BIRATIONAL AUTOMORPHISMS OF NODAL QUARTIC THREEFOLDS

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ABSTRACT. It is well-known that a nonsingular minimal cubic surface is birationally rigid, a group of its birational selfmaps is generated by biregular selfmaps and birational involutions such that all relations between the latter are implied by standard relations between reflections on an elliptic curve. It is also known that a factorial nodal quartic threefold is birationally rigid and its group of birational selfmaps is generated by biregular ones and certain birational involutions. We prove that all relations between these involutions are implied by standard relations on elliptic curves, complete a proof of birational rigidity over a non-closed field and describe the situations when some of the birational involutions in question become regular (and, in particular, complete the proof of the initial theorem on birational rigidity, since some details were not established in the original paper of M. Mella).

1. INTRODUCTION

One of the popular problems of birational geometry is to find all Mori fibrations birational to a given Mori fibration $\mathcal{X} \to T$, and to compute the group of birational automorphisms $\mathrm{Bir}(\mathcal{X})$ of a variety $\mathcal{X}$. The cases when there are few structures of Mori fibrations on $\mathcal{X}$, for example, when there is only one — up to a natural equivalence — structure of Mori fibration, are of special interest; such varieties are called *birationally rigid* (see section 3 for a definition).

The first example of a birationally rigid variety is a minimal cubic surface. Recall that an Eckardt point on a cubic surface S defined over a field $\Bbbk$ is a point contained in three lines lying on $S_{\bar{\Bbbk}}$.

Theorem 1.1 (see [25, Chapter V, Theorems 1.5 and 1.6]). *Let S be a nonsingular minimal cubic surface over a perfect field $\Bbbk$. Then*

1. S is birationally rigid,

2. $\mathrm{Bir}(S)$ is generated by its subgroup $\mathrm{Aut}(S)$, birational involutions t_P centered in non-Eckardt points (Geiser involutions) and birational

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involutions t_{PQ} centered in pairs of conjugate points such that the corresponding line does not intersect any line contained in $S_{\overline{\Bbbk}}$ (Bertini involutions),

3. All relations between these generators are implied by the following ones:

$$t_P^2 = t_{PQ}^2 = \mathrm{id},$$
$$wt_Pw^{-1} = t_{w(P)} \textit{ for } w \in \mathrm{Aut}(S),$$
$$wt_{PQ}w^{-1} = t_{w(P)w(Q)} \textit{ for } w \in \mathrm{Aut}(S),$$
$$(t_{P_1} \circ t_{P_2} \circ t_{P_3})^2 = \mathrm{id} \textit{ for collinear points } P_1, P_2, P_3.$$

Fano threefolds of low degree give examples of birationally rigid varieties with relatively simple groups of birational selfmaps. Birational superrigidity (see section 3 for a definition) of a smooth quartic was proved in [20]; a proof of birational superrigidity of a smooth double cover of $\mathbb{P}^3$ branched over a sextic and birational rigidity of a smooth double cover of a quadric branched over a divisor of degree 4 (together with the calculation of its group of birational automorphisms) can be found in [19] and in [21].

The same questions may be posed (and sometimes solved) for varieties with mild singularities (for example, some nodal varieties, see [28], [8], [17] and [27]).

Theorem 1.2 (see [27, Theorem 2 or Theorem 7])**.** *Let X be a factorial nodal quartic threefold. Then*

1. X is birationally rigid,

2. $\mathrm{Bir}(X)$ is generated by its subgroup $\mathrm{Aut}(X)$, birational involutions τ_P centered in singular points $P \in \mathrm{Sing}\, X$, and birational involutions τ_L centered in lines[1] L containing one or two singular points of X.

Remark 1.3. Note that conditions of Theorem 1.2 are indeed necessary. If one allows more complicated singularities, the statement may fail to hold: for example, a general quartic hypersurface with a single singularity analytically isomorphic to a hypersurface singularity $xy+z^3+t^3=0$ is factorial but not birationally rigid (see [11]). On the other hand, if one releases the factoriality assumption, X may be even rational, like a general determinantal quartic (see [27]). In general factoriality is a global property that depends on the configuration of singular points on X, but there are sufficient conditions for X to be factorial depending only on the number of singular points (see [3, Theorems 1.2 and 1.3],

[1]Description will follow in section 5.

[30, Theorem 1.3]). For a treatment of geometry of non-factorial nodal quartics see [23] (and also [3] and [5]).

Recall that involutions $t_P \in \mathrm{Bir}(S)$ (resp., $t_{PQ} \in \mathrm{Bir}(S)$) are also defined for "bad" points (resp., pairs of points), i. e. Eckardt points P (resp., such pairs $\{P, Q\}$ that the corresponding line intersects some line contained in $S_{\bar{\Bbbk}}$), but such involutions are regular on S.

Motivated by the analogy with a cubic surface, we give the following definitions for a (nodal factorial) quartic threefold X defined over a field $\Bbbk$.

Definition 1.4 (cf., for example, [25, 8.8.3] and [6, Definition 2.3])**.** Let P be a singular point on X. We call P *an Eckardt point* if P is a vertex of some (two-dimensional) cone contained in $X_{\bar{\Bbbk}}$.

Definition 1.5. Let $L \subset X$ be a line. We call L *an Eckardt line* if there are infinitely many lines intersecting L on $X_{\bar{\Bbbk}}$.

We prove the following result that describes regularizations on a quartic threefold.

Proposition 1.6. *Let X be a factorial nodal quartic threefold. Then an involution τ_P is regular on X if and only if P is an Eckardt point, and an involution τ_L is regular on X if and only if L is an Eckardt line.*

Remark 1.7. Actually, Theorem 1.2 is not exactly what is proved in [27]. To derive Theorem 1.2 from the results of [27] one needs to prove that Eckardt points and Eckardt lines cannot be non-canonical centers on X (see Remark 7.12). Still it is not hard to do; it is done in Remark 7.12.

As in Theorem 1.1, one can observe that the involutions τ_P and τ_L may be not independent in $\mathrm{Bir}(X)$ because of relations arising from standard ones for reflections on elliptic curves (see Examples 5.7 and 5.9).

The main goal of this paper is to prove the following result, that may be considered a generalization of the third part of Theorem 1.1.

Theorem 1.8. *In the settings of Theorem 1.2 all relations between the generators of* $\mathrm{Bir}(X)$ *are implied by the following ones:*

$$\tau_P^2 = \tau_L^2 = \mathrm{id},$$
$$w\tau_P w^{-1} = \tau_{w(P)} \textit{ for } w \in \mathrm{Aut}(S),$$
$$w\tau_L w^{-1} = \tau_{w(L)} \textit{ for } w \in \mathrm{Aut}(S),$$
$$(\tau_{P_1}\tau_{P_2}\tau_{P_3})^2 = \mathrm{id} \textit{ for collinear points } P_1, P_2, P_3,$$
$$(\tau_{P_1} \circ \tau_{P_2} \circ \tau_L)^2 = \mathrm{id} \textit{ for } P_1, P_2 \in L.$$

Note that one of possible generalizations of a quartic threefold is a threefold Fano hypersurface of index 1 with terminal singularities in a weighted projective space. There are 95 families of such hypersurfaces. Their birational rigidity is known under some generality assumptions (see [12, Theorem 1.3]), as well as the fact that the groups of their birational automorphisms is generated by involutions centered in points and lines (also known as Geiser and Bertini involutions or quadratic and elliptic involutions, see [12, Remark 1.4]). The relations between these generators are also known and are analogous to those listed in Theorem 1.8 (see [9, Theorem 1.1]). Note that we establish the same results for a quartic without any generality assumptions.

The paper is organized as follows. In section 2 we recall some standard definitions and fix notations that we are going to use throughout the paper. In section 3 we recall standard definitions and constructions related to the method of maximal singularities. Section 4 contains some auxiliary results. Section 5 contains explicit description of involutions τ_P and τ_L and apparent relations between them, and section 6 gathers information about the action of these involutions. Section 7 contains a proof of Proposition 1.6 and a small improvement of the proof of Theorem 1.2 (see Remark 7.12). In section 8 we prove Proposition 8.2 that is a technical counterpart of Theorem 1.8; actually, the method that reduces Theorem 1.8 to Proposition 8.2 is standard (see [25, Chapter V, §7.8] or [21, 3.2.4], so we omit this step. Finally, section 9 contains an improvement of the proof of [27, Theorem 5] (which states that Theorem 1.2 holds over algebraically non-closed fields as well).

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2. Notation and conventions

All varieties throughout the paper are assumed to be defined over a field of complex numbers $\mathbb{C}$, except for section 9 where everything is defined over an arbitrary field $\Bbbk$ of characteristic $\mathrm{char}(\Bbbk) = 0$. On the other hand, all other results hold over $\Bbbk$ as well, with apparent changes in statements.[2]

Let Y be an n-dimensional variety. A singular point $y \in Y$ is called *an ordinary double point* (or *a node*) if its neighborhood is analytically isomorphic to a neighborhood of a vertex of a cone over a nonsingular quadric of dimension $n-1$. If Y is a hypersurface in $\mathbb{P}^{n+1}$ given by an equation $f=0$ in an affine neighborhood of y then this property is equivalent to non-degeneracy of the Hessian matrix $H(f)$ at y. A variety that has only nodes as singularities is called *nodal*.

A variety Y is called *factorial* if any Weil divisor on Y is a Cartier divisor, and $\mathbb{Q}$-*factorial* if an appropriate multiple of any Weil divisor is a Cartier divisor. Factorial varieties enjoy some properties typical for non-singular ones, for example, Lefschetz theorem (see Lemma 4.1). Note that for nodal varieties being factorial is equivalent to being $\mathbb{Q}$-factorial. In the sequel by "divisor" we usually mean "$\mathbb{Q}$-divisor".

We use the following standard notation throughout the paper. If D is a divisor and $\mathcal{D}$ is a linear system on Y, then $\operatorname{supp} D$ denotes the support of D, and $\operatorname{Bs}\mathcal{D}$ — the base locus of $\mathcal{D}$. If Z is a cycle, $\operatorname{mult}_Z D$ denotes the multiplicity of D at Z. In fact we'll use this notion only for the cases when Z is either an ordinary double point or a cycle not contained in the singular locus $\operatorname{Sing} Y$ of Y; under these assumptions $\operatorname{mult}_Z D$ may be defined using the equation

$$\pi^* D = \pi^{-1} D + (\operatorname{mult}_Z D) E,$$

where $\pi : \widetilde{Y} \to Y$ is a blow-up of Z and E is the (unique) exceptional divisor. The multiplicity $\operatorname{mult}_Z \mathcal{D}$ is defined as that of a general divisor $D \in \mathcal{D}$.

The symbol $\equiv$ denotes the numerical equivalence (of Cartier or $\mathbb{Q}$-Cartier divisors). If S is a surface, we write $\mathrm{NS}^1_{\mathbb{Q}}(S)$ for a $\mathbb{Q}$-vector space generated by the Cartier divisors on S modulo numerical equivalence; this space is endowed with a bilinear symmetric intersection form.

If $C \subset \mathbb{P}^2$ is a (nonsingular) cubic curve, *group law on C* means a standard group law on the elliptic curve with an inflection point of C (any of these) as a zero element. Given such a curve C and a point $P \in C$, *reflection with respect to P* means a reflection $R_P : C \to C$ with respect to the group law (i. e. a map $x \mapsto 2P - x$; recall that R_P depends only on the class of P modulo 2-torsion and does not depend on the choice of a zero element). Since a projection from P defines a

[2]These are easy but not completely automatic. For example, in Remark 5.5 the points P_1, P_2 and P_3 are not necessarily defined over $\Bbbk$ and one should assume only that they are contained in a line $L \subset X_{\overline{\Bbbk}}$, in Lemma 8.4 the line L is not necessarily defined over $\Bbbk$ etc.

double cover of $\mathbb{P}^1$, one can also associate to P *a Galois involution* τ_P, i. e. the natural involution of this double cover; note that $\tau_P = R_{-\frac{P}{2}}$.

If $Y_1, \ldots, Y_k$ are subsets of $\mathbb{P}^n$, we denote by $\langle Y_1, \ldots, Y_k \rangle$ the linear span of $Y_1 \cup \ldots \cup Y_k$.

We'll reserve the symbol X to denote a three-dimensional factorial nodal quartic hypersurface throughout the paper.

3. Preliminaries on the method of maximal singularities

We briefly recall the main constructions of the method of maximal singularities and introduce the necessary notation and terminology (see [29] or [10] for details). The basic notions and facts concerning Minimal Model Program and in particular necessary classes of singularities can be found in [26].

Let V be a (three-dimensional) $\mathbb{Q}$-factorial Fano variety with terminal singularities and Picard number $\rho(V) = 1$ (one may assume that V is a Mori fibration over an arbitrary base S as well, but we don't need this level of generality). The variety V is called *birationally rigid* if any birational map $\chi : V \dashrightarrow V'$ to a Mori fibration $V' \to S'$ is an isomorphism, and *birationally superrigid* if $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ (see [29] or [10] for the definitions in a general case).

Let $V' \to S'$ be a Mori fibration. Assume that there is a birational map $\chi : V \dashrightarrow V'$. There is an algorithm to obtain a decomposition of χ into elementary maps (links) of four types, known as Sarkisov program (see, for example, [10] or [26]). Choose a very ample divisor M' on V' and let $\mathcal{M} = \chi_*^{-1}|M'|$ (note that $\mathcal{M}$ is mobile, i. e. has no base components, but in general has base points and is not complete). Let μ be a (rational) number such that $\mathcal{M} \subset |-\mu K_V|$. The Nöether–Fano inequality (see [19], [10], [26] or [29]) implies that if χ is not an isomorphism, then the pair $(V, \frac{1}{\mu}\mathcal{M})$ is not canonical. One can show that there is an extremal contraction (in the sense of a usual Minimal Model Program) $g : \widetilde{V} \to V$, such that the discrepancy of the exceptional divisor of g with respect to the pair $(V, \frac{1}{\mu}\mathcal{M})$ is negative. Furthermore, there exists a link χ_1 of type II or III (a definition can be found, for example, in [10] or [26]) starting with this contraction and decreasing an appropriately defined "degree" of the map χ (i. e. the "degree" of $\chi \circ \chi_1^{-1}$ is less then that of χ). The only fact about this "degree" that we will use is the following: it decreases if the degree μ of the linear system $\mathcal{M}$ does (see [10] or [26] for details).

The previous statements imply the following: to prove that V cannot be transformed to another Mori fibration (i. e. is birationally rigid) it suffices to check that there are no non-canonical centers[3] on V except those that are associated with links that give rise to birational automorphisms of V, and to describe all birational selfmaps $\chi : V \dashrightarrow V$ it is sufficient to classify all non-canonical centers and to find an "untwisting" selfmap for each of them (i. e. a selfmap χ_Z such that the degree μ of $\mathcal{M}$ decreases after one applies χ_Z provided that Z was a non-canonical center).

4. Auxiliary statements

We'll refer to the following lemma as Lefschetz theorem, since it is a straightforward analog for factorial Fano varieties.

Lemma 4.1. *Let $Y \subset \mathbb{P}^n$, $n \geqslant 4$, be a factorial Fano hypersurface with log-terminal singularities. Then any (effective) Weil divisor $D \subset Y$ is cut out by a hypersurface $\widetilde{D} \subset \mathbb{P}^n$. In particular,* $\deg D$ *is divisible by* $\deg Y$.

Proof. A standard argument (see, for example, [14, Theorem 7.7]) shows that a natural map $H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is an isomorphism. On the other hand, since $H^1(Y, \mathcal{O}_Y) = H^2(Y, \mathcal{O}_Y) = 0$ by General Kodaira Vanishing (see [24, Theorem 2.17]), one has $\mathrm{Pic}\,(Y) = H^2(Y, \mathbb{Z})$. Since Y is factorial, any Weil divisor D is Cartier, and the statement follows. □

The following results will be used in section 8.

Theorem 4.2 (see [2, Theorem 1.7.20])**.** *Let V be a variety of dimension* $\dim V \geqslant 3$, *$x \in V$ — an ordinary double point, D — an effective divisor, such that the pair (V, D) is not canonical at x. Then* $\mathrm{mult}_x D > 1$.

Lemma 4.3 (cf. [7, Lemma 0.2.8])**.** *Let S — be a nonsingular surface, Δ — an effective divisor on S, such that*

$$\Delta \equiv \sum_{i=1}^{r} c_i C_i,$$

where $c_i > 0$, and the support of Δ does not contain any of the curves C_i. Assume that the intersection form on the subspace $W \subset \mathrm{NS}^1_{\mathbb{Q}}(S)$, generated by the curves C_i, is negative semidefinite. Then $\Delta^2 = 0$.

[3]To be more accurate, one should speak about non-canonical centers *with respect to* $\frac{1}{\mu}\mathcal{M}$. Still we'll avoid mentioning $\mathcal{M}$ since all arguments would go with a fixed linear system.

Proof. The argument is identical to that of Lemma 0.2.8 in [7]. Let $\Delta = \sum_{j=1}^{k} b_j B_j$, $b_i > 0$. Then

$$0 \geqslant (\sum_{i=1}^{r} c_i C_i)^2 = (\sum_{j=1}^{k} b_j B_j)(\sum_{i=1}^{r} c_i C_i) \geqslant 0,$$

that is

$$0 = (\sum_{i=1}^{r} c_i C_i)^2 = \Delta^2.$$

□

Lemma 4.4. *Let a nodal quartic $Y \subset \mathbb{P}^4$ contain a line L. Then the following conditions are equivalent:*

(i) there is a hyperplane H tangent to Y along L,

(ii) there are infinitely many planes Π such that $Y|_{\Pi} = 2L + Q$ for some (possibly reducible) conic Q,

(iii) L contains three singular points of Y.

Moreover, if one of these conditions holds, then $\mathrm{mult}_L H = 2$, *and any plane Π as in (ii) is contained in H.*

Proof. Easy. □

The following lemmas describe the singularities of general hyperplane sections of a threefold nodal hypersurface.

Lemma 4.5. *Let $Y \subset \mathbb{P}^4$ be a nodal hypersurface, P — a singular point of Y, $\Pi_0 \ni P$ — a two-dimensional plane. Assume that*

$$Y|_{\Pi_0} = \sum m_i C_i + \sum m'_j C'_j,$$

where $P \notin C'_j$, $P \in C_i$, the curves C_i are nonsingular at P, and m_i, m'_j are integers. Let $k = \sum m_i$. Take a general hyperplane section $H \subset Y$ passing through Π_0. Then the singularity $P \in H$ is Du Val of type $A_{k'}$ with $k' \leqslant k - 1$.

Proof. Choose an affine neighborhood U of P with coordinates x, y, z, t so that the hypersurface Y is given by an equation $F(x, y, z, t) = 0$, where

$$F(x, y, z, t) = xz + yt + F_{\geqslant 3}(x, y, z, t),$$

and $\mathrm{ord}_0 F_{\geqslant 3} \geqslant 3$. If the restriction of the polynomial $xz + yt$ to Π_0 is not identically zero, then H has an ordinary double point (that is a Du Val singularity of type A_1) at P. Hence we may assume that Π_0 is given by the equation $z = t = 0$, and H is cut out by a hyperplane $t = \alpha z$. Then H is given by the equation

$$(x + \alpha y)z + \widetilde{F}_{\geqslant 3}(x + \alpha y, y, z) = 0,$$

where $\mathrm{ord}_0\widetilde{F}_{\geqslant 3} \geqslant 3$, and hence H has a Du Val singularity of type $A_{k'}$ at P (see, for example, [1, Chapter II, 11.1]).

Assume that $k' \geqslant 2$. The projectivization of the plane Π_0 gives a line l contained in a nonsingular quadric $Q = (xz + yt = 0) \subset \mathbb{P}(V) \simeq \mathbb{P}^3$, and the projectivization of the hyperplane $t = \alpha z$ gives a plane in $\mathbb{P}(V)$, intersecting Q by a pair of lines $l \cup l'$. Let $\overline{f} : \overline{Y} \to Y$ be a blow-up of the point P with an exceptional divisor E, and $\overline{H} = \overline{f}^{-1}H$. Let $\overline{f}_H$ be a restriction of $\overline{f}$ to $\overline{H}$. Then $\overline{f}$ is a blow-up of H at the point P, and the exceptional locus of $\overline{f}_H$ is identified with $E \cap \overline{H} = l \cup l'$. The surface $\overline{H}$ has a Du Val singularity of type $A_{k'-2}$ at the point $P' = l \cap l'$ and is nonsingular at the points $(l \cup l') \setminus \{P'\}$. The proper transforms $\overline{f}_H^{-1}C_i$ of the curves C_i intersect the line l and do not pass through P'.

Consider a resolution of singularities $f : H' \to H$ that is obtained from $\overline{H}$ by a sequence of blow-ups. Let $l_1, \ldots, l_{k'}$ be exceptional curves of the resolution f that are contracted to P, labelled so that $l_i l_{i+1} = 1$ for $1 \leqslant i \leqslant k' - 1$. According to the above observation, all proper transforms $f^{-1}C_i$ intersect one and the same exceptional curve, which corresponds to one of the ends of the chain of exceptional curves (say, $l_{k'}$).

Let us compute the multiplicities of the exceptional curves l_t in the pull-back of the curve C_i. Let

$$f^*C_i = f^{-1}C_i + \sum_{t=1}^{k'} a_{i,t} l_t.$$

¿From the system of equations

$$0 = l_t f^* C_i = \begin{cases} a_{i,2} - 2a_{i,1} \text{ for } t = 1, \\ a_{i,t+1} - 2a_{i,t} + a_{i,t-1} \text{ for } 1 < t < k', \\ 1 - 2a_{i,k'} + a_{i,k'-1} \text{ for } t = k'; \end{cases}$$

we obtain

$$a_{i,t} = \frac{t}{k'+1}.$$

In particular, for all C_i we have

$$a_{i,1} = \frac{1}{k'+1}.$$

Since $D = \sum m_i C_i + \sum m'_j C'_j$ is a Cartier divisor and hence the divisor f^*D is integral, one has $\frac{k}{k'+1} \in \mathbb{Z}$ and hence $k \geqslant k' + 1$. □

Lemma 4.6. *Let $Y \subset \mathbb{P}^4$ be a nodal hypersurface of degree $\deg Y = d$, $L \subset Y$ — a line, containing exactly n singular points of Y, $n \geqslant 0$. Let Π_0 be a two-dimensional plane such that $Y|_{\Pi_0} = kL + C$, where*

$C \geqslant 0$ and $L \not\subset \operatorname{supp} C$. Assume that $k \geqslant 2$. Take a general hyperplane section $H \subset Y$ passing through Π_0. Let

$$\mathcal{P} = (L \cap \operatorname{Sing} H) \setminus (L \cap \operatorname{Sing} Y).$$

Then

(1) *H has isolated singularities, and for any point $P_0 \in L \setminus \operatorname{Sing} Y$ one can chose H so that H is nonsingular at P_0;*
(2) *$\mathcal{P}$ contains at most $d - n - 1$ points;*
(3) *any point $P \in \mathcal{P}$ is a Du Val singularity of type A_{k-1} on H.*

Proof. The first assertion is obvious: it suffices to choose H so that the three-dimensional subspace $\langle H \rangle \simeq \mathbb{P}^3$ does not coincide with a tangent subspace $T_{P_0}Y \simeq \mathbb{P}^3$ at $P_0 \in L \setminus \operatorname{Sing} Y$.

Now choose homogeneous coordinates $x_0, \ldots, x_4$ in $\mathbb{P}^4$ such that the subspace $\langle H \rangle$ is given by equation $x_4 = 0$, the plane Π_0 — by equations $x_3 = x_4 = 0$, and the line L — by equations $x_2 = x_3 = x_4 = 0$. Then Y is given by an equation of the form

$$x_2^k F(x_0, x_1, x_2) + x_3 G_3(x_0, \ldots, x_3) + x_4 G_4(x_0, \ldots, x_4) = 0,$$

where $\deg F = d - k$, $\deg G_3 = \deg G_4 = d - 1$. The equation of the surface H in $\langle H \rangle \simeq \mathbb{P}^3$ with homogeneous coordinates $x_0, \ldots, x_3$ is

$$x_2^k F(x_0, x_1, x_2) + x_3 G_3(x_0, \ldots, x_3) = 0. \tag{4.7}$$

Note that partial derivatives of the left hand side of 4.7 with respect to x_0, x_1 and x_2 vanish on the line L, hence the set $L \cap \operatorname{Sing} H$ is just a zero locus of the restriction of the polynomial G_3 to L. Moreover, G_3 does not vanish identically on L since otherwise H would be singular along L. This implies the second assertion of the Lemma.

To prove the third assertion consider a point $P \in \mathcal{P}$. We may assume that $P = (1 : 0 : 0 : 0 : 0)$. By the first assertion of the Lemma for any point $P' \in L \setminus \operatorname{Sing} Y$ there is a hyperplane section nonsingular at P'; since H is general, we may assume that the surface H is nonsingular at all the points $P' \in (L \cap C) \setminus \operatorname{Sing} Y$, i. e. P is not contained in $L \cap C$ and hence F is not of the form $F = x_1 F_1 + x_2 F_2$. Since G_3 does not vanish identically on L, it is not of the form $G_3 = x_2 G_{32} + x_3 G_{33}$. Choose an affine neighborhood U of P; let x, y, z be coordinates in U corresponding to (homogeneous) coordinates x_1, x_2, x_3. The surface H in the neighborhood of P is given by

$$y^k(1 + \widetilde{F}(x, y)) + z(c_x x + c_y y + c_z z + \widetilde{G}_3(x, y, z)), \tag{4.8}$$

where $\operatorname{ord}_0 \widetilde{F} \geqslant 1$, $\operatorname{ord}_0 \widetilde{G}_3 \geqslant 2$, c_x, c_y and c_z are constants such that $c_x \neq 0$. It is easy to see that the equation 4.8 defines a Du Val singularity of type A_{k-1}. $\square$

5. Generators and relations

¿From now on we denote by X a nodal factorial quartic threefold. In this section we recall constructions of birational involutions that (together with $\mathrm{Aut}(X)$) generate $\mathrm{Bir}(X)$, and list apparent relations between them. Note that the generators of $\mathrm{Bir}(X)$ are constructed in the same way as in a very standard way (see, for example, [25, Introduction and Chapter V, 1.4], [21, 3.1.2 and 3.1.4], [21, 5.1.2 and 5.1.3], [12, 2.6], [30, Example 4.4] etc).

Example 5.1. Let P be a singular point of X. Projection from P defines a (rational) double cover $\phi : X \dashrightarrow \mathbb{P}^3$; the Galois involution of ϕ gives rise to a birational involution τ_P of X.

Example 5.2. Let P be a singular point of X, and $L \subset X$ — a line containing P and no other singular points of X. Projection from L defines an elliptic fibration $\psi : X \dashrightarrow \mathbb{P}^2$, and a fiberwise reflection[4] in a section of ϕ arising from the point P gives rise to a birational involution τ_L of X.

Example 5.3. Let P_1 and P_2 be singular points of X, and $L \subset X$ — a line passing through P_1 and P_2 but no other singular points of X. As in Example 5.2, define an elliptic fibration ψ, denote by E_1 and E_2 the sections of its regularization corresponding to the points P_2 and P_2, and take a reflection (with respect to the group law on a general fiber) in the section[5] $\frac{E_1+E_2}{2}$; one can also define this involution as a fiberwise Galois involution with respect to the section $-(E_1 + E_2)$, i. e. the section arising from L. We'll also denote the corresponding birational involution by τ_L.

Remark 5.4. Note that the involution φ_2^L defined in [27] in the settings of Example 5.3 is different from the involution τ_L defined in Example 5.3 (in [27] it corresponds to a reflection in E_1). It does not matter if one is interested only in the structure of the group $\mathrm{Bir}(X)$ since $\tau_L = \tau_{P_1} \circ \tau_{P_2} \circ \varphi_2^L$, but our definition is a little bit more natural from the point of view of Sarkisov program, since it is exactly the untwisting involution for L in this case (see Lemma 6.3).

[4]To be more precise one should define the reflection on a general fiber of (a regularization of) ψ and then extend it to an involution of the whole variety.

[5] Actually, since an elliptic curve contains 2-torsion points, $\frac{E_1+E_2}{2}$ is not correctly defined as a section of the elliptic fibration, but the corresponding fiberwise reflection is correctly defined since it does not depend on 2-torsion, so from here on we'll afford such abuse of notation.

Remark 5.5. A quartic with isolated singularities cannot have more than three collinear singular points. The situation of three singular points P_1, P_2 and P_3 contained in some line $L \subset X$ is possible, but such lines do not contribute to $\mathrm{Bir}(X)$ since they cannot be non-canonical centers (see [27] or use Lemma 4.4). Moreover, if one defines an involution τ_L in this situation as in Example 5.3 with respect to the points P_1 and P_2, it will coincide with the involution τ_{P_3}.

Remark 5.6. Note that an involution τ_P also acts as a fiberwise reflection on any elliptic fibration associated with a line $L \subset X$ containing P (one should reflect in the section $-\frac{E_P}{2}$, where E_P is a section corresponding to P).

One of the main results of [27] states (see Theorem 1.2) that the involutions listed in Examples 5.1, 5.2 and 5.3 together with $\mathrm{Aut}(X)$ generate the group $\mathrm{Bir}(X)$. On the other hand, it is easy to see that there may appear relations between these generators.

Example 5.7. Let $P_1, P_2, P_3 \in \operatorname{Sing} X$ be collinear. Then the line $L = \langle P_1, P_2, P_3 \rangle$ is contained in X, and all the involutions τ_{P_i} act fiberwise on the corresponding elliptic fibration. Hence one has

$$(\tau_{P_1} \circ \tau_{P_2} \circ \tau_{P_3})^2 = \mathrm{id} \tag{5.8}$$

by the well-known relation between three reflections on an elliptic curve (see, for example, [25, Chapter I, 2.3]).

Example 5.9. Let $P_1, P_2 \in \operatorname{Sing} X$; let $L \subset X$ be a line containing P_1 and P_2 but no other singular points of X. Then all three involutions τ_L, τ_{P_1} and τ_{P_2} act fiberwise on the elliptic fibration associated to L, so that

$$(\tau_{P_1} \circ \tau_{P_2} \circ \tau_L)^2 = \mathrm{id}. \tag{5.10}$$

Remark 5.11. Note that there are other relations that differ from 5.8 and 5.10 by a permutation of indices, but they are equivalent to 5.8 and 5.10 (modulo trivial relations $\tau_{P_i}^2 = \tau_L^2 = \mathrm{id}$).

One of the main goals of this paper is to show that relations 5.8 and 5.10 imply all relations in $\mathrm{Bir}(X)$ (up to trivial ones, see Theorem 1.8). This will be proved in section 8.

6. Action of birational involutions

In this section we gather information about the action of birational involutions τ_P and τ_L, i. e. describe the way the degrees and multiplicities change under the action of these involutions.

We fix the following notations. Let $\chi : X \dashrightarrow X$ be a birational map, and $\mathcal{M} = \mathcal{M}(\chi)$ be a linear system of degree $\mu(\chi)$ defined as in section 3. For a subvariety $Z \subset X$ we put $\nu_Z(\chi) = \mathrm{mult}_Z \mathcal{M}(\chi)$.

Remark 6.1. Assume that a line $L \subset X$ is not an Eckardt line, contains a singular point P and at most one more singular point of X. Then there is only a finite number of conics and lines in the fibers of a projection ψ from L: if a fiber is reducible, then it either contains lines intersecting L and different from L (by assumption there is only a finite number of fibers of this type), or contains L, i. e. the corresponding plane section has multiplicity at least 2 along L, which is possible for an infinite number of plane sections only if L contains three singular points of X by Lemma 4.4. Moreover, only a finite number of irreducible residual cubic curves in plane sections passing through L has a singular point at P, and in the case of two singular points of X lying on L none of these irreducible cubic curves has a singular point on L outside the singular points on X. Hence birational involutions $\widetilde{\tau_P}$ and $\widetilde{\tau_L}$ (corresponding to $\tau_P, \tau_L \in \mathrm{Bir}(X)$) of a variety $\widetilde{X}$, obtained as a blow-up of X at singular points lying on L and then a strict transform of L, are regular up to codimension 2 since both a reflection and a Galois involution are well defined in a smooth point of an irreducible plane cubic.

Lemma 6.2. *Let a line $L \subset X$ contain the only singular point P of X. Assume that L is not an Eckardt line. Then*

$$\begin{aligned} \mu(\chi \circ \tau_L) &= 11\mu(\chi) - 10\nu_L(\chi), \\ \nu_L(\chi \circ \tau_L) &= 12\mu(\chi) - 11\nu_L(\chi), \\ \nu_P(\chi \circ \tau_L) &= 6\mu(\chi) - 6\nu_L(\chi) + \nu_P(\chi). \end{aligned}$$

Proof. The proof is reduced to the calculation of the action of a birational involution $\widetilde{\tau_L}$ corresponding to τ_L on a Picard group of a variety $\widetilde{X}$ obtained as a blow-up of P and then a strict transform of L. Note that $\widetilde{\tau_L}$ is regular on X up to codimension 2 by Remark 6.1. The rest of the calculation coincides with that of [21, Lemma 5.1.3]. $\square$

Lemma 6.3. *Let a line $L \subset X$ contain exactly two singular points of X, say, P_1 and P_2. Assume that L is not an Eckardt line. Then*

$$\begin{aligned} \mu(\chi \circ \tau_L) &= 5\mu(\chi) - 4\nu_L(\chi), \\ \nu_L(\chi \circ \tau_L) &= 6\mu(\chi) - 5\nu_L(\chi), \\ \nu_{P_1}(\chi \circ \tau_L) &= 3\mu(\chi) - 3\nu_L(\chi) + \nu_{P_2}(\chi), \\ \nu_{P_2}(\chi \circ \tau_L) &= 3\mu(\chi) - 3\nu_L(\chi) + \nu_{P_1}(\chi). \end{aligned}$$

Proof. Analogous to that of Lemma 6.2. □

Lemma 6.4. *Let a line $L \subset X$ contain the only singular point P of X. Assume that L is not an Eckardt line. Then*

$$\mu(\chi \circ \tau_P) = 3\mu(\chi) - 2\nu_P(\chi),$$
$$\nu_P(\chi \circ \tau_P) = 4\mu(\chi) - 3\nu_P(\chi),$$
$$\nu_L(\chi \circ \tau_P) = \mu(\chi) - \nu_P(\chi) + \nu_L(\chi).$$

Proof. Note that τ_P preserves an elliptic fibration associated with L. The rest is analogous to Lemma 6.2. □

Lemma 6.5. *Let a line $L \subset X$ contain exactly two singular points of X, say, P and P_1. Assume that L is not an Eckardt line. Then*

$$\mu(\chi \circ \tau_P) = 3\mu(\chi) - 2\nu_P(\chi),$$
$$\nu_P(\chi \circ \tau_P) = 4\mu(\chi) - 3\nu_P(\chi),$$
$$\nu_{P_1}(\chi \circ \tau_P) = \mu(\chi) - \nu_P(\chi) + \nu_L(\chi),$$
$$\nu_L(\chi \circ \tau_P) = \mu(\chi) - \nu_P(\chi) + \nu_{P_1}(\chi).$$

Proof. Analogous to that of Lemma 6.4. □

Lemma 6.6. *Let a line $L \subset X$ contain three singular points of X, say, P, P_1 and P_2. Assume that P, P_1 and P_2 are not Eckardt points. Then*

$$\mu(\chi \circ \tau_P) = 3\mu(\chi) - 2\nu_P(\chi),$$
$$\nu_P(\chi \circ \tau_P) = 4\mu(\chi) - 3\nu_P(\chi),$$
$$\nu_{P_1}(\chi \circ \tau_P) = \mu(\chi) - \nu_P(\chi) + \nu_{P_2}(\chi),$$
$$\nu_{P_2}(\chi \circ \tau_P) = \mu(\chi) - \nu_P(\chi) + \nu_{P_1}(\chi),$$
$$\nu_L(\chi \circ \tau_P) = 2\mu(\chi) - 2\nu_P(\chi) + \nu_L(\chi).$$

Proof. Analogous to that of Lemma 6.4. We give a sketch to mention some minor differences.

Let $\widetilde{X}$ be a variety obtained as a blow-up of X in P, P_1, P_2 and then a strict transform of L, and $\widetilde{\tau_P}$ — a corresponding (birational) involution of $\widetilde{X}$ (note that $\widetilde{\tau_P}$ is regular up to codimension 2 by 6.1). Let h denote the class of a pull-back of a hyperplane section of X in $\mathrm{Pic}\,(\widetilde{X})$, and let e, e_1, e_2 and e_L denote the classes of (the preimages of) exceptional divisors. Note that $\widetilde{X}$ has a structure of an elliptic fibration $\psi : \widetilde{X} \to \mathbb{P}^2$. Let C be a general fiber of ψ, and S — a preimage of a general line in $\mathbb{P}^2$. Then a kernel K of a restriction map $\mathrm{Pic}\,(\widetilde{X}) \to \mathrm{Pic}\,(C)$ is generated by $h - e - e_1 - e_2 - e_L$ and e_L: indeed, K is generated by a preimage of a general line in $\mathbb{P}^2$ (that is

$h-e-e_1-e_2-e_L$) and divisors swept by the components of reducible fibers; one of the latter is e_L, and another is swept by conics and is equivalent to $h-2e-e_1-e_2-e_L$ since a general conic is contained in a hyperplane section $H \subset X$ tangent to X along L and $\mathrm{mult}_L H = 2$ by Lemma 4.4. The remaining computations are analogous to those of [21, Lemma 5.1.3]. Restricting to C, one gets

$$\begin{aligned} \widetilde{\tau_P}^* h &= 3(e_1+e_2) - h + m_1(h-e-e_1-e_2-e_L) + m_2 e_L, \\ \widetilde{\tau_P}^* e &= e_1+e_2-e+n_1(h-e-e_1-e_2-e_L)+n_2 e_L, \\ \widetilde{\tau_P}^* e_L &= e_L + k_1(h-e-e_1-e_2-e_L) + k_2 e_L, \\ \widetilde{\tau_P}^* e_1 &= e_2 + l_1(h-e-e_1-e_2-e_L) + l_2 e_L, \\ \widetilde{\tau_P}^* e_2 &= e_1 + l_1(h-e-e_1-e_2-e_L) + l_2 e_L. \end{aligned}$$

Computing intersection numbers on S, one obtains that $l_1 = l_2 = 0$, $n_2 = 0$, $n_1 = 2$, $k_1 = k_2 = 0$, $m_1 = 4$, $m_2 = 2$, and the statement follows. □

7. Regularization

In this section we describe the cases when the birational involutions of X become regular. These effects are analogous to regularization of birational involutions of minimal cubic surfaces arising from Eckardt points.

The following example shows that birational involutions of both types may regularize on X.

Example 7.1 (cf. [12, 7.4.2])**.** Let $X \subset \mathbb{P}^4$ be given by equation

$$w^2 q_2(x,y,z,t) + q_4(x,y,z,t) = 0, \tag{7.2}$$

where $(x:y:z:t:w)$ are homogeneous coordinates in $\mathbb{P}^4$ and q_i is a form of degree i. Let $P = (0:0:0:0:1)$; note that P is a singular point on X, and X contains a cone $q_2 = q_4 = 0$ with a vertex at P.

Let $L \subset X$ be a line passing through P such that L contains no singular points of X except P. It is easy to see that the involution τ_P is regular and acts as

$$\iota : (x:y:z:t:w) \mapsto (x:y:z:t:-w).$$

Moreover, let Π be a general plane passing through L, so that $X|_\Pi = L \cup C$; then C is a nonsingular plane cubic, and $P \in C$ is an inflection point, so the involutions τ_L and τ_P coincide on C (and hence on X), and so τ_L is also regular on X.

If q_4 is general enough, P is a node and, moreover, the only singular point on X. The latter implies that X is factorial by [3, Theorem 1.2] (in particular, X is birationally superrigid by Theorem 1.2 and the previous argument).

If X is given by equation

$$w^2(xy+zt)-(x^3y+y^4+z^4+t^4),$$

then X is singular exactly in three collinear (ordinary double) points: $P'=(1:0:0:0:1)$, $P''=(-1:0:0:0:1)$ and P. In particular, X is factorial by [3, Theorem 1.2] (and hence birationally rigid by Theorem 1.2).

Example 7.3. Let $L\subset X$ be a line such that there are infinitely many lines contained in X that intersect L in smooth points of X. Then the involution τ_L is regular (provided that it is defined, i. e. L contains one or two singular points of X). Indeed, assume that τ_L is not regular on X. Then there is a mobile linear system $\mathcal{M}\subset|-\mu K_X|$ such that L is a non-canonical center with respect to $\frac{1}{\mu}\mathcal{M}$ (one can take $\mathcal{M}=(\tau_L)_*^{-1}|\mathcal{O}(1)|$), i. e. $\mathrm{mult}_L\mathcal{M}>\mu$. In particular, $\mathrm{mult}_P\mathcal{M}>\mu$, and hence all lines passing through P are contained in $\mathrm{Bs}\,\mathcal{M}$, a contradiction.

Next example shows that there are factorial nodal quartics containing lines of the type described in Example 7.3.

Example 7.4. Let $X\subset\mathbb{P}^4$ be given by equation

$$w^3x+wx(xy+zt)+(x^4+y^4+z^4+tz^3)=0.$$

Then $P=(0:0:0:0:1)$ is a vertex of a two-dimensional cone contained in X, and a node in $P'=(0:0:0:1:0)$ is the only singular point of X (in particular, X is factorial by [3, Theorem 1.2]). The line $L=\langle P,P'\rangle$ is contained in X and fits into the settings of Example 7.3.

Remark 7.5. If $P\in\mathrm{Sing}\,X$ is a point such that there are infinitely many lines contained in X and passing through P, one could also argue as in Example 7.3 using Theorem 4.2 to show that P cannot be a non-canonical center and hence τ_P is regular.

We'll see below that Examples 7.1 and 7.3 describe (at least to some extent) the general situation.

Lemma 7.6. *Let X have a singular Eckardt point (say, P). Then X is given by equation of type 7.2; moreover, any line $L\subset X$ passing through P contains either one or three singular points of X.*

Proof. Let $(x : y : z : t : w)$ be homogeneous coordinates in $\mathbb{P}^4$ such that $P = (0 : 0 : 0 : 0 : 1)$. Then X is given by equation

$$(7.7) \qquad w^2 q_2(x, y, z, t) + w q_3(x, y, z, t) + q_4(x, y, z, t) = 0,$$

where q_i is a form of degree i.

Assume that q_3 is not divisible by q_2. The equation $q_2 = 0$ defines a nonsingular quadric surface in $\mathbb{P} = (w = 0) \simeq \mathbb{P}^3$. By assumption the curves cut out on this quadric by the equations $q_3 = 0$ and $q_4 = 0$ have a common (irreducible) component F (so that K is a cone over F). By Lefschetz theorem $\deg K$ must be divisible by 4; since $\deg K = \deg F \leqslant 6$, the only possible case is $\deg F = 4$, i. e. F is an irreducible curve of type $(2, 2)$. In the latter case K is cut out on X by a hyperplane (again by Lefschetz theorem), and hence $F \subset \mathbb{P}$ is contained in a plane, a contradiction.

So $q_3 = q_2 \cdot l$ for some linear form l, and replacing w by $w + \frac{l}{2}$ we may assume that $q_3 = 0$ and K is given by equations $q_2 = q_4 = 0$.

Now assume that a line $L \subset X$ passing through P contains a point $P' \in \operatorname{Sing} X$ different from P. Let $P' = (x' : y' : z' : t' : w')$. If $w' = 0$, then we may assume that $P' = (1 : 0 : 0 : 0 : 0)$, so that y, z, t and w are local coordinates in an affine neighborhood of P'. Note that all second partial derivatives of the left hand side of 7.2 with respect to w and some other coordinate of y, z, t, w vanish at P' (since so does q_2), so P' cannot be an ordinary double point of X. Hence $w' \neq 0$, and the point

$$P'' = \tau_P(P') = (x' : y' : z' : t' : -w')$$

is different from P and P', lies on L and is singular on X. $\square$

Now we'll analyze the cases when the involutions τ_P and τ_L are regular.

Lemma 7.8. *Let $L \subset X$ be a line passing through one or two singular points of X. Assume that L is not an Eckardt line. Then the involution τ_L is not regular.*

Proof. It follows from Lemma 6.2 in the case of one singular point on L and from Lemma 6.3 in the case of two singular points. $\square$

Lemma 7.9. *Let $P \in \operatorname{Sing} X$. Then the involution τ_P is regular if and only if P is an Eckardt point on X.*

Proof. If P is an Eckardt point, τ_P is regular by Lemma 7.6 and Example 7.1. Now assume that P is not an Eckardt point. Then a general line $L \subset \mathbb{P}^4$ such that $\operatorname{mult}_P(X|_L) \geqslant 3$ is not contained in X and $\operatorname{mult}_P(X|_L) = 3$. So there is a single intersection point $P_L \in X \cap L$

different from P, and hence τ_P is not regular in P (equivalently, one can see that the divisor D swept by such points P_L maps to P under the involution τ_P). $\square$

Remark 7.10. If P is a point such that there is a non-Eckardt line $L \subset X$ passing through P, then one can use Lemmas 6.4, 6.5 and 6.6 to derive that τ_P is non-regular. Still the direct proof of Lemma 7.9 seems more convenient to avoid looking for such line passing through P.

Combining the previous results we get the following.

Corollary 7.11. *An involution τ_L is regular if and only if L is an Eckardt line.*

Proof. If L is an Eckardt line, then either L contains a singular Eckardt point, or there are infinitely many lines contained in X that intersect L in smooth points of X. In the former case τ_L is regular by Remark 7.5 or by Lemma 7.6 and Example 7.1. In the latter case τ_L is regular by Example 7.3.

If L is not an Eckardt line, then τ_L is not regular by Lemma 7.8. $\square$

Corollary 7.11 and Lemma 7.9 prove Proposition 1.6.

Remark 7.12. In [27] it was proved that a non-canonical center on X is either a singular point or a line containing one or two singular points. As we have seen in this section, the involutions τ_P and τ_L are untwisting involutions for a point P and a line L, respectively, only if P is not an Eckardt point and L is not an Eckardt line. It means that to derive Theorem 1.2 from the results of [27] one should check that Eckardt points and lines cannot be non-canonical centers. This is done below.

An Eckardt point cannot be a maximal center by Remark 7.5. Let L be an Eckardt line. Then either L contains a singular Eckardt point P, or there are infinitely many lines contained in X that intersect L in smooth points of X. Assume that L is a non-canonical center with respect to a normalized mobile linear system $\frac{1}{\mu}\mathcal{M}$. In the former case take a general plane section containing L and some line passing through P. Then a residual conic Q (that is possibly reducible but does not contain L as a component) intersects L in two smooth points of X (since L cannot contain exactly two singular points by Lemma 7.6) and hence is contained in $\operatorname{Bs}\mathcal{M}$, that is a contradiction. In the latter case a general line intersecting L is contained in $\operatorname{Bs}\mathcal{M}$, that is also a contradiction.

8. Non-canonical centers

¿From now on we denote by $\mathcal{M}$ the linear system obtained as in section 3. Recall that by a non-canonical center we mean a non-canonical center of $\frac{1}{\mu}\mathcal{M}$.

Part of the results of [27] can be stated as follows.

Theorem 8.1 (see [27, Theorem 17])**.** *A non-canonical center on X is either a singular point or a line passing through one or two singular points.*

One of the purposes of this section is to prove the following.

Proposition 8.2. *Assume that there are at least two non-canonical centers appearing simultaneously on X. Then there are exactly two of them, and these are either two singular points connected by a line contained in X, or a singular point and a line containing exactly one more singular point.*

Remark 8.3. An ordinary double point P is a non-canonical center with respect to $\frac{1}{\mu}\mathcal{M}$ if and only if $\mathrm{mult}_P\mathcal{M} > \mu$ by Theorem 4.2. The same holds for a line $L \subset X$ (or, more generally, for any curve not contained in a singular locus of an ambient variety), since the only extremal contraction with center in L is isomorphic to a blow-up of X in L in a neighborhood of a general point of L.

Lemma 8.4. *If the points P_1 and P_2 are non-canonical centers, then a line $L = \langle P_1, P_2\rangle$ is contained in X.*

Proof. Assume that $L \not\subset X$. Let H' be a general member of the linear system $|H - P_1 - P_2|$. Then H' does not contain the base curves of $\mathcal{M}$ and for general $D_1, D_2 \in \mathcal{M}$ the local intersection index $(D_1D_2H')_{P_i} > 2\mu^2$ by Theorem 4.2. Hence

$$4\mu^2 = D_1D_2H' \geqslant (D_1D_1H')_{P_1} + (D_1D_2H')_{P_2} > 2\mu^2 + 2\mu^2 = 4\mu^2,$$

a contradiction. □

Lemma 8.5. *If the points P_1, P_2 and P_3 are non-canonical centers then they are not collinear.*

Proof. Assume that they are. By Lemma 8.4 the line $L = \langle P_1, P_2, P_3\rangle$ is contained in X. Let Π be a general two-dimensional plane passing through L, and $X|_\Pi = L \cup C$. Since $C \not\subset \mathrm{Bs}\,\mathcal{M}$, by Theorem 4.2 for a general $D \in \mathcal{M}$

$$3\mu = CD \geqslant \sum_{i=1}^{3} \mathrm{mult}_{P_i}\mathcal{M} > \sum_{i=1}^{3} \mu = 3\mu,$$

a contradiction. □

Lemma 8.6. *If the points P_1 and P_2 are non-canonical centers, then the line $L = \langle P_1, P_2\rangle$ is not a non-canonical center.*

Proof. Similar to that of Lemma 8.5. □

Lemma 8.7. *If a point P and a line $L \ni P$ are non-canonical centers, then L contains exactly one more singular point.*

Proof. Similar to that of Lemma 8.5 (except for "exactly" that is implied by Theorem 8.1). □

Lemma 8.8. *Two skew lines cannot be non-canonical centers.*

Proof. Assume that there exist skew lines L_1 and L_2 that are non-canonical centers. Let Π be a general plane passing through L_1, and $X|_\Pi = L_1 \cup C$. Let $C \cap L_1 = \{P_1, P_2, P_3\}$, $C \cap L_2 = P$. By Theorem 8.1 at least one of the points P_1, P_2, P_3 is a nonsingular point of X. Since P is also nonsingular and $C \not\subset \operatorname{Bs}\mathcal{M}$, for a general $D \in \mathcal{M}$ we have

$$3\mu = CD \geqslant \operatorname{mult}_P\mathcal{M} + \sum\nolimits_{i=1}^{3} \operatorname{mult}_{P_i}\mathcal{M} > \mu + \mu + \frac{\mu}{2} + \frac{\mu}{2} = 3\mu,$$

a contradiction. □

Lemma 8.9. *Let the points P_1 and P_2 be non-canonical centers. Assume that a line $L = \langle P_1, P_2\rangle$ does not pass through other singular points of X. Then L is not an Eckardt line.*

Proof. Assume that it is (note that $L \subset X$ by Lemma 8.4). Let $L' \subset X$ be a general line intersecting L, $\Pi = \langle L, L'\rangle$ and let $\Pi|_X = L + L' + Q$, where $L \not\subset Q$ by Lemma 4.4. Then Q is a (possibly reducible) conic passing through P_1 and P_2, so by Theorem 4.2 it is contained in $\operatorname{Bs}\mathcal{M}$, a contradiction. □

Lemma 8.10. *Let the points P_1 and P_2 be non-canonical centers. Assume that a line $L = \langle P_1, P_2\rangle$ contains a third singular point P_3. Then P_3 is not an Eckardt point.*

Proof. Analogous to that of Lemma 8.9. Note that in this case a general residual conic Q does not contain L because the cone of lines passing through an Eckardt point is not contained in a hyperplane by Lemma 7.6. □

The following statement will be one of the main tools to exclude configurations of non-canonical centers. To state it we'll use the following notations.

Let the lines $C_1, \ldots, C_k \subset X$, $0 \leqslant k \leqslant 4$, and the points $P_1, \ldots, P_l \in \operatorname{Sing} X$, $l \geqslant 0$, be contained in a plane Π_0. Let

$$X|_{\Pi_0} = d_1C_1 + \ldots + d_kC_k + \ldots + d_mC_m$$

for some $m \leqslant 4$, and

$$\Pi_0 \cap \operatorname{Sing} X = \{P_1, \ldots, P_l, P_{l+1}, \ldots, P_n\}.$$

Let H be a general hyperplane section passing through Π_0, so that

$$\operatorname{Sing} H = \{P_1, \ldots, P_n, P_{n+1}, \ldots, P_r\},$$

where $r \geqslant n$ (note that the inequality $r > n$ can hold only if the intersection $X \cap \Pi_0$ has components with multiplicities greater than 1 by Lemma 4.6). Let $\overline{\pi} : \widetilde{X} \to X$ be a sequence of blow-ups with centers lying over the points $P_1, \ldots, P_r$ such that the restriction π of the morphism $\overline{\pi}$ to a strict transform $\widetilde{H}$ of H is a minimal resolution of H. Let $\overline{E_i^t}$ be exceptional divisors of $\overline{\pi}$ such that $\overline{\pi}(\overline{E_i^t}) = P_i$ for $1 \leqslant i \leqslant r$, $1 \leqslant t \leqslant \overline{T_i}$; let E_i^t, $1 \leqslant i \leqslant r$, $1 \leqslant t \leqslant T_i$, be components of restrictions of divisors $\overline{E_i^t}$ to $\widetilde{H}$ (so that E_i^t are prime exceptional divisors of π with $\pi(E_i^t) = P_i$; note that T_i may be different from $\overline{T_i}$); finally, let $\widetilde{C_j}$ be proper transforms of the curves C_j, $1 \leqslant j \leqslant m$.

Lemma 8.11. *Let $(\cdot, \cdot)$ be the intersection form on $\mathrm{NS}_{\mathbb{Q}}(\widetilde{H})$. Let G be a set that consists of all curves E_i^t, $l+1 \leqslant i \leqslant r$, and $\widetilde{C_j}$, $k+1 \leqslant j \leqslant m$, and G' — a set of all curves E_i^t, $1 \leqslant i \leqslant l$, and $\widetilde{C_j}$, $1 \leqslant j \leqslant k$. Assume that the following condition holds:*

$(*)$ the set G splits into a disjoint union $G = G_1 \cup \ldots \cup G_p$ so that for all $1 \leqslant s \leqslant p$ the intersection form $(\cdot, \cdot)$ is negative semi-definite on the subspace W_s generated by G_s, negative definite on each subspace of W_s generated by all elements of G_p except one, and subspaces W_s are pairwise orthogonal with respect to $(\cdot, \cdot)$.

Then all curves from G' cannot appear simultaneously as non-canonical centers on X.

Proof. Assume that they can. Let $\operatorname{mult}_{C_j}\mathcal{M} = \gamma_j$. Let H' be a general hyperplane section passing through Π_0; then $H'|_H = C_1 + \ldots + C_m$. Since the singularities of H are Du Val of type A (see Lemmas 4.5 and 4.6), we have

$$\pi^*(H'|_H) = \pi^{-1}(H'|_H) + \sum_{i=1}^{r} \sum_{t=1}^{T_i} E_i^t.$$

Let $\overline{\mathcal{M}} = \overline{\pi}^{-1}\mathcal{M}$. Define ν_i^t to satisfy

$$\overline{\mathcal{M}} = \overline{\pi}^*\mathcal{M} - \sum_{i=1}^{r} \sum_{t=1}^{\overline{T_i}} \nu_i^t \overline{E_i^t}.$$

Note that since H has only Du Val singularities of type A, all divisors $\overline{E_i^t}\big|_{\widetilde{H}}$ are reduced, and hence

$$\left(\sum_{t=1}^{\overline{T_i}} \overline{E_i^t}\right)\Bigg|_{\widetilde{H}} = \sum_{t=1}^{T_i} E_i^t.$$

Let

$$\overline{\mathcal{M}}\big|_{\widetilde{H}} = F + \sum_{j=1}^{m} \gamma_j \widetilde{C_j},$$

where F is a mobile divisor. Then

$$\text{(8.12)}\quad F + \sum_{j=1}^{m} \gamma_j \widetilde{C_j} = \overline{\mathcal{M}}\big|_{\widetilde{H}} = \left(\overline{\pi}^*\mathcal{M} - \sum_{i=1}^{r}\sum_{t=1}^{\overline{T_i}} \nu_i^t \overline{E_i^t}\right)\Bigg|_{\widetilde{H}} \equiv$$

$$\equiv (\overline{\pi}^*(\mu H'))|_{\widetilde{H}} - \sum_{i=1}^{r}\sum_{t=1}^{\overline{T_i}} \nu_i^t\, \overline{E_i^t}\Big|_{\widetilde{H}} =$$

$$= \pi^*(\mu\, H'|_H) - \sum_{i=1}^{r}\sum_{t=1}^{T_i} \nu_i^t E_i^t =$$

$$= \mu\pi^{-1}(H'|_H) + \mu\sum_{i=1}^{r}\sum_{t=1}^{T_i} E_i^t - \sum_{i=1}^{r}\sum_{t=1}^{T_i} \nu_i^t E_i^t =$$

$$= \mu\sum_{j=1}^{m} \widetilde{C_j} + \sum_{i=1}^{r}\sum_{t=1}^{T_i} (\mu - \nu_i^t) E_i^t.$$

Rewrite the equality 8.12 as

$$\text{(8.13)}\qquad F + \sum_{i,t} \varkappa_i^t E_i^t + \sum_j \theta_j \widetilde{C}_j \equiv \sum_{i',t'} \varkappa_{i'}^{t'} E_{i'}^{t'} + \sum_{j'} \theta_{j'} \widetilde{C}_{j'},$$

where all the coefficients $\varkappa_i^t$, $\varkappa_{i'}^{t'}$, θ_j and $\theta_{j'}$ are positive, and the sets of summation indices of the right hand side and the left hand side are disjoint. By assumption $\mathrm{mult}_{P_i}\mathcal{M} > \mu$ for $1 \leqslant i \leqslant l$; in particular, $\nu_i^t > \mu$ for $1 \leqslant i \leqslant l$. By assumption we also have $\gamma_j > \mu$ for $1 \leqslant j \leqslant k$. (We don't assume a priori that the inequalities $\nu_i^t \leqslant \mu$ for $l+1 \leqslant i \leqslant r$ and $\gamma_j \leqslant \mu$ for $k+1 \leqslant j \leqslant m$ hold.) We do not exclude a possibility that some summations in 8.13 are performed with respect to empty sets of indices, but in any case the set of indices i' (resp., j') that appear on the right hand side of 8.13 is contained in the set $\{l+1, \ldots, r\}$ (resp., $\{k+1, \ldots, m\}$) by the assumption on multiplicities. Condition $(*)$

implies that the intersection form is negative semi-definite on the space $W = \bigoplus_s W_s$, so by Lemma 4.3

$$(8.14) \qquad (F + \sum \varkappa_i^t E_i^t + \sum \theta_j \widetilde{C_j})(\sum \varkappa_{i'}^{t'} E_{i'}^{t'} + \sum \theta_{j'} \widetilde{C}_{j'}) = 0.$$

The right hand side of the equality 8.13 is non-zero since an effective divisor cannot be numerically trivial. By 8.14 self-intersection of the right hand side of 8.13 is zero, so condition $(*)$ for any $1 \leqslant s \leqslant p$ either all curves from G_s appear on the right hand side of 8.13 with non-zero coefficients, or no curve from G_s appears there at all. The union $\bigcup_{i,t} E_i^t \cup \bigcup_j \widetilde{C_j}$ is connected, and by condition $(*)$ any two curves $D_1 \in G_{s_1}$, $D_2 \in G_{s_2}$ are disjoint for $s_1 \neq s_2$. Hence for any $1 \leqslant s \leqslant p$ there are curves $D \in G_s$ and $D' \in G'$ such that D intersects D'. Since all the curves $D' \in G'$ appear on the left hand side of 8.13 with non-zero coefficients, the intersection of the left hand side and the right hand side of 8.13 is strictly positive, hence a contradiction with 8.14. $\square$

Remark 8.15. Lemma 8.11 will be applied to normal crossing configurations of nonsingular rational curves on K3-surfaces. Such curve is a (-2)-curve, so the properties of the corresponding intersection form depend only on the structure of a dual graph (and the condition of Lemma 8.11 is equivalent to the requirement that all connected components of a dual graph are subgraphs of affine Dynkine diagrams). To describe such graphs we'll use the standard notation for usual and affine Dynkine diagrams (see, for example, [22]).

Corollary 8.16. *Three points cannot appear simultaneously as non-canonical centers on X.*

Proof. Assume that the points P_1, P_2 and P_3 are non-canonical centers. By Lemma 8.5 they are not collinear, and by Lemma 8.4 the lines $L_{ij} = \langle P_i, P_j \rangle$ are contained in X. Let $\Pi_0 = \langle P_1, P_2, P_3 \rangle$. Then $X|_{\Pi_0} = L_{12} + L_{23} + L_{13} + L$, where L is a line (possibly coinciding with one of the lines L_{ij}). Let $\pi : \widetilde{H} \to H$ be a minimal resolution of singularities of a general hyperplane section H passing through Π_0. Let a collection G consist of proper transforms of L and L_{ij}, and of all exceptional curves of π except those that lie over the points P_i. Let Γ be a dual graph of G.

If L coincides with one of the lines L_{ij}, say, with L_{12}, then by Lemmas 4.5 and 4.6 the surface H has at worst A_2 singularities at P_1 and P_2 and A_1 singularities at P_3 and possibly at one more point $P \in L_{12}$. One easily checks that the only non single-point component of Γ is of type A_2.

If L coincides with none of the lines L_{ij} but passes through one of their intersection points P_i, say, through P_1, then by Lemma 4.5 the surface H has at worst A_2 singularity at P_1, singularities of type A_1 at the points P_2 and P_3 and possibly one more A_1 singularity at a point $P = L \cap L_{23}$ (if X itself is singular at P). So Γ is a union of two single-point graphs with a graph of type A_3 or A_2, depending on whether X is singular at P or not.

If L passes through none of the points P_i, then by Lemma 4.5 all singularities of H are of type A_1, and Γ is a subgraph of a graph of type $E_6^{(1)}$.

In any case the intersection form on the subspace $W \subset \mathrm{NS}^1_{\mathbb{Q}}(\widetilde{H})$, generated by G, satisfies the conditions of Lemma 8.11, hence P_1, P_2 and P_3 do not appear simultaneously as non-canonical centers. $\square$

Corollary 8.17. *Two lines cannot appear simultaneously as non-canonical centers on X.*

Proof. Assume that the lines L_1 and L_2 are non-canonical centers. By lemma 8.8 they are coplanar. Let $\Pi_0 = \langle L_1, L_2 \rangle$. Then $X|_{\Pi_0} = L_1 + L_2 + Q$, where Q is a (possibly reducible) conic. Let $\pi : \widetilde{H} \to H$ be a minimal resolution of singularities of a general hyperplane section H passing through Π_0. Let a collection G consist of proper transforms of the components of Q and all exceptional curves of π. Let Γ be a dual graph.

If the conic Q is irreducible, then the only non single-point component of Γ (such a component exists if Q contains singularities of X) is a subgraph of a graph of type D_5 or $D_4^{(1)}$, depending on whether Q passes through the point $P = L_1 \cap L_2$ or not (in the former case by Lemma 4.5 there are at most two singularities of type A_1 and one of type A_2 on $Q \subset H$, and in the latter case there are at most four singularities of type A_1).

If $Q = L_3 + L_4$, $L_3 \neq L_4$, $L_3 \not\ni P$, $L_4 \not\ni P$ and the point $P' = L_3 \cap L_4$ lies neither on L_1 nor on L_2, then by Lemma 4.5 the surface H has only A_1 singularities, and the only non single-point component of Γ is a subgraph of a graph of type $D_6^{(1)}$ or $D_5^{(1)}$ depending on whether the point $P' = L_3 \cap L_4$ is singular on X or not.

If $Q = L_3 + L_4$, $L_3 \neq L_4$, $L_3 \not\ni P$, $L_4 \not\ni P$ and the point $P' = L_3 \cap L_4$ lies on L_1, then by Lemma 4.5 the surface H has only A_1 singularities except for a possible A_2 singularity at P', and the only non single-point component of Γ is a subgraph of a graph of type E_6.

If $Q = L_3 + L_4$, $L_3 \neq L_1$, $L_3 \neq L_2$, $L_3 \ni P$, $L_4 \not\ni P$, then by Lemma 4.5 the surface H has only A_1 singularities except for a possible singularity of type A_2 at the point P, and the only non single-point component of Γ is a subgraph of a graph of type D_7.

If $Q = L_3 + L_4$, the lines L_i are pairwise distinct for $1 \leqslant i \leqslant 4$, and $L_3, L_4 \ni P$, then by Lemma 4.5 the surface H has the only singularity at the point P and this singularity is at most A_3, so the only non single-point component of Γ is a subgraph of a graph of type D_5.

If $Q = 2L$, $L \not\ni P$, then by Lemmas 4.5 and 4.6 the surface H has at most A_2 singularities at the points $L \cap L_i$ and possibly one more singularity of type A_1 at some point $P' \in L$; the only non single-point component of Γ is a subgraph of a graph of type E_6.

If $Q = 2L$, $L \neq L_i$, $L \ni P$, then by Lemmas 4.5 and 4.6 the surface H has at most A_3 singularity at the point P and at most two singularities of type A_1 at some points $P', P'' \in L$; the only non single-point component of Γ is a subgraph of a graph of type D_6.

If $Q = L_1 + L$, $L \not\ni P$, then by Lemmas 4.5 and 4.6 the surface H has at most A_2 singularities at the points P and $P' = L \cap L_1$ and possibly one more singularity of type A_1 at some point $P'' \in L_1$; the graph Γ has at most two non single-point components, one of them of type A_2 and the other of type A_k with $k \leqslant 4$.

Finally, if $Q = L_1 + L$, $L \ni P$ (L may coincide with L_1 or L_2), then by Lemmas 4.5 and 4.6 the surface H has at most A_3 singularity at the point P (and at most A_2 singularities on multiple lines, the case of A_2 arising only if $L = L_1$), and all non single-point components of Γ are of type A_k with $k \leqslant 4$.

In any case the intersection form on the subspace $W \subset \mathrm{NS}^1_{\mathbb{Q}}(\widetilde{H})$, generated by G, satisfies the conditions of Lemma 8.11, hence L_1 and L_2 do not appear simultaneously as non-canonical centers. $\square$

Corollary 8.18. *A line and a point outside it cannot appear simultaneously as non-canonical centers on X.*

Proof. Assume that a line L and a point $P \notin L$ are non-canonical centers. Let $\Pi_0 = \langle L, P\rangle$, $X|_{\Pi_0} = L + C$. Let $\pi : \widetilde{H} \to H$ be a minimal resolution of singularities of a general hyperplane section H passing through Π_0. Let a collection G consist of proper transforms of components of C and all exceptional curves of π except those that lie over P. Let Γ be a dual graph.

If C is an irreducible cubic[6] (singular at P), then H has singularities of type A_1, and Γ is a subgraph of a graph of type D_4.

[6]In this case one can also argue as follows, avoiding the use of Lemma 8.11: if L and P are non-canonical centers, after an involution τ_P the curve C becomes a non-canonical center that is impossible by Theorem 8.1.

If $C = Q + L_1$, where Q is an irreducible conic, then $L_1 \ni P$ (in particular, $L_1 \neq L$), and the only non single-point component of Γ (if any) is a subgraph of a graph of type D_6.

If $C = L_1 + L_2 + L_3$ and the lines L, L_1, L_2 and L_3 are pairwise distinct and the latter three lines pass through the point P, then by Lemma 4.5 the surface H has only singularities of type A_1 outside P, and Γ has at most three non single-point components, each of them of type A_2.

If $C = L_1 + L_2 + L_3$, the lines L, L_1, L_2 and L_3 are pairwise distinct, L_1 and L_2 pass through P, while L_3 passes through the intersection point $P_1 = L \cap L_1$, then by Lemma 4.5 the surface H has only A_1 singularities except for a possible A_2 singularity at the point P_1, and the only non single-point component of Γ is a subgraph of a graph of type D_7.

If $C = L_1 + L_2 + L_3$, the lines L, L_1, L_2 are L_3 pairwise distinct, L_1 and L_2 pass through P, and L_3 passes neither through P, nor through the intersection points of the lines L and L_1 or L and L_2, then by Lemma 4.5 the surface H has only A_1 singularities, and the only non single-point component of Γ is a subgraph of a graph of type $E_7^{(1)}$.

If $C = 2L_1 + L_2$, $P \notin L_2$ and $L_2 \neq L$, then the surface H has only A_1 singularities except for possible A_2 singularities at P and $P_1 = L \cap L_1$, and the only component of Γ is a subgraph of a graph of type E_7.

If $C = 2L + L_1$, $P \in L_2$, $L_2 \neq L$, then Γ has at most two non single-point components, each of them of type A_k with $k \leqslant 4$.

If $C = 2L_1 + L$, then the only non single-point component of Γ is of type A_k with $k \leqslant 5$.

If $C = 3L_1$, then the only non single-point component of Γ is of type A_k with $k \leqslant 6$.

In any case the intersection form on the subspace $W \subset \mathrm{NS}^1_{\mathbb{Q}}(\widetilde{H})$, generated by G, satisfies the conditions of Lemma 8.11, hence L and P do not appear simultaneously as non-canonical centers. $\square$

Proof of Proposition 8.2. By Theorem 8.1 all non-canonical centers are either lines or singular points. If one of the centers is a line L, then by Corollary 8.17 all other non-canonical centers are points, and by Corollary 8.18 these points lie on L; finally, by Lemma 8.6 there can be at most one such point, and by Lemma 8.7 the line L contains exactly two singular points. If all non-canonical centers are points, then by Corollary 8.16 there are only two of them, and by Lemma 8.4 they lie on a line contained in X. $\square$

Remark 8.19. The statement of Proposition 8.2 (as well as all previous statements) remains true if instead of two non-canonical centers one considers a center of non-canonical singularities and a center of strictly canonical singularities of $\mathcal{M}$.

Proposition 8.2 (or rather Remark 8.19) implies Theorem 1.8 using the calculations of Lemmas 6.3, 6.5 and 6.6 in a standard way (see [25, Chapter V, §7] or [21, 3.2.4] for a very detailed proof). Note that Lemmas 8.9 and 8.10 ensure that the calculations of the former Lemmas are applicable, i. e. that for two points P_1 and P_2 that are non-canonical centers the line $L = \langle P_1, P_2 \rangle$ is not an Eckardt line if L does not contain a third singular point, and that the third singular point is not an Eckardt point if it does.

9. Algebraically non-closed fields

One of the results of [27] (namely, [27, Theorem 5]) states that the main theorems of [27] (birational rigidity of X and description of generators of $\mathrm{Bir}(X)$) hold over algebraically non-closed field $\Bbbk$ of characteristic 0 as well as over $\mathbb{C}$. Unfortunately, there is a gap in the proof (the fact that three conjugate points cannot form a non-canonical center is derived from the statement that even two points cannot, and this is not true, see Example 9.2 below). The aim of this section is to provide a patch to the proof.

Example 9.1 (cf. [25, Chapter V, 1.4])**.** Let $P_1, P_2 \in \operatorname{Sing} X_{\bar{\Bbbk}}$ be two points contained in a line $L \subset X_{\bar{\Bbbk}}$. Let E be a section of the associated elliptic fibration arising from the line L. Take a fiberwise reflection in the section E, and denote the corresponding birational involution of $X_{\bar{\Bbbk}}$ by $\tau_{P_1P_2}$. If P_1 and P_2 are both non-canonical centers, then $\tau_{P_1P_2}$ untwists both of them (see Lemma 8.9 and Lemma 9.4 below). On the other hand, starting with a linear system $|\mathcal{O}(1)|$ and taking a strict transform with respect to $\tau_{P_1P_2} : X_{\bar{\Bbbk}} \dashrightarrow X_{\bar{\Bbbk}}$, one obtains a mobile linear system $\mathcal{M}$ such that P_1 and P_2 are non-canonical centers with respect to $\frac{1}{\mu}\mathcal{M}$, provided that $\tau_{P_1P_2}$ is not regular. If X is general enough so that L is not an Eckardt line, Lemma 9.4 implies that the involution $\tau_{P_1P_2}$ is indeed non-regular.

Example 9.2. Assume that singular point P_1 and P_2 are conjugate (i. e. $\{P_1, P_2\}$ is a $\Bbbk$-point of X of degree 2), so that the line $L = \langle P_1, P_2 \rangle$ is defined over $\Bbbk$. Then the involution $\tau_{P_1P_2}$ is also defined over $\Bbbk$. In particular, $\{P_1, P_2\}$ can be a non-canonical center on X (provided that X is general enough).

Remark 9.3. In the settings of Example 9.2 the line L is defined over $\Bbbk$, and so is the involution τ_L. One has

$$\tau_{P_1P_2} = \tau_{P_1} \circ \tau_L \circ \tau_{P_2}.$$

Lemma 9.4. *Let a line $L \subset X$ contain exactly two singular points P_1 and P_2 of $X_{\bar{\Bbbk}}$. Assume that L is not an Eckardt line. Then*

$$\mu(\chi \circ \tau_{P_1P_2}) = 13\mu(\chi) - 6\nu_{P_1}(\chi) - 6\nu_{P_2}(\chi),$$
$$\nu_{P_1}(\chi \circ \tau_{P_1P_2}) = 14\mu(\chi) - 7\nu_{P_1}(\chi) - 6\nu_{P_2}(\chi),$$
$$\nu_{P_2}(\chi \circ \tau_{P_1P_2}) = 14\mu(\chi) - 6\nu_{P_1}(\chi) - 7\nu_{P_2}(\chi),$$
$$\nu_L(\chi \circ \tau_{P_1P_2}L) = 8\mu(\chi) - 4\nu_{P_1}(\chi) - 4\nu_{P_2}(\chi) + \nu_L(\chi).$$

Proof. Analogous to that of Lemma 6.2. Note that Remark 6.1 is also applicable in this case. □

Lemma 9.4 implies that a point $\{P_1, P_2\}$ of degree 2 is a non-canonical center with respect to some normalized mobile linear system provided that the corresponding line L is contained in X and L is not an Eckardt line. In this case the involution $\tau_{P_1P_2}$ is an untwisting involution for this center (again by Lemma 9.4). On the other hand, $\{P_1, P_2\}$ cannot be a maximal center if L is not contained in X by Lemma 8.4, and also if L is an Eckardt line by Lemma 8.9. Finally, Corollary 8.16 applied to $X_{\bar{\Bbbk}}$ implies the following.

Corollary 9.5. *A $\Bbbk$-point of degree $d \geqslant 3$ cannot be a non-canonical center.*

So the main statements of [27] (i. e. Theorem 1.2) really hold over $\Bbbk$. Moreover, the involutions $\tau_{P_1P_2}$ described in Example 9.2 are needed only in the proof, while one does not need to add them to the set of generators since they are expressible in terms of the involutions centered in lines and points by Remark 9.3.

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