Birationally rigid Fano fibrations

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Abstract

We prove that a general Fano fibration $\pi: V \to \mathbf{P}^1$, the fiber of which is a Fano hypersurface $W_M \subset \mathbf{P}^M$ of index 1, is birationally superrigid, provided that it is sufficiently twisted over the base $\mathbf{P}^1$. In particular, V has no other structures of a fibration into rationally connected varieties. We also formulate and discuss conjectures on birational rigidity for a large class of Fano varieties and Fano fibrations over a base of arbitrary dimension and related questions.

Introduction

The aim of the present paper is to prove birational (super)rigidity of general Fano fibrations $\pi: V \to \mathbf{P}^1$, the fiber of which is $W_M \subset \mathbf{P}^M$, a Fano hypersurface of the index 1. This result is known for $M = 3, 4$ [P1], and can be looked at as a particular case of the following general principle [P3]:

> if a Fano fibration is "sufficiently twisted" over the base, then birational geometry of the entire variety V reduces to the birational geometry of the generic fiber F_η, that is, a Fano variety over a non-closed field $\mathbf{C}(S)$.

A general hypersurface $W_M \subset \mathbf{P}^M$ is a birationally superrigid variety, which has neither non-trivial birational selfmaps, nor non-trivial structures of a Fano fibration.

Similarly, if $\pi: V \to \mathbf{P}^1$ is a general Fano fibration, any fiber of which is a hypersurface of degree M in $\mathbf{P}^M$, and V is sufficiently "twisted" over $\mathbf{P}^1$, then the pencil $|\pi|$ is the only pencil of rationally connected divisors on V.

The proof is obtained by combining the techniques of [P2] and [P1]. The main ingredient of our arguments is formed by the method of maximal singularities, which goes back to the paper of V.A.Iskovskikh and Yu.I.Manin [IM]. The last section of the paper (Section 5) contains a brief discussion of some natural conjectures.

The integer $M \geq 5$ is fixed throughout the paper. The symbol $\mathbf{P}$ stands for the M-dimensional projective space $\mathbf{P}^M$.The ground field is always $\mathbf{C}$.

I started this work in the spring of 1997, during my stay at Max-Planck-Institut für Mathematik in Bonn. After proving the superrigidity of Fano hypersurfaces [P2], it seemed that the new ideas which made it possible would almost automatically give a much wider class of generalizations. The reality proved much more complicated. Nevertheless, the present attempt has come to a successful termination. This means that the method of maximal singularities has far from exhausted its potential field of application.

I would like to use this opportunity to thank Max-Planck-Institut for hospitality and stimulating creative atmosphere.

1 Algebraic varieties with a pencil of Fano hypersurfaces

1.1 An explicit construction of a family of Fano fibrations

Let $\mathcal{E}$ be a locally free sheaf of rank $M+1$ on the projective line $\mathbf{P}^1$,

$$\mathcal{E} = \bigoplus_{i=0}^{M} \mathcal{O}_{\mathbf{P}^1}(a_i),$$

$a_0 = 0 \leq a_1 \leq \ldots \leq a_M$, $X = \mathbf{P}(\mathcal{E})$ be the corresponding projective bundle, so that $\mathbf{P} = \mathbf{P}^M$ is the fiber of the canonical projection

$$\pi: X \to \mathbf{P}^1.$$

Let $L \in \operatorname{Pic} X$ be the class of the tautological sheaf and $G \in \operatorname{Pic} X$ be the class of a fiber $\cong \mathbf{P}$. For an arbitrary $m \in \mathbf{Z}$ define the family $\mathcal{V}(m)$ of codimension 1 subvarieties in X as the subset of the linear system $|ML + mG|$, consisting of smooth divisors. Here the parameter m measures "the degree of twistedness" of a Fano fibration $V \in \mathcal{V}(m)$ over the base $\mathbf{P}^1$. Projection of V onto $\mathbf{P}^1$ is denoted by the same symbol π. The generic fiber F_η of a Fano fibration $\pi: V \to \mathbf{P}^1$ is a Fano hypersurface of index 1 in $\mathbf{P}^M_{\mathbf{C}(t)}$, where t is a linear coordinate on the line. For $m \geq 1$ by the Lefschetz theorem $\operatorname{Pic} V = \operatorname{Pic} X|_V$, so that

$$\operatorname{Pic} V = \mathbf{Z}K_V \oplus \mathbf{Z}F,$$

where $F = G|_V$ is the class of a fiber. A particular fiber $\pi^{-1}(t)$ over a point $t \in \mathbf{P}^1$ is denoted by the symbol F_t.

Obviously,

$$K_V = -L_V + (a_1 + \ldots + a_M - 2 + m)F,$$

so that the higher is m, the "smaller" is the anticanonical linear system $|-K_V|$.

1.2 The K^2-condition and the regularity condition

Let $g \in A^2(V)$ be the class of a hyperplane section of a fiber $F_t \subset \mathbf{P}$. As usual [P1,P3], we say that the Fano fibration $\pi\colon V \to \mathbf{P}^1$ satisfies the *K^2-condition*, if

(K^2) the numerical class of codimension 2 cycles on V

$$MK_V^2 - g$$

is not effective for any $M \in \mathbf{Z}$.

Since the linear system $|L_V|$ is free, the K^2-condition holds for V, provided that the following inequality is true:

$$\left(K_V^2 \cdot L_V^{M-2}\right) \leq 0.$$

The latter intersection index is easy to compute: it is equal to $M(4 - a_1 - \ldots - a_M) - m(2M - 1)$. Thus the inequality

$$m \geq \frac{M}{2M-1}\left(4 - \sum_{i=1}^{M} a_i\right)$$

guarantees that the K^2-condition is satisfied.

Consider an arbitrary hypersurface $W \subset \mathbf{P}$ of degree M. Let $p \in W$, $z_* = (z_1, \ldots, z_M)$ be a system of linear coordinates on $\mathbf{P}$ with the origin at the point p,

$$q = q_1 + q_2 + \ldots + q_M$$

be the equation of the hypersurface W with respect to the coordinates z_*, decomposed into homogeneous polynomials q_i of degree i.

Definition 1. The point $p \in W$ is said to be a *regular smooth (non-singular) point* of the hypersurface W, if the sequence

$$(q_1, \ldots, q_{M-1})$$

is regular. The point p is said to be a *regular singular point* of the hypersurface W, if $q_1 \equiv 0$, the sequence

$$(q_2, \ldots, q_M)$$

is regular, the quadric $q_2 = 0$ in $\mathbf{P}^{M-1} = T_p\mathbf{P}$ is smooth and, moreover, the system of equations

$$q_2 = \ldots = q_M = 0$$

defines in $\mathbf{P}^{M-1} = T_p\mathbf{P}$ a finite set, consisting of exactly $M!$ distinct points, any M of which are linearly independent.

Definition 2. A Fano fibration $\pi: V \to \mathbf{P}^1$, constructed above, is said to be *regular*, if any fiber F_t of the projection π is a $\mathbf{Q}$-factorial variety (so that the group of classes of Weil divisors tensored by $\mathbf{Q}$ is isomorphic to $\mathbf{Q}$ and generated by a hyperplane section of $F_t \subset \mathbf{P}$), any point of which $p \in F_t$ is regular (either as a smooth, or as a singular point).

In [P2] it was shown that a general smooth hypersurface $W \subset \mathbf{P}$ of degree M is regular. Modifying the arguments of [P2] a bit, we show below that the complement to the set of hypersurfaces W with at most double points, all the points of which are regular, is of codimension ≥ 2. A Fano fibration $\pi: V \to \mathbf{P}^1$ can be looked at as a curve in the space $\mathbf{P}(H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(M)))$; thus, if the system $|ML + mG|$ is sufficiently big, its general member is a regular Fano fibration.

1.3 Formulation of the principal result

Recall that the *threshold of canonical adjunction* of a divisor D on a Fano fibration $\rho: W \to S$ is equal to the non-negative number

$$c(D, W) = \sup\{ \frac{b}{a} \,\Big|\, a, b \in \mathbf{Z}_+ \backslash \{0\}, |aD + bK_W| \neq \emptyset \}.$$

A Fano fibration $\rho: W \to S$ is said to be *birationally superrigid*, if for any Fano fibration $\rho': W' \to S'$, $\dim W' = \dim W$, any birational map $\chi: W - - \to W'$ (provided there are any), and any linear system $|D'|$ on W', free in codimension 1 (i.e. $\operatorname{codim} \operatorname{Bs} |D'| \geq 2$), the following inequality holds:

$$c(D, W) \leq c(D', W'),$$

where $|D| = (\chi^{-1})_* |D'|$ is the proper inverse transform $|D'|$ on W with respect to the map χ.

The principal result of the present paper is given by the

Theorem. *Let $\pi: V \to \mathbf{P}^1$ be a Fano fibration, constructed in Section 1.1 above. Assume that it is regular and satisfies the K^2-condition. Then it is birationally superrigid.*

Corollary. (i) *Any birational map $\chi: V - - \to W$ onto another Fano fibration $\rho: W \to S$ is fiber-wise, i.e. $S = \mathbf{P}^1$ or a point, and in the first case there exists an isomorphism $\alpha: \mathbf{P}^1 \to S$ such that $\alpha \circ \pi = \rho \circ \chi$.*

(ii) *The group of birational automorphisms of the variety V coincides with the group of biregular automorphisms $\operatorname{Bir} V = \operatorname{Aut} V$. For a general member of the family $\mathcal{V}(m)$ this group is trivial.*

(iii) *The variety V is non-rational.*

Proof of the corollary. The part (i) follows immediately from the theorem, taking into account the fact that if $|D|$ is a linear system on V, which is free in codimension 1, then $D \sim -nK_V + lF$, where $n \geq 0$ and $l \geq 0$. The first of these inequalities is obvious, while the second one can be proved as in [P1]. Combining (i) with the main theorem of [P2], we get (ii). The part (iii) is obvious.

Example. Let us consider a very particular family of varieties, which corresponds to the values $a_i = 0$, $1 \le i \le M$, that is, $X = \mathbf{P}^1 \times \mathbf{P}$ is a direct product. The varieties from $\mathcal{V}(m)$ are given by bihomogeneous polynomials

$$p_{m,M}(u, v; x_0, \ldots, x_M).$$

If $m \ge 3$ and the polynomial p is sufficiently general, then it satisfies both the K^2-condition and the regularity condition. Therefore, the variety $\{p = 0\}$ is birationally superrigid as a Fano fibration over $\mathbf{P}^1$. For a high $M \gg 0$ the K^2-condition is "almost true" for $m = 2$. However, we can subtract "a little bit" from the numerical class K_V^2 so that the result $NK_V^2 - g$ is still effective on $\mathbf{P}^1 \times \mathbf{P}$. In this connection we set up two questions. Firstly, is the K^2-condition true for a general $V \in \mathcal{V}(2)$ (the class $NK_V^2 - g$ may well be non-effective on V; the condition above is just a *sufficient* numerical condition)? Secondly, even if the K^2-condition is not true, may be the arguments, given below (Sections 2 and 3), can be somewhat strengthened and, by doing so, extended to a slightly larger class of varieties, for which the K^2-condition is "almost true", that is, K_V^2 is very near to the boundary of the set of effective classes (see [G]). It is possible, and even rather probable, that a general $V \in \mathcal{V}(2)$ is also superrigid. For $m = 1$ this is certainly not true: on hypersurfaces of degree M in $\mathbf{P}^{M+1}$ the pencils of Fano hypersurfaces form a $2M$-dimensional family.

Remark. It can be seen from the proof, given below, that the regularity condition, especially with respect to singular points of the fibers, is unnecessarily strong and can be somewhat weakened without any harm to the main result.

2 Proof of the main theorem

2.1 Maximal singularities

Following the traditional scheme of arguments [IM,P1-P4], let us fix a map $\chi: V - - \rightarrow W$, where $\rho: W \rightarrow S$ is a Fano fibration, and assume that for some system $|D'|$ on W, which is free in codimension 1, the monotonicity condition does not hold: $n = c(D, V) > c(D', W) \ge 0$. The linear system $|\chi| = |D|$, $D \sim -nK_V + lF$, $l \ge 0$, has no fixed components. The threshold of canonical adjunction in this case can be interpreted in a simple geometric way: the divisors in $|\chi|_{F_t}$ on a fiber $F_t \subset \mathbf{P}$ are cut out by hypersurfaces of degree n in $\mathbf{P}$.

In the notations of [P1-P4], a geometric discrete valuation $\nu \in \mathcal{N}(V)$ is said to be a *maximal singularity*, if the *Nöther-Fano(-Manin-Iskovskikh)* inequality $e(\nu) = \nu(|\chi|) - na(\nu, V) > 0$ is satisfied, where $a(\cdot)$ is the discrepancy of a valuation with respect to the given model.

Proposition 1. *One of the following two cases holds:*

(i) *there is a maximal singularity $\nu \in \mathcal{N}(V)$, the centre of which covers the base: $\pi(\text{centre}(\nu)) = \mathbf{P}^1$;*

(ii) *the centres of all the maximal singularities lie in the fibers of the projection π, and there exists a finite set of maximal singularities $\mathcal{M} \subset \mathcal{N}(V)$, such that the* *following inequality is satisfied*

$$\sum_{t\in\mathbf{P}} \max_{\{\nu\in\mathcal{M}|\ \mathrm{centre}(\nu)\subset F_t\}} \left(\frac{e(\nu)}{\nu(F_t)}\right) > l$$

Proof: see [P1,P3].

First of all, note that the possibility (i) can not realize: if the centre of a maximal singularity ν covers the base, then ν induces a maximal singularity $\nu_t \in \mathcal{N}(F_t)$ of the linear system $|\chi|_t = |\chi|_{F_t}$ on a fiber F_t of general position. However, we know from [P2], that a linear system of divisors without fixed components on a smooth regular hypersurface F_t of degree M in $\mathbf{P}$ can not have a maximal singularity. Note also that the centre of any maximal singularity $\nu \in \mathcal{M}$ is of codimension ≥ 3 with respect to V: otherwise $B = \mathrm{centre}(\nu)$ is a divisor on F_t, that is, a fixed component of the linear system $|\chi|_t$. However,

$$D_t = D|_{F_t} \sim nh, \quad B \sim bh,$$

where $h \in \mathrm{Pic}\, F_t$ is the class of a hyperplane section in $\mathbf{P}$, $b \geq 1$, and thus

$$\mathrm{mult}_B\, |\chi| \leq \mathrm{mult}_B\, |\chi|_t \leq n,$$

which contradicts the Nöther-Fano inequality.

2.2 Supermaximal singularity

Let $D_i \in |\chi|$ be general divisors, $Z = (D_1 \bullet D_2)$ be the algebraic cycle of their scheme-theoretic intersection, $Z = Z^v + Z^h$ be its decomposition into the vertical (i.e., contained in fibers) and horizontal (i.e., covering the base $\mathbf{P}^1$) components, respectively. For Z^v we have a further decomposition:

$$Z^v = \sum_{t\in\mathbf{P}^1} Z^v_t, \quad \mathrm{Supp}\, Z^v_t \subset F_t.$$

By the degree of a horizontal cycle we mean the degree of its intersection with a fiber F of general position (with respect to $\mathbf{P}$). By the degree of a vertical cycle, the support of which is contained in the fiber F_t, we mean its degree in $\mathbf{P} \supset F_t$. Obviously, the K^2-condition implies that

$$\sum_{t\in\mathbf{P}^1} \deg Z^v_t \leq 2nlM.$$

Combining this inequality with the inequality of the part (ii) of Proposition 1, we conclude that there exists a maximal singularity $\nu \in \mathcal{M}$, such that $\mathrm{centre}(\nu) \subset F = F_t$ and

$$e = e(\nu) > \frac{\nu(F)}{2Mn} \deg Z^v_t.$$

Following [P1,P3], we call this singularity *supermaximal*. Fix it. For simplicity of notations we write Z^v instead of Z^v_t and so on.

2.3 The techniques of counting multiplicities

Let

$$\begin{array}{cccc} \varphi_{i,i-1}: & V_i & \to & V_{i-1} \\ & \cup & & \cup \\ & E_i & \to & B_{i-1} \end{array}$$

$i=1,\dots,K$, be the resolution of the valuation ν, where the first $L\le K$ blow ups correspond to the cycles B_{i-1} of codimension ≥ 3, whereas the first $N\le L$ ones correspond to the cycles $B_{i-1}\subset F^{i-1}$ (recall that the operation of taking the proper inverse image on the j-th storey of the resolution is denoted by adding the upper index j), $B_0=\mathrm{centre}(\nu)$, $B_j=\mathrm{centre}(V_j,\nu)$. For $i=1,\dots,L$ set

$$m_i^h=\mathrm{mult}_{B_{i-1}}(Z^h)^{i-1},\quad m_i^v=\mathrm{mult}_{B_{i-1}}(Z^v)^{i-1}.$$

Obviously, $m_i^v=0$ for $i\ge N+1$. Set also $\mu_i=\mathrm{mult}_{B_{i-1}}F^{i-1}$ for $i=1,\dots,N$. Obviously,

$$m_i^h\le m_h=m_1^h=\mathrm{mult}_{\mathrm{centre}(\nu)}Z^h.$$

The integer $p_i\ge 1$ stands for the number of paths in the graph $\mathcal{N}(V)$ connecting the vertices ν and E_i, see [P4]. Set

$$\Sigma_l=\sum_{i=1}^{L}p_i,\quad \Sigma_u=\sum_{i=L+1}^{K}p_i,\quad \Sigma_f=\sum_{i=1}^{N}p_i.$$

Obviously, $\nu(F)\ge\sum\limits_{i=1}^{N}p_i\mu_i$. Setting $d_h=\deg Z^h$, $d_v=\deg Z_t^v$, we introduce the coefficients

$$k_h=\frac{m_h}{d_h},\quad k_v=\frac{1}{\nu(F)d_v}\sum_{i=1}^{N}p_im_i^v,\quad k_f=\frac{1}{\nu(F)}\sum_{i=1}^{N}p_i.$$

Proposition 2. *The following inequality holds:*

$$(4-Mk_h)\Sigma_l(\Sigma_l+\Sigma_u)n^2+\Sigma_u^2n^2+e^2+$$
$$+2(2-k_vM)\Sigma_lne+2(1-k_vM)\Sigma_une<0.$$

For the **proof**, see [P1,P3]. This is the standard techniques of counting multiplicities, which goes back to the paper of V.A.Iskovskikh and Yu.I.Manin [IM].

Proposition 3. *The following estimates are true:*

$$k_h\le\frac{4}{M},\quad k_v\le\frac{2}{M}.$$

Proof is given below, in Section 3. In fact, these estimates make the crucial step of our arguments (modulo the known facts).

Using these estimates, we get a contradiction: Proposition 2 implies that

$$\Sigma_u^2n^2-2\Sigma_une+e^2<0,$$

which is impossible. Consequently, the inequality

$$c(D,V)>c(D',W)$$

can not be true. Thus birational superrigidity of the Fano fibration $\pi\colon V\to\mathbf{P}^1$ is proved.

3 Singular points of algebraic cycles of small codimension on the regular Fano hypersurfaces

3.1 Horizontal cycles

Let $p \in \mathrm{centre}(\nu) \subset F$ be an arbitrary point of general position on the irreducible cycle $B_0 = \mathrm{centre}(\nu)$. Let

$$\begin{array}{cccc} \sigma\colon & \widetilde{F} & \to & F \\ & \cup & & \cup \\ & E & \to & \{p\} \end{array}$$

be its blow up, considered as the restriction of the blow up of the point p on $\mathbf{P} \supset F$, $\sigma\colon \widetilde{\mathbf{P}} \to \mathbf{P}$, with the exceptional divisor $\widetilde{E} \cong \mathbf{P}^{M-1}$. If p is smooth on the fiber F, then the arguments of [P2] imply that for any cycle $Z \subset F$ of codimension 2 we have the inequality

$$\frac{\mathrm{mult}_p}{\deg} Z \leq \frac{4}{M}.$$

Consequently, $k_h \leq \frac{4}{M}$, as required. Now let p be a non-degenerate double point of the fiber F.

Lemma 1. *For any effective cycle $Z \subset F$ of codimension 2 the following inequality holds*

$$\frac{\mathrm{mult}_p}{\deg} Z \leq \frac{4}{M}.$$

Proof. The regularity condition for the point p makes it possible to repeat the "smooth" arguments word for word.

Thus $k_h \leq \frac{4}{M}$ always. Proposition 2.3 is proved for the horizontal cycles.

3.2 Vertical cycles. The non-singular case

Set $m_v = m_1^v$ and note that $k_v \leq k_f \frac{m_v}{d_v}$.

Let us first consider the case when the fiber F is smooth at the point p.

Lemma 2. *Let $Y \subset F$ be a prime Weil divisor. Then we have the estimate*

$$\frac{\mathrm{mult}_p}{\deg} Y \leq \frac{2}{M},$$

where the equality is attained at a unique cycle — that is, the intersection $F \cap T_p F$ with the tangent hyperplane.

Proof. The estimate itself is very easy to obtain. For $T = F \cap T_p F$ it is true by the regularity condition. Assume that $Y \neq T$. Then the effective cycle $(Y \bullet T)$ has the multiplicity $\geq 2\,\mathrm{mult}_p Y$ at the point p. In accordance with [P2],

$$\frac{2}{M} \geq \frac{1}{2} \frac{\mathrm{mult}_p}{\deg} (Y \bullet T) \geq \frac{\mathrm{mult}_p}{\deg} Y,$$

which is exactly what we need.

The second assertion of the lemma, that is, the extremality of the cycle T is of no use for the proof of Proposition 2.3. However, the arguments which yield this assertion will be employed below in the hard singular case. That is why we prove the second claim of the lemma, too.

Consider the linear systems

$$\Lambda_k = |kH - (k+1)E|,$$

$k = 3, \dots, M-2$. By the regularity condition, the codimension $\mathrm{codim}_T \mathrm{Bs}\, \Lambda_k|_T \geq k-1$, so that for general divisors $D_k \in \Lambda_k$ the algebraic cycle

$$Z = (T \bullet D_3 \bullet \dots \bullet D_{M-2})$$

is effective, and its components form an algebraic family of surfaces, sweeping out T. At the same time

$$\frac{\mathrm{mult}_p}{\deg} Z \geq \frac{2(M-1)}{3M} > \frac{1}{2},$$

so that (taking a hyperplane section, containing the point p), we conclude that T is swept out by an algebraic family of curves $\mathcal{C} = (C_s, s \in S)$, for which the ratio $\mathrm{mult}_p / \deg$ is strictly higher than $1/2$. Now, if

$$\frac{\mathrm{mult}_p}{\deg} Y = \frac{2}{M},$$

then $Y \supset C_s$ for any $s \in S$ and thus $Y = T$. Q.E.D. for Lemma 2.

Remark. In fact, as we see from the proof given above, the following estimate is true:

$$\frac{\mathrm{mult}_p}{\deg} Y \leq \frac{3}{2(M-1)}$$

for any prime Weil divisor $Y \neq T$.

Since $k_f \leq 1$ and $k_v \leq \frac{m_v}{d_v} \leq \frac{2}{M}$, Proposition 2.3 is proved for any non-singular point p.

3.3 Vertical cycles. The singular case, $N = 1$

Now let $B_0 = \mathrm{centre}(\nu) = p \in F$ be a regular singular point. Let us first consider the more elementary case $N = 1$, i.e. either $\mathrm{codim}_{V_1} B_1 = 2$, or $B_1 \not\subset F^1$. Here $k_f \leq 1/2$ and accordingly $k_v \leq \frac{1}{2}\frac{m_v}{d_v}$.

Lemma 3. *For a prime Weil divisor $Y \subset F$ the following estimate is true:*

$$\frac{\mathrm{mult}_p}{\deg} Y \leq \frac{3}{M}.$$

Moreover, the equality takes place for a unique cycle, that is, the intersection $Q = T_pF \cap F$, where T_pF is the projectivized tangent cone to the hypersurface $F \subset \mathbf{P}$ at the point p.

Proof. Obviously, we get $\widetilde{Q} \in |2H - 3E|$, so that for Q we have the equality $(\mathrm{mult}_p / \mathrm{deg}) Q = 3/M$. Assume that $Y \neq Q$ is a prime Weil divisor. Then $(Y \bullet Q)$ is an effective cycle of codimension 2 on F. Applying Lemma 1 to this cycle, we get

$$\frac{4}{M} \geq \frac{\mathrm{mult}_p}{\mathrm{deg}}(Y \bullet Q) \geq \frac{3}{2} \frac{\mathrm{mult}_p}{\mathrm{deg}} Y,$$

so that $(\mathrm{mult}_p / \deg) Y \leq (8/3M)$, as required.

Thus in the case under consideration $k_v \leq 3/2M$: Q.E.D. for Proposition 2.3.

3.4 Vertical cycles. The singular case, $N \geq 2$

This is the hardest case. Here $\mathrm{codim}\, B_1 \geq 3$ and $B_1 \subset F^1$. Choose an arbitrary point $x \in B_1 \subset E$ of general position.

We have the obvious estimates:

$$k_v \leq \frac{1}{\nu(F)} \left[\left(\frac{m_v}{d_v} \right) p_1 + \left(\frac{m_2^v}{d_v} \right) \sum_{i=2}^{N} p_i \right],$$

$$\nu(F) \geq 2p_1 + \sum_{i=2}^{N} p_i.$$

We already know (Lemma 3) that $(\frac{m_v}{d_v}) \leq \frac{3}{M} < \frac{4}{M}$.

The crucial fact is given by the following

Lemma 4. *For any prime Weil divisor $Y \subset F$ we have the estimate*

$$\frac{\mathrm{mult}_x \widetilde{Y}}{\deg Y} \leq \frac{2}{M},$$

where $\widetilde{Y} \subset \widetilde{F}$ is the proper inverse image of Y with respect to the blow up of the point p.

Now let us complete the proof of Proposition 2.3. Obviously,

$$\frac{m_2^v}{d_v} \leq \frac{\mathrm{mult}_x (Z^v)^1}{\deg Z^v} \leq \frac{2}{M},$$

which implies that $k_v < \frac{2}{M}$. Q.E.D.

3.5 Proof of Lemma 4.

Assume the converse and fix a prime Weil divisor $Y \subset F$, satisfying the inequality

$$\frac{\mathrm{mult}_x \widetilde{Y}}{\deg Y} > \frac{2}{M}.$$

Since F is a $\mathbf{Q}$-factorial variety, Y is a $\mathbf{Q}$-Cartier divisor and so for certain $d, \nu \in \mathbf{Q}_+$ we have $\widetilde{Y} \in |dH - \nu E|$. Obviously, $\nu = \frac{1}{2} \mathrm{mult}_p Y$. Set $\mu = \mathrm{mult}_x \widetilde{Y}$. Recall that

$E \subset \widetilde{E} = \mathbf{P}^{M-1}$ is a non-singular quadric hypersurface. Then $B = E \cap T_x E$ is a non-degenerate quadric cone with the vertex at the point x.

Lemma 5. *The following estimate holds:*

$$\operatorname{mult}_B \widetilde{Y} \geq \frac{1}{2}(\mu - \nu).$$

Proof is given below.

Consider the hyperplane $H \subset \mathbf{P}$, containing the point p, such that $T_p H = T_x E \subset \widetilde{E}$. In accordance with Lemma 5 (and elementary intersection theory [F], see also [P1,P2,P4]), we get

$$\operatorname{mult}_p(Y \bullet H) \geq 2\nu + (\mu - \nu) > \frac{3}{M} \deg Y$$

(since $2\nu > \mu$). However, this is impossible by

Lemma 6. *For any hyperplane $H \subset \mathbf{P}$, containing the point p, and any effective $\mathbf{Q}$-Cartier divisor W on $F_H = F \cap H$ the following inequality holds:*

$$\frac{\operatorname{mult}_p}{\deg} W \leq \frac{3}{M},$$

whereas the equality is attained at the divisors mS_H, $m \in \mathbf{Z}_+$, only, where $S_H = F_H \cap T_p F = H \cap Q$.

Proof of the lemma is given below.

It remains to notice that Lemmas 5 and 6 give a contradiction. Consequently, the initial assumption was wrong. Lemma 4 is proved (modulo Lemmas 5 and 6).

3.6 Proof of Lemma 5

We give two independent proofs. The first one is more explicit and well explains the essence of the matter, the second one is considerably shorter.

To begin with, let us introduce a series of notations:
$X = \mathbf{C}^M$, $(z_*) = (z_1, \dots, z_M)$ are linear coordinates on X;

$$\begin{array}{rccc} \varphi\colon & \widetilde{X} & \to & X \\ & \bigcup & & \\ & \widetilde{E} & \cong \mathbf{P}^{M-1} & \end{array}$$

is the blow up of the origin $p = (0, \dots, 0)$;
$(1 : 0 : \dots : 0)$ is the direction of the infinitely near point $x \in \widetilde{E}$;
$(y_1 = z_1, y_2 = z_2/z_1, \dots, y_M = z_M/z_1)$ is a system of coordinates on an open affine set in $\widetilde{X}$, containing the point x, with the origin at this point;
$q = q_2 + \dots$ is the equation of the hypersurface F;
$q_2 = q_E$, $c = \frac{\partial^3}{\partial z_1^3} q|_{(0,\dots,0)} = \frac{\partial^3}{\partial z_1^3} q_3 \in \mathbf{C}$;
$E \subset \widetilde{E}$ is the exceptional divisor of the blow up of the point p on F, that is, a

non-singular quadric in $\widetilde{E} = \mathbf{P}^{M-1}$,

$$\begin{array}{cccc} \varphi\colon & \widetilde{F} & \to & F \\ & \bigcup & & \\ & E & \to & p; \end{array}$$

l_E is the equation of the hyperplane $T_x E$ in $\widetilde{E}$, that is, the linear part of the polynomial q_E with respect to the coordinates $(y_2, \dots, y_M) = (y'_*)$;
$l_E + cy_1$ is the equation of the tangent hyperplane $T_x \widetilde{F}$ in the standard affine set $\mathbf{C}^M_{(y_1,\dots,y_M)}$;
$|-\alpha E - \beta x|$ is the linear space of formal analytic series, or functions, analytic around zero, or algebraic functions of the form $f(z_*)$, regular at zero, such that y_1^α restricted to $\widetilde{F}$ divides $\varphi^* f|_{\widetilde{F}}$ and $\mathrm{mult}_x(\varphi^* f|_{\widetilde{F}}/y_1^\alpha) \geq \beta$;
$\mathcal{H} = |-\nu E - \mu x|$;
$\mathcal{H}_k = \mathcal{H} \cap |-(\nu+k)E|$, $k \in \mathbf{Z}_+$.

Now, if f is an analytic (or regular at zero algebraic) function, then $f \in \mathcal{H} \Leftrightarrow (\varphi^* f|_{\widetilde{F}}) - \nu E \geq 0$ and $\mathrm{mult}_x(\varphi^* f|_{\widetilde{F}}) \geq \nu + \mu$. If, moreover, $f \in \mathcal{H}_k$, then $(\varphi^* f|_{\widetilde{F}}) - (\nu+k)E \geq 0$ into the bargain. Let $B = E \cap T_x E$ be our quadric cone on E.

Obviously, Lemma 5 follows from the inequality (*)

$$\mathrm{mult}_B(\varphi^* f|_{\widetilde{F}}) \geq \max(\nu, \frac{\nu+\mu}{2})$$

for any $f \in \mathcal{H}$. The inequality $\mathrm{mult}_B(\varphi^* f|_{\widetilde{F}}) \geq \nu$ is trivial. Thus it is sufficient to consider the case $\mu > \nu$. Let us prove (*) by descending induction on

$$k = \max\{i \in \mathbf{Z}_+ | f \in \mathcal{H}_i\}.$$

If $k \geq \dfrac{\mu-\nu}{2}$, then (*) holds in an obvious way. Therefore assume that it is proved for all $g \in \mathcal{H}_{k+1}$ and $f \in \mathcal{H}_k \setminus \mathcal{H}_{k+1}$.

Let $f = f_a + f_{a+1} + \dots$ be the Taylor series of the function f, where f_i are homogeneous in z_* of degree i. The pull-back $\varphi^* f$ has the form

$$y_1^a(f_a(1, y'_*) + y_1 f_{a+1}(1, y'_*) + \dots),$$

where we may assume that q_E does not divide f_a, i.e. $\mathrm{mult}_E(\varphi^* f|_{\widetilde{F}}) = a$. (Indeed, for any function $g \in \mathcal{H}$ such that $g|_F = f|_F$, we have the estimate $\mathrm{mult}_p\, g \leq \nu + k$, since $f \notin \mathcal{H}_{k+1}$. Thus the function f can be chosen from the start to satisfy the condition

$$a = \mathrm{mult}_p\, f = \max\{\mathrm{mult}_p\, g | g - f \equiv 0 \quad \text{on} \quad F\}.$$

Now if q_E divides f_a, then we get a contradiction, replacing f by $f - q\dfrac{f_a}{q_E}$.)

Thus $a = \nu + k$. Therefore the multiplicity of the function $\dfrac{\varphi^* f}{y_1^a}|_{\widetilde{F}}$, which does not vanish identically on E, at the point x is at least $\mu - k$. The more so $\mathrm{mult}_x\, f_a|_E \geq \mu - k$.

Consequently, the polynomial f_a cuts out on the quadric E a (non-trivial) divisor of multiplicity at least $\mu - k$ at the point x. If $a \leq \mu - k - 1$, then the cycle $B \subset E$

makes a component of this divisor, of multiplicity $\mu-k-a+\beta\geq 1$, where $\beta\in\mathbf{Z}_+$. In other words, there exist polynomials g,h such that

$$f_a=l_E^{\mu-k-a+\beta}g+q_Eh.$$

At the same time B is not a component of the divisor $g|_E=0$ and therefore $\mathrm{mult}_x\, g|_E\leq\deg g$. We may assume that g has the full degree, that is, $\deg g+\mu-k-a+\beta=a$.

Lemma 7. *Let $P\in H^0(\mathbf{P}^{M-1},\mathcal{O}(b))$ be a homogeneous polynomial of degree b, and assume that* $\mathrm{mult}_x\, P|_E\leq b$. *Then there exists a homogeneous polynomial P^* of the same degree such that $P-P^*\equiv 0$ on E and* $\mathrm{mult}_x\, P^*=\mathrm{mult}_x\, P|_E$.

Proof is obtained by descending induction on $\mathrm{mult}_x\, P\leq\deg P$. If $\mathrm{mult}_x\, P=b$, then take $P^*=P$. Assume that the statement of the lemma is true for all the polynomials with $\mathrm{mult}_x\geq j+1$ and $\mathrm{mult}_x\, P=j\leq b-1$. With respect to a system of affine coordinates with the origin at the point x write down $P=P_j+\ldots+P_b$, where P_i are homogeneous of degree i. If $j=\mathrm{mult}_x\, P|_E$, then there is nothing to prove: take $P^*=P$. Assume therefore that $\mathrm{mult}_x\, P|_E>j$. This implies that l_E divides P_j, i.e. $P_j=l_ER$, where R is a homogeneous polynomial of degree $j-1\leq b-2$. Now let us replace P by $P'=P-q_ER\in H^0(\mathbf{P}^{M-1},\mathcal{O}(b))$ (where we identify homogeneous polynomials with their non-homogeneous presentations). Since $\mathrm{mult}_x\, P'\geq\mathrm{mult}_x\, P+1$, the desired claim is true for P'. Consequently, it is true for P, either. Q.E.D. for the lemma.

Let us come back to the proof of Lemma 5.

In accordance with Lemma 7, we may assume that

$$\mathrm{mult}_x\, g|_E=\mathrm{mult}_x\, g=\mathrm{mult}_x\, g|_{\widetilde{F}}.$$

Consider now the polynomial $\xi(z)=l_E(z_*)+cz_1^2$. By construction, $\mathrm{mult}_E\,\varphi^*\xi|_{\widetilde{F}}=1$, and with respect to the coordinates (y_*) the hyperplane

$$0=\frac{\varphi^*\xi}{y_1}=l_E(y'_*)+cy_1$$

is nothing else than the hyperplane, tangent to $\widetilde{F}$ at the point x. In other words, $\xi\in|-E-2x|$. On the quadric E the function $\varphi^*\xi/y_1$ cuts out the cycle B. Finally, consider the polynomial

$$\Delta=\xi^{\mu-a-k+\beta}g(z_*),$$

g is homogeneous degree $\deg g$. Obviously, $\mathrm{mult}_E\,\varphi^*\Delta=a$, and the multiplicity of the function $(\varphi^*\Delta/y_1^a)_{\widetilde{F}}$ at the point x equals $\mathrm{mult}_x\, f_a|_E\geq\mu-k$ exactly. Therefore, $\Delta\in\mathcal{H}_k$. Furthermore,

$$\frac{\varphi^*\Delta}{y_1^a}|_E\equiv\frac{\varphi^*f}{y_1^a}|_E$$

by construction, so that $f-\Delta\in\mathcal{H}_{k+1}$. Note, finally, that

$$\mathrm{mult}_B\,\varphi^*\Delta|_{\widetilde{F}}\geq\max(a,a+\mu-k-a+\beta)=$$

$$=\max(\nu+k,\mu-k+\beta)\geq\frac{\mu+\nu}{2}.$$

The proof of Lemma 5 is complete.

3.7 A short proof of Lemma 5

Let $P \ni p$ be a general 4-plane in $\mathbf{P}$, the proper inverse image $\widetilde{P}$ of which on $\widetilde{\mathbf{P}}$ contains the point x. Set $Y_P = Y \cap P$, $F_P = F \cap P$, $\widetilde{E}_P = \widetilde{E} \cap \widetilde{P} \cong \mathbf{P}^3$, $E_P = E \cap \widetilde{P} \cong \mathbf{P}^1 \times \mathbf{P}^1$ and so on. Obviously, $B_P = T_x E_P \cap E_P = L_1 + L_2$ is a pair of lines. Set $L = L_1$. Obviously, it is sufficient to prove that

$$\operatorname{mult}_L \widetilde{Y}_P \geq \frac{1}{2}(\mu - \nu).$$

Let $\varphi_L \colon P_L \to \widetilde{P}$ be the blow up of the line L, F_L be the proper inverse image of the (smooth) 3-fold hypersurface F_P on P_L, $E_L \subset F_L$ be the exceptional divisor. It is easy to see that

$$\mathcal{N}_{L/F_P} \cong \mathcal{O}_L \oplus \mathcal{O}_L(-1),$$

so that E_L is a ruled surface of the $\mathbf{F}_1$ type. Let s and f be the classes of the exceptional section and the fiber in $\operatorname{Pic} E_L$, respectively. It is easy to see that $(E_L \cdot E_L) = -s - f$, since $(E_L \cdot s) = (E_L \cdot \varphi_L^* L) = 0$. Set $\gamma = \operatorname{mult}_L \widetilde{Y}_P$. The class of the proper inverse image of the divisor Y on F_L is obviously equal to $dH - \nu E_P - \gamma E_L$, whereas its restriction onto E_L is equal to $\gamma s + (\gamma + \nu) f$. This is the class of an effective curve on E_L, which contains the fiber over the point x with the multiplicity $\mu - \gamma$. Consequently,

$$\gamma + \nu \geq \mu - \gamma.$$

Thus $2\gamma \geq \mu - \nu$, as required. Q.E.D.

3.8 Proof of Lemma 6

Note, first of all, that $\deg S_H = 2M$ and $\operatorname{mult}_p S_H = 6$, so that the divisor S_H does realize the limit case of our inequality.

First assume that $M \geq 9$. Suppose that a prime Weil divisor W on F_H satisfies the inequality

$$\frac{\operatorname{mult}_p}{\deg} W > \frac{3}{M}.$$

Then $W \neq S_H$ and the effective cycle $C = (W \bullet S_H)$ of codimension 2 (on F_H) satisfies the estimate

$$\frac{\operatorname{mult}_p}{\deg} C > \frac{9}{2M}.$$

Now arguing as in [P2], consider the hypertangent linear systems

$$\Lambda_k = |kL - (k+1)E|,$$

$k = 2, \dots, M-1$, where $L \in \operatorname{Pic} F_H$ is the class of a hyperplane section, and $E \subset \widetilde{F}_H \to F_H \ni p$ is the exceptional divisor of the blow up of the point p. Obviously, $\operatorname{codim}_{F_H} \operatorname{Bs} \Lambda_k \geq k-1$ for $2 \leq k \leq M-2$ by the regularity condition. Choose general divisors $D_k \in \Lambda_k$ for $k = 4, \dots, M-2$ and look at the effective 1-cycle

$$Z^* = (C \bullet D_4 \bullet \ldots \bullet D_{M-2}) \subset F_H.$$

This cycle satisfies the inequality

$$\frac{\mathrm{mult}_p}{\deg} Z^* > \frac{9}{2M} \cdot \frac{M-1}{4} \geq 1,$$

which is impossible. This contradiction proves the desired inequality for $M \geq 9$ (and also the claim on the extremality of the cycle S_H for $M \geq 10$).

Remark. F_H is a hypersurface of degree M in $\mathbf{P}^{M-1}$. It is exactly for this reason that, repeating the scheme of arguments from [P2], we have to perform one step less, intersecting the cycle C with $D_4, \dots, D_{M-2}$, but not with D_{M-1}: the dimension does not allow to do it. However, for M sufficiently high, this makes no difference.

To obtain Lemma 6 in full, we need a bit more delicate arguments. The proof given below works well for all the values $M \geq 5$. Note that $m\widetilde{W} \in |\alpha L - \beta E|$ for some $m, \alpha, \beta \in \mathbf{Z}_+ \setminus \{0\}$ with $\beta/\alpha > 3/2$. Thus any curve $C \subset F_H$, satisfying the inequality (†)

$$\frac{\mathrm{mult}_p}{\deg} C > \frac{2}{3},$$

is contained in W. Consequently the lemma is implied by the following

Lemma 8. *The cycle S_H is swept out by a family of curves $\mathcal{C} = (C_t, t \in T)$, the general member of which is ireducible and satisfies the inequality* (†).

Proof of the lemma. As an example consider the most transparent case $M = 5$. The linear system of curves $\Lambda_4|_{S_H}$ on the surface S_H has by the regularity condition γ, $0 \leq \gamma \leq 4$, base lines of multiplicity one (since any 5 lines from the set of 5! solutions of the system of equations $q_2 = \ldots = q_5 = 0$ are linearly independent and therefore can not lie in H). Throwing away these lines and taking an irreducible component of a general curve of the residual family (which has no fixed components), we get a family $\mathcal{C} = (C_t, t \in T)$, for which

$$\frac{\mathrm{mult}_p}{\deg} C_t = \frac{30 - \gamma}{40 - \gamma} > \frac{2}{3},$$

as we need.

Now take an arbitrary $M \geq 6$. We argue in a similar way (see also the proof of Lemma 2 above). Let γ, $0 \leq \gamma \leq M - 1$ be the number of solutions of the system of equations $q_2 = \ldots = q_M = 0$, which lie in H. Set

$$\beta_k = \frac{3(k+4)}{4M}.$$

Let us construct by induction on $k = 1, \dots, M-5$, a family of irreducible cycles $\mathcal{R}_k = (R_t^{(k)}, t \in T_k)$, which sweep out S_H, are of codimension k in S_H, and satisfy the inequality

$$\frac{\mathrm{mult}_p}{\deg} R_t^{(k)} \geq \beta_k.$$

Assume that the family $\mathcal{R}_k$ has already been constructed and $k \leq M - 6$. Let $R = R_t^{(k)}$ be a sufficiently general member of this family. Then by the regularity condition the linear system $\Lambda_{k+4}|_R$ has no fixed components, since

$$\operatorname{codim} \operatorname{Bs} \Lambda_{k+4}|_{S_H} \geq k + 2 = \operatorname{codim} R + 2.$$

A general divisor of the system $\Lambda_{k+4}|_R$ is irreducible, unless it is composed of a pencil; in any case the irreducible components of a general divisor, taken over all $R^{(k)}_t$, $t\in T_k$, form the desired family $\mathcal{R}_{k+1}$, since

$$\frac{\mathrm{mult}_p}{\mathrm{deg}}R^{(k+1)}_t \geq \beta_k\frac{k+5}{k+4}=\beta_{k+1}.$$

Now take a general surface $R=R^{(M-5)}_t$ from the family $\mathcal{R}_{M-5}$. Restricting the linear system Λ_{M-1} onto R, we obtain a linear system of curves. It may have lines on F_H as fixed components. Throwing away these components (each of which is of multiplicity exactly 1), we get a family of curves $\mathcal{C}=(C_t, t\in T)$, sweeping out S_H. A general curve of this family is an irreducible component of a curve of the family $\mathcal{C}^*=(C^*_t, t\in T^*)$, which can be constructed directly as follows. Take the family of 1-cycles

$$(S_H\bullet D_4\bullet\ldots\bullet D_{M-1}),$$

where $D_i\in\Lambda_i$ are general divisors, and throw away γ lines of multiplicity one, lying in F_H, $0\leq\gamma\leq M-1$. Consequently,

$$\frac{\mathrm{mult}_p}{\mathrm{deg}}C_t=\frac{\mathrm{mult}_p}{\mathrm{deg}}C^*_t\geq\frac{\frac{M!}{4}-\gamma}{\frac{M!}{3}-\gamma}>\frac{2}{3},$$

which is what we need. Q.E.D. for Lemma 6.

Proof of the main theorem of the present paper is complete.

Remark. For M sufficiently large the ratio $(\mathrm{mult}_p/\mathrm{deg})$ for the curves of the system $\mathcal{C}$ tends to 3/4. This means that, similarly to the smooth case (Lemma 2), there is a considerable gap in the set of possible values of the ratio $(\mathrm{mult}_p/\mathrm{deg})$ for S_H and all the other prime Weil divisors $W\neq S_H$. For these cycles for $M\gg 0$ we get

$$\max\left\{\frac{\mathrm{mult}_p}{\mathrm{deg}}W|W\neq S_H\right\}\approx\frac{8}{3M}.$$

4 Regular hypersurfaces: dimension count

4.1 Existence of regular hypersurfaces

Our arguments here are a modification of the proof of Proposition 1.1 in [P2]. The notations of the present section and Section 1 in [P2] are compatible.

Let $\mathcal{P}_{k,K}$ be the space of homogeneous polynomials of degree k in K variables, $\dim\mathcal{P}_{k,K}=\binom{k+K-1}{k}$. Consider the space of pairs

$$S=\{(x,F)|x\in\mathbf{P},F\in H^0(\mathbf{P},\mathcal{O}_{\mathbf{P}}(M))\cong\mathcal{P}_{M,M+1}\}$$

with the natural projections $p\colon S\to\mathbf{P}$ and $q\colon S\to H^0(\mathbf{P},\mathcal{O}_{\mathbf{P}}(M))$, the incidence hypersurface $I=\{(x,F)|\ F(x)=0\}\subset S$ and set

$$Y^*=\{(x,F)|x\in F\quad\text{is a non-regular smooth point}\},$$

$$Y^{\text{sing}} = \{(x,F) | x \in F \quad \text{is a non-regular singular point}\},$$

$$Y = Y^* \bigcup Y^{\text{sing}}.$$

Proposition 1. *The set $q(Y)$ is a proper closed subset of codimension 2 in the space $H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(M))$.*

Proof. Set $I(x) = I \cap p^{-1}(x)$ and $Y(x) = Y \cap p^{-1}(x)$. It is sufficient to show that

$$\operatorname{codim}_{I(x)} Y(x) = M + 1.$$

Let us consider the smooth and singular points separately.

4.2 Smooth non-regular points

Following [P2], set
$Y^*(x) = Y^* \cap p^{-1}(x)$,
$Y_k^*(x) = \left\{ (q_1, \ldots, q_k) \in \prod_{i=1}^k \mathcal{P}_{i,M} \middle| q_1 \neq 0, \operatorname{codim}\{q_1 = \ldots = q_k = 0\} < k \right\}$,
where we identify

$$p^{-1}(x) = \prod_{i=1}^M \mathcal{P}_{i,M} = \{(q_1, \ldots, q_M)\}$$

via a system of linear coordinates $(z_1, \ldots, z_M)$ with the origin at x. Set also $R_1 = \{q_1 \neq 0\}$,

$$R_k = \prod_{i=1}^k \mathcal{P}_{i,M} \setminus Y_k^*, k \geq 2,$$

and (abusing the notations)

$$\pi \colon \prod_{i=1}^{k+1} \mathcal{P}_{i,M} = \prod_{i=1}^k \mathcal{P}_{i,M} \times \mathcal{P}_{k+1,M} \to \prod_{i=1}^k \mathcal{P}_{i,M}$$

to be the natural projection.

It was shown in [P2] that for $k = 2, \ldots, M-2$ the following estimate holds

$$\operatorname{codim} Y_k^* \bigcap \pi^{-1}(R_{k-1}) \geq \binom{M}{k} > M + 1,$$

where the codimension is taken with respect to the space of polynomials $\prod_{i=1}^k \mathcal{P}_{i,M}$. Thus, the set of sequences $(q_1, \ldots, q_M)$ with $q_1 \neq 0$, which fail to be regular at the k-th step, $k \leq M-2$, has a high codimension. Consider the set R_{M-2}. The system of equations

$$q_1 = \ldots = q_{M-2} = 0$$

defines a curve $C(q_*) \subset \mathbf{P}^{M-1} = \mathbf{P}(T_x\mathbf{P})$.

Lemma 1. *The set $\mathcal{L} \subset R_{M-2}$, for the points of which the curve $C(q_*)$ contains a line in $\mathbf{P}^{M-1}$ as a component, is of codimension $\frac{1}{2}(M-2)(M-3)$.*

Proof. Obvious computations.

For our purposes it is sufficient that $\mathcal{L} \subset R_{M-2}$ is a proper closed subset (i.e. is of codimension ≥ 1), since by [P2]

$$\operatorname{codim} Y^*_{M-1} \bigcap \pi^{-1}(\mathcal{L}) \geq M+1.$$

Therefore we may assume that $C(q_*)$ does not contain a line. To complete the proof (in the non-singular case) it is sufficient to show that

$$\operatorname{codim}_{\pi^{-1}(q_*)}(Y^*_{M-1} \bigcap \pi^{-1}(q_*)) \geq M+1.$$

Let $P \subset \mathbf{P}^{M-1}$ be a general $(M-4)$-plane, so that the restriction of the linear projection

$$\pi_P \colon \mathbf{P}^{M-1} - - \to \mathbf{P}^2$$

onto each component C of the curve $C(q_*)$ is a birational morphism, and, moreover, $d = \deg \pi_P(C) = \deg C \geq 2$. Consequently, the kernel of the restriction map

$$\operatorname{res}_C \colon H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(M-1)) \to H^0(C, \mathcal{O}_C(M-1))$$

is of codimension $Md - \dfrac{d(d-1)}{2}$ for $d \leq M-1$ (for $d \geq M$ the kernel is trivial). Thus there exists a subspace $W \subset \mathcal{P}_{3,M-1}$ of dimension $M+1$, intersecting $Y^* = Y^*_{M-1}$ by zero only. Therefore

$$\operatorname{codim}_{I(x)} Y^*(x) \geq M+1,$$

which is what we need.

4.3 Singular non-regular points

Set $I^{\mathrm{sing}}(x) = \{(x,F) \in I(x) | \ \dfrac{\partial F}{\partial z_*}|_x = 0\}$. Since obviously $\operatorname{codim}_{I(x)} I^{\mathrm{sing}}(x) = M$, to complete the proof of Proposition 1 it suffices to show that $Y^{\mathrm{sing}}(x) \subset I^{\mathrm{sing}}(x)$ is a proper closed subset. But this is obvious, since the regularity condition is an open one, whereas for a general polynomial of the form

$$q_2 + \ldots + q_M$$

zero is a regular double point.

Q.E.D. for Proposition 1.

5 Conjectures

5.1 Fano complete intersections

Let $V = V_{m_1 \cdot \ldots \cdot m_k} \subset \mathbf{P}$, $m_1 + \ldots + m_k = M$, be a Fano complete intersection of index 1.

Conjecture 1. *Smooth V is birationally superrigid.*

The techniques, which made it possible to prove that general Fano hypersurfaces (corresponding to the case $k = 1$) are birationally superrigid [P2], is not strong enough for the complete intersections. However, the general scheme of arguments, which was introduced in [P2] and used in the present paper above, works partially for the complete intersections, either, and excludes an infinitely near singularity $\nu \in \mathcal{N}(V)$ with the centre at a *general* point $p = \mathrm{centre}(\nu) \in V$. Indeed, assume that $k \geq 2$ and

$$V = H_1 \bigcap \ldots \bigcap H_k,$$

where the hypersurface H_i is given by the equation

$$f_i = q_{i,1} + q_{i,2} + \ldots + q_{i,m_i} = 0$$

with respect to a system of linear coordinates $z_1, \ldots, z_M$ with the origin at p. Taking the truncated polynomials

$$f_{i,j} = q_{i,1} + \ldots + q_{i,j},$$

we get the hypertangent linear systems $\Lambda_k = |kH - (k+1)E|$, where, if the point p is sufficiently general, the following estimate holds:

$$\mathrm{codim}_V \mathrm{Bs}\,\Lambda_l \geq \sum_{\alpha=1}^{k} \min\{m_\alpha, l\}.$$

For $l \geq 2$ set

$$n_l = \sharp\{\alpha |\, 1 \leq \alpha \leq k, m_\alpha \geq l+1\},$$
$$n_1 = k - 2 \geq 0,$$

and choose in each system Λ_l precisely n_l general divisors $F_{l,1}, \ldots, F_{l,n_l}$. It is easy to see that the multiplicity of the zero-dimensional scheme

$$\left(Z \bullet \prod_{l=1}^{\max\{m_i\}} \prod_{\alpha=1}^{n_l} F_{l,\alpha} \right)$$

at the point p, where $Z = (D_1 \bullet D_2)$, $D_i \in |\chi|$ are general divisors, is strictly greater than $d = n^2 \prod\limits_{l=1}^{k} m_l!$, whereas the global degree of this scheme is equal to d. This contradiction excludes the maximal singularity $\nu \in \mathcal{N}(V)$.

Unfortunately, this scheme does not work, if there are lines on V, passing through the point p. Nevertheless, there can be no doubt that Conjecture 1 (and even essentially more strong claim, permitting V to be singular) is true.

5.2 Fibrations into Fano complete intersections

Let S be a smooth variety, $\mathcal{E}$ be a locally free sheaf of rank $M+1$, $X = \mathbf{P}(\mathcal{E})$ be a locally trivial $\mathbf{P}^M$-bundle over S. Consider smooth subvarieties $V \subset X$ of codimension $k \geq 1$, which are locally realizable as complete intersections of the type $(m_1, \ldots, m_k)$ in $\mathbf{P} \times U$, where $U \subset S$ are open affine subsets, over which the fibration $\pi\colon X \to S$ is trivial, $m_1 + \ldots + m_k = M$. Obviously, $\mathrm{Pic}\, V = \mathbf{Z} K_V \oplus \pi^* \mathrm{Pic}\, S$. Let $f \in A^{\dim S+1}(V)$ be the class of a hyperplane section of a general fiber $V \cap \pi^{-1}(s)$, which is a cycle of codimension $\dim S + 1$ on V.

Conjecture 2. *Assume that the following K^2-condition holds: for any class $C \in A^{\dim S-1}(S)$ of moving rational curves, sweeping out S, the numerical class*

$$M(K_V^2 \cdot \pi^* C) - f$$

is not effective on V. Then the Fano fibration $\pi: V \to S$ is birationally superrigid.

Twisting $\mathcal{E}$, if necessary, we may assume that this sheaf is generated by global sections, so that the class L of the tautological sheaf on X gives a free linear system $|L|$. The K^2-condition holds, if the following inequality is satisfied

$$(K_V^2 \cdot \pi^* C \cdot L_V^{M-1-k}) \leq 0.$$

This estimate can be easily verified in terms of invariants of the base S and the Chern classes of the sheaf $\mathcal{E}$.

The main obstruction to proving Conjecture 2 is the same as for Conjecture 1: the regularity conditions do not hold at the points and on the fibers of non-general position. If there were no these problems, we could argue in the following way. Let

$$\begin{array}{cccc} \chi\colon & V & \dashrightarrow & V' \\ & \downarrow & & \downarrow \\ & S & & S' \end{array}$$

be a birational map of Fano fibrations. Consider a linear system $|D'|$ on V' and a dense free algebraic family of rational curves $\mathcal{C}' = (C'_t, t \in T)$ on V' (for instance, contracted by the projection $\pi': V' \to S'$ to points). Take their proper inverse images

$$|D| \subset |-nK_V + \pi^* A|,$$

$A \in \operatorname{Pic} S$ and

$$\mathcal{C} = (C_t, t \in T)$$

on V, and argue as above (Sections 2-3), restricting the linear system $|D|$ onto the preimage $\pi^{-1}(\pi(C_t))$ of a general curve of the family $\pi_* \mathcal{C}$ on S. If all the fibers of π were sufficiently good, the K^2-condition would have provided a complete analogy to the case of one-dimensional base. At the moment, this scheme of arguments works well for conic bundles only [P4] (the Sarkisov theorem [S]).

5.3 Further conjectures

The experience of working with Fano complete intersections in a weighted projective space $\mathbf{P}(a_0, a_1, \ldots, a_n)$ [CPR] shows that, under appropriate assumptions, the results on complete intersections in $\mathbf{P}^N$ can be carried over to the weighted case (and, moreover, the case of trivial weights $a_0 = \ldots = a_N = 1$ seems to be the hardest one). Thus Conjectures 1 and 2 should be true (with the necessary modificatons of smoothness etc.) for the weighted complete intersections and fibrations into weighted complete intersections, respectively.

Finally, let us formulate one more conjecture on maximal singularities of linear systems of divisors on Fano fibrations. It is true in all the cases where investigation was successful. Unfortunately, here we can not give even a promising scheme of its proof.

Conjecture 3 (Basic conjecture) *Let $\pi\colon V \to S$ be a Fano fibration, $\nu \in \mathcal{N}(V)$ be a maximal singularity of a linear system Λ. Then there exists an irreducible cycle $B \subset V$, $B \cap \mathrm{centre}(\nu) \neq \emptyset$, which is maximal for Λ (i.e. $\nu_B = \mathrm{mult}_B$ is a maximal singularity).*

The hope that Conjecture 3 is true is confirmed by the papers of Kawamata [K] and Corti [C] (and of course, by all that we know about maximal singularities).

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