Generic coverings of plane with A-D-E-singularities.

V.S. Kulikov [*] and Vic.S. Kulikov [†]

Abstract

We investigate a presentation of an algebraic surface X with A-D-E-singularities as a generic covering $f : X \to \mathbb{P}^2$, i.e. a finite morphism, having at most folds and pleats apart from singular points, isomorphic to a projection of a surface $z^2 = h(x, y)$ onto the plane x, y in neighbourhoods of singular points, and the branch curve $B \subset \mathbb{P}^2$ of which has only nodes and ordinary cusps except singularities originated from the singularities of X. It is deemed that classics proved that a generic projection of a non-singular surface $X \subset \mathbb{P}^r$ is of such form. In this paper this result is proved for an embedding of a surface X with A-D-E-singularities, which is a composition of the given one and a Veronese embedding. We generalize results of the paper [K] , in which a Chisini conjecture on the unique reconstruction of f by the curve B is investigated. For this fibre products of generic coverings are studied. The main inequality bounding the degree of a covering in the case of existence of two nonequivalent coverings with the branch curve B. This inequality is used for the proof of the Chisini conjecture for m-canonical coverings of surfaces of general type for $m \geq 5$.

Introduction

Let $S \subset \mathbb{P}^r$ be a non-singular projective surface, $f : S \to \mathbb{P}^2$ be its generic projection to the plane, $B \subset \mathbb{P}^2$ be the branch curve, which we call the discriminant curve. It is deemed that classics proved (see [Z] , p.104) that (i) the map f is a finite covering, which has as singularities at most double points (folds), or singular points of cuspidal type (pleats); (ii) with this $f^*(B) = 2R + C$, where the double curve R is non-singular and irreducible, and the curve C is reduced; (iii) the curve B is cuspidal, i.e. has at most nodes and ordinary cusps; over a node there lie two double points, and over a cusp – one point of cuspidal type; (iv) the restriction of f to R is of degree one. Any finite morphism $f : S \to \mathbb{P}^2$ is called a *generic* (or *simple*) *covering*, if it possesses the same properties as a generic projection. Two coverings of plane (S_1, f_1) and (S_2, f_2) are called equivalent, if there is a morphism $\varphi : S_1 \to S_2$ such that $f_1 = f_2 \circ \varphi$.

In this paper we consider a generalization of the notion of a generic covering to the case of surfaces with A-D-E-singularities. First of all we want to explain why we need such a

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generalization. A presentation of an algebraic variety as a finite covering of the projective space is one of the affective ways of studying projective varieties as well as their moduli. To compare we recall what such an approach gives in the case of curves. For a curve C of genus g a generic covering $f : C \to \mathbb{P}^1$ is such a covering that in every fibre there is at most one ramification point which is a double point (or a singular point of f of type A_1). Let $B \subset \mathbb{P}^1$ be the set of branch points, and $d = \deg B$, i.e. $d = \sharp(B)$. Then according to the Hurwitz formula $d = 2N + 2g - 2$, where $N = \deg f$. If $N \geq g + 1$, then any curve of genus g can be presented as a simple covering of $\mathbb{P}^1$ of degree N. The set of all simple coverings (up to equivalence) $f : C \to \mathbb{P}^1$ of degree N with d branch points is parametrized by a Hurwitz variety $H = H^{N,d}$. Let $\mathbb{P}^d \setminus \Delta$ (Δ – discriminant) be the projective space parametrizing the sets of d different points of $\mathbb{P}^1$, and let M_g be the moduli space of curves of genus g. There are two maps: a map $h : H \to \mathbb{P}^d \setminus \Delta$, sending f to the set of branch points $B \subset \mathbb{P}^1$, and a map $\mu : H \to M_g$, sending f to the class of curves isomorphic to C. Hurwitz introduced and investigated the variety H in 1891. He proved that the variety H is connected, and h is a finite unramified covering. In modern functorial language H was studied also by W.Fulton in 1969. The map μ is surjective (and has fibres of dimension $N + (N - g + 1)$). This gives one of the proofs ofirreducibility of the moduli space M_g.

In the case of surfaces we also can consider an analog of Hurwitz variety H of all generic coverings (up to equivalence) $f : S \to \mathbb{P}^2$ of degree N and with discriminant curve B of degree d with given number n of nodes and given number c of cusps. Let $\mathbb{P}^\nu$, $\nu = \frac{d(d+3)}{2}$, be a projective space parametrizing curves of degree d, and $h : H \to \mathbb{P}^\nu$ be a map sending a covering f to its discriminant curve B. In [K] a Chisini conjecture is studied. It claims that if B is the discriminant curve of a generic covering f of degree $N \geq 5$, then f is uniquely up to equivalence defined by the curve B. In other words, it means that the map h is injective (and, besides, $N = \deg f$ is determined by B). In [K] it is proved that the Chisini conjecture is true for almost all generic coverings. In particular, it is true for generic coverings defined by a multiple canonical class. A construction of the moduli space of surfaces of general type uses pluricanonical maps. As is known [BPV] , if S is a minimal surface of general type, then for $m \geq 5$ the linear system $|mK_S|$ blows down only (-2)-curves and gives a birational map of S to a surface $X \subset \mathbb{P}^r$ (the canonical model) with at most A-D-E-singularities (in other terms, rational double points, Du Val singularities, simple singularities of Arnol'd and etc.). This requires a generalization of the notion of a generic covering to the case of surfaces with A-D-E-singularities.

In this paper we, firstly, generalize a classical result on singularities of generic projections of non-singular surfaces to the case of surfaces with A-D-E-singularities. We prove that if a surface $X \subset \mathbb{P}^r$ has at most A-D-E-singularities, then (may be after a "twist") for a generic projection $f : X \to \mathbb{P}^2$ the discriminant curve B also has at most A-D-E-singularities. It follows from a slightly more general theorem.

Theorem 0.1 *Let $X \subset \mathbb{P}^r$ be a surface with at most isolated singularities of the form $z^2 = h(x, y)$ (= "double planes"), $X \to \mathbb{P}^2$ be the restriction to X of a generic projection $\mathbb{P}^r \setminus L \to \mathbb{P}^2$ from a generic linear subspace L of dimension $r - 3$. Then*

(i) f is a finite covering;

(ii) at non-singular points of X the covering f has as singularities at most either double points (folds), or singular points of cuspidal type (pleats); in a neighbourhood of these points f is equivalent to a projection of a surface $x = z^2$, respectively $y = z^3 + xz$, to the plane x, y;

(iii) in a neighbourhood of a point $s \in Sing\ X$ the covering f is analytically equivalent to a projection of a surface $z^2 = h(x, y)$ to the plane x, y; from (ii) and (iii) it follows that the ramification divisor is reduced, i.e. $f^*(B) = 2R + C$, where $B = f(R)$, and R and C are reduced curves;

(iv) except singular points $f(Sing\, X)$ the discriminant curve B is cuspidal;

(v) the restriction of f to R is of degree 1.

Actually, the main difficulty in the proof of this theorem lies in the classical case, when the surface X is non-singular. Unfortunately, authors do not know a complete (and mordern) proof of this theorem, and it seams that such a proof does not exist. Thus, the proof, even in the case of a non-singular surface, take interest. In this paper we prove a weakened version of Theorem 0.1, in which the initial embedding is 'twisted' by a Veronese embedding. This is quite enough for the purposes described above.

Thus, the curve B has, firstly, 'the same' singularities as the surface X (and as the curve R), which are locally defined by the equation $h(x, y) = 0$. These singularities on B we call *s-singularities*, in particular, *s-nodes* and *s-cusps*. Besides, there are nodes and cusps on B originated from singularities of the map f, which we call *p-nodes* and *p-cusps*. There are two double points of f over a p-node, at which f is defined locally as a projection of surfaces $z_1 = x^2$ and $z_2 = y^2$ to the plane x, y.

If S is a surface with A-D-E-singularities, then a covering $f : S \to \mathbb{P}^2$ is called *generic*, if it satisfies the properties of Theorem 0.1.

Secondly, we generalize the central result of [K] to the case of surfaces with A-D-E-singularities. It is proved there that if a generic covering $f : S \to \mathbb{P}^2$ of a non-singular surface S with discriminant curve B is of sufficiently big degree $\deg f = N$, namely under condition

$$N > \frac{4(3\bar{d} + g - 1)}{2(3\bar{d} + g - 1) - c}\,, \tag{1}$$

where $2\bar{d} = \deg B$, g be the geometric genus of B, and c be the number of cusps, then B is the discriminant curve of a unique generic covering (the Chisini conjecture holds for B).

We can't expect an analogous result in the case of singular surfaces, because for a curve B of even degree with at most A-D-E-singularities there always exists a double covering, which is generic. But if two generic coverings with given discriminant curve B are coverings of sufficiently big degree, then they are equivalent. More exactly, we prove the following theorem. Let there are two generic coverings $f_1 : X_1 \to \mathbb{P}^2$ and $f_2 : X_2 \to \mathbb{P}^2$ of surfaces with A-D-E-singularities and with the same discriminant curve $B \subset \mathbb{P}^2$. Let $f_i^*(B) = 2R_i + C_i$, $i = 1, 2$. With respect to a pair of coverings f_1 and f_2 nodes and cusps of B are partitioned into four types: ss-, sp-, ps- and pp-nodes and cusps. For example, a sp-node $b \in B$ is a node, which is a s-node for f_1 and a p-node for f_2. The number of sp-nodes is denoted by n_{sp}. Then $n = n_{ss} + n_{sp} + n_{ps} + n_{pp}$. The analogous terminology is used for cusps.

Theorem 0.2 *If f_1 and f_2 are nonequivalent generic coverings, then*

$$\deg f_2 \leq \frac{4(3\bar{d}+g_1-1)}{2(3\bar{d}+g_1-1)-\iota_1}\ , \qquad (2)$$

where $g_1 = p_a(R_1)$ is the arithmetic genus of the curve R_1, and $\iota_1 = 2n_{sp} + 2c_{sp} + c_{pp}$.

We apply the main inequality (2) to the proof of the Chisini cojecture in the case of generic pluricanonical coverings. Let S be a minimal model of a surface of general type. According to a theorem of Bombieri [BPV] , if $m \geq 5$, then the m-canonical map $\varphi_m : S \to \mathbb{P}^{p_m-1}$, defined by the complete linear system numerically equivalent to $|mK_S|$, is a birational morphism, which blows down (-2)-curves on S. Then the canonical model $X = \varphi_m(S)$ has at most A-D-E-singularities. A generic projection $f : X \to \mathbb{P}^2$ is called a *generic m-canonical covering for S*. We prove the following theorem.

Theorem 0.3 *Let S_1 and S_2 be minimal models of surfaces of general type with the same (K_S^2) and $\chi(S)$, and let $f_1 : X_1 \to \mathbb{P}^2$, $f_2 : X_2 \to \mathbb{P}^2$ be generic m-canonical coverings with the same discriminant curve. Then for $m \geq 5$ the coverings f_1 and f_2 are equivalent.*

Consider a subvariety $\mathcal{H} \subset Hilb \times Gr$, parametrizing m-canonical coverings. Here $Hilb$ is a subscheme of the Hilbert scheme, parametrizing numerically m-canonical embeddings $X \subset \mathbb{P}^M$ of surfaces with A-D-E-singularities and fixed (K_S^2) and $\chi(S)$, Gr is the Grassmann variety of projection centres from $\mathbb{P}^M$ to $\mathbb{P}^2$, and $\mathcal{H}$ consists of pairs $(X \subset \mathbb{P}^M, L)$) such, that a restriction to X of a projection with centre L is a generic covering. By theorem 0.1 there is a one-to-one correspondence between the set of irreducible (respectively, connected) components of $Hilb$ and $\mathcal{H}$. Let $h : \mathcal{H} \to \mathbb{P}^\nu$ be a map, taking a covering to its discriminant curve. Denote by $\mathcal{D}$ a variety of plane curves of degree d with A-D-E-singularities, among which the number of nodes $\geq n_p$, and the number of cusps $\geq {}_p$, where d, n_p, c_p are defined by invariants of S (see §6). By theorem 0.3 it follows (cf. [K] , §5)

Corollary. The map, induced by h, from the set of irreducible (respectively, connected) components of the variety $\mathcal{H}$ to the set of irreducible (respectively, connected) components of the variety $\mathcal{D}$ is injective.

The proof of the main inequality (2) in [K] in the case of non-singular surfaces runs as follows. To compare two coverings f_1 and f_2, a normalization X of the fibre product $X_1 \times_{\mathbb{P}^2} X_2$ is considered. Let $g_i : X \to X_i$, $i = 1, 2$, be the corresponding mappings to the factors. The preimage $g_1^{-1}(R_1) = R + C$ falls into two parts, where R is the curve mapped by g_2 to R_2, and C is the curve mapped by g_2 to C_2. If f_1 and f_2 are nonequivalent, then the surface X is irreducible, and if X_i are non-singular, then X is non-singular too. The main inequality is obtained by applying the Hodge index theorem to the pair of divisors R and C on X. We use the same idea also in the case of surfaces with A-D-E-singularities. For this we carry out the local analysis of the normalization of the fibre product X in the case of generic coverings of surfaces with A-D-E-singularities.

In §1 we generalize to the case of surfaces with A-D-E-singularities the theorem on generic projections. In §2 a local analysis of a normalization of the fibre product X is carried out.

In §3 we investigate the canonical cycle of an A-D-E-singularity, with the help of which we compute numerical invariants of a generic covering in §4. In §5 the main inequality (2) is proved. Finally, in §6 the Chisini conjecture for generic m-canonical coverings of surfaces of general type is proved.

1 Singularities of a generic projection of a surface with A-D-E-singalarities.

In this section we prove Theorem 0.1.

1.1. *A generic projection to* $\mathbb{P}^3$. Let $X \subset \mathbb{P}^r$ be a surface of degree $\deg X = N$ with at most isolated hypersurface singularities $x_1, \dots, x_k$, i.e. such that the dimension of the tangent spaces $dim\ T_{X,x_i} = 3$. Denote by $\pi_L : \mathbb{P}^r \setminus L \to \mathbb{P}^{e-1}$ a projection from a linear subspace L of codimension e. It can be obtained as a composition of projections with centers at points. The Theorem 0.1 on projections of X to the plane ($e = 3$) is one of a series of theorems on generic projections for different e, beginning with projections from points ($e = r$) and finishing by projections to the line ($e = 2$), i.e. Lefschetz pencils.

A classical result is that, if $r > 5$ ($= 2\dim X + 1$), then the projection from a generic point gives an isomorphic embedding of X into $\mathbb{P}^{r-1}$. It follows that, if $e \geq 6$, then the projection from a generic subspace L gives an isomorphic embedding of X into $\mathbb{P}^{e-1}$. In particular, by a generic projection the surface X is embedded into $\mathbb{P}^5$. When projecting to $\mathbb{P}^4$, $e = 5$, there appears isolated singularities on $\pi_L(X)$, which is not difficult to describe. To prove Theorem 0.1 we are going to consider a generic projection of X into $\mathbb{P}^3$, $e = 4$, and to take advantage of the following theorem.

Theorem 1.1 *If $X \subset \mathbb{P}^r$ is a surface with at most isolated hypersurface singularities x_i, then the restriction of a projection $\pi_L : \mathbb{P}^r \setminus L \longrightarrow \mathbb{P}^3$ with the centre in a generic subspace $L \subset \mathbb{P}^r$ of codimension 4 gives a birational map of X onto a surface $Y \subset \mathbb{P}^3$, which is an isomorphism outside the double curve $D \subset X$ not passing through the points x_i, and Y has, except the points $\pi_L(x_i)$, at most ordinary singularities – the double curve $\Delta = \pi_L(D)$, on which there lie a finite number of ordinary triple points and a finite number of pinches. In neighbourhoods of these points in appropriate local analytic coordinates Y has normal forms as follows: $uv = 0$ for ordinary double points, $uvw = 0$ for ordinary triple points, $u^2 - vw^2 = 0$ for pinches (or "Whitney umbrellas").*

The contemporary proof of this theorem one can find in the textbook [G-H]. The presence of singular points x_i do not add extra troubles: we need only to see to the centre of the projection L not to intersect the tangent spaces T_{X,x_i}, $dim\ T_{X,x_i} = 3$. A proof of this theorem one can find also in [M] .

We want to prove that for a generic point $\xi \in \mathbb{P}^3$ the composition of projections π_L and $\pi_\xi : \mathbb{P}^3 \setminus \xi \to \mathbb{P}^2$, i.e. the projection $\mathbb{P}^r \setminus \pi_L^{-1}(\xi) \to \mathbb{P}^2$ with the centre $\pi_L^{-1}(\xi)$, restricted to X, $f = \pi_\xi \circ \pi_{L|_X} : X \to \mathbb{P}^2$, gives a covering satisfying the properties stated in Theorem 0.1 .

1.2. *The disposition of lines with respect to a surface* $\mathbb{P}^3$. To describe a projection π_ξ we need to investigate the disposition of lines $l \subset \mathbb{P}^3$ with respect to the surface Y. A line l is called *transversal* to Y at a point y, if it is transversal to the tangent cone to Y at this point. It means that $(l \cdot Y)_y = 1$, if $y \notin Sing\ Y$; $(l \cdot Y)_y = 2$, if $y \in \Delta \setminus \Delta_t$ and $(l \cdot Y)_y = 3$, if $y \in \Delta_t$. We denote by Δ_t and Δ_p the set of triple points and the set of pinches. If l is not transversal to Y at a point y, we say that it is tangent to Y at this point. A line l is called a *simple tangent* to Y at y, if $y \notin Sing\ Y$ and $(l \cdot Y)_y = 2$, or if $y \in \Delta \setminus (\Delta_t \cup \Delta_p)$ and $(l \cdot Y)_y = 3$, i.e. $(l \cdot Y_i)_y = 2$ for one of two branches Y_i at the point y. A line l is called *stationary* tangent, respectively *simple stationary* tangent to Y at y, if $y \notin Sing\ Y$ and $(l \cdot Y)_y \geq 3$, respectively $= 3$. A line l is called *stationary tangent*, respectively *simple stationary tangent* to Y, if l is transversal to Y at all points, except one, at which l is stationary tangent, respectively simple stationary tangent, and, besides the other points of intersection $l \cap Y$ are non-singular on Y. Finally, l is called *simple bitangent*, if l is transversal to Y at all points, except two of them, at which the contact is simple, the tangent planes at them are distinct, and, besides, $l \cap Sing\ Y = \emptyset$. We want to prove that for a generic point $\xi \in \mathbb{P}^3$ all lines $l \ni \xi$ are at most simple bitangents and simple stationary tangents with respect to Y.

To study the disposition of lines $l \subset \mathbb{P}^3$ with respect to Y, we consider the Grassmann variety $G = G(1,3)$ and the flag variety $\mathbb{F} = \{(\xi, l) \in \mathbb{P}^3 \times G \mid \ \xi \in l\}$. There are two projections $pr_1 : \mathbb{F} \to \mathbb{P}^3$ and $pr_2 : \mathbb{F} \to G$, which are $\mathbb{P}^2$- and $\mathbb{P}^1$-bundles respectively; $dim\ \mathbb{F} = 5$, and $dim\ G = 4$. In the sequal we consider points $\xi \in \mathbb{P}^3$ as centres of projection $\pi_\xi : \mathbb{P}^3 \setminus \xi \to \mathbb{P}^2$. The fibre $pr_1^{-1}(\xi) \simeq \mathbb{P}^2$ is mapped by the projection pr_2 isomorphically onto $\mathbb{P}^2_\xi \subset G$. For $\xi \in \mathbb{P}^3$ there is a section $s_\xi : \mathbb{P}^3 \setminus \xi \to \mathbb{F}$ of the projection pr_1, $y \longmapsto (y, \overline{\xi y})$. Then π_ξ coincides with the restriction of the projection pr_2 to $s_\xi(\mathbb{P}^3 \setminus \xi)$.

Firstly, we consider the case, when a surface Y is non-singular, and then we describe the necessary modifications and supplements in the case, when there is a double curve Δ and isolated singularities s_i on Y.

Consider a filtration of the variety $\mathbb{F}$ by subvarieties

$$Z_k = \{(\xi, l) \in \mathbb{F} \mid \ (l \cdot Y)_\xi \geq k\}.$$

Then $Z_1 = pr_1^{-1}(Y), dim\ Z_1 = 4$. Over a generic point $l \in G$ the map $\varphi = pr_{2|Z_1} : Z_1 \to G$ is an unramified covering of degree N. If there are no lines on Y, then φ is a finite covering, and Z_2 is the ramification divisor of the covering.

Now consider restrictions of the projection pr_1. The variety Z_2 is isomorphic to a projectivized tangent bundle, $Z_2 \simeq \mathbb{P}(\Theta_Y)$, and $Z_2 \to Y$ is a $\mathbb{P}^1$-fibre bundle, $dim\ Z_2 = 3$. At a generic point $y \in Y$ there are two asymptotic directions l_1 and l_2 in $T_{Y,y}$, for which $(l_1 \cdot Y)_y$ and $(l_2 \cdot Y)_y \geq 3$. Therefore, over a generic point the restriction of pr_1 onto Z_3, $\psi : Z_3 \to Y$, is a two-sheeted covering, the branch curve of which $P \subset Y$ is the parabolic curve consisting of points with coinciding asymptotic directions. Some fibres of the projection pr_1 are exceptional curves of the map ψ. Their images on Y are points y, at which the restriction of the second differential of the local equation of Y onto the tangent plane $T_{Y,y}$ vanishes. Such points y are called the *planar points* of the surface Y. The curve $H = \psi(Z_4) \subset Y$ consists of points y, at which at least one of the numbers $(l_i \cdot Y)_y \geq 4$ (H is a curve, if the surface Y is not a quadric).

1.3. *Absence of non simple stationary tangents.* Consider a product $Y \times \mathbb{F} \subset Y \times \mathbb{P}^3 \times G$ and projections pr'_1 and pr'_2 onto $Y \times \mathbb{P}^3$ and $Y \times G$. We can consider the varieties Z_k as subvarieties in $Y \times G \subset \mathbb{P}^3 \times G$. Consider a variety

$$I_4 = \{(y; \xi, l) \in Y \times \mathbb{F} \mid \ (l \cdot Y)_y \geq 4\} = (pr_2')^{-1}(Z_4).$$

The projection $pr_2' = id_Y \times pr_2$, as well as $pr_2 : \mathbb{F} \to G$, is a $\mathbb{P}^1$-bundle. Therefore, $dim\ I_4 = 2$ and $dim\ \Sigma_4 \leq 2$, where $\Sigma_4 = p_2(I_4)$, and p_2 is a projection of $Y \times \mathbb{P}^3 \times G$ to $\mathbb{P}^3$. Then, if $\xi \in \mathbb{P}^3 \setminus \Sigma_4$, we have that $(l \cdot Y)_y \leq 3$ for any line $l \ni \xi$ at any point $y \in Y$.

1.4. *Absence of non simple bitangents.* Consider a variety $\Sigma_{2,3} \subset \mathbb{P}^3$, made up of non simple bitangents, and show that $\Sigma_{2,3} \leq 2$. Consider a product $Y \times Y \times \mathbb{F} \subset Y \times Y \times \mathbb{P}^3 \times G$ and subvarieties $I_{i,j}$, which are closures of

$$I^0_{i,j} = \{(y_1, y_2; \xi, l) \in Y \times Y \times \mathbb{F} : (Y \cdot l)_{y_1} \geq i, (Y \cdot l)_{y_2} \geq j, y_1 \neq y_2\}.$$

Denote a projection of $Y \times Y \times \mathbb{F}$ to $Y \times Y \times G$ by pr_2'', and let $pr_2''(I_{i,j}) = \tilde{I}_{i,j}$. The projection pr_2'' and its restriction to $I_{i,j}$, $I_{i,j} \to \tilde{I}_{i,j}$ are $\mathbb{P}^1$-bundles.

Lemma 1.1

$$\dim\ I_{2,3} \leq 2.$$

Proof. Consider subvarieties

$$Y \times Y \times G \supset \tilde{I}_{1,1} \supset \tilde{I}_{2,1} \supset \tilde{I}_{2,2} \supset \tilde{I}_{2,3}\ ,$$

and let q_1 be a projection onto the first factor. Obviously, $\tilde{I}_{1,1}$ is an irreducible variety of dimension $dim\ \tilde{I}_{1,1} = 4$, birationally isomorphic to $Y \times Y$. The projection $q_1 : \tilde{I}_{2,1} \to Y$ is a fibration, fibers of which are curves $q_1^{-1}(y) \simeq C_y$, where

$$C_y = Y \cap T_{Y,y}.$$

The curve C_y has a singularity at the point y, which is a node for a generic point y.

Furthermore, the ristriction of the projection to $\tilde{I}_{2,2}$, $q_1 : \tilde{I}_{2,2} \to Y$, is surjective, and its fibre over a point $y \in Y$ is

$$q_1^{-1}(y) = \{(y, y', l) \mid \ l \subset T_{Y,y} \text{ and } l \text{ is tangent to } {}_y \text{ at } y'\}.$$

We want to prove that $q_1(\tilde{I}_{2,3})$ doesn't coincide with Y, i.e. the embedding $Y \subset \mathbb{P}^3$ possesses the following property (L_1) : there exists a point $y \in Y$ such that all lines $l \subset T_{Y,y}$, passing through y, have at most simple contact with $C_y \setminus \{y\}$. We prove this below in 1.6 (Proposition 1.2) under the assumption that the embedding $Y \subset \mathbb{P}^3$ is obtained by a projection of an embedding "improved" by a Veronese embedding v_k, $k \geq 2$.

Thus, dim $q_1(\tilde{I}_{2,3}) \leq 1$. A generic fibre of the map $q_1 : \tilde{I}_{2,3} \to Y$ is of dimension zero (it being one, Y is a ruled surface and we obtain a contradiction to the property (L_1)), therefore, dim $\tilde{I}_{2,3} \leq 1$ and, consequently, dim $I_{2,3} \leq 2$. ■

Set $\Sigma_{2,3} = p_3(I_{2,3})$, where p_3 is a projection of $Y \times Y \times \mathbb{P}^3 \times G$ to $\mathbb{P}^3$. It follows from Lemma 1.1 that $dim\ \Sigma_{2,3} \leq 2$. If $\xi \notin \Sigma_{2,3}$, then any line $l \ni \xi$, touching Y at two points y_1 and y_2, has a simple contact at these points.

1.5. *Absence of 3-tangents.* Consider a product $Y \times Y \times Y \times \mathbb{F} \subset Y \times Y \times Y \times \mathbb{P}^3 \times G$ and subvarieties $I_{i,j,k}$, which are closures of

$$I^0_{i,j,k} = \{(y_1, y_2, y_3; \xi, l) \in Y \times Y \times Y \times \mathbb{F} \mid \ (Y \cdot l)_{y_1} \geq i, (Y \cdot l)_{y_2} \geq j, (Y \cdot l)_{y_3} \geq k\},$$

where $y_1 \neq y_2 \neq y_3 \neq y_1$. Denote a projection of $Y \times Y \times Y \times \mathbb{F}$ onto $Y \times Y \times Y \times G$ by $pr_2^{(3)}$, and let $\tilde{I}_{i,j,k} = pr_2^{(3)}(I_{i,j,k})$. As above, it is clear that *dim* $\tilde{I}_{1,1,1} = 4$, and $pr_1^{(3)}$ being a $\mathbb{P}^1$-bundle, we have *dim* $I_{1,1,1} = 5$.

Lemma 1.2

$$\dim\ I_{2,2,2} \leq 2.$$

Proof. Again consider a projection of $Y \times Y \times Y \times G$ and of its sibvarieties

$$Y \times Y \times Y \times G \supset \tilde{I}_{1,1,1} \supset \tilde{I}_{2,1,1} \supset \tilde{I}_{2,2,1} \supset \tilde{I}_{2,2,2},$$

to the first factor. Consider $q_1 : \tilde{I}_{2,2,2} \to Y$. For a point $y \in Y$ we have $q_1^{-1}(y) = \{(y, y_2, y_3; l) \mid l \subset T_{Y,y},\ l$ is tangent to $_y$ at points y_2 and $y_3 \in l\}$. Just as in Lemma 1.1 it is sufficient to prove that $q_1(\tilde{I}_{2,2,2})$ doesn't coincide with Y. It means that there exists a point $y \in Y$, possessing the following property (L_2): none of the lines $l \subset T_{Y,y}$, passing through y, is not a bitangent, i.e. can't touch $C_y \setminus \{y\}$ at two different points. We prove this below in the following 1.6 (Proposition 1.2) under the assumption that the embedding $Y \subset \mathbb{P}^3$ is obtained by a projection of an embedding "improved" by a Veronese embedding v_k. ■

Set $\Sigma_{2,2,2} = p_4(I_{2,2,2})$, where p_4 is a projection of $Y \times Y \times Y \times \mathbb{P}^3 \times G$ to $\mathbb{P}^3$. Then *dim* $\Sigma_{2,2,2} \leq 2$ and if $\xi \notin \Sigma_{2,2,2}$, then any line $l \ni \xi$ touches Y at most at two points.

1.6. *Embeddings with a Lefschetz property.* The properties (L_1) and (L_2) in the two previous subsections mean that there exists a point $y \in Y$, for which the projection π_y of the curve $C_y \setminus \{y\} \subset T_{Y,y} \simeq \mathbb{P}^2$ from the point y is a Lefschetz pencil. Thus, to prove Lemmas 1.1 and 1.2 it is necessary to prove the existence of a point $y \in Y$ possessing the following "Lefschetz property" (L) with respect to the embedding into $\mathbb{P}^3$. We formulate it for a surface X embedded into a projective space of any dimension.

Let $X \subset \mathbb{P}^r$ be an embedding into the projective space. We say, that *a hyperplane* $L_1 \subset \mathbb{P}^r$ *possesses a property* (L), if the curve $X \cap L_1$ has at most one node, i.e. L_1 touches X at a unique point x, at which the curve $X \cap L_1$ has an ordinary quadratic singularity. In other words, the point $[L_1] \in \check{\mathbb{P}}^r$, corresponding to L_1, is a non-singular point of the dual variety $X^\vee$. We say that a *pair* (L_1, L_3), where $L_3 \subset L_1$ is a linear subspace of dimension $r - 3$, *possesses a property* (L) , if : L_1 possesses the property (L) , $x \in L_3$, and a projection of the curve $X \cap L_1 - \to \mathbb{P}^1$ from the centre L_3 is a Lefschetz pencil, i.e. any fibre of this (rational) mapping contains one singular point, and this point is at most quadratic (is of multiplicity 2). We say that an *embedding* $X \subset \mathbb{P}^r$ *possesses a property* (L) , if $\exists x \in X$, for which $L_1 = T_{X,x}$ possesses the property (L), and L_1 can be added to a pair (L_1, L_3) with the property (L).

It is clear that, if a pair (L_1, L_3) possesses the property (L) and $Y \subset \mathbb{P}^3$ is obtained from X by projection from a centre $L_4 \subset L_3$, *dim* $L_4 = r - 4$, then the embedding $Y \subset \mathbb{P}^3$ possesses the property (L).

Proposition 1.1 *If $S \subset \mathbb{P}^q$ is an embedding of a non-singular surface, and $X \subset \mathbb{P}^{r_k}$ is its composition with the Veronese embedding v_k defined by polynomials of degree k, then the embedding $X \subset \mathbb{P}^{r_k}$ possesses the property (L).*

Proof. Consider the hyperplane L_1 corresponding to a point $[L_1] \in X^\vee \setminus Sing\ X^\vee$. Then the curve $C = X \cap L_1$ contains a unique singular point – a node $x \in C$. Let $i : C \to X$ be the embedding. Consider a projection $\pi_{k,x} : \mathbb{P}^{r_k} \setminus x \to \mathbb{P}^{r_k - 1}$ from the point x. To prove Proposition 1.1 it is enough to show that the image $\pi_{k,x}(C)$ is a non-singular curve in $\mathbb{P}^{r_k-1}$. For then, if L'_3, $dim\ L'_3 = r_k - 3$, is a centre of projection $\mathbb{P}^{r_k-1} \setminus L'_3 \to \mathbb{P}^1$, which is a Lefshetz pencil for $\pi_{k,x}(C)$, then, obviously, the pair (L_1, L_3), where $L_3 = \pi_{k,x}^{-1}(L'_3) \cap L_1$, possesses the property (L).

Let I_x be the ideal sheaf of the point x on S, and $\mathcal{O}_S(1)$ be the sheaf of hyperplane sections. Under the identification $v_k : S \simeq X$, the map $\pi_{k,x}$ is given by sections of $H^0(S, \mathcal{O}_S(k) \otimes I_x)$. Let $k = 2$ and let $\sigma : \overline{S} \to S$ be a σ-process with centre at the point x. We can assume that $\overline{S}$ is embedded into $\mathbb{P}^{r_2-1}$, where $r_2 - 1 = q(q+3)/2$, and the rational map $\sigma^{-1} : S \to \overline{S}$ is given by sections of $H^0(S, \mathcal{O}_S(2) \otimes I_x)$, i.e. it coincides with $\pi_{2,x}$. Since the proper transform $\overline{C} = \sigma^{-1}(C) \subset \overline{S}$ is a non-singular curve, we obtain Proposition 1.1 in the case $k = 2$. Besides, note that sections of $i^*(H^0(S, \mathcal{O}_S(2) \otimes I_x))$ give an embedding of $\overline{C}$ into $\mathbb{P}^{r_2-1}$. Consequently, for $k > 2$ sections of $i^*(H^0(S, \mathcal{O}_S(k) \otimes I_x))$ also give an embedding of $\overline{C}$ into $\mathbb{P}^{r_k-1}$, since there is a natural injection $H^0(S, \mathcal{O}_S(k-2)) \otimes H^0(S, \mathcal{O}_S(2) \otimes I_x) \subset H^0(S, \mathcal{O}_S(k) \otimes I_x)$, and sections of $H^0(S, \mathcal{O}_S(k-2))$ have no base points and fixed components. Therefore, sections of $i^*(H^0(S, \mathcal{O}_S(k) \otimes I_x))$ separate points and tangent directions on $\overline{C}$. ■

We say that a *linear subspace* L_4 of dimension $r - 4$ *possesses a property* (L) *with respect to an embedding* $X \subset \mathbb{P}^r$, if the projection π_{L_4} to $\mathbb{P}^3$ from the centre L_4 maps X onto a surface $Y = \pi_{L_4}(X)$ with ordinary singularities.

As is known (see [G-H]), there is an open subset U in the Grassmannian $G_4 = Gr(r-4, r)$, points of which correspond to linear subspaces with the property (L).

Proposition 1.2 *If an embedding $X \subset \mathbb{P}^r$ possesses the property (L), then there exists a linear subspace L_4 with the property (L), which can be added to a flag $L_1 \supset L_3 \supset L_4$ such that the pair (L_1, L_3) possesses the property (L). In other words, there exists a projection to $\mathbb{P}^3$, for which the embedding $Y \subset \mathbb{P}^3$, where Y is the image of X, possesses the property (L).*

Proof. Let $G_1 = \check{\mathbb{P}}^r$ be the dual space to $\mathbb{P}^r$, $G_3 = G(r-3, r)$ be the Grassmann variety of linear subspaces L_3 of dimension $r - 3$, and $\mathbb{F} = \mathbb{F}_{1,3,4} \subset \check{\mathbb{P}}^r \times G_3 \times G_4$ be the variety of flags $L_1 \supset L_3 \supset L_4$. Let $X^\vee$ be the dual variety, $W \subset X \times X^\vee \subset \mathbb{P}^r \times \check{\mathbb{P}}^r$ be a closed subvariety $W = \{(x, L_1) : L_1 \supset T_{X,x}\}$. Then the projection of $W \to X^\vee$ is an isomorphism over $X_0^\vee = X^\vee \setminus Sing\ X^\vee$, $W_0 \simeq X_0^\vee$.

Denote by $Z \subset X \times \mathbb{F}$ a closed subvariety

$$Z = \{(x; L_1 \supset L_3 \supset L_4) \mid (x, L_1) \in W, L_3 \ni x\},$$

and by $Z_0 \subset Z$ an open subset: $(x, L_1) \in W_0$. Then Z is an irreducible variety. Consider a projection $Z_0 \to W_0$. The fibres are not empty by the previous proposition, and each of the fibres contains an open set of points z, for which the pair (L_1, L_3) possesses the property (L) (because the centres of projections for Lefschetz pencils form an open set). Therefore, Z contains an open set Z_L, for points of which the pair (L_1, L_3) possesses the Lefschetz property.

Obviously, the map $Z \to G_4$ is surjective. Therefore, $p_4^{-1}(U)$, where p_4 is a projection of Z to G_4, is a non empty Zariski open set. Then $Z_L \cap p_4^{-1}(U)$ is not empty, and if $(x; L_1 \supset L_3 \supset L_4)$ is a point of this set, then L_4 possesses the desired property. ■

1.7. *Projecting in a neighbourhood of a generic point of the double curve* Δ. Now let $Y \subset \mathbb{P}^3$ has ordinary singularities along the double curve Δ and isolated singularities $s_i \in Y \setminus \Delta, i = 1, \dots, k$, which are double planes. Under the incidence varieties, defined in the previous subsections, we mean the closures of the corresponding varieties, initially defined for an open surface $Y \setminus Sing\, Y$.

Consider $Y \times \mathbb{F}$. In addition to notations in 1.3, let q_1 and q_2 the projections of $Y \times G$ to Y and G. Consider the intersection $\tilde{A} = (\Delta \times G) \cap Z_3$. Then the restriction of the projection $Y \times G \to Y$ to $\tilde{A}$, $\tilde{A} \to \Delta$, is a covering of degree 4 over a generic point : at a point $y \in \Delta$ there are two asymptotic directions for each of two branches of Y at y. Therefore, $\tilde{A}$ is a curve. Set $A = (pr'_2)^{-1}(\tilde{A})$. It is a ruled surface. Set $\Sigma_\Delta = p_2(A)$. Then, if $\xi \notin \Sigma_\Delta$, we have that for a generic point $y \in \Delta$ the lines $l \ni \xi$ have at most simple contact with branches of Y.

Denote by Σ_0 the union of planes in $\mathbb{P}^3$ composing the tangent cones at the rest points of Δ, including triple points and pinches, and also at singular points $s_i \in Y$.

1.8. *Projecting in a neighbourhood of a triple point.* If $\xi \notin \Sigma_0$, then in a neighbourhood of a point $y \in \Sigma_t$ all lines $l \ni \xi$ are transversal to each of the three branches of Y at y, and therefore, locally these branches are mapped isomorphically onto $\mathbb{P}^2$.

1.9. *Projecting in a neighbourhood of a pinch.* In a neighbourhood of a pinch $y \in Y$ there are coordinates, by which Y is locally defined by an equation $u^2 = vw^2$. The double curve $\Delta \subset Y$ is a line $u = w = 0$, and the tangent cone $C_{Y,y}$ to Y at y has an equation $u = 0$. In a neighbourhood of a pinch a normalization $\mathbb{C}^2 \to Y$ is defined by formulae $u = tw, v = t^2, w = w$. Since X is non-singular and π_L is a finite map, we can assume that the projection π_L is the normalization. If a point ξ does not belong to the tangent cone $C_{Y,y}$, then the projection π_ξ locally is a map of gedrr 2. A projection $f : X \to \mathbb{P}^2$ a neighbourhood of the preimage of a pinch is a two-sheeted covering of non-singular varieties, and, hence, locally is defined by equations $v = t^2, w = w$. Thus, the curve $\bar{R} \subset Y$ goes through the pinch, and pinches are projected to non-singular points of the discriminant curve B.

1.10. *Normal forms of a generic projection at points of the ramification curve.*

Lemma 1.3 ([A]) *Let* $(X, 0) \subset (\mathbb{C}^3, 0)$ *be a non-singular surface, and* $(\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ *be a smooth morphism, the restriction of which* $f : X \to \mathbb{C}^2$ *is a finite covering of degree* μ. *Then one can choose local coordinates* x, y *in* $\mathbb{C}^2$ *and* x, y, z *in* $\mathbb{C}^3$ *such, that* X *is defined by an equation*

$$y = z^\mu + \lambda_1(x) z^{\mu-2} + \ldots + \lambda_{\mu-2}(x) z,$$

and f *is a projection along* z *axis.*

Proof. This is Lemma 1 in Arnol'd paper [A]. It is obtained, if we consider the covering

f as a 2-paremeter family of 0-dimensional hypersurface singularities of multiplicity μ, and, consequently, f is induced by the miniversal deformation of the singularity of type $A_{\mu-1}$. ■

We proved that at a generic point P of the ramification curve a projection $f : X \to \mathbb{P}^2$ is of degree $\mu = 2$, and at isolated points it is of degree $\mu = 3$. By Lemma 1.3 for $\mu = 2$ we obtain that at a generic point of the ramification curve a generic projection is equivalent to a projection of the surface $X : x = z^2$ to the x, y-plane, i.e. it is a fold. For $\mu = 3$ we obtain

Corollary 1.1 *For a non-singular surface X a finite covering $X \to \mathbb{C}^2$ of degree 3 locally is a projection to the x, y-plane of one of the surfaces*

$$y = z^3 + x^k z,\ k = 1, 2, \ldots,\ \textit{or}\ \ y = z^3\ (k = \infty).$$

In the case $k \neq \infty$ the ramification curve C is reduced and has an equation $3z^2 + x^k = 0$ in local coordinates x, z on X. The curve C is non-singular only for $k = 1$. The discriminant curve B has an equation $4x^{3k} + 27y^2 = 0$, i.e. B is a cusp. It is an ordinary cusp only for $k = 1$.

Proof. By lemma 1.3 we have that X is defined by an equation $y = z^3 + \lambda_1(x)z$. We obtain the stated normal form of the covering f, where k is the order of vanishing of $\lambda_1(x)$ at the point $x = 0$. The ramification curve C is defined by equation $J = 0$, where $J = 3z^2 + x^k$ is the Jacobian of the covering f. The discriminant curve B is defined by 0th Fitting ideal $F_0\,(f_*\mathcal{O}_C)$ of the sheaf $f_*\mathcal{O}_C$. To obtain an equation of B — the generator of the Fitting ideal, we need to take a finite presentation $f_*\mathcal{O}_X \xrightarrow{J} f_*\mathcal{O}_X \to f_*\mathcal{O}_C \to 0$ of the sheaf $f_*\mathcal{O}_C$, where $(f_*\mathcal{O}_X)_0 = \mathbb{C}\{x, z\} = \mathbb{C}\{x, y\} \cdot 1 \oplus \mathbb{C}\{x, y\}z \oplus \mathbb{C}\{x, y\}z^2$, and to compute a determinant of the $\mathbb{C}\{x, y\}$-linear map J, which is a multiplication by the Jacobian J. ■

Now we show that for a generic projection the discriminant curve B has at most ordinary nodes and cusps. Let $b \in B$ be a point corresponding to a bitagent l under projecting $\pi_\xi : \mathbb{P}^3 \backslash \xi \to \mathbb{P}^2$ from a point ξ. Let l touches Y at points P_1 and P_2, to which correspond branches B_1 and B_2 of the discriminant curve B at a point b. We have to show that for a generic projection the point b is a node, i.e. the branches B_1 and B_2 have different tangents. Determine where does the centres ξ of "bad" projections lie. Let a line $\lambda \subset \mathbb{P}^2$ is a common tangent to branches B_1 and B_2 at a point b. Then the plane $\pi_\xi^{-1}(\lambda)$ is bitangent – it touches the surface Y at points P_1 and P_2. Consider the dual surface $Y^\vee \subset \check{\mathbb{P}}^3$. Then the point $[\pi_\xi^{-1}(\lambda)] \in \mathit{Sing}\, Y^\vee = \gamma^\vee$. Set $\gamma = \tau^{-1}(\gamma^\vee)$, where $\tau : Y \to Y^\vee$ is the Gauss map. Let $\Sigma_u \subset \mathbb{P}^3$ be a ruled surface composed by lines P_1P_2, where $P_1, P_2 \in \gamma$, $\tau(P_1) = \tau(P_2) = [\pi_\xi^{-1}(\lambda)]$. Then, if $\xi \notin \Sigma_u$, then at points b, corresponding to bitangents l, the curve B has at most nodes.

Now let $b \in B$ be a point corresponding to a stationary tangent l at a point $P \in Y$. As was noted above, in a neighbourhood of P the projection π_ξ is equivalent to a projection of a surface $y = z^3 + x^k z$ to the x, y-plane. We have to show that for a generic projection the exponent $k = 1$. The fact is that, if $k > 1$, then the point P is a planar point of the surface Y. Excepting the centres of projection lying in tangent planes to Y at planar points, we obtain that in a neighbourhood of a point with $\mu = 3$ the projection f is equivalent to a projection of a surface $y = z^3 + xz$ to the x, y-plane, i.e. it is a pleat.

1.11. *Projecting in a neighbourhood of an isolated double plane singularity.*

Lemma 1.4 *If* $(X,0) \subset (\mathbb{C}^3,0)$ *is an (isolated) double plane singularity* $z^2 = h(x,y)$, $\pi : X \longrightarrow \mathbb{C}^2$ *be a projection from any point* $p \in \mathbb{C}^3$*, not lying in the tangent cone* $z=0$*, then the ramification curve of* π *is reduced, and the discriminant curve* $B \subset \mathbb{C}^2$ *is locally analytically isomorphic to the singularity* $h(x,y)=0$.

Proof. The singularity $(X,0)$ is of multiplicity 2. Therefore, π is a covering of degree 2, and, consequently, is locally a projection of a double plane $w^2 = g(u,v)$ to the (u,v)-plane. Thus, the germs of singularities $h(x,y)$ and $g(u,v)$ are stably isomorphic, and hence isomorphic ([AGV] , vol.1). ∎

∎

2 Local structure of fibre products.

2.1. *Local structure of a generic covering.* Let $f : X \to \mathbb{P}^2$ be a generic covering of the plane by a surface X with A-D-E-singularities, and let $B \subset \mathbb{P}^2$ be the discriminant curve, $f^*(B) = 2R + C$. Singular points $o \in Sing\, X$ will be called *s-points* of the surface X (from the word singularity). In a neighbourhood of a s-point o the covering f is isomorphic to the projection to x,y-plane of a surface $z^2 = h(x,y)$, where $h(x,y)$ has one of the A-D-E-singularities. Singular points o on X correspond to singular points of the same type as o on B. With respect to f non-singular points of X are partitioned into *r-points* (from the word regular), at which the morphism f is étale, and *p-points* (from the words singularity of projection) – they are points of the ramification curve R. A p-point is either a fold (or a singular p-point of type A_1), in a neighbourhood of which $f : (x,z) \longmapsto (x,y),\ y = z^2$, or a pleat $o \in R \cap C$ (or a singular p-point of type A_2), in a neighbourhood of which $f : (x,z) \longmapsto (x,y),\ y = z^3 - 3xz$ (more details about this see in section 2.4 below).

The singular points of B 'originated' from singular points $Sing\, X$ we call *s-points* . There are additional singular points of type A_1 (nodes) and of type A_2 (cusps), which we call *p-nodes* and *p-cusps*. Over a generic point $b \in B$ there lie one fold and $N-2$ r-points; over p-node there lie two folds and $N-4$ r-points; over a p-cusp there lie one pleat and $N-3$ r-points; over a s-node or a s-cusp, as over any s-point $b \in B$, there lie one singular point of X and $N-2$ r-points.

2.2. *Types of points on the fibre product.* With respect to a pair of generic coverings $f_1 : X_1 \to \mathbb{P}^2$ and $f_2 : X_2 \to \mathbb{P}^2$ with the same discriminant curve $B \subset \mathbb{P}^2$ nodes and cusps on B are partitioned by this time into 4 types: *ss-* , *sp-* , *ps-* and *pp-nodes* and *cusps*. For example, a ps-node it is a node $b \in B$, such that there are two folds on X_1 over b, and on X_2 over b there is a singular point of type A_1. The analogous terminology is used for the classification of points $\bar{x} = (x_1, x_2)$ on the fibre product $X^\times = X_1 \times_{\mathbb{P}^2} X_2$: we say about rs-points, ss-points, sp-points, etc. For example, we say that $\bar{x}$ is a ps-point of type A_2, if $x_1 \in X_1$ is a p-point of type A_2, and $x_2 \in Sing\, X_2$ is a singular s-point of type A_2.

In this section we describe the structure of a normalization $\nu : X = (X^\times)^{(\nu)} \to X^\times$ of the fibre product $X^\times$. Denote by g_1, g_2 and f the morphisms of X to X_1, X_2 and $\mathbb{P}^2$. Since the normalization is defined locally, we can replace $\mathbb{P}^2$ by a neighbourhood of the point $0 \in \mathbb{C}^2$ and to assume that X_1 and X_2 are neighbourhoods of points $x_1 \in X_1$ and $x_2 \in X_2$. We pass on to an item-by-item examination of all possible types of points $\bar{x} = (x_1, x_2) \in X^{\times}$. We do it up to the permutation of factors X_1 and X_2.

At first we consider quite trivial cases.

2.2.1. If $\bar{x}$ is a $r*$-point (where $* = r, s, p$), then $X^{\times}$ at the point $\bar{x}$ is locally the same as X_2 at the point x_2, and $f^{\times} : X^{\times} \to \mathbb{C}^2$ is locally the same as $f_2 : X_2 \to \mathbb{C}^2$.

2.2.2. If $\bar{x}$ is a 2×2-*point*, i.e. x_1 and x_2 are points of 'double planes', $z_1^2 = h(x,y)$, $z_2^2 = h(x,y)$, then $X^{\times} = X_1 \times_{\mathbb{C}^2} X_2$ in a neighbourhood of the point $\bar{x} = (x_1, x_2)$ is a surface in $\mathbb{C}^4 \ni (x, y, z_1, z_2)$, defined by equations $z_1^2 = h(x,y)$, $z_2^2 = h(x,y)$. We obtain that $z_1^2 = z_2^2$ and hence $X^{\times} = X_1^{\times} \cup X_2^{\times}$, where $X_1^{\times} : z_1^2 = h(x,y), z_2 = z_1$, $X_2^{\times} : z_2^2 = h(x,y)$, $z_1 = -z_2$. The surfaces $X_1^{\times}$ and $X_2^{\times}$ meet along a curve $z_1 = z_2 = 0$, $h(x,y) = 0$. We obtain that a normalization $X = \widetilde{X^{\times}} = X_1^{\times} \sqcup X_2^{\times}$ locally consists of two disjoint components $X_1^{\times}$ and $X_2^{\times}$ isomorphic to X_1 and X_2.

In particular, we obtain a description of the normalization in a neighbourhood of a pp-point $(x_1, x_2) \in X^{\times}$ lying over a non-singular point of B, $B : x = 0$,