PLURI-CANONICAL DIVISORS ON KÄHLER MANIFOLDS II

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INTRODUCTION

In our paper [L] we proved the following theorem:

Theorem. Let $p: X \to D$ be a smooth and proper map of a complex manifold X to a disk D, with connected fibers. Fix a positive interger m. Suppose that the fiber $X_0 = p^{-1}(0)$ is in the class strict C of Fujiki. Suppose further that the general element s of $H^0(X_0, \mathcal{O}_{X_0}(mK))$ has smooth divisor. Then the m-genus $P_m(p^{-1}(t))$ is constant over a neighborhood of 0 in D.

Here we will prove an extension of the above theorem to the casein which the general m-canonical divisor has "mild" singularities. More precisely, let s be a general element of $H^0(X_0, \mathcal{O}_{X_0}(mK))$, let Y be the m-fold covering of X_0, contained in the canonical line bundle, and branched over (s). Let $f: Y^* \to Y$ be a resolution of singularities of Y such that the exceptional locus E is a divisor with normal crossing. Let ω be the dualizing sheaf on Y. Then Y has "mild" singularities if

1) Y is smooth in codimension one (i.e. (s) is reduced)

2) $f^*(\omega)$ is contained in the sheaf of forms with log poles $\Omega^r_{Y^*}\langle E\rangle$, $r = \dim(Y)$

For example, if (s) is a reduced divisor with normal crossing, then Y has "mild" singularities.

We prove here the following theorem:

<u>Theorem</u>. Let $p : X \to D$ be a smooth and proper map of a complex manifold X to a disk D, with connected fibers. Fix a positive integer m. Suppose that X_0 is in $\overset{\text{script } C}{C}$, and suppose further that for a general element s of $H^0(X_0, \mathcal{O}_{X_0}(mK))$, the m-fold covering Y of X_0 branched along (s) has "mild" singularities. Then the m-genus $P_m(p^{-1}(t))$ is constant over a neighborhood of 0 in D.

Let $p:Z \to D$ be a smooth map of a complex manifold Z to a disk D, and let t be a parameter on D. We denote by Z_n the reduction of Z mod t^{n+1},

$$\mathcal{O}_{Z_n} = \mathcal{O}_Z/(t^{n+1}) \quad . \tag{+1}$$

We define the sheaf of relative C^∞ functions on Z_n, $C^{(0,0)}_{Z_n}$, by

$$C^{(0,0)}_{Z_n} = C^\infty_Z/(t^{n+1}, \bar{t}) \quad .$$

If U is a coordinate patch on Z, with coordinates (z,t), $z=(z_1,\ldots,z_d)$, and x is a point of U_0, then by Taylor's theorem, every element f of $C^{(0,0)}_{Z_n,x}$ can be written uniquely as

$$f = \sum_{i=0}^{n} f_i(z,\bar{z})t^i \quad ,$$

where the f_i are germs of C^∞ functions at x. We define the sheaf of relative C^∞ forms of type (p,q) on Z_n, $C^{(p,q)}_{Z_n}$, by

$$C^{(p,q)}_{Z_n} = C^{(p,q)}_{Z/D} \;/\; (t^{n+1},\bar{t})C^{(p,q)}_{Z/D} \quad .$$

If Y_n is a closed subscheme of Z_n, flat over $\mathrm{Spec}(\mathbb{C}[t]/t^{n+1})$, defined by a sheaf of ideals I, then we set

$$C^{(0,0)}_{Y_n} = C^{(0,0)}_{Z_n}/I+\bar{I}$$

$$C^{(1,0)}_{Y_n} = C^{(1,0)}_{Z_n}/(I+\bar{I})C^{(1,0)}_{Z_n} + dI$$

$$C^{(0,1)}_{Y_n} = C^{(0,1)}_{Z_n}/(I+\bar{I})C^{(0,1)}_{Z_n} + d\bar{I}$$

and

$$C^{(p,q)}_{Y_n} = \Lambda^p C^{(1,0)}_{Y_n} \otimes \Lambda^q C^{(0,1)}_{Y_n} \quad .$$

The operators ∂ and $\bar{\partial}$ on $\oplus C_Z^{(p,q)}$ descend to operators ∂ and $\bar{\partial}$ on $\oplus C_{Y_n}^{(p,q)}$, and at each smooth point of Y_0, the sheaf sequence

$$0 \to \Omega^p_{Y_n/D} \to C_{Y_n}^{(p,0)} \xrightarrow{\bar{\partial}} C_{Y_n}^{(p,1)} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} C_{Y_n}^{(p,r)} \to 0$$

($r=\dim(Y_0)$) is exact. In addition, the sheaves $C_{Y_n}^{(p,q)}$ are fine sheaves, hence, if Y_0 is smooth, we have

$$H^q(Y_n,\Omega^p_{Y_n/D}) = H^0(Y_n,C_{Y_n}^{(p,q)})^{\bar{\partial}\text{closed}}/\bar{\partial}\, H^0(Y_n,C_{Y_n}^{(p,q-1)}) \quad .$$

We also define the sheaf of C^∞ forms of degree m on Y_n, $C^m_{Y_n}$, to be the direct sum $\bigoplus\limits_{p+q=m} C_{Y_n}^{(p,q)}$, and we let d be the operator $\partial + \bar{\partial}$ on $\bigoplus\limits_m C^m_{Y_n}$.

In general, let U_0 be the smooth locus of Y_0, and let U_n be the respective open subscheme of Y_n. We define the respective sheaves for U_n to be the restriction to U_n of the above sheaves defined for Y_n. Let γ be a (singular) cycle on U_0, of real dimension r+q. We define a $\mathbb{C}[t]/(t^{n+1})$ linear map

$$\int_\gamma : H^q(U_n,\Omega^r_{U_n/D}) \to \mathbb{C}[t]/(t^{n+1})$$

as follows:

Since γ is compact, and contained in the smooth locus U_0, and since $H^1(U_0, \Theta_{U_0}\otimes C^\infty_{U_0}) = 0$, there is a neighborhood V of in U_0, and isomorphisms

$$a_m : C^m_{V_n} \to C^m_{V_0} \otimes \mathbb{C}[t]/(t^{n+1}) \qquad ,$$

commuting with d. Since $r=\dim(U_0)$, the operator $\bar{\partial} : C^{(r,q)}_{U_n} \to C^{(r,q+1)}_{U_n}$ is the restriction of the operator $d : C^m_{U_n} \to C^{m+1}_{U_n}$ where $m=r+q$. If w is an element of $H^0(U_n, C^{(r,q)}_{U_n})$, we can write $a_m(w|_V)$ as

$$a_m(w\,|\,V) = \sum_{i=0}^{n} w_i t^i \quad , \qquad w_i \text{ in } H^0(V_0, C^m_{V_0}) \quad .$$

If w and w' are two elements of $H^0(U_n, C^{(r,q)}_{Y_n})$ representing an element z of $H^q(U_n, \Omega^r_{U_n/D})$, then w and w' differ by $\bar{\partial}u = du$, for some u in $H^0(U_n, C^{(r,q-1)}_{U_n})$, hence

$$\begin{aligned} a_m(w-w') &= d(a_{m-1}(u)) \\ &= \sum_{i=0}^{n} du_i t^i \quad . \end{aligned}$$

Thus $\sum_{i=0}^{n} (\int_\gamma w_i) t^i$ is independent of the choice of w representing z, so we can define $\int_\gamma z$ as

$$\int_\gamma z = \sum_{i=0}^{n} (\int_\gamma w_i) t^i \qquad .$$

<u>Note</u>: We do not check that $\int_\gamma z$ is independent of the isomorphisms a_m.

Since the a_m commute with d, we have

<u>Lemma 1</u>. If μ is an element of $H^q(U_n, \Omega^{r-1}_{U_n/D})$ then $\int_\gamma d\mu = 0$ for every cycle γ of dimension q+r on U_0.

<u>Proof</u>. We can represent μ by a $\bar\partial$ closed element u* of $H^0(U_n, C^{(q,r-1)}_{U_n})$. Then $d\mu$ is represented by $\partial u^* = du^*$, whence the lemma.

q.e.d.

<u>Proposition 2</u>. Let Z_n, Y_n, U_n be as above. Suppose that Y_0 is a hypersurface in Z_n, smooth in codimension one, and in the

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class $\mathcal{C}$ of Fujiki(Y_0 is dominated by a compact Kähler manifold). Let $f: Y^* \to Y_0$ be a resolution of singularities of Y_0 such that the exceptional locus E of f is a divisor with normal crossing. Suppose further that $f^*(\omega_{Y_0})$ is contained in the sheaf of differentials with logarithmic poles, $\Omega^r_{Y^*}\langle E\rangle$, where ω_{Y_0} is the dualizing sheaf on Y_0. Then the map

$$d: H^1(Y_n, \Omega^{r-1}_{Y_n/D}) \to H^1(Y_n, \omega_{Y_n})$$

is the zero map.

<u>Proof</u>. We consider U_0 as an open subset of Y*, with complement E. The following diagram commutes:

$$\begin{tikzcd} & H^1(Y^*, \Omega^r_{Y^*}\langle E\rangle) \arrow[dr, "j^*"] & \\ H^1(Y_0, \omega_{Y_0}) \arrow[ur, "f^*"] \arrow[rr, "i^*"'] & & H^1(U_0, \Omega^r_{U_0}) \end{tikzcd},$$

where i and j are the relevant inclusions. Let μ be an element of $H^1(Y_n, \Omega^{r-1}_{Y_n/D})$. By induction, we may assume that $d\mu = t^n w$

~~for some element w of $H^1(Y_n, \omega_{Y_n})$~~

for some element w of $H^1(Y_0, \omega_{Y_0})$. Thus

$$0 = \int_\gamma d\mu \quad = t^n \int_\gamma w$$

for every r+1 cycle γ on U_0. On the other hand, by Deligne (Hodge II), the map $H^1(Y^*, \Omega^r_{Y^*}\langle E\rangle) \to H^{r+1}(U_0, \mathbb{C})$ is injective. If $f^*(w)$ is not zero, then by duality there is a cycle γ on U_0 with $\int_\gamma w \neq 0$, contradicting the above equation. Thus $f^*(w) = 0$. Finally, since the complement of U_0 in Y_0 is of codimension at least two, a local cohomology argument shows that the map i^* is injective, hence f^* is also injective, and $w = 0$ as desired.

q.e.d.

Let now $p: X \to D$ be a smooth , proper map of a complex manifold X to D, with connected fibers of dimension r, with X_0 in the class $\mathcal{C}$. Let Z_n be the relative canonical line bundle over X_n. Fix a positive integer m, and let s be a section in $H^0(X_n, \mathcal{O}_{X_n}(mK))$. We then let Y(s) denote the m-fold covering of X_n, contained in Z_n, branched over the divisor (s). Let $g: Z_n \to X_n$ denote the projection.

<u>Corollary 3</u>. Let X, Z_n be as above. Suppose that for a general s_0 in $H^0(X_0, \mathcal{O}_{X_0}(mK))$, the variety $Y(s_0)$ satifie. the conditions of lemma 2 (e.g. s_0 is a reduced divisor with normal crossing, or more generally, $Y(s_0)$ has only quotient singularities). Then the m-genus $P_m(p^{-1}(t))$ is constant over a neighborhood of 0 in D.

<u>Proof</u>. We let s_n denote a general section of $H^0(X_n, \mathcal{O}_{X_n}(mK))$, and $\bar{s}_n$ its reduction mod t. From [L] the obstruction to lifting $\bar{s}_n$ to an element s_{n+1} of $H^0(X_{n+1}, \mathcal{O}_{X_{n+1}}(mK))$ is given by

$$\mathrm{Tr}_{Y(s_n)/X_n}(w^{m-1}d\mu)$$

where w is the restriction to $Y(s_n)$ of the canonical section of $g^*(K)$ over K ($K = Z_n$), and μ is a certain element of $H^1(Y(s_n), \Omega^{r-1}_{Y(s_n)/D})$. By induction, we may suppose that $\bar{s}_n$ is a general section in $H^0(X_0, \mathcal{O}_{X_0}(mK))$, and so, by proposition2, $d\mu = 0$. Thus the obstruction vanishes, and we may continue the induction. The result then follows from the formal function theorem for coherent sheaves (Grauert)

q.e.d.

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