$$\begin{array}{ccc} X(f)^{-1}\mathcal{A}^p & \xrightarrow{f_*} & f^\sharp \mathcal{A}^q \\ {\scriptstyle X(f)^*F^p}\downarrow & & \downarrow{\scriptstyle f^\sharp F^q} \\ X(f)^{-1}\mathcal{B}^p & \xrightarrow{f_*} & f^\sharp \mathcal{B}^q \end{array}$$

commutes. The composition of 1-morphisms is given by the composition of their respective components.

The identity 1-morphism $\mathtt{Id}_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}$ is given by $\mathtt{Id}^p_{\mathcal{A}} = \mathtt{Id}$.

Suppose that $F_1, F_2\colon \mathcal{A} \to \mathcal{B}$ are two 1-morphisms. A 2-*morphism* $b\colon F_1 \to F_2$ is given by a collection of 2-morphisms $b^p\colon F_1^p \to F_2^p$ satisfying

$$f_*(X(f)^* b^p) = f^\sharp b^q$$

for any morphism $f\colon [p] \to [q]$ in Δ. The compositions of 2-morphisms are again componentwise.

Let $\mathrm{CMA}_K^\Psi(X)$ denote the category of cosimplicial matrix K-algebras with 1- and 2-morphisms defined as above. In the case when $\Psi = \mathrm{Sh}_k(X)$, i.e. no restrictions are imposed, we will simply write $\mathrm{CMA}_K(X)$.

4.2. Deformations of cosimplicial matrix algebras.

4.2.1. DGLA from cosimplicial matrix algebras. Suppose that X is a simplicial space and K is a special cosimplicial sheaf of commutative k-algebras on X. To $\mathcal{A} \in \mathrm{CMA}_K^\Psi(X)$ we associate a cosimplicial sheaf of DGLA $\mathfrak{g}(\mathcal{A})$ on $|X|$ (see 2.1.5).

Let $\mathfrak{g}^n(\mathcal{A})$ denote the sheaf on $|X|_n$ whose restriction to $|X|_\lambda$ is equal to $X(\lambda(0n))^{-1} C^\bullet(\mathcal{A}^{\lambda(0)})^{loc}[1]$.

For a morphism $f\colon [m]\to[n]$ in Δ let

$$f_*\colon |X|(f)^{-1}\mathfrak{g}^m(\mathcal{A})\to\mathfrak{g}^n(\mathcal{A})$$

denote the map whose restriction to $|X|_\lambda$ is equal to the composition

$$X(\Upsilon(f)^\lambda)^{-1}X(f^*(\lambda)(0m))^{-1}C^\bullet(\mathcal{A}^{f^*(\lambda)(0)})^{loc}[1]\cong$$
$$X(\Upsilon(f)^\lambda\circ f^*(\lambda)(0m))^{-1}C^\bullet(\mathcal{A}^{f^*(\lambda)(0)})^{loc}[1]\xrightarrow{f^\sharp}$$
$$X(\lambda(0n))^{-1}C^\bullet(f^\sharp\mathcal{A}^{\lambda(0)})^{loc}[1]\cong$$
$$X(\lambda(0n))^{-1}C^\bullet(\mathcal{A}^{\lambda(0)})^{loc}[1]$$

in the notations of 2.1.2.

We leave it to the reader to check that the assignment $[n]\mapsto\mathfrak{g}^n(\mathcal{A})$, $f\mapsto f_*$ is a cosimplicial sheaf of DGLA on $|X|$.

We will denote by $\mathfrak{d}(\mathcal{A})$ the cosimplicial subDGLA $[n]\mapsto X(\lambda(0n))^{-1}\mathfrak{d}(\mathcal{A}^{\lambda(0)})$.

For each $i=0,1,\dots$ we have the cosimplicial vector space of degree i local cochains $\Gamma(|X|;\mathfrak{g}^{\bullet,i}(\mathcal{A}))$, $[n]\mapsto\Gamma(|X|_n;\mathfrak{g}^{n,i}(\mathcal{A}))$. The following theorem was proved in [7] in the special case when X is the nerve of an open cover of a manifold. The proof of Theorem 5.2 in [7] extends verbatim to give the following result.

Theorem 4.3. *For each $i,j\in\mathbb{Z}$, $j\neq 0$, $H^j(\Gamma(|X|;\mathfrak{g}^{\bullet,i}(\mathcal{A})))=0$.*

Let

$$\mathfrak{G}(\mathcal{A})=\mathtt{Tot}(\Gamma(|X|;\mathfrak{g}(\mathcal{A}))). \tag{4.2.1}$$

As will be shown in Theorem 4.5, the DGLA $\mathfrak{G}(\mathcal{A})$ plays the role of a deformation complex of $\mathcal{A}$.

4.2.2. The deformation functor. Suppose that X is a simplicial space, Ψ is as in 4.1 subject to the additional condition as in 3.2.5. For $\mathcal{A}\in\mathrm{CMA}_k^\Psi(X)$ we define the deformation functor $\mathrm{Def}(\mathcal{A})$ on ArtAlg_k (see 3.2.5 for Artin algebras).

Definition 4.4. An *R-deformation $\mathbf{B}_R$ of $\mathcal{A}$* is a cosimplicial matrix R-algebra structure on $\mathcal{A}_R:=\mathcal{A}\otimes_k R$ with the following properties:

(1) $\mathbf{B}_R\in\mathrm{CMA}_R^\Psi(X)$
(2) for all $f\colon[p]\to[q]$ the structure map $f_*^{\mathbf{B}}\colon X(f)^{-1}\mathbf{B}^p\to f^\sharp\mathbf{B}^q$ is equal to the map $f_*^{\mathcal{A}}\otimes\mathtt{Id}_R\colon X(f)^{-1}\mathcal{A}^p\otimes_k R\to f^\sharp\mathcal{A}^q\otimes_k R$.
(3) the identification $\mathbf{B}_R\otimes_R k\cong\mathcal{A}$ is compatible with the respective cosimplicial matrix algebra structures.

A 1-morphism of R-deformations of $\mathcal{A}$ is a 1-morphism of cosimplicial matrix R-algebras which reduces to the identity 1-morphism $\mathtt{Id}_{\mathcal{A}}$ modulo the maximal ideal.

A 2-morphism between 1-morphisms of deformations is a 2-morphism which reduces to the identity endomorphism of $\mathtt{Id}_{\mathcal{A}}$ modulo the maximal.

We denote by $\mathrm{Def}(\mathcal{A})(R)$ the 2-category with objects R-deformations of $\mathcal{A}$, 1-morphisms and 2-morphisms as above. It is clear that the assignment $R \mapsto \mathrm{Def}(\mathcal{A})(R)$ is natural in R.

4.2.3. Suppose given an R-deformation $\mathbf{B}$ of $\mathcal{A}$ as above. Then, for each $p = 0, 1, 2, ...$ we have the Maurer-Cartan (MC) element $\gamma^p \in \Gamma(X_p; C^\bullet(\mathcal{A}^p)^{loc}[1]) \otimes_k \mathfrak{m}_R$ which corresponds to the R-deformation $\mathbf{B}^p$ of the matrix algebra $\mathcal{A}^p$ on X_p. This collection defines a MC element $\gamma \in \Gamma(|X|_0, \mathfrak{g}^0(\mathcal{A}))$. It is clear from the definition that $\gamma \in \ker(\Gamma(|X|_0, \mathfrak{g}^0(\mathcal{A})) \rightrightarrows \Gamma(|X|_1, \mathfrak{g}^1(\mathcal{A}))) \otimes_k \mathfrak{m}_R$. This correspondence induces a bijection between the objects of $\mathrm{Def}(X; \mathcal{A})(R)$ and those of $\mathrm{MC}^2(\ker(\Gamma(|X|_0, \mathfrak{g}^0(\mathcal{A})) \rightrightarrows \Gamma(|X|_1, \mathfrak{g}^1(\mathcal{A}))) \otimes_k \mathfrak{m}_R)$, which can clearly be extended to an isomorphism of 2-groupoids. Recall that we have a morphism of DGLA

$$\ker(\Gamma(|X|_0, \mathfrak{g}^0(\mathcal{A})) \rightrightarrows \Gamma(|X|_1, \mathfrak{g}^1(\mathcal{A}))) \to \mathfrak{G}(\mathcal{A}) \tag{4.2.2}$$

where $\mathfrak{G}(\mathcal{A}) = \mathtt{Tot}(\Gamma(|X|; \mathfrak{g}(\mathcal{A})))$.

This morphism induces a morphism of the corresponding Deligne 2-groupoids. Therefore we obtain a morphism of 2-groupoids

$$\mathrm{Def}(\mathcal{A})(R) \to \mathrm{MC}^2(\mathfrak{G}(\mathcal{A}) \otimes_k \mathfrak{m}_R) \tag{4.2.3}$$

Theorem 4.5. *The map* (4.2.3) *is an equivalence of* 2*-groupoids.*

Proof. Theorem 4.3 shows that the condition (3.1.5) of the Proposition 3.2 is satisfied. The statement of the Theorem then follows from the conclusion of the Proposition 3.2. □

4.3. Deformations of cosimplicial matrix Azumaya algebras. Suppose that X is a *Hausdorff* étale simplicial manifold.

4.3.1. Cosimplicial matrix Azumaya algebras.

Definition 4.6. A *cosimplicial matrix Azumaya K-algebra* $\mathcal{A}$ on X is a cosimplicial matrix K-algebra on X (see 4.1) such that for every $p = 0, 1, 2, \ldots$ the matrix algebra $\mathcal{A}^p$ is a matrix Azumaya K-algebra on X_p (see Definition 3.15).

Suppose that $\mathcal{A}$ is a cosimplicial matrix $\mathcal{O}_X$-Azumaya algebra. Recall that, by convention we treat such as objects of `DIFF` (see 3.2.2).

Then, $\mathcal{J}_X(\mathcal{A})$ is a cosimplicial matrix $\mathcal{J}_X$-Azumaya algebra, which we view as an object of `JET`, (see 3.2.2) equipped with the canonical flat connection $\nabla^{can}_{\mathcal{A}}$. Therefore, we have the cosimplicial DGLA

with flat connection $\mathfrak{g}(\mathcal{J}_X(\mathcal{A}))$ on $|X|$, hence the cosimplicial DGLA $\mathtt{DR}(\mathfrak{g}(\mathcal{J}_X(\mathcal{A}))$. Let

$$\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X(\mathcal{A})) = \mathtt{Tot}(\Gamma(|X|; \mathtt{DR}(\mathfrak{g}(\mathcal{J}_X(\mathcal{A})))))$$

The inclusion of the subsheaf of horizontal sections is a quasi-isomorphism of DGLA $\mathfrak{g}(\mathcal{A}) \to \mathtt{DR}(\mathfrak{g}(\mathcal{J}_X(\mathcal{A})))$ which induces the quasi-isomorphism of DGLA

$$\mathfrak{G}(\mathcal{A}) \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X(\mathcal{A})) \tag{4.3.1}$$

4.3.2. Cosimplicial splitting. Suppose that $\mathcal{A}$ is a cosimplicial matrix $\mathcal{O}_X$-Azumaya algebra.

According to 3.3.2, for each $p = 0, 1, 2, \ldots$ we have the gerbe $S(\mathcal{A}^p)$ on X_p. Moreover, for each morphism $f\colon [p] \to [q]$ in Δ we have the morphism $f_*\colon X(f)^{-1}S(\mathcal{A}^p) \to S(\mathcal{A}^q)$ defined as the composition

$$X(f)^{-1}S(\mathcal{A}^p) \cong S(X(f)^{-1}\mathcal{A}^p) \cong S(f^\sharp \mathcal{A}^q) \overset{S(\mathcal{A}^q_f)}{\longrightarrow} S(\mathcal{A}^q)$$

It is clear that the assignment $[p] \mapsto S(\mathcal{A}^p)$, $f \mapsto f_*$ is a cosimplicial gerbe on X.

Let $\mathcal{E}^p := \oplus_{j=0}^n \mathcal{A}^p_{j0}$. There is a natural isomorphism $\mathcal{A}^p \cong \underline{\mathrm{End}}(\mathcal{E}^p)$, the action given by the isomorphism $\mathcal{A}_{ij} \otimes \mathcal{A}_{j0} \to \mathcal{A}_{i0}$. In other words, $\mathcal{E}^p$ is a splitting of the Azumaya algebra $\mathcal{A}^p$, i.e. a morphism $\mathcal{E}^p\colon \mathcal{O}^\times_{X_p}[1] \to S(\mathcal{A}^p)$.

We are going to extend the assignment $p \mapsto \mathcal{O}^\times_{X_p}[1]$ to a cosimplicial gerbe $\mathcal{S}_\mathcal{A}$ on X so that $\mathcal{E}$ is a morphism of cosimplicial gerbes $\mathcal{S}_\mathcal{A} \to S(\mathcal{A})$. To this end, for $f\colon [p] \to [q]$ let $(\mathcal{S}_\mathcal{A})_f = \mathcal{A}^q_{0f(0)}$. For a pair of composable arrows $[p] \overset{f}{\to} [q] \overset{g}{\to} [r]$ let $(\mathcal{S}_\mathcal{A})_{f,g}$ be defined by the isomorphism $X(g)^{-1}(\mathcal{A}^q_{0f(0)}) \otimes \mathcal{A}^r_{0g(0)} \cong \mathcal{A}^r_{0g(f(0))}$.

Since $f_*\mathcal{A}^p_{i0} \cong \mathcal{A}^q_{f(i)f(0)} \cong \mathcal{A}^q_{f(i)0} \otimes \mathcal{A}^q_{0f(0)}$ for every i we obtain a canonical chain of isomorphisms

$$\mathcal{A}^q_f \otimes_{f^\sharp \mathcal{A}^q} f_*\mathcal{E}^p \cong \bigoplus_{i=0}^q \mathcal{A}^q_{if(0)} \cong \bigoplus_{i=0}^q \mathcal{A}^q_{i0} \otimes \mathcal{A}^q_{0f(0)} \cong \mathcal{E}^q \otimes \mathcal{A}^q_{0f(0)}. \tag{4.3.2}$$

Let $\mathcal{E}_f$ be the 2-morphism induced by the isomorphism (4.3.2). We than have the following:

Lemma 4.7. *$(\mathcal{E}^p, \mathcal{E}_f)$ is a morphism of cosimplicial gerbes $\mathcal{S}_\mathcal{A} \to S(\mathcal{A})$.*

4.3.3. Twisted DGLA of jets.

Definition 4.8. For a DGLA $(\mathfrak{r}, d)$ and a Maurer-Cartan element $\gamma \in \mathtt{Der}(\mathfrak{r})$ we define the *γ-twist of* $\mathfrak{r}$, denoted $\mathfrak{r}_\gamma$, to be the DGLA

whose underlying graded Lie algebra coincides with that of $\mathfrak{r}$ and whose differential is equal to $d + \gamma$.

In 4.3.2 we associated with $\mathcal{A}$ a cosimplicial gerbe $\mathcal{S}_{\mathcal{A}}$ on X. The construction of 2.5.6 associates to $\mathcal{S}_{\mathcal{A}}$ the characteristic class $[\mathcal{S}_{\mathcal{A}}] \in H^2(|X|; |\mathtt{DR}(\overline{\mathcal{J}}_X)|)$ represented by a 2-cocycle $\overline{B} \in \mathtt{Tot}(\Gamma(|X|; |\mathtt{DR}(\overline{\mathcal{J}}_X)|))$ dependent on appropriate choices.

Recall that under the standing assumption that X is a Hausdorff étale simplicial manifold, the cosimplicial sheaves $\mathcal{O}_X$, $\mathcal{J}_X$, $\Omega^{\bullet}_X$ are special.

We have special cosimplicial sheaves of DGLA $\overline{C}^{\bullet}(\mathcal{O}_X)[1]$ and $\mathtt{DR}(\overline{C}^{\bullet}(\mathcal{J}_X)[1])$. The inclusion of the subsheaf of horizontal sections is a quasi-isomorphism of DGLA $\overline{C}^{\bullet}(\mathcal{O}_X)[1] \to \mathtt{DR}(\overline{C}^{\bullet}(\mathcal{J}_X))[1]$. Let

$$\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X) = \mathtt{Tot}(\Gamma(|X|; |\mathtt{DR}(\overline{C}^{\bullet}(\mathcal{J}_X)[1])|'))$$

(see 2.2.4 for $|\bullet|'$).

The canonical isomorphism $|\mathtt{DR}(\overline{\mathcal{J}}_X)|' \cong |\mathtt{DR}(\overline{\mathcal{J}}_X)|$ (see 2.2.4) induces the isomorphism of complexes $\mathtt{Tot}(\Gamma(|X|; |\mathtt{DR}(\overline{\mathcal{J}}_X)|)) \cong \mathtt{Tot}(\Gamma(|X|; |\mathtt{DR}(\overline{\mathcal{J}}_X)|'))$. Thus, we may (and will) consider $\overline{B}$ as a 2-cocycle in the latter complex.

The adjoint action of the abelian DGLA $\mathcal{J}_X[1]$ on $\overline{C}^{\bullet}(\mathcal{J}_X)[1]$ induced an action of $\overline{\mathcal{J}}_X[1]$ on $\overline{C}^{\bullet}(\mathcal{J}_X)[1]$. The latter action gives rise to an action of the DGLA $\mathtt{Tot}(\Gamma(|X|; |\mathtt{DR}(\overline{\mathcal{J}}_X[1])|'))$ on the DGLA $\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X)$ by derivations. Since the cocycle $\overline{B}$ is a Maurer-Cartan element in $\mathtt{Der}(\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X))$, the DGLA $\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X)_{\overline{B}}$ is defined.

4.4. Construction. Let $\mathcal{A}$ be a cosimplicial matrix $\mathcal{O}_X$-Azumaya algebra on X. This section is devoted to the construction and uniqueness properties of the isomorphism (4.4.22).

To simplify the notations we will denote $\mathcal{O}_X$ (respectively, $\mathcal{O}_{X_p}$, $\mathcal{J}_X$, $\mathcal{J}_{X_p}$) by $\mathcal{O}$ (respectively $\mathcal{O}^p$, $\mathcal{J}$, $\mathcal{J}^p$).

4.4.1. Construction: step one. We have the cosimplicial matrix $\mathcal{J}$-Azumaya algebra $\mathcal{A} \otimes \mathcal{J}$, hence the cosimplicial graded Lie algebras $\mathfrak{g}(\mathcal{A} \otimes \mathcal{J})$ and $\Omega^{\bullet}_{|X|} \otimes \mathfrak{g}(\mathcal{A} \otimes \mathcal{J})$. Let $\mathfrak{H}$ denote the graded Lie algebra $\mathtt{Tot}(\Gamma(|X|; \Omega^{\bullet}_{|X|} \otimes \mathfrak{g}(\mathcal{A} \otimes \mathcal{J})))$.

We begin by constructing an isomorphism of graded Lie algebras.

$$\Sigma \colon \mathfrak{H} \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}(\mathcal{A})) \tag{4.4.1}$$

For each $p = 0, 1, 2, \ldots$ we choose

$$\sigma^p \in \mathrm{Isom}_0(\mathcal{A}^p \otimes \mathcal{J}^p, \mathcal{J}(\mathcal{A}^p))^{loc} .$$

Consider a simplex $\lambda \colon [n] \to \Delta$.

For $0 \leq i \leq n$ the composition

$$X(\lambda(0n))^{-1}\mathcal{A}^{\lambda(0)} \to$$
$$X(\lambda(in))^{-1}X(\lambda(0i))^{-1}\mathcal{A}^{\lambda(0)} \xrightarrow{\lambda(0i)_*} X(\lambda(in))^{-1}\lambda(0i)^\sharp\mathcal{A}^{\lambda(i)}$$

defines an isomorphism

$$X(\lambda(0n))^{-1}\mathcal{A}^{\lambda(0)} \to X(\lambda(in))^{-1}\lambda(0i)^\sharp\mathcal{A}^{\lambda(i)} \tag{4.4.2}$$

The composition of the map $X(\lambda(in))^*\lambda(0i)^\sharp$ with the isomorphism (4.4.2) defines the following maps, all of which we denote by $\tau^*_{\lambda,i}$:

$$\tau^*_{\lambda,i}\colon \mathrm{Isom}_0(\mathcal{A}^{\lambda(i)} \otimes \mathcal{J}^{\lambda(i)}, \mathcal{J}(\mathcal{A}^{\lambda(i)}))^{loc} \to$$
$$\mathrm{Isom}_0(X(\lambda(0n))^{-1}\mathcal{A}^{\lambda(0)} \otimes \mathcal{J}^{\lambda(n)}, \mathcal{J}(X(\lambda(0n))^{-1}\mathcal{A}^{\lambda(0)}))^{loc}$$

$$\tau^*_{\lambda,i}\colon \Omega^\bullet_{X_{\lambda(i)}} \otimes C^\bullet(\mathcal{A}^{\lambda(i)})^{loc} \otimes \mathcal{J}^{\lambda(i)} \to$$
$$\Omega^\bullet_{X_{\lambda(n)}} \otimes C^\bullet((X(\lambda(0n))^{-1}\mathcal{A}^{\lambda(0)})^{loc} \otimes \mathcal{J}^{\lambda(n)}$$

etc. Define $\sigma^\lambda_i = \tau^*_{\lambda,i}\sigma^{\lambda(i)}$.

Recall that $\mathtt{pr}_X\colon X_p \times \Delta^n \to X_p$ denotes the projection. Let σ^λ be an element in $\mathrm{Isom}_0(\mathtt{pr}_X^{-1}(X(\lambda(0n))^{-1}\mathcal{A}^{\lambda(0)} \otimes \mathcal{J}^{\lambda(0)}),\ \mathtt{pr}_X^{-1}\mathcal{J}(X(\lambda(0n))^{-1}\mathcal{A}^{\lambda(0)}))^{loc}$ defined as follows. For each morphism $f\colon [p] \to [q]$ in Δ there is a unique

$$\vartheta(f) \in \Gamma(X_q; \mathtt{Der}_{\mathcal{O}^q_X}(X(f)^{-1}\mathcal{A}^p)^{loc} \otimes \mathcal{J}^q_0)$$

such that the composition

$$X(f)^{-1}\mathcal{A}^p \otimes \mathcal{J}^q \xrightarrow{\exp(\vartheta(f))} X(f)^{-1}\mathcal{A}^p \otimes \mathcal{J}^q \xrightarrow{X(f)^{-1}\sigma^p} X(f)^{-1}\mathcal{J}(\mathcal{A}^p)$$

is equal to the composition

$$X(f)^{-1}\mathcal{A}^p \otimes \mathcal{J}^q \xrightarrow{f_*} f^\sharp\mathcal{A}^q \otimes \mathcal{J}^q \xrightarrow{f^\sharp(\sigma^q)}$$
$$f^\sharp\mathcal{J}(\mathcal{A}^q) \xrightarrow{(f_*)^{-1}} X(f)^{-1}\mathcal{J}(\mathcal{A}^p).$$

Let

$$\sigma^\lambda = (\mathtt{pr}_X^*\sigma^\lambda_n) \circ \exp(-\sum_{i=0}^n t_i \cdot \lambda(0i)^\sharp\vartheta(\lambda(in))). \tag{4.4.3}$$

In this formula and below we use the isomorphism (4.4.2) to view $\lambda(0i)^\sharp\vartheta(\lambda(in))$ as an element of $\Gamma(X_{\lambda(n)}; \mathtt{Der}_{\mathcal{O}^{\lambda(n)}}(X(\lambda(0n))^{-1}\mathcal{A}^{\lambda(0)})^{loc} \otimes \mathcal{J}^{\lambda(n)})$.

The isomorphism of algebras σ^λ induces the isomorphism of graded Lie algebras

$$\Omega^\bullet_{X_{\lambda(n)}\times\Delta^n}\otimes \mathtt{pr}_X^{-1}(\mathfrak{g}^n_\lambda(\mathcal{A}\otimes\mathcal{J})) \xrightarrow{(\sigma^\lambda)^*} \Omega^\bullet_{X_{\lambda(n)}\times\Delta^n}\otimes \mathtt{pr}_X^{-1}(\mathfrak{g}^n_\lambda(\mathcal{J}(\mathcal{A})))$$

Lemma 4.9. *The map σ^λ induces an isomorphism*

$$\Omega_n\otimes_{\mathbb{C}}\Omega^\bullet_{X_{\lambda(n)}}\otimes\mathfrak{g}^n_\lambda(\mathcal{A}\otimes\mathcal{J}) \xrightarrow{(\sigma^\lambda)^*} \Omega_n\otimes_{\mathbb{C}}\Omega^\bullet_{X_{\lambda(n)}}\otimes\mathfrak{g}^n_\lambda(\mathcal{J}(\mathcal{A}))$$

Proof. It is sufficient to check that $\exp(-\sum_{i=0}^n t_i\cdot\lambda(0i)^\sharp\vartheta(\lambda(in)))$ maps $\mathfrak{g}^n_\lambda(\mathcal{A}\otimes\mathcal{J})$ into $\Omega_n\otimes_{\mathbb{C}}\mathfrak{g}^n_\lambda(\mathcal{A}\otimes\mathcal{J})$. Since the Lie algebra $\mathtt{Der}_{\mathcal{O}^q}(X(f)^{-1}\mathcal{A}^p)^{loc}$ is commutative, this is a consequence of the following general statement: if $\mathcal{A}$ is an Azumaya algebra on X, $D\in C^\bullet(\mathcal{A}\otimes\mathcal{J})[1]$ and $\vartheta\in\Gamma(X;\mathtt{Der}_{\mathcal{O}_X}(\mathcal{A})\otimes\mathcal{J}_{X,0})$, then $(\exp(t\,\mathtt{ad}\,\vartheta))^*D$ is polynomial in t. But this is so because ϑ is inner and D is section of a sheaf of jets of *multidifferential* operators. □

It is clear that the collection of maps $(\sigma^\lambda)^*$ is a morphism of cosimplicial graded Lie algebras; the desired isomorphism Σ (4.4.1) is, by definition the induced isomorphism of graded Lie algebras. We now describe the differential on $\mathfrak{H}$ induced by the differential on $\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}(\mathcal{A}))$ via the isomorphism (4.4.1).

Recall that the differential in $\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}(\mathcal{A}))$ is given by $\delta+\widetilde{\nabla}^{can}$. It is easy to see that δ induces the Hochschild differential δ on $\mathfrak{H}$. The canonical connection $\widetilde{\nabla}^{can}$ induces the differential whose component corresponding to the simplex λ is the connection $(\sigma^\lambda)^{-1}\circ X(\lambda(0n))^*\widetilde{\nabla}^{can}\circ\sigma^\lambda$.

To get a more explicit description of this connection choose for each $p=0,1,2,\dots$ a connection $\nabla^p\in\mathcal{C}(\mathcal{A}^p)^{loc}(X_p)$; it gives rise to the connection $\nabla^p\otimes\mathtt{Id}+\mathtt{Id}\otimes\nabla^{can}$ on $\mathcal{A}^p\otimes\mathcal{J}^p$. Let

$$\Phi^p=(\sigma^p)^{-1}\circ\nabla^{can}\circ\sigma^p-(\nabla^p\otimes\mathtt{Id}+\mathtt{Id}\otimes\nabla^{can})\ ,$$

$\Phi^p\in\Gamma(X_p;\Omega^1_{X_p}\otimes\mathtt{Der}_{\mathcal{O}^p}(\mathcal{A}^p)^{loc}\otimes\mathcal{J}^p)\subset\Gamma(X_p;\Omega^1_{X_p}\otimes C^1(\mathcal{A}^p\otimes\mathcal{J}^p)^{loc})$.

Let

$$\nabla^\lambda=\sum_{i=0}^n t_i\tau^*_{\lambda,i}\nabla^{\lambda(i)}\otimes\mathtt{Id}+\mathtt{Id}\otimes\widetilde{\nabla}^{can}$$

be the induced derivation of $\Omega_n\otimes_{\mathbb{C}}\Gamma(X_{\lambda(n)};X(\lambda(0n))^{-1}\mathcal{A}^{\lambda(0)})$. It is easy to see that the collection ∇^λ induces a derivation on $\mathfrak{H}$ which we denote by ∇.

Let

$$\Phi^\lambda=(\sigma^\lambda)^{-1}\circ X(\lambda(0n))^*\widetilde{\nabla}^{can}\circ\sigma^\lambda-\nabla^\lambda\ ,$$

$\Phi^\lambda \in \Omega_n \otimes \Gamma(X_{\lambda(n)}; \Omega^\bullet_{X_{\lambda(n)}} \otimes \mathtt{Der}_{\mathcal{O}^{\lambda(n)}}(X(\lambda(0n))^{-1}\mathcal{A}^{\lambda(0)})^{loc} \otimes \mathcal{J}^{\lambda(n)})$. Using the formula (4.4.3) we obtain:

$$\Phi^\lambda = \sum_{i=0}^{n} t_i \cdot \tau^*_{\lambda,i}\Phi^{\lambda(i)} - \sum_{i=0}^{n} dt_i \wedge \lambda(0i)^\sharp \vartheta(\lambda(in)) \ . \tag{4.4.4}$$

It is easy to see that the collection Φ^λ defines an element total degree one in $\mathfrak{H}$. The differential induced on $\mathfrak{H}$ via the isomorphism (4.4.1) can therefore be written as

$$\delta + \nabla + \mathtt{ad}\,\Phi \tag{4.4.5}$$

4.4.2. Now we construct an automorphism of the graded Lie algebra $\mathfrak{H}$ which conjugates the differential given by the formula (4.4.5) into a simpler one. This is achieved by constructing $F \in \mathtt{Tot}(\Gamma(|X|; \Omega^\bullet_{|X|} \otimes (\mathfrak{d}(\mathcal{A}) \otimes \mathcal{J}))) \subset \mathfrak{H}$ with components F^λ such that $\Phi^\lambda = -\delta F^\lambda$. This construction requires the following choices:

(1) for each p we chose

$$F^p \in \Gamma(X_p; \Omega^1_{X_p} \otimes (\mathfrak{d}(\mathcal{A}^p) \otimes \mathcal{J}^p))$$

such that $\Phi^p = -\delta F^p$, and

(2) for each morphism $f\colon [p] \to [q]$ in Δ we chose

$$D(f) \in \Gamma(X_q; X(f)^{-1}\mathfrak{d}(\mathcal{A}^p) \otimes \mathcal{J}_0^q)$$

such that $\vartheta(f) = \delta D(f)$.

The unit map

$$\mathcal{J}^r \to X(g \circ f)^{-1}\mathcal{A}^p \otimes \mathcal{J}^r \tag{4.4.6}$$

gives the canonical identification of the former with the center of the latter.

Lemma 4.10. *For a pair of composable arrows $[p] \xrightarrow{f} [q] \xrightarrow{g} [r]$ the section $X(g)^*D(f) + f^\sharp D(g) - D(g \circ f) \in \Gamma(X_r; X(g \circ f)^{-1}\mathcal{A}^p \otimes \mathcal{J}_0^r)$ is the image of a unique section*

$$\beta(f, g) \in \Gamma(X_r; \mathcal{J}_0^r)$$

under the map (4.4.6)*.(Here we regard $f^\sharp D(g)$ as an element of $\Gamma(X_r; X(g \circ f)^{-1}\mathcal{A}^p \otimes \mathcal{J}^r)$ using* (4.4.2)*.)*

Proof. Follows from the identity $X(g)^*\vartheta(f) + f^\sharp\vartheta(g) - \vartheta(g \circ f) = 0$. □

Let

$$F^\lambda = \sum_{i=0}^{n} t_i \tau_{\lambda,i}^* F^{\lambda(i)} - \sum_{i=0}^{n} dt_i \wedge \lambda(0i)^\sharp D(\lambda(in)) + \sum_{0\le i\le j\le n} (t_i dt_j - t_j dt_i) \wedge \beta(\lambda(ij), \lambda(jn)) \ ,$$

where we regard $\lambda(0i)^\sharp D(\lambda(in))$ as an element of $\Gamma(X_{\lambda(n)}; X(\lambda(0n))^{-1}\mathfrak{d}(\mathcal{A}^{\lambda(0)})\otimes \mathcal{J}^{\lambda(n)})$, so that

$$F^\lambda \in \Omega_n \otimes \Gamma(X_{\lambda(n)}; \Omega^\bullet_{X_{\lambda(n)}} \otimes (X(\lambda(0n))^{-1}\mathfrak{d}(\mathcal{A}^{\lambda(0)}) \otimes \mathcal{J}^{\lambda(n)})) \ .$$

Lemma 4.11.

(1) *The collection* $\{F^\lambda\}$ *defines an element* $F \in \mathfrak{H}$
(2) $\Phi = -\delta F$

Proof. Direct calculation using Lemma 2.10. □

As before, $\overline{F}$ denotes the image of F in $\mathtt{Tot}(|X|; \Omega^\bullet_{|X|}\otimes(\mathfrak{d}(\mathcal{A})\otimes\overline{\mathcal{J}}_{|X|}))$.

The unit map $\mathcal{O} \to \mathcal{A}$ (inclusion of the center) induces the embedding

$$\mathtt{Tot}(\Gamma(|X|; \Omega^\bullet_{|X|} \otimes \overline{\mathcal{J}}_{|X|})) \hookrightarrow \mathtt{Tot}(\Gamma(|X|; \Omega^\bullet_{|X|} \otimes (\mathfrak{d}(\mathcal{A}) \otimes \overline{\mathcal{J}}_{|X|})) \tag{4.4.7}$$

Lemma 4.12. *The element*

$$-\nabla\overline{F} \in \mathtt{Tot}(\Gamma(|X|; \Omega^\bullet_{|X|} \otimes (\mathfrak{d}(\mathcal{A}) \otimes \overline{\mathcal{J}}_{|X|})))$$

is the image of a unique closed form

$$\omega \in \mathtt{Tot}(\Gamma(|X|; \Omega^\bullet_{|X|} \otimes \overline{\mathcal{J}}_{|X|}))$$

under the inclusion (4.3.1)*. The cohomology class of* ω *is independent of the choices made in its construction.*

Proof. Using $\Phi = -\delta F$ (Lemma 4.11) and the fact that (4.4.5) is a differential, one obtains by direct calculation the identity $\delta(\nabla F)+\nabla^2 = 0$. Note that $\nabla^2 = \delta\theta$ for some $\theta \in \mathtt{Tot}(\Gamma(|X|; \Omega^\bullet_{|X|} \otimes \mathfrak{d}(\mathcal{A})))$. Hence $\nabla F + \theta$ is a central element of $\mathtt{Tot}(\Gamma(|X|; \Omega^\bullet_{|X|} \otimes (\mathfrak{d}(\mathcal{A}) \otimes \mathcal{J})))$, and the first statement follows. □

Following longstanding traditions we denote by ι_G the adjoint action of $G \in \mathtt{Tot}(\Gamma(|X|; \Omega^\bullet_{|X|} \otimes (\mathfrak{d}(\mathcal{A}) \otimes \mathcal{J}))) \subset \mathfrak{H}$.

Proposition 4.13.

$$\exp(\iota_F) \circ (\delta + \nabla + \Phi) \circ \exp(-\iota_F) = \delta + \nabla - \iota_{\nabla F}. \tag{4.4.8}$$

Proof. Analogous to the proof of Lemma 16 of [5]. Details are left to the reader. □

4.4.3. It follows from (4.4.8) that the map

$$\exp(-\iota_F)\colon \mathfrak{H} \to \mathfrak{H} \tag{4.4.9}$$

is an isomorphism of DGLA where the differential in the left hand side is given by $\delta + \nabla - \iota_{\nabla F}$ and the differential in the right hand side is given by $\delta + \nabla + \Phi$, as in(4.4.5).

Consider the map

$$\mathrm{cotr}\colon \overline{C}^\bullet(\mathcal{J}^p)[1] \to C^\bullet(\mathcal{A}^p \otimes \mathcal{J}^p)[1] \tag{4.4.10}$$

defined as follows:

$$\mathrm{cotr}(D)(a_1 \otimes j_1, \dots, a_n \otimes j_n) = a_0 \dots a_n D(j_1, \dots, j_n). \tag{4.4.11}$$

The map (4.4.10) is a quasiisomorphism of DGLAs (cf. [21], section 1.5.6; see also [5] Proposition 14).

The maps (4.4.10) induce the map of graded Lie algebras

$$\mathrm{cotr}\colon \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}) \to \mathfrak{H} \tag{4.4.12}$$

Lemma 4.14. *The map* (4.4.12) *is a quasiisomorphism of DGLA, where the source and the target are equipped with the differentials* $\delta + \widetilde{\nabla}^{can} + \iota_\omega$ *and* $\delta + \nabla - \iota_{\nabla F}$ *respectively, i.e.* (4.4.12) *is a morphism of DGLA*

$$\mathrm{cotr}\colon \mathfrak{G}_{\mathtt{DR}}(\mathcal{J})_\omega \to \mathfrak{H}$$

4.4.4. Recall that in the section 4.3.2 we introduced the bundles $\mathcal{E}^p = \bigoplus_{j=0}^p \mathcal{A}^p_{j0}$ over X_p. For $f\colon [p] \to [q]$ there is a canonical isomorphism $X(f)^{-1}\mathcal{E}^p \cong f^\sharp \mathcal{E}^q \otimes \mathcal{A}^q_{0f(0)}$ which we use to identify the former with the latter.

We make the following choices of additional structure:

(1) for each $p = 0, 1, 2, \dots$ an isomorphism $\sigma^p_{\mathcal{E}}\colon \mathcal{E}^p \otimes \mathcal{J}^p \to \mathcal{J}(\mathcal{E}^p)$ such that $\sigma^p_{\mathcal{E}}(\mathcal{A}^p_{j0} \otimes \mathcal{J}^p) \subset \mathcal{J}(\mathcal{A}^p_{j0})$
(2) for each $p = 0, 1, 2, \dots$ a connection $\nabla^p_{\mathcal{E}}$ on $\mathcal{E}^p$
(3) for every $f\colon [p] \to [q]$ an isomorphism $\sigma_f\colon \mathcal{A}^q_{0f(0)} \otimes \mathcal{J}^q \to \mathcal{J}(\mathcal{A}^q_{0f(0)})$
(4) for every $f\colon [p] \to [q]$ a connection ∇_f on $\mathcal{A}^q_{0f(0)}$

Let $\sigma^p \in \mathrm{Isom}_0(\mathcal{A}^p \otimes \mathcal{J}^p, \mathcal{J}(\mathcal{A}^p))^{loc}$ denote the isomorphism induced by $\sigma^p_{\mathcal{E}}$. Let $\nabla^p \in \mathcal{C}(\mathcal{A}^p)^{loc}(X_p)$ denote the connection induced by $\nabla^p_{\mathcal{E}}$.

Let

$$\begin{aligned} F^p &:= (\sigma^p_{\mathcal{E}})^{-1} \circ \nabla^{can}_{\mathcal{E}^p} \circ \sigma^p_{\mathcal{E}} - (\nabla^p_{\mathcal{E}} \otimes \mathtt{Id} + \mathtt{Id} \otimes \nabla^{can}) \\ F_f &:= (\sigma_f)^{-1} \circ \nabla^{can}_{\mathcal{A}^q_{0f(0)}} \circ \sigma_f - (\nabla_f \otimes \mathtt{Id} + \mathtt{Id} \otimes \nabla^{can}) \end{aligned}$$

$F^p \in \Gamma(X_p; \Omega^1_{X_p} \otimes (\mathcal{A}^p \otimes \mathcal{J}^p))$. We define $D(f) \in \Gamma(X_q; X(f)^{-1}\mathfrak{d}(\mathcal{A}^p) \otimes \mathcal{J}^q)$ by the equation

$$X(f)^* \sigma^p_{\mathcal{E}} \circ \exp(D(f)) = f^\sharp \sigma^q_{\mathcal{E}} \otimes \sigma_f$$

In 4.3.2 we constructed the cosimplicial gerbe $\mathcal{S}_{\mathcal{A}}$ on X such that $\mathcal{S}^p_{\mathcal{A}}$ is trivialized. Starting with the choices of σ_f, ∇_f as above we calculate the representative $\overline{B}$ of the characteristic class of $\mathcal{S}_{\mathcal{A}}$ using 2.5.6. By Lemma 4.7 the collection of bundles $\mathcal{E}^p$ establishes an equivalence between $\mathcal{S}_{\mathcal{A}}$ and the cosimplicial gerbe $S(\mathcal{A})$ (of splittings of $\mathcal{A}$). Hence, $\overline{B}$ is a representative of the characteristic class of $\mathcal{S}_{\mathcal{A}}$.

In the notations of 2.5, for $f\colon [p] \to [q]$, we have

$$\mathcal{G}_f = \underline{\mathrm{Isom}}_0(\mathcal{A}^q_{0f(0)} \otimes \mathcal{J}^q, \mathcal{J}(\mathcal{A}^q_{0f(0)}))$$

and, under this identification, $\beta_f = \sigma_f^{-1} \circ \nabla^{can} \circ \sigma_f$.

Then, $\beta(f,g)$ is uniquely determined by the equation

$$(\sigma_g \otimes X(g)^* \sigma_f) = \sigma_{g\circ f} \circ \exp \beta(f,g)$$

which implies

(4.4.13)

$$\nabla^{can}\beta(f,g) = \sigma_g^{-1} \circ \nabla^{can}_{\mathcal{S}_{\mathcal{A}g}} \circ \sigma_g + \sigma_f^{-1} \circ \nabla^{can}_{\mathcal{S}_{\mathcal{A}f}} \circ \sigma_f - \sigma^{-1}_{g\circ f} \circ \nabla^{can}_{\mathcal{S}_{\mathcal{A}g\circ f}} \circ \sigma_{g\circ f} \ .$$

Equations (4.4.13) and (2.5.6) show that $\beta(f,g)$ coincides with $\beta_{f,g}$ defined by (2.5.2).

Lemma 4.15. *The form $\overline{B}$ of 2.5.6 coincides with the form ω of Lemma 4.12*

Proof. For $f\colon [p] \to [q]$ in Δ the following identities hold:

$$(\nabla^p \otimes \mathtt{Id} + \mathtt{Id} \otimes \nabla^{can})\overline{F}^p = 0 \tag{4.4.14}$$

$$\nabla^{can}\overline{F}_f = 0 \tag{4.4.15}$$

$$\nabla^{can}\overline{D}(f) + X(f)^*\overline{F}^p = f^\sharp \overline{F}^q + \overline{F}_f \tag{4.4.16}$$

Using these identities we compute :

$$\nabla^{\lambda}\overline{F}^{\lambda} = \sum_{i=0}^{n} t_i \cdot \tau_{\lambda,i}^{*}(\nabla^{\lambda(i)} \otimes \mathtt{Id} + \mathtt{Id} \otimes \nabla^{can})\overline{F}^{\lambda(i)} +$$
$$\sum_{i=0}^{n} dt_i \wedge \tau_{\lambda,i}^{*}\overline{F}^{\lambda(i)} + \sum_{i=0}^{n} dt_i \wedge \lambda(0i)^{\sharp}\nabla^{can}(\overline{D}(\lambda(in))) +$$
$$\widetilde{\nabla}^{can}(\sum_{0\leq i\leq j\leq n}(t_i dt_j - t_j dt_i) \wedge \bar{\beta}(\lambda(ij),\lambda(jn))) =$$
$$\sum_{i=0}^{n} dt_i \wedge \lambda(0i)^{\sharp}(\nabla^{can}(\overline{D}(\lambda(in)) + X(\lambda(in))^{*}\overline{F}^{\lambda(i)}) +$$
$$\widetilde{\nabla}^{can}(\sum_{0\leq i\leq j\leq n}(t_i dt_j - t_j dt_i) \wedge \bar{\beta}(\lambda(ij),\lambda(jn))) =$$
$$\sum_{i=0}^{n} dt_i \wedge \lambda(0i)^{\sharp}(\lambda(in)^{\sharp}\overline{F}^{n} + \overline{F}_{\lambda(in)}) +$$
$$\widetilde{\nabla}^{can}(\sum_{0\leq i\leq j\leq n}(t_i dt_j - t_j dt_i) \wedge \bar{\beta}(\lambda(ij),\lambda(jn))) =$$
$$(\sum_{i=0}^{n} dt_i) \wedge \lambda(0n)^{\sharp}\overline{F}^{n} + \sum_{i=0}^{n} dt_i \wedge \overline{F}_{\lambda(in)} +$$
$$\widetilde{\nabla}^{can}(\sum_{0\leq i\leq j\leq n}(t_i dt_j - t_j dt_i) \wedge \bar{\beta}(\lambda(ij))) =$$
$$\sum_{i=0}^{n} dt_i \wedge \overline{F}_{\lambda(in)} + \widetilde{\nabla}^{can}(\sum_{0\leq i\leq j\leq n}(t_i dt_j - t_j dt_i) \wedge \bar{\beta}(\lambda(ij)))$$

The result is identical to the formula (2.5.7). □

4.4.5. In what follows we will denote a choice of auxiliary data as in 4.4.4 by ϖ.

Given a choice of auxiliary data ϖ we denote by $\overline{B}_{\varpi}$ the corresponding characteristic form, by Σ_{ϖ} the corresponding map (4.4.1), etc., and by

$$\Upsilon_{\varpi} \colon \mathfrak{G}_{\mathtt{DR}}(\mathcal{J})_{\overline{B}_{\varpi}} \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}(\mathcal{A}))$$

the quasiisomorphism of DGLA defined as the composition

$$\mathfrak{G}_{\mathtt{DR}}(\mathcal{J})_{\overline{B}_{\varpi}} \xrightarrow{\mathrm{cotr}} \mathfrak{H} \xrightarrow{\exp \iota_{F_{\varpi}}} \mathfrak{H} \xrightarrow{\Sigma_{\varpi}} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}(\mathcal{A}))$$

To finish the construction, we will "integrate" the "function" $\varpi \mapsto \Upsilon_{\varpi}$ over the "space" of choices of auxiliary data in order to produce

- the DGLA $\mathfrak{G}_{\mathtt{DR}}(\mathcal{J})_{[S(\mathcal{A})]}$
- for each choice of auxiliary data ϖ a quasiisomorphism

$$\mathtt{pr}_{\varpi}\colon \mathfrak{G}_{\mathtt{DR}}(\mathcal{J})_{[S(\mathcal{A})]} \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J})_{\overline{B}_{\varpi}}$$

- the morphism in the derived category of DGLA

$$\overline{\Upsilon}\colon \mathfrak{G}_{\mathtt{DR}}(\mathcal{J})_{[S(\mathcal{A})]} \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}(\mathcal{A}))$$

such that the morphism in the derived category given by the composition

$$\mathfrak{G}_{\mathtt{DR}}(\mathcal{J})_{\overline{B}_{\varpi}} \xrightarrow{(\mathtt{pr}_{\varpi})^{-1}} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J})_{[S(\mathcal{A})]} \xrightarrow{\overline{\Upsilon}} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}(\mathcal{A}))$$

coincides with Υ_{ϖ}.

4.4.6. Integration. To this end, for a cosimplicial matrix Azumaya algebra $\mathcal{A}$ on X (respectively, a cosimplicial $\mathcal{O}_X^{\times}$-gerbe $\mathcal{S}$ as in 2.2.7) let $\mathtt{AD}(\mathcal{A})$ (respectively, $\mathtt{AD}(\mathcal{S})$) denote the category with objects choices of auxiliary data (1) – (4) as in 4.4.4 (respectively, (i)–(iii) as in 2.5.2 with $\mathcal{G} = \nabla^{can}\log \mathbf{1}(\mathcal{S})$) and one-element morphism sets. Thus, $\mathtt{AD}(\mathcal{A})$ (respectively, $\mathtt{AD}(\mathcal{S})$) is a groupoid such that every object is both initial and final.

For a cosimplicial matrix Azumaya algebra $\mathcal{A}$ we have the functor

$$\pi\colon \mathtt{AD}(\mathcal{A}) \to \mathtt{AD}(S(\mathcal{A})) \tag{4.4.17}$$

which associates to $\varpi \in \mathtt{AD}(\mathcal{A})$ (in the notations of 4.4.5) the auxiliary data as in 2.5.6, items (1)–(3). Here we use the equivalence of Lemma 4.7.

For a category $\mathcal{C}$ we denote by $\mathrm{Sing}(\mathcal{C})$ denote the category, whose objects are "singular simplices" $\mu\colon [m] \to \mathcal{C}$. For $\mu\colon [m] \to \mathcal{C}$, $\nu\colon [n] \to \mathcal{C}$, a morphism $f\colon \mu \to \nu$ is an *injective* (on objects) morphism $f\colon [m] \to [n]$ such that $\mu = \nu \circ f$. The functor (4.4.17) induces the functor

$$\pi\colon \mathrm{Sing}(\mathtt{AD}(\mathcal{A})) \to \mathrm{Sing}(\mathtt{AD}(S(\mathcal{A})))$$

For $\mu\colon [m] \to \mathtt{AD}(\mathcal{A})$ (respectively, $\mu\colon [m] \to \mathtt{AD}(\mathcal{S})$) , $0 \leq i \leq m$, the choice of auxiliary data $\mu(i)$ consists of $\sigma^p_{\mathcal{E}}(\mu(i))$, $\nabla^p_{\mathcal{E}}(\mu(i))$, $\sigma_f(\mu(i))$, $\nabla_f(\mu(i))$ (respectively, $\eth^p(\mu(i))$, $B^p(\mu(i))$, $\beta_f(\mu(i))$) for all objects $[p]$ and morphisms f in Δ.

In either case, let $X(\mu)\colon = \Delta^m \times X$. Then, $X(\mu)$ is an étale simplicial manifold. Let $\mathcal{A}(\mu) = \mathtt{pr}_X^*\mathcal{A}$ (respectively, $\mathcal{S}(\mu) = \mathtt{pr}_X^*\mathcal{S}$). Then, $\mathcal{A}(\mu)$ is a cosimplicial matrix $\mathcal{O}_{X(\mu)}$-Azumaya algebra on $X(\mu)$ (respectively, a cosimplicial $\mathcal{O}^{\times}_{X(\mu)}$-gerbe).

For $\mu\colon [m] \to \mathtt{AD}(\mathcal{A})$ let $\sigma^p_{\mathcal{E}}(\mu)$ (respectively, $\nabla^p_{\mathcal{E}}$, $\sigma_f(\mu)$, $\nabla_f(\mu)$) denote the convex combination of $\mathtt{pr}_X^*\sigma^p_{\mathcal{E}}(\mu(i))$ (respectively, $\mathtt{pr}_X^*\nabla^p_{\mathcal{E}}(\mu(i))$, $\mathtt{pr}_X^*\sigma_f(\mu(i))$, $\mathtt{pr}_X^*\nabla_f(\mu(i))$), $i = 0, \ldots, m$. The collection consisting of

$\sigma^p_{\mathcal{E}}$, $\nabla^p_{\mathcal{E}}$, $\sigma_f(\mu)$, $\nabla_f(\mu)$ for all objects $[p]$ and morphisms f in Δ constitutes a choice of auxiliary data, denoted $\widetilde{\mu}$.

Similarly, for $\mu\colon [m] \to \mathtt{AD}(\mathcal{S})$ one defines $\widetilde{\mu}$ as the collection of auxiliary data $\eth^p(\mu)$ (respectively, $B^p(\mu)$, $\beta_f(\mu)$) consisting of convex combinations of $\eth^p(\mu(i))$ (respectively, $B^p(\mu(i))$, $\beta_f(\mu(i))$), $i = 0, \dots, m$. The construction of 2.5 apply with $X\colon = X(\mu)$, $\nabla^{can}\colon = \mathtt{pr}_X^*\nabla^{can}$, $\mathcal{S}\colon = \mathcal{S}(\mu)$ yielding the cocycle $\overline{B}_{\widetilde{\mu}} \in \Gamma(|X(\mu)|; \mathtt{DR}(\overline{\mathcal{J}}_{|X(\mu)|}))$. A morphism $f\colon \mu \to \nu$ in $\mathrm{Sing}(\mathtt{AD}(\mathcal{S}))$ induces a quasiisomorphism of complexes

$$f^*\colon \Gamma(|X(\nu)|; \mathtt{DR}(\overline{\mathcal{J}}_{|X(\nu)|})) \to \Gamma(|X(\mu)|; \mathtt{DR}(\overline{\mathcal{J}}_{|X(\mu)|})).$$

Moreover, we have $f^*(\overline{B}_{\widetilde{\nu}}) = \overline{B}_{\widetilde{\mu}}$.

Hence, as explained in 4.3.3, for $\mu\colon [m] \to \mathtt{AD}(\mathcal{S})$, we have the DGLA $\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)})_{\overline{B}_{\widetilde{\mu}}}$. Moreover, a morphism $f\colon \mu \to \nu$ in $\mathrm{Sing}(\mathtt{AD}(\mathcal{S}))$ induces a quasiisomorphism of DGLA

$$f^*\colon \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\nu)})_{\overline{B}_{\widetilde{\nu}}} \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)})_{\overline{B}_{\widetilde{\mu}}}.$$

Let

$$\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X)_{[\mathcal{S}]}\colon = \varprojlim_{\mu} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)})_{\overline{B}_{\widetilde{\mu}}} \tag{4.4.18}$$

where the limit is taken over the category $\mathrm{Sing}(\mathtt{AD}(\mathcal{S}))$.

For $\mu\colon [m] \to \mathtt{AD}(\mathcal{A})$ the constructions of 4.4.1 – 4.4.3 apply with $X\colon = X(\mu)$, $\mathcal{J}\colon = \mathcal{J}_{X(\mu)}$, $\nabla^{can}\colon = \mathtt{pr}_X^*\nabla^{can}$, $\mathcal{A}\colon = \mathcal{A}(\mu)$ and the choice of auxiliary data $\widetilde{\mu}$ yielding the quasiisomorphism of DGLA

$$\Upsilon_{\widetilde{\mu}}\colon \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)})_{\overline{B}_{\widetilde{\pi(\mu)}}} \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)}(\mathcal{A}(\mu)))$$

For a morphism $f\colon \mu \to \nu$ in $\mathrm{Sing}(\mathtt{AD}(\mathcal{A}))$, the diagram

$$\begin{array}{ccc} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\nu)})_{\overline{B}_{\widetilde{\pi(\nu)}}} & \xrightarrow{\Upsilon_{\widetilde{\nu}}} & \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\nu)}(\mathcal{A}(\nu))) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)})_{\overline{B}_{\widetilde{\pi(\mu)}}} & \xrightarrow{\Upsilon_{\widetilde{\mu}}} & \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)}(\mathcal{A}(\mu))) \end{array}$$

is commutative. Thus, we have two functors, $\mathrm{Sing}(\mathtt{AD}(\mathcal{A}))^{op} \to$ DGLA, namely,

$$\mu \mapsto \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)})_{\overline{B}_{\widetilde{\pi(\mu)}}}$$

and

$$\mu \mapsto \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)}(\mathcal{A}(\mu))),$$

and a morphism of such, namely, $\mu \mapsto \Upsilon_{\widetilde{\mu}}$. Since the first functor factors through $\mathrm{Sing}(\mathtt{AD}(S(\mathcal{A})))$ there is a canonical morphism of DGLA

$$\varprojlim_{\mu} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)})_{\overline{B}_{\widetilde{\pi(\mu)}}} \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X)_{[S(\mathcal{A})]}. \tag{4.4.19}$$

On the other hand, Υ induces the morphism

$$\varprojlim_{\mu} \Upsilon_{\widetilde{\mu}} \colon \varprojlim_{\mu} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)})_{\overline{B}_{\widetilde{\pi(\mu)}}} \to \varprojlim_{\mu} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)}(\mathcal{A}(\mu))). \tag{4.4.20}$$

Lemma 4.16. *The morphisms* (4.4.19) *and* (4.4.20) *are quasiisomorphisms.*

For each $\mu \in \mathrm{Sing}(\mathtt{AD}(\mathcal{A}))$ we have the map

$$\mathtt{pr}_X^*(\mu) \colon \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X(\mathcal{A})) \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)}(\mathcal{A}(\mu))).$$

Moreover, for any morphism $f \colon \mu \to \nu$ in $\mathrm{Sing}(\mathtt{AD}(\mathcal{A}))$, the diagram

$$\begin{array}{ccc} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X(\mathcal{A})) & \xrightarrow{\mathtt{pr}_X^*(\nu)} & \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\nu)}(\mathcal{A}(\nu))) \\ {\scriptstyle \mathtt{Id}}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X(\mathcal{A})) & \xrightarrow{\mathtt{pr}_X^*(\mu)} & \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)}(\mathcal{A}(\mu))) \end{array}$$

is commutative. Therefore, we have the map

$$\mathtt{pr}_X^* \colon \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X(\mathcal{A})) \to \varprojlim_{\mu} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)}(\mathcal{A}(\mu))) \tag{4.4.21}$$

Note that, for $\nu \colon [0] \to \mathtt{AD}(\mathcal{A})$, we have $\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\nu)}(\mathcal{A}(\nu))) = \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X(\mathcal{A}))$, and the composition

$$\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X(\mathcal{A})) \xrightarrow{\mathtt{pr}_X^*} \varprojlim_{\mu} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)}(\mathcal{A}(\mu))) \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\nu)}(\mathcal{A}(\nu)))$$

is equal to the identity.

Lemma 4.17. *For each $\nu \colon [0] \to \mathtt{AD}$ the morphism in the derived category induced by the canonical map*

$$\varprojlim_{\mu} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)}(\mathcal{A}(\mu))) \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\nu)}(\mathcal{A}(\nu)))$$

is inverse to $\mathtt{pr}_X^*$.

Corollary 4.18. *The morphism*

$$\varprojlim_{\mu} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)}(\mathcal{A}(\mu))) \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\nu)}(\mathcal{A}(\nu))) = \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X(\mathcal{A}))$$

in derived category does not depend on $\nu \colon [0] \to \mathtt{AD}$.

Let

$$\overline{\Upsilon}\colon \mathfrak{G}_{\mathtt{DR}}(\mathcal{J})_{[S(\mathcal{A})]} \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X(\mathcal{A})) \tag{4.4.22}$$

denote the morphism in the derived category represented by

$$\mathfrak{G}_{\mathtt{DR}}(\mathcal{J})_{[S(\mathcal{A})]} \xleftarrow{(4.4.19)} \varprojlim_{\mu} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)})_{\overline{B}_{\widetilde{\pi(\mu)}}} \xrightarrow{(4.4.20)}$$
$$\varprojlim_{\mu} \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\mu)}(\mathcal{A}(\mu))) \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{X(\nu)}(\mathcal{A}(\nu))) = \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_X(\mathcal{A}))$$

for any $\nu\colon [0] \to \mathtt{AD}$.

5. Applications to etale groupoids

5.1. Algebroid stacks. In this section we review the notions of algebroid stack and twisted form. We also define the notion of descent datum and relate it with algebroid stacks.

5.1.1. Algebroids. For a category $\mathcal{C}$ we denote by $i\mathcal{C}$ the subcategory of isomorphisms in $\mathcal{C}$; equivalently, $i\mathcal{C}$ is the maximal subgroupoid in $\mathcal{C}$.

Suppose that R is a commutative k-algebra.

Definition 5.1. An *R-algebroid* is a *nonempty* R-linear category $\mathcal{C}$ such that the groupoid $i\mathcal{C}$ is connected

Let Algd_R denote the 2-category of R-algebroids (full 2-subcategory of the 2-category of R-linear categories).

Suppose that A is an R-algebra. The R-linear category with one object and morphisms A is an R-algebroid denoted A^+.

Suppose that $\mathcal{C}$ is an R-algebroid and L is an object of $\mathcal{C}$. Let $A = \mathrm{End}_{\mathcal{C}}(L)$. The functor $A^+ \to \mathcal{C}$ which sends the unique object of A^+ to L is an equivalence.

Let ALG^2_R denote the 2-category of with

- objects R-algebras
- 1-morphisms homomorphism of R-algebras
- 2-morphisms $\phi \to \psi$, where $\phi, \psi\colon A \to B$ are two 1-morphisms are elements $b \in B$ such that $b \cdot \phi(a) = \psi(a) \cdot b$ for all $a \in A$.

It is clear that the 1- and the 2- morphisms in ALG^2_R as defined above induce 1- and 2-morphisms of the corresponding algebroids under the assignment $A \mapsto A^+$. The structure of a 2-category on ALG^2_R (i.e. composition of 1- and 2- morphisms) is determined by the requirement that the assignment $A \mapsto A^+$ extends to an embedding $(.)^+\colon \mathrm{ALG}^2_R \to \mathrm{Algd}_R$.

Suppose that $R \to S$ is a morphism of commutative k-algebras. The assignment $A \to A \otimes_R S$ extends to a functor $(.)\otimes_R S\colon \mathrm{ALG}_R^2 \to \mathrm{ALG}_S^2$.

5.1.2. Algebroid stacks.

Definition 5.2. A stack in R-linear categories $\mathcal{C}$ on a space Y is an *R-algebroid stack* if it is locally nonempty and locally connected by isomorphisms, i.e. the stack $i\widetilde{\mathcal{C}}$ is a gerbe.

Example 5.3. Suppose that $\mathcal{A}$ is a sheaf of R-algebras on Y. The assignment $X \supseteq U \mapsto \mathcal{A}(U)^+$ extends in an obvious way to a prestack in R-algebroids denoted $\mathcal{A}^+$. The associated stack $\widetilde{\mathcal{A}^+}$ is canonically equivalent to the stack of locally free $\mathcal{A}^{op}$-modules of rank one. The canonical morphism $\mathcal{A}^+ \to \widetilde{\mathcal{A}^+}$ sends the unique object of $\mathcal{A}^+$ to the free module of rank one.

1-morphisms and 2-morphisms of R-algebroid stacks are those of stacks in R-linear categories. We denote the 2-category of R-algebroid stacks by $\mathrm{AlgStack}_R(Y)$.

Suppose that G is an étale category.

Definition 5.4. An R-algebroid stack on G is a stack $\underline{\mathcal{C}} = (\mathcal{C}, \mathcal{C}_{01}, \mathcal{C}_{012})$ on G such that $\mathcal{C} \in \mathrm{AlgStack}_R(N_0G)$, $\mathcal{C}_{01}$ is a 1-morphism in $\mathrm{AlgStack}_R(N_1G)$ and $\mathcal{C}_{012}$ is a 2-morphism in $\mathrm{AlgStack}_R(N_2G)$.

Definition 5.5. A 1-morphism $\underline{\phi} = (\phi_0, \phi_{01})\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ of R-algebroid stacks on G is a morphism of stacks on G such that ϕ_0 is a 1-morphism in $\mathrm{AlgStack}_R(N_0G)$ and ϕ_{01} is a 2-morphism in $\mathrm{AlgStack}(N_1G)$.

Definition 5.6. A 2-morphism $b\colon \underline{\phi} \to \underline{\psi}$ between 1-morphisms $\underline{\phi}, \underline{\psi}\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a 2-morphism $b\colon \phi_0 \to \psi_0$ in $\mathrm{AlgStack}_R(N_0G)$.

We denote the 2-category of R-algebroid stacks on G by $\mathrm{AlgStack}_R(G)$.

5.2. Base change for algebroid stacks. For an R-linear category $\mathcal{C}$ and homomorphism of commutative k-algebras $R \to S$ we denote by $\mathcal{C} \otimes_R S$ the category with the same objects as $\mathcal{C}$ and morphisms defined by $\mathrm{Hom}_{\mathcal{C}\otimes_R S}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B) \otimes_R S$.

For an R-algebra A the categories $(A \otimes_R S)^+$ and $A^+ \otimes_R S$ are canonically isomorphic.

For a prestack $\mathcal{C}$ in R-linear categories we denote by $\mathcal{C} \otimes_R S$ the prestack associated to the fibered category $U \mapsto \mathcal{C}(U) \otimes_R S$.

For $U \subseteq X$, $A, B \in \mathcal{C}(U)$, there is an isomorphism of sheaves $\underline{\mathrm{Hom}}_{\mathcal{C}\otimes_R S}(A, B) = \underline{\mathrm{Hom}}_{\mathcal{C}}(A, B) \otimes_R S$.

The proof of the following lemma can be found in [7] (Lemma 4.13 of loc. cit.)

Lemma 5.7. *Suppose that $\mathcal{A}$ is a sheaf of R-algebras on a space Y and $\mathcal{C} \in \mathrm{AlgStack}_R(Y)$ is an R-algebroid stack.*

(1) $\widetilde{(\widetilde{\mathcal{A}^+} \otimes_R S)}$ *is an algebroid stack equivalent to* $(\widetilde{\mathcal{A} \otimes_R S})^+$.

(2) $\widetilde{\mathcal{C} \otimes_R S}$ *is an algebroid stack.*

The assignment $\mathcal{C} \mapsto \widetilde{\mathcal{C} \otimes_R S}$ extends to a functor denoted

$$(.)\widetilde{\otimes}_R S\colon \mathrm{AlgStack}_R(Y) \to \mathrm{AlgStack}_S(Y)$$

and, hence, for an étale category G, to a functor

$$(.)\widetilde{\otimes}_R S\colon \mathrm{AlgStack}_R(G) \to \mathrm{AlgStack}_S(G)$$

There is a canonical R-linear morphism $\mathcal{C} \to \mathcal{C}\widetilde{\otimes}_R S$ (where the target is considered as a stack in R-linear categories by restriction of scalars) which is characterized by an evident universal property.

5.3. The category of trivializations. Let $\mathrm{Triv}_R(G)$ denote the 2-category with

- objects the pairs $(\underline{\mathcal{C}}, L)$, where $\underline{\mathcal{C}}$ is an R-algebroid stack on G such that $\mathcal{C}(N_0G) \neq \emptyset$, and $L \in \mathcal{C}(N_0G)$
- 1-morphism $(\underline{\mathcal{C}}, L) \to (\underline{\mathcal{D}}, M)$ the pairs $(\underline{\phi}, \phi_\tau)$, where $\underline{\phi}\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a morphism in $\mathrm{AlgStack}_R(G)$ and $\phi_\tau\colon \phi_0(L) \to M$ is an *isomorphism* in $\mathcal{D}(N_0G)$.
- 2-morphisms $(\underline{\phi}, \phi_\tau) \to (\underline{\psi}, \psi_\tau)$ are the 2-morphisms $\underline{\phi} \to \underline{\psi}$.

The composition of 1-morphisms is defined by $(\underline{\phi}, \phi_\tau) \circ (\underline{\psi}, \psi_\tau) = (\underline{\phi} \circ \underline{\psi}, \phi_\tau \circ \phi_0(\psi_\tau))$.

The assignment $(\underline{\mathcal{C}}, L) \mapsto \underline{\mathcal{C}}$, $(\underline{\phi}, \phi_\tau) \mapsto \underline{\phi}$, $b \mapsto b$ extends to a functor

$$\mathrm{Triv}_R(G) \to \mathrm{AlgStack}_R(G) \tag{5.3.1}$$

For a homomorphism of algebras $R \to S$ and $(\underline{\mathcal{C}}, L) \in \mathrm{Triv}_R(G)$ we denote by $(\underline{\mathcal{C}}, L)\widetilde{\otimes}_R S$ the pair which consists of $\underline{\mathcal{C}}\widetilde{\otimes}_R S \in \mathrm{AlgStack}_S(G)$ and the image of L, denoted $L \otimes_R S$, in $\underline{\mathcal{C}}\widetilde{\otimes}_R S$.

It is clear that the forgetful functors (5.3.1) commute with the base change functors.

5.3.1. Algebroid stacks from cosimplicial matrix algebras. Suppose that G is an étale category and $\mathcal{A}$ is a cosimplicial matrix R-algebra on NG.

Let

$$\mathcal{C} = \widetilde{(\mathcal{A}^{0op})^+} \ . \tag{5.3.2}$$

In other words, $\mathcal{C}$ is the stack of locally free $\mathcal{A}$-modules of rank one. There is a canonical isomorphism $\mathcal{C}_i^{(1)} = \widetilde{(\mathcal{A}_{ii}^{1op})^+}$.

Let

$$\mathcal{C}_{01} = \mathcal{A}^1_{01} \otimes_{\mathcal{A}^1_{11}} (.) \colon \mathcal{C}^{(1)}_1 \to \mathcal{C}^{(1)}_0. \tag{5.3.3}$$

The multiplication pairing $\mathcal{A}^2_{01} \otimes \mathcal{A}^2_{12} \to \mathcal{A}^2_{02}$ and the isomorphisms $\mathcal{A}^2_{ij} = (\mathcal{A}^1_{01})^{(2)}_{ij}$ determine the morphism

$$\mathcal{C}_{012} \colon \mathcal{C}^{(2)}_{01} \circ \mathcal{C}^{(2)}_{12} = \mathcal{A}^2_{01} \otimes_{\mathcal{A}^2_{11}} \mathcal{A}^2_{12} \otimes_{\mathcal{A}^2_{22}} (.) \to \mathcal{A}^2_{02} \otimes_{\mathcal{A}^2_{22}} (.) = \mathcal{C}^{(2)}_{02} \tag{5.3.4}$$

Let $\underline{\mathcal{C}} = (\mathcal{C}, \mathcal{C}_{01}, \mathcal{C}_{012})$.

Lemma 5.8. *The triple $\underline{\mathcal{C}}$ defined by* (5.3.2), (5.3.3), (5.3.4) *is an algebroid stack on G.*

We denote by $\mathtt{st}(\mathcal{A})$ the algebroid stack $\underline{\mathcal{C}}$ associated to the cosimplicial matrix algebra $\mathcal{A}$ by the above construction.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are cosimplicial matrix R-algebras on NG and $F \colon \mathcal{A} \to \mathcal{B}$ is a 1-morphism of such. Let $\underline{\mathcal{C}} = \mathtt{st}(\mathcal{A})$, $\underline{\mathcal{D}} = \mathtt{st}(\mathcal{B})$. The map $F^0 \colon \mathcal{A}^0 \to \mathcal{B}^0$ of sheaves of R-algebras on N_0G induces the morphism $\phi_0 \colon \mathcal{C} \to \mathcal{D}$ of R-algebroid stacks on N_0G. Namely, we have

$$\phi_0 = \mathcal{B}^0 \otimes_{\mathcal{A}^0} (.) \tag{5.3.5}$$

The map $F^1 \colon \mathcal{A}^1 \to \mathcal{B}^1$ restricts to the map $F^1_{01} \colon \mathcal{A}^1_{01} \to \mathcal{B}^1_{01}$ which induces the map

$$\mathcal{B}^1_{00} \otimes_{\mathcal{A}^1_{00}} \mathcal{A}^1_{01} \to \mathcal{B}^1_{01} = \mathcal{B}^1_{01} \otimes_{\mathcal{B}^1_{11}} \mathcal{B}^1_{11}$$

of $\mathcal{B}^1_{00} \otimes (\mathcal{A}^1_{11})^{op}$-modules, hence the 2-morphism

$$\phi_{01} \colon \phi^{(1)}_0 \circ \mathcal{C}_{01} \to \mathcal{D}_{01} \circ \phi^{(1)}_1 \tag{5.3.6}$$

Let $\underline{\phi} = (\phi_0, \phi_{01})$.

Lemma 5.9. *The pair $\underline{\phi}$ defined by* (5.3.5), (5.3.6) *is a* 1*-morphism in* $\mathrm{AlgStack}_R(G)$.

For a 1-morphism of cosimplicial matrix R-algebras $F \colon \mathcal{A} \to \mathcal{B}$ on NG we denote by $\mathtt{st}(F) \colon \mathtt{st}(\mathcal{A}) \to \mathtt{st}(\mathcal{B})$ the 1-morphism in $\mathrm{AlgStack}_R(G)$ given by the above construction.

Suppose that $F_1, F_2 \colon \mathcal{A} \to \mathcal{B}$ are 1-morphisms of cosimplicial matrix R-algebras on NG and $b \colon F_1 \to F_2$ is a 2-morphism of such. The 2-morphism $b^0 \colon F^0_1 \to F^0_2$ induces the 2-morphism of functors $\widetilde{(\mathcal{A}^{0op})}^+ \to \widetilde{(\mathcal{B}^{0op})}^+$.

Lemma 5.10. *The* 2*-morphism b^0 is a* 2*-morphism in* $\mathrm{AlgStack}_R(G)$.

For a 2-morphism $b\colon F_1 \to F_2$ we denote by $\mathtt{st}(b)\colon \mathtt{st}(F_1) \to \mathtt{st}(F_2)$ the corresponding 2-morphism in $\mathrm{AlgStack}_R(G)$ given by the above construction.

We denote the canonical trivialization of the algebroid stack $\mathtt{st}(\mathcal{A})$ by $\mathbb{1}_{\mathcal{A}}$. Let $\mathtt{triv}(\mathcal{A})$ denote the object of $\mathrm{Triv}_R(G)$ given by the pair $(\mathtt{st}(\mathcal{A}), \mathbb{1}_{\mathcal{A}})$.

If $F\colon \mathcal{A} \to \mathcal{B}$ is a 1-morphism in $\mathrm{CMA}_R(NG)$, then $\mathtt{st}(F)(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{B}}$. Let $\mathtt{st}(F)_\tau$ denote this identification; let $\mathtt{triv}(F) = (\mathtt{st}(F), \mathtt{st}(F)_\tau)$.

For a 2-morphism b in $\mathrm{CMA}_R(G)$ let $\mathtt{triv}(b) = \mathtt{st}(b)$.

Proposition 5.11.

(1) *The assignments* $\mathcal{A} \mapsto \mathtt{st}(\mathcal{A})$, $F \mapsto \mathtt{st}(F)$, $b \mapsto \mathtt{st}(b)$ *define a functor*

$$\mathtt{st}\colon \mathrm{CMA}_R(NG) \to \mathrm{AlgStack}_R(G)\ .$$

(2) *The assignments* $\mathcal{A} \mapsto \mathtt{triv}(\mathcal{A})$, $F \mapsto \mathtt{triv}(F)$, $b \mapsto \mathtt{triv}(b)$ *define a functor*

$$\mathtt{triv}\colon \mathrm{CMA}_R(NG) \to \mathrm{Triv}_R(G)\ .$$

which lifts the functor $\mathtt{st}$.

5.3.2. Base change for `st` and `triv`. A morphism $f\colon R \to S$ of commutative k-algebras induces the R-linear morphism $\mathtt{Id} \otimes f\colon \mathcal{A} \to \mathcal{A} \otimes_R S$ in $\mathrm{CMA}_R(NG)$, hence, the morphism $\mathtt{st}(\mathtt{Id} \otimes f)\colon \mathtt{st}(\mathcal{A}) \to \mathtt{st}(\mathcal{A} \otimes_R S)$ in $\mathrm{AlgStack}_R(G)$. By the universal property of the base change morphism, the latter factors canonically through a unique morphism $\mathtt{st}(\mathcal{A})\widetilde{\otimes}_R S \to \mathtt{st}(\mathcal{A} \otimes_R S)$ such that $\mathbb{1}_{\mathcal{A}} \otimes_R S \mapsto \mathbb{1}_{\mathcal{A}\otimes_R S}$ and the induced map $\mathcal{A}\otimes_R S = \underline{\mathrm{End}}_{\mathtt{st}(\mathcal{A}\widetilde{\otimes}_R S}(\mathbb{1}_{\mathcal{A}}\otimes_R S) \to \underline{\mathrm{End}}_{\mathtt{st}(\mathcal{A}\otimes_R S)}(\mathbb{1}_{\mathcal{A}\otimes_R S}) = \mathcal{A} \otimes_R S$ is the identity map. In particular, it is an equivalence. The inverse equivalence $\mathtt{st}(\mathcal{A}\otimes_R S) \to \mathtt{st}(\mathcal{A})\widetilde{\otimes}_R S$ is induced by the canonical morphism $(\mathcal{A} \otimes_R S)^+ = \underline{\mathrm{End}}_{\mathtt{st}(\mathcal{A}\otimes_R S)}(\mathbb{1}_{\mathcal{A}\otimes_R S})^+ \to \mathtt{st}(\mathcal{A})\widetilde{\otimes}_R S$. Thus, we have the canonical mutually inverse equivalences $\mathtt{triv}(\mathcal{A})\widetilde{\otimes}_R S \leftrightarrows \mathtt{triv}(\mathcal{A} \otimes_R S)$.

A morphism $F\colon \mathcal{A} \to \mathcal{B}$ in $\mathrm{CMA}_R(G)$ gives rise to the diagram

$$\begin{array}{ccc} \mathtt{triv}(\mathcal{A})\widetilde{\otimes}_R S & \xrightarrow{\mathtt{triv}F\widetilde{\otimes}_R S} & \mathtt{triv}(\mathcal{A})\widetilde{\otimes}_R S \\ \downarrow & & \downarrow \\ \mathtt{triv}(\mathcal{A} \otimes_R S) & \xrightarrow{\mathtt{triv}(F\otimes_R S)} & \mathtt{triv}(\mathcal{A} \otimes_R S) \end{array}$$

which is commutative up to a unique 2-morphism.

5.3.3. Cosimplicial matrix algebras from algebroid stacks. We fix $(\underline{\mathcal{C}}, L) \in \mathrm{Triv}_R(G)$ and set, for the time being, $\mathcal{A}^0 = \underline{\mathrm{End}}_{\mathcal{C}}(L)^{op}$. The assignment $L' \to \underline{\mathrm{Hom}}_{\mathcal{C}}(L, L')$ defines an equivalence $\mathcal{C} \to \widetilde{(\mathcal{A}^{0op})^+}$. The inverse equivalence is determines by the assignment $\mathbb{1}_{\mathcal{A}} \mapsto L$.

For $n \geq 1$, $0 \leq i, j \leq n$ let

$$\mathcal{A}^n_{ij} = \begin{cases} \underline{\mathrm{Hom}}_{\mathcal{C}_i^{(n)}}(L_i^{(n)}, \mathcal{C}_{ij}^{(n)}(L_j^{(n)})) & \text{if} \quad i \leq j \\ \underline{\mathrm{Hom}}_{\mathcal{C}_j^{(n)}}(\mathcal{C}_{ji}^{(n)}(L_i^{(n)}), L_j^{(n)}) & \text{if} \quad j \leq i \end{cases} \tag{5.3.7}$$

where $\mathcal{C}_{ii}^{(n)} = \mathtt{Id}_{\mathcal{C}_i^{(n)}}$. Then, the sheaf $\mathcal{A}^n_{ij}$ has a canonical structure of a $(\mathcal{A}^0)_i^{(n)} \otimes (\mathcal{A}^0)_j^{(n)op}$-module. Let

$$\mathcal{A}^n = \bigoplus_{i,j=0}^{n} \mathcal{A}^n_{ij}$$

The definition of the multiplication of matrix entries comprises several cases. Suppose that $i \leq j \leq k$. We have the map

$$\begin{aligned} \mathcal{A}^n_{jk} = \underline{\mathrm{Hom}}_{\mathcal{C}_j^{(n)}}(L_j^{(n)}, \mathcal{C}_{jk}^{(n)}(L_k^{(n)})) \xrightarrow{\mathcal{C}_{ij}^{(n)}} \\ \underline{\mathrm{Hom}}_{\mathcal{C}_i^{(n)}}(\mathcal{C}_{ij}^{(n)}(L_j^{(n)}), (\mathcal{C}_{ij}^{(n)} \circ \mathcal{C}_{jk}^{(n)})(L_k^{(n)})) \xrightarrow{\mathcal{C}_{ijk}^{(n)}} \\ \underline{\mathrm{Hom}}_{\mathcal{C}_i^{(n)}}(\mathcal{C}_{ij}^{(n)}(L_j^{(n)}), \mathcal{C}_{ik}^{(n)}(L_k^{(n)})) \end{aligned} \tag{5.3.8}$$

The multiplication of matrix entries is given by

$$\begin{aligned} \mathcal{A}^n_{ij} \otimes \mathcal{A}^n_{jk} \xrightarrow{\mathtt{Id} \otimes (5.3.8)} \\ \underline{\mathrm{Hom}}_{\mathcal{C}_i^{(n)}}(L_i^{(n)}, \mathcal{C}_{ij}^{(n)}(L_j^{(n)})) \otimes \underline{\mathrm{Hom}}_{\mathcal{C}_i^{(n)}}(\mathcal{C}_{ij}^{(n)}(L_j^{(n)}), \mathcal{C}_{ik}^{(n)}(L_k^{(n)})) \\ \xrightarrow{\circ} \underline{\mathrm{Hom}}_{\mathcal{C}_i^{(n)}}(L_i^{(n)}, \mathcal{C}_{ik}^{(n)}(L_k^{(n)})) = \mathcal{A}^n_{ik} \end{aligned} \tag{5.3.9}$$

Suppose that $j \leq i \leq k$. We have the map

$$\begin{aligned} \mathcal{A}^n_{ik} = \underline{\mathrm{Hom}}_{\mathcal{C}_i^{(n)}}(L_i^{(n)}, \mathcal{C}_{ik}^{(n)}(L_k^{(n)})) \xrightarrow{\mathcal{C}_{ji}^{(n)}} \\ \underline{\mathrm{Hom}}_{\mathcal{C}_j^{(n)}}(\mathcal{C}_{ji}^{(n)}(L_i^{(n)}), (\mathcal{C}_{ji}^{(n)} \circ \mathcal{C}_{ik}^{(n)})(L_k^{(n)})) \xrightarrow{\mathcal{C}_{jik}^{(n)}} \\ \underline{\mathrm{Hom}}_{\mathcal{C}_j^{(n)}}(\mathcal{C}_{ji}^{(n)}(L_i^{(n)}), \mathcal{C}_{jk}^{(n)}(L_k^{(n)})) \end{aligned} \tag{5.3.10}$$

The multiplication of matrix entries is given by

$$\begin{aligned}(5.3.11)\quad \mathcal{A}^n_{ij}\otimes\mathcal{A}^n_{jk} = \\ \underline{\mathrm{Hom}}_{\mathcal{C}^{(n)}_j}(\mathcal{C}^{(n)}_{ji}(L^{(n)}_i),L^{(n)}_j)\otimes\underline{\mathrm{Hom}}_{\mathcal{C}^{(n)}_j}(L^{(n)}_j,\mathcal{C}^{(n)}_{jk}(L^{(n)}_k))\xrightarrow{\circ} \\ \underline{\mathrm{Hom}}_{\mathcal{C}^{(n)}_j}(\mathcal{C}^{(n)}_{ji}(L^{(n)}_i),\mathcal{C}^{(n)}_{ik}(L^{(n)}_i))\xrightarrow{(5.3.10)^{-1}}\mathcal{A}^n_{ik}\end{aligned}$$

We leave the remaining cases and the verification of the associativity of multiplication on $\mathcal{A}^n$ to the reader.

Lemma 5.12. *The collection of matrix algebras $\mathcal{A}^n$, $n=0,1,2,\ldots$, has a canonical structure of a cosimplicial matrix R-algebra.*

We denote by $\mathtt{cma}(\underline{\mathcal{C}},L)$ the cosimplicial matrix algebra associated to $(\underline{\mathcal{C}},L)$ by the above construction.

Suppose that $(\underline{\phi},\phi_\tau)\colon(\underline{\mathcal{C}},L)\to(\underline{\mathcal{D}},M)$ is a 1-morphism in $\mathrm{Triv}_R(G)$. Let $\mathcal{A}=\mathtt{cma}(\underline{\mathcal{C}},L)$, $\mathcal{B}=\mathtt{cma}(\underline{\mathcal{D}},M)$.

Let $F^0\colon\mathcal{A}^0\to\mathcal{B}^0$ denote the composition

$$\mathcal{A}^0=\underline{\mathrm{End}}_{\mathcal{C}}(L)^{op}\xrightarrow{\phi_0}\underline{\mathrm{End}}_{\mathcal{D}}(\phi_0(L))^{op}\xrightarrow{\phi_\tau}\underline{\mathrm{End}}_{\mathcal{D}}(M)^{op}=\mathcal{B}^0$$

For $n\geq 1$, $0\leq i\leq j\leq n$ let $F^n_{ij}\colon\mathcal{A}^n_{ij}\to\mathcal{B}^n_{ij}$ (see (5.3.7)) denote the composition

$$\begin{aligned}&\underline{\mathrm{Hom}}_{\mathcal{C}^{(n)}_i}(L^{(n)}_i,\mathcal{C}^{(n)}_{ij}(L^{(n)}_j))\xrightarrow{\phi^{(n)}_i} \\ &\qquad\underline{\mathrm{Hom}}_{\mathcal{D}^{(n)}_i}(\phi^{(n)}_i(L^{(n)}_i),(\phi^{(n)}_i\circ\mathcal{C}^{(n)}_{ij})(L^{(n)}_j))\xrightarrow{\phi^{(n)}_{ij}} \\ &\qquad\underline{\mathrm{Hom}}_{\mathcal{D}^{(n)}_i}(\phi^{(n)}_i(L^{(n)}_i),(\mathcal{D}^{(n)}_{ij}\circ\phi^{(n)}_j)(L^{(n)}_j))\xrightarrow{\phi^{(n)}_\tau} \\ &\qquad\qquad\underline{\mathrm{Hom}}_{\mathcal{D}^{(n)}_i}(M^{(n)}_i,\mathcal{D}^{(n)}_{ij}(M^{(n)}_j)).\end{aligned}$$

The construction of F^n_{ij} in the case $j\leq i$ is similar and is left to the reader. Let $F^n=\oplus F^n\colon\mathcal{A}^n\to\mathcal{B}^n$.

Lemma 5.13. *The collection of maps $F^n\colon\mathcal{A}^n\to\mathcal{B}^n$ is a 1-morphism of cosimplicial matrix R-algebras.*

We denote by $\mathtt{cma}(\underline{\phi},\phi_\tau)\colon\mathtt{cma}(\underline{\mathcal{C}},L)\to\mathtt{cma}(\underline{\mathcal{D}},M)$ the 1-morphism of cosimplicial matrix algebras associated to the 1-morphism $(\underline{\phi},\phi_\tau)$ in $\mathrm{Triv}(G)$ by the above construction.

Suppose that $(\underline{\phi},\phi_\tau),(\underline{\psi},\psi_\tau)\colon(\underline{\mathcal{C}},L)\to(\underline{\mathcal{D}},M)$ are 1-morphisms in $\mathrm{Triv}(G)$ and $b\colon(\underline{\phi},\phi_\tau)\to(\underline{\psi},\psi_\tau)$ is a 2-morphism.

Let $\mathtt{cma}(b)^0\colon \mathtt{cma}(\underline{\phi},\phi_\tau)^0 \to \mathtt{cma}(\underline{\psi},\psi_\tau)^0$ denote the composition

$$M \xrightarrow{(\phi_\tau)^{-1}} \phi(L) \xrightarrow{b(B)} \psi(L) \xrightarrow{\psi_\tau} M$$

(i.e. $\mathtt{cma}(b)^0 \in \Gamma(N_0G; \underline{\mathrm{End}}_{\mathcal{D}}(M))$.)

Lemma 5.14. *The section* $\mathtt{cma}(b)^0 \in \Gamma(N_0G; \underline{\mathrm{End}}_{\mathcal{D}}(M))$ *is a* 2*-morphism* $\mathtt{cma}(\underline{\phi},\phi_\tau)^0 \to \mathtt{cma}(\underline{\psi},\psi_\tau)^0$.

Proof. For $f \in \underline{\mathrm{End}}_{\mathcal{C}}(L)$ we have

$$\begin{aligned}
\mathtt{cma}(b)^0 \circ \mathtt{cma}(\underline{\phi},\phi_\tau)^0(f) &= \psi_\tau \circ b(L) \circ (\phi_\tau)^{-1} \circ \phi_\tau \circ \phi(f) \circ (\phi_\tau)^{-1} \\
&= \psi_\tau \circ b(L) \circ \phi(f) \circ (\phi_\tau)^{-1} \\
&= \psi_\tau \circ \psi(f) \circ b(L) \circ (\phi_\tau)^{-1} \\
&= \psi_\tau \circ \psi(f) \circ (\psi_\tau)^{-1} \circ \psi_\tau \circ b(L) \circ (\phi_\tau)^{-1} \\
&= \mathtt{cma}(\underline{\psi},\psi_\tau)^0(f) \circ \mathtt{cma}(b)^0
\end{aligned}$$

□

For $n \geq 1$ let $\mathtt{cma}(b)^n = (\mathtt{cma}(b)^n_{ij}) \in \Gamma(N_nG; \mathtt{cma}(\underline{\mathcal{D}}, M)^n)$ denote the "diagonal" matrix with $\mathtt{cma}(b)^n_{ii} = (\mathtt{cma}(b)^0)^{(n)}_i$.

Lemma 5.15. *The collection* $\mathtt{cma}(b)$ *of sections* $\mathtt{cma}(b)^n$, $n = 0, 1, 2, \ldots$ *is a* 2*-morphism* $\mathtt{cma}(\underline{\phi},\phi_\tau) \to \mathtt{cma}(\underline{\psi},\psi_\tau)$.

Proposition 5.16.

(1) *The assignments* $(\underline{\mathcal{C}}, L) \mapsto \mathtt{cma}(\underline{\mathcal{C}}, L)$, $(\underline{\phi},\phi_\tau) \mapsto \mathtt{cma}(\underline{\phi},\phi_\tau)$, $b \mapsto \mathtt{cma}(b)$ *define a functor*

$$\mathtt{cma}\colon \mathrm{Triv}_R(G) \to \mathrm{CMA}_R(NG)$$

(2) *The functors* cma *commute with the base change functors.*

For $\mathcal{A} \in \mathrm{CMA}(NG)$ we have

$$\mathtt{cma}(\mathtt{triv}(\mathcal{A}))^0 = \underline{\mathrm{End}}_{\widetilde{(\mathcal{A}^{0op})^+}}(\mathbb{1}_{\mathcal{A}})^{op} = \mathcal{A}^0$$

For $0 \leq i \leq j \leq n$ there is a canonical isomorphism

$$\underline{\mathrm{Hom}}_{\widetilde{(\mathcal{A}^{nop}_{ii})^+}}(\mathbb{1}^{(n)}_i, \mathtt{triv}(\mathcal{A})^{(n)}_{ij}(\mathbb{1}^{(n)}_j)) \cong \mathcal{A}^n_{ij}\ .$$

For $0 \leq j \leq i \leq n$ we have

$$\underline{\mathrm{Hom}}_{\widetilde{(\mathcal{A}^{nop}_{jj})^+}}(\mathtt{triv}(\mathcal{A})^{(n)}_{ji}(\mathbb{1}^{(n)}_i), \mathbb{1}^{(n)}_j) \cong \underline{\mathrm{Hom}}_{\mathcal{A}^n_{jj}}(\mathcal{A}^n_{ji}, \mathcal{A}^n_{jj}) \cong \mathcal{A}^n_{ij}\ ,$$

where the last isomorphism comes from the (multiplication) pairing $\mathcal{A}^n_{ji} \otimes \mathcal{A}^n_{ij} \to \mathcal{A}^n_{jj}$. These isomorphisms give rise to the canonical isomorphism of CMA

$$\mathtt{cma}(\mathtt{triv}(\mathcal{A})) \cong \mathcal{A}$$

On the other hand, given $(\underline{\mathcal{C}}, L) \in \mathrm{Triv}(G)$, we have the canonical equivalence $(\widetilde{\underline{\mathrm{End}}_{\mathcal{C}}(L)^{op}})^+ \to \mathcal{C}$ determined by $\mathbb{1}_{(\widetilde{\underline{\mathrm{End}}_{\mathcal{C}}(L)^{op}})^+} \mapsto L$.

To summarize, we have the following proposition.

Proposition 5.17. *The functors* `triv` *and* `cma` *are mutually quasi-inverse equivalences of categories.*

5.3.4. Ψ-algebroid stacks. Suppose that X is a space and Ψ is a pseudo-tensor subcategory of $\mathrm{Sh}_k(X)$ as in 3.2.1.

An algebroid stack $\mathcal{C} \in \mathrm{AlgStack}_k(X)$ is called a Ψ-*algebroid stack* if for any open subset $U \subseteq X$ and any three objects $L_1, L_2, L_3 \in \mathcal{C}(U)$ the composition map

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(L_1, L_2) \otimes_k \underline{\mathrm{Hom}}_{\mathcal{C}}(L_2, L_3) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(L_1, L_2)$$

is in Ψ.

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are Ψ-algebroid stacks. A 1-morphism $\phi\colon \mathcal{C} \to \mathcal{D}$ is called a Ψ-1-*morphism* if for any open subset $U \subseteq X$ and any two objects $L_1, L_2 \in \mathcal{C}(U)$ the map

$$\phi\colon \underline{\mathrm{Hom}}_{\mathcal{C}}(L_1, L_2) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(\phi(L_1), \phi(L_2))$$

is in Ψ.

Suppose that $\phi, \psi\colon \mathcal{C} \to \mathcal{D}$ are Ψ-1-morphisms. A 2-morphism $b\colon \phi \to \psi$ is called a Ψ-2-*morphism* if for any open subset $U \subseteq X$ and any two objects $L_1, L_2 \in \mathcal{C}(U)$ the map

$$b\colon \underline{\mathrm{Hom}}_{\mathcal{D}}(\phi(L_1), \phi(L_2)) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(\psi(L_1), \psi(L_2))$$

is in Ψ.

We denote by $\mathrm{AlgStack}_k^{\Psi}(X)$ the subcategory of $\mathrm{AlgStack}_k(X)$ whose objects, 1-morphisms and 2-morphisms are, respectively, the Ψ-algebroid stacks, the Ψ-1- and the Ψ-2-morphisms.

Suppose that G is an étale category and let $\Psi\colon p \mapsto \Psi^p$ be a cosimplicial pseudo-tensor subcategory of $\mathrm{Sh}_k(NG)$ as in 4.1. An algebroid stack $\underline{\mathcal{C}} = (\mathcal{C}, \mathcal{C}_{01}, \mathcal{C}_{012}) \in \mathrm{AlgStack}_k(G)$ is called a Ψ-*algebroid stack* if $\mathcal{C}$ is a Ψ-algebroid stack on N_0G, $\mathcal{C}_{01}$ is a Ψ-1-morphism and $\mathcal{C}_{012}$ is a Ψ-2-morphism. Similarly, one defines Ψ-1-morphism and Ψ-2-morphism of Ψ-algebroid stacks on G; the details are left to the reader. We denote the resulting subcategory of $\mathrm{AlgStack}_k(G)$ by $\mathrm{AlgStack}_k^{\Psi}(G)$.

The subcategory $\mathrm{Triv}_k^{\Psi}(G)$ of $\mathrm{Triv}_k(G)$ has objects pairs $(\underline{\mathcal{C}}, L)$ with $\underline{\mathcal{C}} \in \mathrm{AlgStack}_k^{\Psi}(G)$, and 1-morphisms and 2-morphisms restricted accordingly.

5.3.5. Deformations of Ψ-algebroid stacks. Recall (3.2.2, Example DEF) that, for $R \in \mathrm{ArtAlg}_k$ we have the pseudo-tensor category $\Psi(R)$.

Suppose that G is an étale category and $\underline{\mathcal{C}}$ is a Ψ-algebroid stack on G.

We denote by $\widetilde{\mathrm{Def}}(\underline{\mathcal{C}})(R)$ the following category:

- an object of $\widetilde{\mathrm{Def}}(\underline{\mathcal{C}})(R)$ is a pair $(\underline{\mathcal{D}}, \underline{\pi})$ where $\underline{\mathcal{D}}$ is a $\widetilde{\Psi(R)}$-algebroid stack on G and $\underline{\pi}\colon \underline{\mathcal{D}}\widetilde{\otimes}_R k \to \underline{\mathcal{C}}$ is an equivalence of Ψ-algebroid stacks,
- a 1-morphism $(\underline{\mathcal{D}}, \underline{\pi}) \to (\underline{\mathcal{D}}', \underline{\pi}')$ of $\widetilde{\Psi(R)}$-deformations of $\underline{\mathcal{C}}$ is a pair $(\underline{\phi}, \beta)$, where $\underline{\phi}\colon \underline{\mathcal{D}} \to \underline{\mathcal{D}}'$ is a 1-morphism of $\widetilde{\Psi(R)}$-algebroid stacks and $\beta\colon \underline{\pi} \to \underline{\pi}' \circ (\underline{\phi}\widetilde{\otimes}_R k)$ is 2-isomorphism,
- a 2-morphism $b\colon (\underline{\phi}_1, \beta_1) \to (\underline{\phi}_2, \beta_2)$ is a 2-morphism $b\colon \underline{\phi}_1 \to \underline{\phi}_2$ such that $\beta_2 = (\mathtt{Id}_{\pi'} \otimes b) \circ \beta_1$

Suppose that $(\underline{\phi}, \beta)\colon (\underline{\mathcal{D}}, \underline{\pi}) \to (\underline{\mathcal{D}}', \underline{\pi}')$ and $(\underline{\phi}', \beta')\colon (\underline{\mathcal{D}}', \underline{\pi}') \to (\underline{\mathcal{D}}'', \underline{\pi}'')$ are 1-morphisms of R-deformations of $\underline{\mathcal{C}}$. We have the composition

$$\pi \xrightarrow{\beta} \pi' \circ (\underline{\phi}\widetilde{\otimes}_R k) \xrightarrow{\beta' \otimes \mathtt{Id}_{\underline{\phi}\widetilde{\otimes}_R k}} \pi'' \circ (\underline{\phi}'\widetilde{\otimes}_R k) \circ (\underline{\phi}\widetilde{\otimes}_R k) \tag{5.3.12}$$

Then, the pair $(\underline{\phi}' \circ \underline{\phi}, (5.3.12))$ is a 1-morphism of R-deformations $(\underline{\mathcal{D}}, \underline{\pi}) \to (\underline{\mathcal{D}}'', \underline{\pi}'')$ defined to be the composition $(\underline{\phi}', \beta') \circ (\underline{\phi}, \beta)$.

Vertical and horizontal compositions of 2-morphisms of deformation are those of 2-morphisms of underlying algebroid stacks.

Lemma 5.18. *The 2-category $\widetilde{\mathrm{Def}}(\underline{\mathcal{C}})(R)$ is a 2-groupoid.*

We denote by $\mathrm{Def}(\underline{\mathcal{C}})(R)$ the full subcategory of $\Psi(R)$-algebroid stacks.

For $(\underline{\mathcal{C}}, L) \in \mathrm{Triv}^{\Psi}(G)$ we define the 2-category $\mathrm{Def}(\underline{\mathcal{C}}, L)(R)$ as follows:

- the objects are quadruples $(\underline{\mathcal{D}}, \underline{\pi}, \pi_\tau, M)$ such that
  $(\underline{\mathcal{D}}, \underline{\pi}) \in \mathrm{Def}(\underline{\mathcal{C}})(R)$,
  $(\underline{\mathcal{D}}, M) \in \mathrm{Triv}^{\Psi(R)}(G)$,
  $(\underline{\pi}, \pi_\tau)\colon (\underline{\mathcal{D}}, M)\otimes_R k \to (\underline{\mathcal{C}}, L)$ is a 1-morphism in $\mathrm{Triv}^{\Psi}(G)$;
- a 1-morphism $(\underline{\mathcal{D}}, \underline{\pi}, \pi_\tau, M) \to (\underline{\mathcal{D}}', \underline{\pi}', \pi'_\tau, M')$ is a triple $(\underline{\phi}, \phi_\tau, \beta)$ where
  $(\underline{\phi}, \beta)\colon (\underline{\mathcal{D}}, \underline{\pi}) \to (\underline{\mathcal{D}}', \underline{\pi}')$ is a 1-morphisms in $\mathrm{Def}(\underline{\mathcal{C}})(R)$,
  $(\underline{\phi}, \phi_\tau)\colon (\underline{\mathcal{D}}, M) \to (\underline{\mathcal{D}}', M')$ is a 1-morphism in $\mathrm{Triv}^{\Psi(R)}(G)$,
- a 2-morphism $(\underline{\phi}, \phi_\tau, \beta) \to (\underline{\phi}', \phi'_\tau, \beta')$ is a 2-morphisms $\underline{\phi} \to \underline{\phi}'$.

5.3.6. `triv` for deformations. Let $\mathcal{A} \in \mathrm{CMA}_k^{\Psi}(NG)$, $R \in \mathrm{ArtAlg}_k$. The functors `st` and `triv` were defined in 5.3.1 (see Proposition 5.11).

Lemma 5.19.

(1) *The functor* `st` *restricts to the functor* $\mathtt{st}\colon \mathrm{Def}(\mathcal{A})(R) \to \mathrm{AlgStack}^{\Psi(R)}(G)$.
(2) *The functor* `triv` *restricts to the functor* $\mathtt{triv}\colon \mathrm{Def}(\mathcal{A})(R) \to \mathrm{Triv}^{\Psi(R)}(G)$.

For $\mathbf{B} \in \mathrm{Def}(\mathcal{A})(R)$ the identification $\mathbf{B} \otimes_R k = \mathcal{A}$ induces the equivalence $\underline{\pi_{\mathbf{B}}}\colon \mathtt{triv}(\mathbf{B})\widetilde{\otimes}_R k \cong \mathtt{triv}(\mathbf{B} \otimes_R k) = \mathtt{triv}(\mathcal{A})$. Let

$$\widetilde{\mathtt{triv}}(\mathbf{B})\colon = (\mathtt{st}(\mathbf{B}), \underline{\pi_{\mathbf{B}}}, \mathtt{Id}, \mathbb{1}_{\mathbf{B}}) \ .$$

A morphism $F\colon \mathbf{B} \to \mathbf{B}'$ in $\mathrm{Def}(\mathcal{A})(R)$ induces the morphism $\mathtt{triv}(F)\colon \mathtt{triv}(\mathbf{B}) \to \mathtt{triv}(\mathbf{B}')$. The diagram

$$\begin{array}{ccc} \mathtt{triv}(\mathbf{B})\widetilde{\otimes}_R k & \xrightarrow{\mathtt{triv}(F)\widetilde{\otimes}_R k} & \mathtt{triv}(\mathbf{B}')\widetilde{\otimes}_R k \\ \Big\downarrow{\scriptstyle \underline{\pi_{\mathbf{B}}}} & & \Big\downarrow{\scriptstyle \underline{\pi'_{\mathbf{B}}}} \\ \mathtt{triv}(\mathcal{A}) & = & \mathtt{triv}(\mathcal{A}) \end{array}$$

commutes up to a unique 2-morphism, hence, $\mathtt{triv}(F)$ gives rise in a canonical way to a morphism in $\mathrm{Def}(\mathtt{triv}(\mathcal{A}))(R)$.

Let $F_i\colon \mathbf{B} \to \mathbf{B}'$, $i = 1, 2$ be two 1-morphisms in $\mathrm{Def}(\mathcal{A})(R)$. A 2-morphism $b\colon F_1 \to F_2$ in $\mathrm{Def}(\mathcal{A})(R)$ induces the 2-morphism $\mathtt{triv}(b)\colon \mathtt{triv}(F_1) \to \mathtt{triv}(F_2)$ in $\mathrm{Triv}_R(G)$ which is easily seen to be a 2-morphism in $\mathrm{Def}(\mathtt{triv}(\mathcal{A}))(R)$.

Lemma 5.20. *The assignment* $\mathbf{B} \mapsto \widetilde{\mathtt{triv}}(\mathbf{B})$ *extends to a functor*

$$\widetilde{\mathtt{triv}}\colon \mathrm{Def}(\mathcal{A})(R) \to \mathrm{Def}(\mathtt{triv}(\mathcal{A}))(R) \tag{5.3.13}$$

naturally in $R \in \mathrm{ArtAlg}_k$.

Theorem 5.21. *The functor* (5.3.13) *is an equivalence.*

Proof. First, we show that, for $\mathbf{B}, \mathbf{B}' \in \mathrm{Def}(\mathcal{A})(R)$ the functor

$$\mathrm{Hom}_{\mathrm{Def}(\mathcal{A})(R)}(\mathbf{B}, \mathbf{B}') \to \mathrm{Hom}_{\mathrm{Def}(\mathtt{triv}(\mathcal{A}))(R)}(\widetilde{\mathtt{triv}}(\mathbf{B}), \widetilde{\mathtt{triv}}(\mathbf{B}')) \tag{5.3.14}$$

induced by the functor $\widetilde{\mathtt{triv}}$ is an equivalence.

For 1-morphism $(\underline{\phi}, \phi_\tau, \beta) \in \mathrm{Hom}_{\mathrm{Def}(\mathtt{triv}(\mathcal{A}))(R)}(\widetilde{\mathtt{triv}}(\mathbf{B}), \widetilde{\mathtt{triv}}(\mathbf{B}'))$ consider the composition

$$\mathbf{B} \cong \mathtt{cma}(\mathtt{triv}(\mathbf{B})) \xrightarrow{\mathtt{cma}(\underline{\phi})} \mathtt{cma}(\mathtt{triv}(\mathbf{B}')) \cong \mathbf{B}' \tag{5.3.15}$$

It is easy to see that β induces an isomorphism (5.3.15) $\otimes_R k \cong \mathtt{Id}_{\mathcal{A}}$. Thus, the functor (5.3.14) is essentially surjective.

Let $F_i\colon \mathbf{B} \to \mathbf{B}'$, $i = 1, 2$ be two 1-morphisms in $\mathrm{Def}(\mathcal{A})(R)$. It is easy to see that the isomorphism

$$\mathrm{Hom}(\mathtt{triv}(F_1), \mathtt{triv}(F_2)) \to \mathrm{Hom}(\mathtt{cma}(\mathtt{triv}(F_1)), \mathtt{cma}(\mathtt{triv}(F_2))) \cong \mathrm{Hom}(F_1, F_2)$$

induced by the functor $\mathtt{cma}$ restricts to an isomorphism of respective space of morphisms in $\mathrm{Hom}_{\mathrm{Def}(\mathtt{triv}(\mathcal{A}))(R)}(\widetilde{\mathtt{triv}}(\mathbf{B}), \widetilde{\mathtt{triv}}(\mathbf{B}'))$ and $\mathrm{Hom}_{\mathrm{Def}(\mathcal{A})(R)}(\mathbf{B}, \mathbf{B}')$.

It remains to show that the functor (5.3.13) is essentially surjective. Consider $(\underline{\mathcal{D}}, \underline{\pi}, \pi_\tau, M) \in \mathrm{Def}(\mathtt{triv}(\mathcal{A}))(R)$. Let $\mathcal{B} = \mathtt{cma}(\underline{\mathcal{D}}, M)$. The morphism $(\underline{\pi}, \pi_\tau)$ induces the isomorphism $\mathtt{cma}(\underline{\pi}, \pi_\tau)\colon \mathcal{B} \otimes_R k \cong \mathcal{A}$.

We choose isomorphisms in $\Psi(R)$ $\mathcal{B}^0 \cong \mathcal{A}^0 \otimes_k R$, $\mathcal{B}^1_{01} \cong \mathcal{A}^1_{01} \otimes_k R$ and $\mathcal{B}^1_{10} \cong \mathcal{A}^1_{10} \otimes_k R$ which induce respective restrictions of the isomorphism $\mathtt{cma}(\underline{\pi}, \pi_\tau)$. The above choices give rise in a canonical way to isomorphisms $\mathcal{B}^n_{ij} \cong \mathcal{A}^n_{ij} \otimes_k R$ for $n = 0, 1, 2, \ldots$, $0 \le i, j \le n$. Let $\mathbf{B}$ denote the cosimplicial matrix algebra structure on $\mathcal{A} \otimes_k R$ induce by that on $\mathcal{B}$ via the above isomorphisms. It is easy to see that $\mathbf{B} \in \mathrm{Def}(\mathcal{A})(R)$. The isomorphism of cosimplicial matrix algebras $\mathcal{B} \cong \mathbf{B}$ induces the equivalence $(\underline{\mathcal{D}}, \underline{\pi}, \pi_\tau, M) \cong \widetilde{\mathtt{triv}}(\mathbf{B})$. □

Theorem 5.22. *Let $(\underline{\mathcal{C}}, L) \in \mathrm{Triv}^\Psi(G)$. Then we have a canonical equivalence of 2-groupoids*

$$\mathrm{Def}(\underline{\mathcal{C}}, L) \cong \mathrm{MC}^2(\mathfrak{G}(\mathtt{cma}(\mathcal{C}, L)) \otimes_k \mathfrak{m}_R)$$

Proof. This is a direct consequence of the Proposition 5.17, Theorem 5.21 and Theorem 4.5. □

5.3.7. Deformation theory of twisted forms of $\mathcal{O}$. Let G be an étale groupoid. We now apply the results of the preceding sections with the $\Psi = \mathtt{DIFF}$ (see 3.2.2) and omit it from notations.

Suppose that $\underline{\mathcal{S}} = (\mathcal{S}, \mathcal{S}_{01}, \mathcal{S}_{012})$ is a twisted form of $\mathcal{O}_G$, i.e. and an algebroid stack $\underline{\mathcal{S}}$ on G such that $\mathcal{S}$ is locally equivalent to $\widetilde{\mathcal{O}^+_{N_0G}}$.

Let $\mathbb{B}$ be a basis of the topology on N_0G. Let $\mathcal{E} := \mathcal{E}_{\mathbb{B}}(G)$ denote the corresponding étale category of embeddings, $\lambda\colon \mathcal{E} \to G$ the canonical map (see 2.3.6).

The functors λ^{-1} and $\lambda_!$ (see 2.3.4, 2.3.10) restrict to mutually quasi-inverse equivalences of 2-categories

$$\lambda^{-1}_R\colon \mathrm{AlgStack}_R(G) \rightleftarrows \mathrm{AlgStack}_R(\mathcal{E})\colon \lambda^R_!$$

natural in R. The explicit construction of 2.3.7, 2.3.8 shows that the equivalence of respective categories of sheaves on G and $\mathcal{E}$ induces an

equivalence of respective $\widetilde{\mathtt{DIFF}(R)}$ categories (which, however, do not preserve the respective $\mathtt{DIFF}(R)$ subcategories).

In particular, for $(\underline{\mathcal{D}},\underline{\pi}) \in \widetilde{\mathrm{Def}}(\underline{\mathcal{S}})(R)$, $\lambda^{-1}\underline{\mathcal{D}}$ is a $\widetilde{\mathtt{DIFF}(R)}$-algebroid stack on $\mathcal{E}$, there is a natural equivalence $\lambda^{-1}\underline{\mathcal{D}}\widetilde{\otimes}_R k \cong \lambda^{-1}(\underline{\mathcal{D}}\widetilde{\otimes}_R\mathbb{C})$, hence $\underline{\pi}$ induces the equivalence $\lambda^*(\underline{\pi})\colon \lambda^{-1}\underline{\mathcal{D}}\widetilde{\otimes}_R\mathbb{C} \to \lambda^{-1}\underline{\mathcal{S}}$. The assignment

$$(\underline{\mathcal{D}},\underline{\pi}) \mapsto \lambda_R^{-1}(\underline{\mathcal{D}},\underline{\pi})\colon = (\lambda^{-1}\underline{\mathcal{D}}, \lambda^*(\underline{\pi}))$$

extends to a functor

$$\lambda^{-1}\colon \widetilde{\mathrm{Def}}(\underline{\mathcal{S}}) \xrightarrow{\cong} \widetilde{\mathrm{Def}}(\lambda^{-1}\underline{\mathcal{S}}). \tag{5.3.16}$$

Lemma 5.23. *The functor* (5.3.16) *is an equivalence of* 2*-groupoids.*

Proof. Follows from the properties of $\lambda_!$. □

Theorem 5.24. *Suppose that $\mathbb{B}$ is a basis of N_0G which consists of Hausdorff contractible open sets. Let $R \in \mathrm{ArtAlg}_{\mathbb{C}}$.*

(1) *$\lambda^{-1}\mathcal{S}(N_0\mathcal{E}_{\mathbb{B}})$ is nonempty and connected by isomorphisms.*
(2) *Let $L \in \lambda^{-1}\mathcal{S}(N_0\mathcal{E}_{\mathbb{B}})$ be a trivialization. The functor*

$$\Xi\colon \mathrm{Def}(\lambda^{-1}\underline{\mathcal{S}}, L)(R) \to \mathrm{Def}(\lambda^{-1}\underline{\mathcal{S}})(R) \tag{5.3.17}$$

defined by $\Xi(\underline{\mathcal{D}},\underline{\pi},\pi_\tau,M) = (\underline{\mathcal{D}},\underline{\pi})$, $\Xi(\underline{\phi},\phi_\tau,\beta) = (\underline{\phi},\beta)$, $\Xi(b) = b$ is an equivalence.

(3) *The functor $\mathrm{Def}(\lambda^{-1}\underline{\mathcal{S}})(R) \to \widetilde{\mathrm{Def}}(\lambda^{-1}\underline{\mathcal{S}})(R)$ is an equivalence.*

Proof. Since $H^l(N_0\mathcal{E}_{\mathbb{B}}; \mathcal{O}^\times_{N_0\mathcal{E}_{\mathbb{B}}})$ is trivial for $l \neq 0$ the first statement follows.

Consider $(\underline{\mathcal{D}},\underline{\pi},\pi_\tau,M), (\underline{\mathcal{D}}',\underline{\pi}',\pi'_\tau,M') \in \mathrm{Def}(\lambda^{-1}\underline{\mathcal{S}}, L)(R)$, and 1-morphisms $(\underline{\phi},\phi_\tau,\beta), (\underline{\psi},\psi_\tau,\gamma)\colon (\underline{\mathcal{D}},\underline{\pi},\pi_\tau,M) \to (\underline{\mathcal{D}}',\underline{\pi}',\pi'_\tau,M')$.

It is clear from the definition of Ξ that the induced map

$$\Xi\colon \mathrm{Hom}((\underline{\phi},\phi_\tau,\beta),(\underline{\psi},\psi_\tau,\gamma)) \to \mathrm{Hom}((\underline{\phi},\beta),(\underline{\psi},\gamma))$$

is an isomorphism, i.e. the functor

$$\Xi\colon \mathrm{Hom}_{\mathrm{Def}(\lambda^{-1}\mathcal{S},L)(R)}((\underline{\phi},\phi_\tau,\beta),(\underline{\psi},\psi_\tau,\gamma)) \to \mathrm{Hom}_{\mathrm{Def}(\lambda^{-1}\mathcal{S})(R)}((\underline{\phi},\beta),(\underline{\psi},\gamma)) \tag{5.3.18}$$

is fully faithful.

Consider a 1-morphism

$$(\underline{\phi},\beta)\colon \Xi(\underline{\mathcal{D}},\underline{\pi},\pi_\tau,M) = (\underline{\mathcal{D}},\underline{\pi}) \to (\underline{\mathcal{D}}',\underline{\pi}') = \Xi(\underline{\mathcal{D}}',\underline{\pi}',\pi'_\tau,M')$$

in $\mathrm{Def}(\lambda^{-1}\mathcal{S})$. We have the isomorphism

$$\underline{\pi}'\colon \mathrm{Hom}_{\underline{\mathcal{D}}'\widetilde{\otimes}_R\mathbb{C}}((\underline{\phi}\otimes_R\mathbb{C})(M\otimes_R\mathbb{C}), M'\otimes_R\mathbb{C}) \to \mathrm{Hom}_{\lambda^{-1}\underline{\mathcal{S}}}(\underline{\pi}'(\underline{\phi}\otimes_R\mathbb{C})(M\otimes_R\mathbb{C}), \underline{\pi}'(M'\otimes_R\mathbb{C}))$$

Let $\varpi\colon (\underline{\phi}\otimes_R\mathbb{C})(M\otimes_R\mathbb{C})\to M'\otimes_R\mathbb{C}$ denote the map such that $\underline{\pi}'(\varpi)=(\pi'_\tau)^{-1}\circ\pi_\tau\circ\beta^{-1}$. Note that ϖ is an isomorphism. Since the map

$$\mathrm{Hom}_{\underline{\mathcal{D}}'}(\phi(M),M')\to\mathrm{Hom}_{\underline{\mathcal{D}}'\widetilde{\otimes}_R\mathbb{C}}((\underline{\phi}\otimes_R\mathbb{C})(M\otimes_R\mathbb{C}),M'\otimes_R\mathbb{C})$$

is surjective there exists $\widetilde{\varpi}\colon\phi(M)\to M'$ which lifts ϖ. Since the latter is an isomorphism and $\mathfrak{m}_R$ is nilpotent, the map $\widetilde{\varpi}$ is an isomorphism. The triple $(\underline{\phi},\widetilde{\varpi},\beta)$ is a 1-morphism $(\underline{\mathcal{D}},\underline{\pi},\pi_\tau,M)\to(\underline{\mathcal{D}}',\underline{\pi}',\pi'_\tau,M')$ such that $\Xi(\underline{\phi},\widetilde{\varpi},\beta)=(\underline{\phi},\beta)$.This shows that the functor (5.3.18) is essentially surjective, hence an equivalence.

It remains to show that (5.3.17) is essentially surjective. It suffices to show that, for any deformation $(\mathcal{D},\pi)\in\mathrm{Def}((N_0\lambda)^{-1}\mathcal{S},L)(R)$, there exists an object $M\in\mathcal{D}(N_0\mathcal{E}_{\mathbb{B}})$ and an isomorphism $M\otimes_R\mathbb{C}\cong L$. This is implied by the following fact: if X is a *Hausdorff* manifold, any deformation of $\widetilde{\mathcal{O}_X^+}$ is a star-product. In other words, for any open covering $\mathcal{U}$ of X, denoting the corresponding étale groupoid by $\mathcal{U}$ and by $\epsilon\colon\mathcal{U}\to X$ the canonical map, the functor

$$\mathrm{MC}^2(\Gamma(X;\mathfrak{g}(\mathcal{O}_X))\otimes_{\mathbb{C}}\mathfrak{m}_R)\to\mathrm{Def}(\widetilde{\mathcal{O}_{\mathcal{U}}^+},\mathbb{1})(R)\tag{5.3.19}$$

is an equivalence. Let $\mathcal{A}_X:=\mathtt{cma}(\widetilde{\mathcal{O}^+})$, $\mathcal{A}_{\mathcal{U}}:=\mathtt{cma}(\widetilde{\mathcal{O}_{\mathcal{U}}^+},\mathbb{1})$. We have the commutative diagram

$$\begin{array}{ccccc}\Gamma(X;\mathfrak{g}(\mathcal{O}_X)) & \longrightarrow & \mathfrak{G}(\mathcal{O}_X) & \xrightarrow{\epsilon^*} & \mathfrak{G}(\mathcal{O}_{\mathcal{U}})\\ & & \downarrow{\scriptstyle\mathrm{cotr}} & & \downarrow{\scriptstyle\mathrm{cotr}}\\ & & \mathfrak{G}(\mathcal{A}_X) & \xrightarrow{\epsilon^*} & \mathfrak{G}(\mathcal{A}_{\mathcal{U}}).\end{array}$$

After the identifications $\mathrm{Def}(\widetilde{\mathcal{O}_{\mathcal{U}}^+},\mathbb{1})(R)\cong\mathrm{Def}(\mathcal{A})(R)\cong\mathrm{MC}^2(\mathfrak{G}(\mathcal{A})\otimes_{\mathbb{C}}\mathfrak{m}_R)$ the functor (5.3.19) is induced by the composition $\Gamma(X;\mathfrak{g}(\mathcal{O}_X))\to\mathfrak{G}(\mathcal{A}_{\mathcal{U}})$ (of morphisms in the above diagram), hence it is sufficient to show that the latter is a quasi-isomorphism. The cotrace (vertical) maps are quasi-isomorphisms by [21] ; the top horizontal composition is the canonical map $\Gamma(X;\mathcal{F})\to\Gamma(|N\mathcal{U}|;|\epsilon^*\mathcal{F}|)$ (with $\mathcal{F}=\mathfrak{g}(\mathcal{O}_X)$) which is a quasi-isomorphism for any bounded below complex of sheaves $\mathcal{F}$ which satisfies $H^i(X;\mathcal{F}^j)=0$ for all $i\neq 0$ and all j. This finishes the proof of the second claim.

Suppose given $(\underline{\mathcal{D}},\underline{\pi})\in\widetilde{\mathrm{Def}}(\lambda^{-1}\mathcal{S})(R)$, i.e., in particular, $\mathcal{D}$ is a $\widetilde{\mathtt{DIFF}}(R)$-stack. In order to establish the last claim we need to show that, in fact, $\mathcal{D}$ is a $\mathtt{DIFF}(R)$-stack. Suppose that L_1 and L_2 are two (locally defined) objects in $\mathcal{D}$; let $\mathcal{F}:=\underline{\mathrm{Hom}}_{\mathcal{D}}(L_1,L_2)$, $\mathcal{F}_0:=\mathcal{F}\otimes_R\mathbb{C}$. By assumption, $\mathcal{F}$ is *locally* isomorphic to $\mathcal{F}_0\otimes_{\mathbb{C}}R$ in $\mathtt{DIFF}$ by a map

which induces the identification $\mathcal{F}_0 = (\mathcal{F}_0 \otimes_{\mathbb{C}} R) \otimes_R \mathbb{C}$. We need to establish the existence of a global such an isomorphism $\mathcal{F} \cong \mathcal{F}_0 \otimes_{\mathbb{C}} R$. Let $\underline{\mathrm{Isom}}_0(\mathcal{F}_0 \otimes_{\mathbb{C}} R, \mathcal{F})$ denote the sheaf of locally defined R-linear morphisms $\mathcal{F}_0 \otimes_{\mathbb{C}} R \to \mathcal{F}$ in $\widetilde{\mathtt{DIFF}}$ which reduce to the identity modulo $\mathfrak{m}_R$. Let $\underline{\mathrm{Aut}}_0(\mathcal{F}_0 \otimes_{\mathbb{C}} R)$ denote the similarly defined sheaf of groups of locally defined automorphisms of $\mathcal{F}_0 \otimes_{\mathbb{C}} R$. Then, $\underline{\mathrm{Isom}}_0(\mathcal{F}_0 \otimes_{\mathbb{C}} R, \mathcal{F})$ is a torsor under $\underline{\mathrm{Aut}}_0(\mathcal{F}_0 \otimes_{\mathbb{C}} R)$.

Arguing as in Lemma 6 and Corollary 7 of [5] using the exponential map $\mathrm{Diff}(\mathcal{F}_0, \mathcal{F}_0) \otimes_{\mathbb{C}} \mathfrak{m}_R \to \underline{\mathrm{Aut}}_0(\mathcal{F}_0 \otimes_{\mathbb{C}} R)$ one shows that the sheaf of groups $\underline{\mathrm{Aut}}_0(\mathcal{F}_0 \otimes_{\mathbb{C}} R)$ is soft, hence the torsor $\underline{\mathrm{Isom}}_0(\mathcal{F}_0 \otimes_{\mathbb{C}} R, \mathcal{F})$ is trivial, i.e. admits a global section. □

5.3.8. Here we obtain the main results of this paper – classification of the deformation groupoid of twisted form of $\mathcal{O}_G$ in terms of the twisted DGLA of jets (cf. (4.4.18)).

Theorem 5.25. *Let G be an etale groupoid and $\underline{\mathcal{S}}$ - a twisted form of $\mathcal{O}_G$. Suppose that $\mathbb{B}$ is a basis of N_0G which consists of Hausdorff contractible open sets, and let $\mathcal{E} = \mathcal{E}_{\mathbb{B}}(G)$ be the corresponding embedding category. Let $R \in \mathrm{ArtAlg}_{\mathbb{C}}$. Then there exists an equivalence of 2-groupoids*

$$\widetilde{\mathrm{Def}}(\underline{\mathcal{S}})(R) \cong \mathrm{MC}^2(\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{N\mathcal{E}})_{[\lambda^{-1}\underline{\mathcal{S}}]} \otimes \mathfrak{m}_R).$$

Proof. Note that we have the following equivalences:

$$\widetilde{\mathrm{Def}}(\underline{\mathcal{S}})(R) \cong \widetilde{\mathrm{Def}}(\lambda^{-1}\underline{\mathcal{S}})(R) \cong \mathrm{Def}(\lambda^{-1}\underline{\mathcal{S}})(R)$$

Here the first equivalence is the result of Lemma (5.23) and the second is a part of the Theorem 5.24. By the Theorem 5.24 $\lambda^{-1}\underline{\mathcal{S}}(N_0\mathcal{E}_{\mathbb{B}})$ is nonempty. Let $L \in \lambda^{-1}\mathcal{S}(N_0\mathcal{E}_{\mathbb{B}})$ be a trivialization. Then the functor

$$\Xi\colon \mathrm{Def}(\lambda^{-1}\underline{\mathcal{S}}, L)(R) \to \mathrm{Def}(\lambda^{-1}\underline{\mathcal{S}})(R) \tag{5.3.20}$$

is an equivalence. By the Theorem 5.22

$$\mathrm{Def}(\lambda^{-1}\underline{\mathcal{S}}, L)(R) \cong \mathrm{MC}^2(\mathfrak{G}(\mathcal{A}) \otimes \mathfrak{m}_R)$$

where $\mathcal{A} = \mathtt{cma}(\lambda^{-1}\underline{\mathcal{S}}, L)$. Finally, we have equivalences

$$\mathrm{MC}^2(\mathfrak{G}_{\mathtt{DR}}(\mathcal{A}) \otimes \mathfrak{m}_R) \cong \mathrm{MC}^2(\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{N\mathcal{E}}(\mathcal{A})) \otimes \mathfrak{m}_R) \cong \mathrm{MC}^2(\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{N\mathcal{E}})_{[S(\mathcal{A})]} \otimes \mathfrak{m}_R)$$

induced by the quasiisomorphisms (4.4.7) and (4.4.22) respectively (with $X = N\mathcal{E}$). Recall the morphism of cosimplicial $\mathcal{O}^\times$-gerbes $\mathcal{S}_{\mathcal{A}} \to S(\mathcal{A})$ defined in 4.3.2. The proof the theorem will be finished if we construct a morphism of cosimplicial gerbes $\mathcal{S}_{\mathcal{A}} \to (\lambda^{-1}\underline{\mathcal{S}})_\Delta$.

For each $n = 0, 1, 2, \ldots$ we have $L_0^{(n)} \in (\lambda^{-1}\mathcal{S})_0^{(n)} = (\lambda^{-1}\underline{\mathcal{S}})_\Delta^n$, i.e. a morphism $L_\Delta^n\colon \mathcal{O}_{N_nG}^\times[1] \to (\lambda^{-1}\underline{\mathcal{S}})_\Delta^n$. For $f\colon [p] \to [q]$ in Δ, we have

canonical isomorphisms $L_{\Delta f}\colon (\lambda^{-1}\underline{\mathcal{S}})_{\Delta f} = \mathcal{A}^{(q)}_{0f(0)}$. It is easy to see that $(L^n_\Delta, L_{\Delta f})$ defines a morphism $\mathcal{S}_{\mathcal{A}} \to (\lambda^{-1}\underline{\mathcal{S}})_\Delta$.

□

In certain cases we can describe a solution to the deformation problem in terms of the nerve of the groupoid without passing to the embedding category.

Theorem 5.26. *Let G be an etale* Hausdorff *groupoid and $\underline{\mathcal{S}}$ - a twisted form of $\mathcal{O}_G$. Then we have a canonical equivalence of* 2*-groupoids*

$$\widetilde{\mathrm{Def}}(\underline{\mathcal{S}}) \cong \mathrm{MC}^2(\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{NG})_{[\underline{\mathcal{S}}_\Delta]} \otimes \mathfrak{m}_R).$$

Proof. Suppose that $\mathbb{B}$ is a basis of N_0G which consists of contractible open sets, and let $\mathcal{E} = \mathcal{E}_{\mathbb{B}}(G)$ be the corresponding embedding category. The map of simplicial spaces $\lambda\colon N\mathcal{E} \to NG$ induces the map of subdivisions $|\lambda|\colon |N\mathcal{E}| \to |NG|$. It induces a map $\mathtt{Tot}(|\lambda|^*)\colon \mathtt{Tot}(\Gamma(|X|; \mathtt{DR}(\overline{\mathcal{J}}_{|NG|})) \to \mathtt{Tot}(\Gamma(|X|; \mathtt{DR}(\overline{\mathcal{J}}_{|N\mathcal{E}|}))$ Let $\overline{B} \in \mathtt{Tot}(\Gamma(|NG|; \mathtt{DR}(\overline{\mathcal{J}}_{|NG|}))$ be a cycle defined in Proposition 2.11 and representing the lift of the class of $\underline{\mathcal{S}}$ in $H^2(\mathtt{Tot}(\Gamma(|NG|; \mathtt{DR}(\overline{\mathcal{J}}_{|NG|}))))$. This form depends on choices of several pieces of data described in 2.5.2. Then $\mathtt{Tot}(|\lambda|^*)\overline{B} \in \mathtt{Tot}(\Gamma(|NG|; \mathtt{DR}(\overline{\mathcal{J}}_{|N\mathcal{E}|}))$ would be the form representing the class of $\lambda^{-1}\underline{\mathcal{S}}$ constructed using pullbacks of the data used to construct $\overline{B}$. We therefore obtain a morphism of DGLA $|\lambda|^*\colon \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{NG})_{\overline{B}} \to \mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{N\mathcal{E}})_{\mathtt{Tot}(|\lambda|^*)\overline{B}}$. It is enough to show that this morphism is a quasiisomorphism. To see this filter both complexes by the (total) degree of differential forms. The map $|\lambda|$ respects this filtration and therefore induces a morphism of the corresponding spectral sequences. The E_1 terms of these spectral sequences are $\mathtt{Tot}(\Gamma(|NG|; |\mathtt{DR}(HH^\bullet(\mathcal{J}_{NG})[1])|'))$ and $\mathtt{Tot}(\Gamma(|N\mathcal{E}|; |\mathtt{DR}(HH^\bullet(\mathcal{J}_{N\mathcal{E}})[1])|'))$ respectively, and the second differential in these spectral sequences is given by $\widetilde{\nabla}$. Here $HH^\bullet(.)$ is the cohomology of the complex $\overline{C}^\bullet(.)$ Since both NG and $N\mathcal{E}$ are Hausdorff, the complexes $\mathtt{DR}(HH^\bullet(\mathcal{J}_{N\mathcal{E}}))$ $\mathtt{DR}(HH^\bullet(\mathcal{J}_{NG}))$ are resolutions of the soft sheaves $HH^\bullet(\mathcal{O}_{N\mathcal{E}})$, $HH^\bullet(\mathcal{O}_{NG})$ respectively.

By the results of 2.3.9

$$\lambda^*\colon C(\Gamma(NG; HH^\bullet(\mathcal{O}_{NG})[1])) \to C(\Gamma(N\mathcal{E}; HH^\bullet(\mathcal{O}_{N\mathcal{E}})[1]))$$

is a quasiisomorphism. Hence

$$|\lambda|^*\colon C(\Gamma(|NG|; |HH^\bullet(\mathcal{O}_{NG})[1]|')) \to C(\Gamma(|N\mathcal{E}|; |HH^\bullet(\mathcal{O}_{N\mathcal{E}})[1]|'))$$

is a quasiisomorphism, by the Lemma 2.1. From this one concludes that

$$\mathtt{Tot}(|\lambda|^*)\colon \mathtt{Tot}(\Gamma(|NG|; |HH^\bullet(\mathcal{O}_{NG})[1]|')) \to \mathtt{Tot}(\Gamma(|N\mathcal{E}|; |HH^\bullet(\mathcal{O}_{N\mathcal{E}})[1]|'))$$

is a quasiisomorphism. Hence

$$\mathtt{Tot}(|\lambda|^*)\colon \mathtt{Tot}(\Gamma(|NG|;|\mathtt{DR}(HH^\bullet(\mathcal{J}_{NG})[1])|')) \to \mathtt{Tot}(\Gamma(|N\mathcal{E}|;|\mathtt{DR}(HH^\bullet(\mathcal{J}_{N\mathcal{E}})[1])|'))$$

is a quasiisomorphism. Hence $\mathtt{Tot}(|\lambda|^*)$ induces a quasiisomorphism of the E_1 terms of the spectral sequence, and therefore is a quasiisomorphism itself. □

5.4. Deformations of convolution algebras. Assume that G is an etale groupoid with N_0G Hausdorff. In the following we will treat N_0G as a subset of N_1G.

With an object $(\underline{\mathcal{C}}, L) \in \mathrm{Triv}_R(G)$ one can canonically associate the nonunital algebra $\mathtt{conv}(\underline{\mathcal{C}}, L)$, called the convolution algebra, cf. [30]. Let $\mathcal{A} = \mathtt{cma}(\underline{\mathcal{C}}, L)$, see 5.3.3. The underlying vector space of this algebra is $\Gamma_c(N_1G; \mathcal{A}^1_{01})$. Here we use the definition of the compactly supported sections as in [9]. The product is defined by the composition

$$\begin{aligned}(5.4.1)\quad &\Gamma_c(N_1G;\mathcal{A}^1_{01})\otimes\Gamma_c(N_1G;\mathcal{A}^1_{01}) \to \Gamma_c(N_2G;\mathcal{A}^2_{01}\otimes\mathcal{A}^2_{12}) \to \\ &\Gamma_c(N_2G;\mathcal{A}^2_{02}) \cong \Gamma_c(N_1G,(d^1_1)_!\mathcal{A}^2_{02}) \to \Gamma_c(N_1G,\mathcal{A}^1_{01}).\end{aligned}$$

Here the first arrow maps $f\otimes g$ to $f^2_{01}\otimes g^2_{12}$, the second is induced by the map (5.3.9). Finally the last arrow is induced by the "summation along the fibers" morphism $(d^1_1)_!\mathcal{A}^2_{02} \to A^1_{01}$.

Recall that a multiplier for a nonunital R-algebra A is a pair (l, r) of R-linear maps $A \to A$ satisfying

$$l(ab) = l(a)b,\ r(ab) = ar(b),\ r(a)b = al(b) \text{ for } a, b, c \in A$$

Multipliers of a given algebra A form an algebra denoted $M(A)$ with the operations given by $\alpha\cdot(l,r)+\alpha'\cdot(l',r') = (\alpha l+\alpha' l', \alpha r+\alpha' r')$, α, $\alpha' \in R$ and $(l,r)\cdot(l',r') = (l'\circ l, r\circ r')$. The identity is given by $(\mathtt{Id},\mathtt{Id})$. For $x = (l,r) \in M(A)$ we denote $l(a)$, $r(a)$ by xa, ax respectively.

Similarly to the 2-category ALG^2_R (see 5.1.1) we introduce the 2-category $(\mathrm{ALG}^2_R)'$ with

- objects – nonunital R-algebras
- 1-morphisms – homomorphism of R-algebras
- 2-morphisms $\phi \to \psi$, where $\phi, \psi\colon A \to B$ are two 1-morphisms – elements $b \in M(B)$ such that $b\cdot\phi(a) = \psi(a)\cdot b$ for all $a \in A$.

Suppose that $(\underline{\phi}, \phi_\tau)\colon (\underline{\mathcal{C}}, L) \to (\underline{\mathcal{D}}, M)$ is a 1-morphism in $\mathrm{Triv}_R(G)$. Let $F = \mathtt{cma}(\underline{\phi}, \phi_\tau)$, see 5.3.3. Let $\mathtt{conv}(\underline{\phi}, \phi_\tau)\colon \mathtt{conv}(\underline{\mathcal{C}}, L) \to \mathtt{conv}(\underline{\mathcal{D}}, M)$ be the morphism induced by F^1_{01}. Suppose that $b\colon (\underline{\phi}, \phi_\tau) \to (\underline{\psi}, \psi_\tau)$ is a $2-morphism$, where $(\underline{\phi}, \phi_\tau), (\underline{\psi}, \psi_\tau)\colon (\underline{\mathcal{C}}, L) \to (\underline{\mathcal{D}}, M)$. Then $\mathtt{cma}(b)^0$ defines a 2-morphism in $(\mathrm{ALG}^2_R)'$ between $\mathtt{conv}(\underline{\phi}, \phi_\tau)$ and $\mathtt{conv}(\underline{\psi}, \psi_\tau)$. We denote this 2-morphism by $\mathtt{conv}(b)$.

Lemma 5.27.

(1) *The assignments* $(\underline{\mathcal{C}}, L) \mapsto \mathtt{conv}(\underline{\mathcal{C}}, L)$, $(\underline{\phi}, \phi_\tau) \mapsto \mathtt{conv}(\underline{\phi}, \phi_\tau)$, $b \mapsto \mathtt{conv}(b)$ *define a functor*

$$\mathtt{conv}\colon \mathrm{Triv}_R(G) \to (\mathrm{ALG}^2_R)'$$

(2) *The functors* conv *commute with the base change functors.*

Assume that $(\underline{\mathcal{S}}, L) \in \mathrm{Triv}_{\mathbb{C}}(G)$ where $\mathcal{S}$ is a twisted form of $\mathcal{O}_G$. Let R be a Artin $\mathbb{C}$-algebra. An R-deformation of $\mathtt{conv}(\underline{\mathcal{S}}, L)$ is an associative R-algebra structure $\star$ on the R-module $B = \mathtt{conv}(\underline{\mathcal{S}}, L) \otimes_{\mathbb{C}} R$ with the following properties:

(1) The product induced on $B \otimes_R \mathbb{C} \cong \mathtt{conv}(\underline{\mathcal{S}}, L)$ is the convolution product defined above.
(2) $\mathrm{Supp}(f * g) \subset\ d^1_1((d^1_0)^{-1}\,\mathrm{Supp}(f) \cap (d^1_2)^{-1}\,\mathrm{Supp}(g))$.

A 1-morphism between two such deformations B_1 and B_2 is an R-algebra homomorphism $F\colon B_1 \to B_2$ such that

(1) The morphism $F \otimes_R \mathbb{C}\colon \mathtt{conv}(\underline{\mathcal{S}}, L) \to \mathtt{conv}(\underline{\mathcal{S}}, L)$ is equal to $\mathtt{Id}$.
(2) $\mathrm{Supp}(F(f)) \subset \mathrm{Supp}(f)$ for any $f \in B_1$.

Given two deformations B_1 and B_2 and two 1-morphisms $F_1, F_2\colon B_1 \to B_2$ a 2 morphism between them is given by a 2-morphism $b = (l, r)$ in $(\mathrm{ALG}^2_R)'$ such that

(1) The 2-morphism $b \otimes_R \mathbb{C}\colon F_1 \otimes_R \mathbb{C} \to\colon F_1 \otimes_R \mathbb{C}$ is equal to $\mathtt{Id}$.
(2) $\mathrm{Supp}(l(f)) \subset \mathrm{Supp}(f)$, $\mathrm{Supp}(r(f)) \subset \mathrm{Supp}(f)$.

Thus, given $(\underline{\mathcal{S}}, L) \in \mathrm{Triv}_{\mathbb{C}}(G)$, we obtain a two-subgroupoid $\mathrm{Def}(\mathtt{conv}(\underline{\mathcal{S}}, L))(R) \subset (\mathrm{ALG}^2_R)'$ of deformations of $\mathtt{conv}(\underline{\mathcal{S}}, L)$.

Let $\mathcal{A} = \mathtt{cma}(\underline{\mathcal{S}}, L)$ and let $\mathbf{B} \in \mathrm{Def}(\mathcal{A})(R)$. Notice that for any $B \in \mathrm{Def}(\mathtt{conv}(\underline{\mathcal{S}}, L))(R)$ we have a canonical isomorphism of vector spaces

$$i\colon B \to \Gamma_c(N_1 G; \mathbf{B}^1_{01}) \tag{5.4.2}$$

Lemma 5.28. *Suppose that* $B \in \mathrm{Def}(\mathtt{conv}(\underline{\mathcal{S}}, L))(R)$. *There exists a unique up to unique isomorphism* $\mathbf{B} \in \mathrm{Def}(\mathcal{A})(R)$ *such that the isomorphism* (5.4.2) *is an isomorphism of algebras.*

Proof. Let $U \subset N_2 G$ be a Hausdorff open subset such that $d^1_i|_U$, $i = 0$, 1, 2 is a diffeomorphism. Define an R-bilinear map

$$m_U\colon \Gamma_c(U; \mathcal{A}^2_{01} \otimes R) \otimes_R \Gamma_c(U; \mathcal{A}^2_{12} \otimes R) \to \Gamma_c(U; \mathcal{A}^2_{02} \otimes R)$$

by

$$m_U(f, g) = (d^1_1|_U)^*(((d^1_0|_U)^{-1})^* f \star ((d^1_2|_U)^{-1})^* g).$$

Here, we view $((d_0^1|_U)^{-1})^* f$, $((d_2^1|_U)^{-1})^* g$ as elements of B. It follows from the locality property of the product $\star$ that $\mathrm{Supp}(m_U(f,g)) \subset \mathrm{Supp}(f) \cap \mathrm{Supp}(g)$. Peetre's theorem [26] implies that m_U is a bidifferential operator. If $V \subset N_2 G$ is another Hausdorff open subset, then clearly $(m_V)|_{U\cap V} = (m_U)|_{U\cap V}$. Therefore, there exists a unique element $m \in \underline{\mathrm{Hom}}_R((\mathcal{A}^2_{01} \otimes R) \otimes_R (\mathcal{A}^2_{12} \otimes R), (\mathcal{A}^2_{02} \otimes R))$ given by a bidifferential operator, such that $m|_U = m_U$ for every Hausdorff open subset of $U \subset N_2 G$. Note that $m \otimes_R \mathbb{C}$ is the map $\mathcal{A}^2_{01} \otimes_{\mathbb{C}} \mathcal{A}^2_{12} \to \mathcal{A}^2_{02}$.

Now define for $i \leq j$ $\mathbf{B}^n_{ij} = \mathcal{A}^n_{ij} \otimes_{\mathbb{C}} R$. For $i \leq j \leq k$ $\mathbf{B}^n_{ij} \otimes_R \mathbf{B}^n_{jk} \to \mathbf{B}^n_{ik}$ is given by $m^{(n)}_{ijk}$. In particular this endows $\mathbf{B}^n_{ij}$, $i \leq j$, with the structure of $\mathbf{B}^n_{ii} - \mathbf{B}^n_{jj}$ bimodule. The map $m^{(n)}_{ijk}$ induces an isomorphism of $\mathbf{B}^n_{kk}$-bimodules $\mathbf{B}^n_{ij} \otimes_{\mathbf{B}^n_{jj}} \mathbf{B}^n_{jk} \xrightarrow{\cong} \mathbf{B}^n_{ik}$. For $i > j$ set $\mathbf{B}^n_{ij} = \underline{\mathrm{Hom}}_{\mathbf{B}^n_{ii}}(\mathbf{B}^n_{ji}, \mathbf{B}^n_{ii})$. We then have a canonical isomorphism $\mathbf{B}^n_{ij} \otimes_{\mathbf{B}^n_{jj}} \mathbf{B}^n_{ji} \xrightarrow{\cong} \mathbf{B}^n_{ii}$. Therefore we have a canonical isomorphism

$$\mathbf{B}^n_{ij} = \mathbf{B}^n_{ij} \otimes_{\mathbf{B}^n_{jj}} \mathbf{B}^n_{ji} \otimes_{\mathbf{B}^n_{ii}} \underline{\mathrm{Hom}}_{\mathbf{B}^n_{jj}}(\mathbf{B}^n_{ji}, \mathbf{B}^m_{jj}) = \underline{\mathrm{Hom}}_{\mathbf{B}^n_{jj}}(\mathbf{B}^n_{ji}, \mathbf{B}^m_{jj}).$$

With this definition we extend the pairing $\mathbf{B}^n_{ij} \otimes_R \mathbf{B}^n_{jk} \to \mathbf{B}^n_{ik}$ to all values of i,j, k. For example for $i \geq j \leq k$ this pairing is defined as the inverse of the isomorphism

$$\mathbf{B}^n_{ik} = \mathbf{B}^n_{ij} \otimes_{\mathbf{B}^n_{jj}} \mathbf{B}^n_{ji} \otimes_{\mathbf{B}^n_{ii}} \mathbf{B}^n_{ik} \xrightarrow{\cong} \mathbf{B}^n_{ij} \otimes_{\mathbf{B}^n_{jj}} \mathbf{B}^n_{jk}.$$

We leave the definition of this pairing in the remaining cases to the reader. Choice of an R-linear differential isomorphism $\mathbf{B}^1_{10} \cong \mathcal{A}^1_{10} \otimes_{\mathbb{C}} R$ induces isomorphisms $\mathbf{B}^n_{ij} \cong \mathcal{A}^n_{ij} \otimes_{\mathbb{C}} R$ for all n and $i > j$. We thus obtain an object $\mathbf{B} \in \mathrm{Def}(\mathcal{A})(R)$. It is clear from the construction that the map (5.4.2) is an isomorphism of algebras. We leave the proof of the uniqueness to the reader.

□

We denote the cosimplicial matrix algebra $\mathbf{B}$ constructed in Lemma 5.28 by $\mathtt{mat}(B)$. By similar arguments using Peetre's theorem one obtains the following two lemmas.

Lemma 5.29. *Let $B_k \in \mathrm{Def}(\mathtt{conv}(\underline{\mathcal{S}}, L))(R)$, $k = 1, 2$, and let i_k be the corresponding isomorphisms defined in* (5.4.2)*. Let $F\colon B_1 \to B_2$ be a* 1*-morphism. Then, there exists a unique* 1*-morphism $\phi\colon \mathtt{mat}(B_1) \to \mathtt{mat}(B_2)$ such that $\phi^1 \circ i_1 = i_2 \circ F$.*

We will denote the 1-morphism ϕ constructed in Lemma 5.29 by $\mathtt{mat}(F)$.

Lemma 5.30. *Let $B_k \in \mathrm{Def}(\mathtt{conv}(\underline{\mathcal{S}}, L))(R)$, $k = 1, 2$, and let i_k be the corresponding isomorphisms defined in* (5.4.2)*. Let F_1, $F_2\colon B_1 \to B_2$ be*

two 1*-morphisms. Let* $b\colon F_1 \to F_2$ *be a* 2*-morphism. Then, there exists a unique* 2*-morphism* $\beta\colon \mathtt{mat}(F_1) \to \mathtt{mat}(F_2)$ *such that* $b\cdot a = \beta^1_{00}\cdot i_2(a)$, $a\cdot b = i_2(a)\cdot\beta^1_{11}$ *for every* $a \in B$.

We will denote the 2-morphism β constructed in Lemma 5.30 by $\mathtt{mat}(b)$.

Lemma 5.31.

(1) *The assignments* $B \mapsto \mathtt{mat}(B)$, $F \mapsto \mathtt{mat}(F)$, $b \mapsto \mathtt{mat}(b)$ *define a functor*

$$\mathtt{mat}\colon \mathrm{Def}(\mathtt{conv}(\underline{\mathcal{S}}, L))(R) \to \mathrm{Def}(\mathtt{cma}(\underline{\mathcal{S}}, L))(R) \tag{5.4.3}$$

(2) *The functors* mat *commute with the base change functors.*

Proposition 5.32. *The functor* (5.4.3) *induces an equivalence*

$$\mathrm{Def}(\mathtt{conv}(\underline{\mathcal{S}}, L))(R) \cong \mathrm{Def}(\mathtt{cma}(\underline{\mathcal{S}}, L))(R).$$

Theorem 5.33. *Assume that* G *is a Hausdorff etale groupoid. Then, there exists a canonical equivalence of categories*

$$\mathrm{Def}(\mathtt{conv}(\underline{\mathcal{S}}, L))(R) \cong \mathrm{MC}^2(\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{NG})_{[\underline{\mathcal{S}_\Delta}]} \otimes \mathfrak{m}_R)$$

Proof. Let $\mathcal{A} = \mathtt{cma}(\underline{\mathcal{S}}, L)$. Then, $\mathrm{Def}(\mathtt{conv}(\underline{\mathcal{S}}, L))(R) \cong \mathrm{MC}^2(\mathfrak{G}(\mathcal{A}) \otimes \mathfrak{m}_R)$ by Proposition 5.32 and Theorem 5.22. Then, as in the proof of Theorem 5.25, we have the equivalences

$$\mathrm{MC}^2(\mathfrak{G}(\mathcal{A})\otimes\mathfrak{m}_R) \cong \mathrm{MC}^2(\mathfrak{G}(\mathcal{J}_{N\mathcal{E}}(\mathcal{A}))\otimes\mathfrak{m}_R) \cong \mathrm{MC}^2(\mathfrak{G}_{\mathtt{DR}}(\mathcal{J}_{N\mathcal{E}})_{[\underline{\mathcal{S}_\Delta}]}\otimes\mathfrak{m}_R)$$

induced by the quasiisomorphisms (4.4.7) and (4.4.22) respectively. □

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