Hitchin systems on ll-curves

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To Prof. Friedrich Hirzebruch on his 75th birthday

1 Introduction

Recall (see, for example [T1] and [T2]) that the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ of the moduli space $\mathcal{M}_g$ of smooth curves of genus g containes a finite configuration of points $\mathcal{P} \subset \overline{\mathcal{M}_g}$ corresponding to the large limit curves and enumerated by 3-valent graphs. Given a 3-valent graph Γ, we write:

- $V(\Gamma) = \{v_i\}$ for the set of its vertices;
- $E(\Gamma) = \{e\}$ for the set of its edges;
- $\vec{E}(\Gamma) = \{\vec{e}\}$ for the set of all orientation choices for all the edges;
- $v_s(\vec{e})$, $v_t(\vec{e})$ for the source and target vertices of an oriented edge $\vec{e} \in \vec{E}$;
- $S(v) \subset E(\Gamma)$ for the star of a vertex $v \in V(\Gamma)$, i. e. for the set of all (non oriented) edges incident to v;

Topologically, each 3-valent graph Γ is equivalent to some 3-dimensional handlebody H_Γ whose boundary $\partial H_\Gamma = \Sigma_\Gamma$ is a smooth compact Riemann surface obtained from Γ by pumping its edges to tubes and vertices to *trinions*, that is 2-spheres with 3 holes. It is easy to see that the cardinalities

$|E(\Gamma)| = 3g-3$, $|V(\Gamma)| = 2g-2$, where g is the genus of Σ_Γ. By the construction, Σ_Γ is equipped with a *trinion decomposition*, i. e. it is sliced into 'pairs of pants' obtained by cutting some meridian C_e out of each tube $e \in E(\Gamma)$. Contracting these $3g-3$ meridians to points, we obtain a connected reducible algebraic curve P_Γ glued from $2g-2$ Riemann spheres $P_v = \mathbb{CP}^1$, $v \in V(\Gamma)$, along triples of points $(p_{e_1}, p_{e_2}, p_{e_3}) \in P_v$ corresponding to the edges of the star $S(v) = \{e_1, e_2, e_3\}$. This reducible algebraic curve P_Γ is Deligne - Mumford stable and has the arithmetical genus g.

The restriction of the canonical sheaf $\mathcal{O}(K_\Gamma)$ to each component P_v coincides with the sheaf of meromorphic differentials ω with simple poles at $(p_{e_1}, p_{e_2}, p_{e_3}) \in P_v$:

$$\mathcal{O}_{P_\Gamma}(K_\Gamma)|_{P_v} = \mathcal{O}_{P_v}(K_{P_v} + p_{e_1} + p_{e_2} + p_{e_3}) = \mathcal{O}_{P_v}(1)\ .$$

Thus, a global holomorphic section $s \in H^0(P_\Gamma, \mathcal{O}(K_\Gamma))$ is a collection of meromorphic differentials $\{\omega_v\}$ on P_v whose residues at the poles satisfy the following compatibility relations:

$$\begin{gathered} \mathrm{res}_{p_e}\omega_{v_s(e)} + \mathrm{res}_{p_e}\omega_{v_t(e)} = 0\ ,\ \forall\ e \in E(\Gamma) \\ \sum_{e \in S(v)} \mathrm{res}_{p_e}\omega(v) = 0\ ,\ \forall\ v \in V(\Gamma) \end{gathered} \tag{1.1}$$

These linear constraints lead to the right value for $\dim H^0(P_\Gamma, \mathcal{O}(K_\Gamma)) = g$, the dimension of the space of global holomorphic differentials. This agrees with the Deligne - Mumford prediction that the vector bundle $\pi : V_1 \to \overline{\mathcal{M}_g}$, whose fiber at a smooth $C \in \mathcal{M}_g$ is $H^0(C, \mathcal{O}(K_C))$, can be continued into each ll-curve point as a vector bundle of rank g. However, the properties of the complete canonical linear system $|K_\Gamma|$ depend on the topology of Γ (more precisely, on the thickness of the graph, see [A]).

The double canonical system $|2K_\Gamma|$ of an ll-curve is much more regular. Namely, a holomorphic quadratic differential $\Omega \in H^0(P_\Gamma, \mathcal{O}(2K_\Gamma)$ is given by a collection of meromorphic quadratic differentials $\{\Omega_v\}$ on the components P_v such that their bi-residues satisfy the following condition:

$$\mathrm{bi\,res}_{p_e}\omega_{v_1} = \mathrm{bi\,res}_{p_e}\omega_{v_2}\ , \quad \forall\ e = (v_1, v_2) \in E(\Gamma) \tag{1.2}$$

(see [T4],[T5]). The system of nodes $\{p_e\} \in P_\Gamma$, $e \in E(\Gamma)$, gives an identification

$$H^0(P_\Gamma, \mathcal{O}(2K_\Gamma))^* = \mathbb{C}^{E(\Gamma)} \tag{1.3}$$

under which the basic linear form H_e, which corresponds to $e \in E(\Gamma)$, takes quadratic differential Ω to $H_e(\Omega) = \mathrm{bi\,res}_{p_e}\Omega$. So, the moduli space $\overline{\mathcal{M}_g}$ has a smooth orbifold structure at each ll-point $P_\Gamma \in \overline{\mathcal{M}_g}$ and the fiber of the tangent bundle $T_{P_\Gamma}\overline{\mathcal{M}_g}$ at P_Γ equals

$$T_{P_\Gamma}\overline{\mathcal{M}_g} = H^0(P_\Gamma, \mathcal{O}_{P_\Gamma}(2K_{P_\Gamma})^* = \mathbb{C}^{E(\Gamma)} \tag{1.4}$$

Moreover, each basic tangent direction $\mathbb{C} \cdot H_e$, $e \in E(\Gamma)$, can be integrated to some rational curve $\mathcal{C}_{e\subset\Gamma} = \psi_{e\subset\Gamma}(\mathbb{P}^1) \subset \overline{\mathcal{M}_g}$ (see (3.1) – (3.12) in [T2]).

So, the Deligne - Mumford compactification $\overline{\mathcal{M}_g}$ contains the configuration of points $\mathcal{P} = \{P_\Gamma\}$ parameterized by 3-valent graphs Γ of genus g and the configuration of rational curves[1] $\mathcal{C} = \bigcup\limits_{e\subset\Gamma} \mathcal{C}_{e\subset\Gamma}$ corresponding to flags $e \subset \Gamma$ in such a way that three flags from the same nest (see (2.17), (2.20) in [T2]) lead to the same curve $\mathcal{C}_{e\subset\Gamma}$. In the previous paper [T2] we described the complex gauge theory on ll-curves $\mathcal{P} \subset \overline{\mathcal{M}_g}$. Here we investigate the Hitchin systems on ll-curves.

2 Framed vector bundles on ll-curves

We write $\mathcal{M}^{\mathrm{ss}}(P_\Gamma)$ for the moduli space of topologically trivial semistable holomorphic vector bundles of rank 2 on P_Γ. By the definition, $\mathcal{M}^{\mathrm{ss}}(P_\Gamma)$ parameterizes rank 2 vector bundles E that have no positive line sub bundles $L \subset E$ and are restricted to trivial holomorphic vector bundles $E|_{P_v}$ over each sphere P_v, $v \in V(\Gamma)$. Any topologically trivial rank 2 vector bundle E on P_Γ can be *framed* by fixing some trivialization

$$E|_{P_v} = P_v \times \mathbb{C}^2 = P_v \times V_0$$

over each component of $P_v \subset P_\Gamma$ (here V_0 is some fixed 2-dimensional vector space). We write $\mathcal{F}(P_\Gamma)$ for the space of framed topologically trivial vector bundles on P_Γ. The framing forgetful map

$$f \;:\; \mathcal{F}(\Gamma) \to \mathcal{M}^{\mathrm{ss}}(P_\Gamma) \tag{2.1}$$

gives a principal bundle with the 'complex gauge' group

$$Map(V(\Gamma), SL(2,\mathbb{C})) \;, \tag{2.2}$$

[1]we consider $\mathcal{C} = \cup \mathcal{C}_{e\subset\Gamma}$ as one reducible curve

where $SL(2,\mathbb{C}) = SL(V_0)$ consists of linear automorphisms of V_0 with determinant 1.

Each framing of E can be considered as a function

$$a : \vec{E}(\Gamma) \to SL(2,\mathbb{C}) \text{ such that } a(\overleftarrow{e}) = a(\overrightarrow{e})^{-1} \tag{2.3}$$

where $\overleftarrow{e}$, $\overrightarrow{e}$ are two opposite orientation choices for edge $e \in E(\Gamma)$. In these terms, complex gauge transformation $g \in Map(V(\Gamma), SL(2,\mathbb{C}))$ acts on a framing a by the formula

$$g(a)(\vec{e}) = g(v_s) \cdot a(\vec{e}) \cdot g(v_t)^{-1}. \tag{2.4}$$

Factorizing (2.1) through this action, we get

$$\mathcal{M}^{\mathrm{ss}}(P_\Gamma) = \mathcal{F}(\Gamma)/Map(V(\Gamma), SL(2,\mathbb{C})). \tag{2.5}$$

At the same time, (2.3) and (2.4) show that a function a from (2.3) is nothing but a flat $SL(2,\mathbb{C})$-connection on the graph Γ. Moreover, the reframing group (2.2) is really the complex gauge group acting on these connections. Thus, the quotient (2.5) coincides with the space of the gauge orbits of flat connections, which, in its own turn, is the same as the space $\mathfrak{Rep}(\pi_1(\Gamma), SL(2,\mathbb{C}))$, of equivalence classes of representations of the fundamental group $\pi_1(\Gamma)$. In other words,

$$\mathcal{M}^{\mathrm{ss}}(P_\Gamma) = \mathfrak{Rep}(\pi_1(\Gamma), SL(2,\mathbb{C})) \ . \tag{2.6}$$

Let us write F_g for the free group with g generators. Then there is an isomorphism of fundamental groups

$$\pi_1(\Gamma) = \pi_1(H_\Gamma) = F_g$$

where H_Γ is the handlebody whose boundary $\partial H_\Gamma = \Sigma_\Gamma$ is the pumped graph. So, the space of representation classes (2.6) is described as the factor of the direct product of g copies of $SL(2,\mathbb{C})$ through diagonal action of $SL(2,\mathbb{C})$ by conjugations:

$$\mathfrak{Rep}(\pi_1(\Gamma), SL(2,\mathbb{C})) = (SL(2,\mathbb{C}))^g / \operatorname{ad}_{\mathrm{diag}} SL(2,\mathbb{C}) \tag{2.7}$$

The right hand side here is called *the Schottki space* of genus g and denoted by S_g. So, $\mathcal{M}^{\mathrm{ss}}(P_\Gamma) = S_g$.

On the other side, we have the canonical surjection

$$r \ : \ \pi_1(\Sigma_\Gamma) \to \pi_1(H_\Gamma) = \pi_1(\Gamma)\, . \tag{2.8}$$

Let us choose some standard presentation

$$\pi_1(\Sigma_\Gamma) = \big\langle\, a_1, \, \dots\, , \, a_g, \, b_1, \, \dots\, , \, b_g \mid \prod_{i=1}^{g}[a_i, b_i] = 1 \big\rangle \tag{2.9}$$

such that

$$\begin{aligned} \ker(r) &= \langle a_1, \, \dots\, , \, a_g \rangle \simeq F_g \\ \pi_1(\Gamma) = \pi_1(H_\Gamma) &= \langle r(b_1), \, \dots\, , \, r(b_g) \rangle \simeq F_g \end{aligned} \tag{2.10}$$

Then there are two Schottki subspaces $S_g^a, S_g^b \subset \mathfrak{Rep}(\pi_1(\Sigma_\Gamma), SL(2,\mathbb{C}))$ associated with the presentation (2.9):

$$\begin{aligned} S_g^a &= \{\varrho \in \mathfrak{Rep}(\pi_1(\Sigma_\Gamma), SL(2,\mathbb{C})) \mid \varrho(a_i) = 1 \text{ for } 1 \le i \le g\} \\ S_g^b &= \{\varrho \in \mathfrak{Rep}(\pi_1(\Sigma_\Gamma), SL(2,\mathbb{C})) \mid \varrho(b_i) = 1 \text{ for } 1 \le i \le g\} \end{aligned} \tag{2.11}$$

Further, each oriented edge $\vec{e} \in \vec{E}(\Gamma)$ gives a homotopy class of oriented 1-cycle $\vec{C}_e$ presented by some (oriented) meridian rounding about the tube corresponding to this edge. Joining these meridians with some fixed base point $p \in P_{v_0} \subset \Sigma_\Gamma$, which lives on the pumped vertex $v_0 \in V(\Gamma)$ say, we get elements $[C_{\vec{e}}]$ of the pointed fundamental group $\pi_1(\Sigma_\Gamma)_p$. These $3g-3$ cycles $[C_{\vec{e}}]$ obviously lay in the kernel $\ker(r)$ of the projection (2.8).

Let us realize $\pi_1(\Gamma)_{v_0}$ as the group of oriented cycles $(\vec{e}_1, \, \dots\, , \, \vec{e}_d)$, $v_t(\vec{e}_i) = v_s(\vec{e}_{i+1})$ based at $v_0 = v_s(\vec{e}_1) = v_t(\vec{e}_d)$. Then there is a map

$$\mathrm{Int} : \pi_1(\Gamma)_{v_0} \to \pi_1(\Sigma_\Gamma)_p$$

defined as $\mathrm{Int}(\vec{e}_1, \, \dots\, , \, \vec{e}_d) \overset{\mathrm{def}}{=} [C_{\vec{e}_1}] \circ [C_{\vec{e}_2}] \circ \cdots \circ [C_{\vec{e}_d}]$. Clearly, $\mathrm{Int}(\pi_1(\Gamma)) = \ker(r)$ and the exact triple of fundamental groups:

$$1 \to \pi_1(\Gamma) \overset{\mathrm{Int}}{\to} \pi_1(\Sigma_\Gamma) \overset{r}{\to} \pi_1(\Gamma) \to 1.$$

induces a chain of maps between the corresponding spaces of equivalence classes of $SL(2,\mathbb{C})$-representations:

$$\mathfrak{Rep}(\pi_1(\Gamma)) \overset{r^*}{\to} \mathfrak{Rep}(\pi_1(\Sigma_\Gamma)) \overset{\mathrm{Int}^*}{\to} \mathfrak{Rep}(\pi_1(\Gamma)) \tag{2.12}$$

which is 'exact' in the sense that each composition $r^* \circ \mathrm{Int}^*$ is the constant map into the identity. The middle space $\mathfrak{Rep}(\pi_1(\Sigma_\Gamma), SL(2,\mathbb{C}))$ in (2.12) parameterizes topologically trivial *flat* holomorphic bundles on P_Γ, that is the pairs (E, h), where $E \in \mathcal{M}^{\mathrm{ss}}(P_\Gamma)$ and h is a holomorphic flat $SL(2,\mathbb{C})$-connection on E (see formula (4.16) and Proposition 4.1 from [T2]). Moreover, the following identification list can be checked at once:

Proposition 2.1

(1) *The coincidence* $\mathfrak{Rep}(\pi_1(\Gamma)) = \mathcal{M}^{ss}(P_\Gamma)$ *described in* (2.6) *identifies the second map of* (2.12)*:* $\mathfrak{Rep}(\pi_1(\Sigma_\Gamma)) \stackrel{\mathrm{Int}^*}{\to} \mathfrak{Rep}(\pi_1(\Gamma))$ *with the forgetful map*

$$\mathfrak{Rep}(\pi_1(\Sigma_\Gamma)) \stackrel{f}{\to} \mathcal{M}^{ss}(P_\Gamma)\ , \tag{2.13}$$

which takes $(E,h) \longmapsto E$*. In particular, the fibers of* $f = \mathrm{Int}^*$ *have the structure of affine spaces associated with the vector spaces of Higgs fields (see for example* [T4], [T5]*).*

(2) The first map of (2.12): $\mathcal{M}^{ss}(P_\Gamma) = \mathfrak{Rep}(\pi_1(\Gamma)) \stackrel{r^*}{\to} \mathfrak{Rep}(\pi_1(\Sigma_\Gamma))$ gives a section for the affine bundle (2.13). So, each fiber of $f = \mathrm{Int}^*$ is naturally identified with the vector space of the Higgs fields on the corresponding vector bundle E.

There is quite simple geometrical reason for why $\mathfrak{Rep}(\pi_1(\Sigma_\Gamma)) \to \mathcal{M}^{ss}(P_\Gamma)$ has a vector bundle structure: the boundary $\overline{\mathcal{M}^{ss}(P_\Gamma)} \setminus \mathcal{M}^{ss}(P_\Gamma)$ of the compactified moduli space contains an effective theta divisor (see [T1]); so, the obstruction for lifting of affine structure to vector one vanishes over $\mathcal{M}^{ss}(P_\Gamma)$.

3 Framed Higgs fields and spectral curves

Let us write $\widetilde{E}_a$ for a framed vector bundle obtained from E by a framing function a as in (2.3). Then a Higgs field $\phi : \widetilde{E}_a \to \widetilde{E}_a \otimes K_\Gamma$ on $\widetilde{E}_a$ is given by a collection of traceless 2×2-matrices $\omega(v)|$, $v \in V(\Gamma)$, whose entries $\omega_{ij}(v)$ are meromorphic differentials on P_v with poles at $p_{e_1}, p_{e_2}, p_{e_3}$, $\{e_1, e_2, e_3\} = S(v)$. Their residue matrices $\mathrm{res}_{p_e}(\omega(v)) = ||\mathrm{res}_{p_e}\omega_{ij}(v)||$ satisfy the relations:

$$\begin{gathered} \mathrm{res}_{p_e}\omega(v_s(\vec{e})) + a(\vec{e}) \circ \mathrm{res}_{p_e}\omega(v_t(\vec{e})) \circ a(\vec{e})^{-1} = 0\ ,\ \forall\, \vec{e} \in \vec{E}(\Gamma) \\ \sum_{e \in S(v)} \mathrm{res}_{p_e}\omega(v) = 0\ ,\ \forall\, v \in V(\Gamma)\ . \end{gathered} \tag{3.1}$$

These constraints are gauge invariant[2] and linear in the Higgs fields. In particular, the dimension of the space of Higgs fields equals $3g-3$.

[2]with respect to the complex reframing group $Map(V(\Gamma), SL(2,\mathbb{C}))$ from (2.2)

Now, following the Hitchin program (see [H]) and sending a Higgs field

$$\phi \,:\, \widetilde{E}_a \to \widetilde{E}_a \otimes K_\Gamma \tag{3.2}$$

to the corresponding quadratic differential[3] $\det \omega(v) = -\omega_{11}^2(v) - \omega_{12}(v) \cdot \omega_{21}(v)$, we get a map

$$\pi \,:\, H^0(P_\Gamma, \mathrm{ad}\, \widetilde{E}_a \otimes K_\Gamma) \to H^0(P_\Gamma, \mathcal{O}_{P_\Gamma}(2\, K_\Gamma)) \tag{3.3}$$

The restriction of the projective line bundle $\mathbb{P}(E)$ onto each P_v is the quadric $\mathbb{P}(E)|_{P_v} = \mathbb{P}_1 \times P_v$, which admits two projections

$$\mathbb{P}_1 \xleftarrow{p_f} \mathbb{P}(E)|_{P_v} \xrightarrow{p_v} P_v \,.$$

We write $H = p_f^* \mathcal{O}_{\mathbb{P}_1}(1)$ for the Grothendieck generator such that $R^0 p_{v*} H = E|_{P_v}$. Then any Higgs field (3.2) is obtained as the direct image $R^0 p_{v*}$ of some homomorphism

$$\Phi \,:\, H \to H \otimes p_v^* \mathcal{O}_{P_v}(1)\,.$$

Since ϕ is traceless, its restriction $\phi_p : \mathbb{P}(E_p) \to \mathbb{P}(E_p)$ onto a fiber over $p \in P_v$ is either a linear automorphism $\mathbb{P}(E_p) \xrightarrow{\sim} \mathbb{P}(E_p)$, which has two distinct fixed points $p_1, p_2 \in \mathbb{P}(E_p)$, or a degenerated map ϕ_p, which contracts $\mathbb{P}(E_p)$ to one point $p_0 \in \mathbb{P}(E_p)$. Thus, we get a ramified double covering

$$w \,:\, \widetilde{P}_v \to P_v \tag{3.4}$$

which depends on and encodes the initial Higgs field ϕ.

Let us suppose for simplicity that for each component P_v, $v \in V(\Gamma)$, the quadratic differential $\det \omega(v)$ has two distinct zero points $z_1(v), z_2(v) \in P_v \setminus \{p_{e_1}, p_{e_1}, p_{e_1}\}$, where $\{e_1, e_2, e_3\} = S(v)$. Then the ramification divisor of the double covering (3.4) is

$$(\det \omega(v))_0 = z_1(v) + z_2(v) \subset P_v \tag{3.5}$$

and the intersection number $\widetilde{P}_v \cdot p_v^{-1}(p_e) = 2$ on the quadric $\mathbb{P}_1 \times P_v$.

Definition 3.1 The pair $(\widetilde{P}_v, L_\phi)$, where $L_\phi = H|_{\widetilde{P}_v} \in \mathrm{Pic}\, \widetilde{P}_v$, is called a *spectral data* of the Higgs field (3.2).

[3]note that the constrains (3.1) automatically imply the constrains (1.2)

The Higgs field ϕ is reconstructed from its spectral data as follows. Twisting the standard adjoint sequence of sheaves

$$0 \to \mathcal{O}(-\widetilde{P}_v) \to \mathcal{O} \to \mathcal{O}_{\widetilde{P}_v} \to 0$$

on the quadric $\mathbb{P}_1 \times P_v$ by the Grothendieck line bundle H, we get an exact triple

$$0 \to \mathcal{O}(-\widetilde{P}_v) \otimes H \to H \to \mathcal{O}_{\widetilde{P}_v}(H) \to 0 . \tag{3.6}$$

Restricting the first term to the fiber, we get

$$\mathcal{O}(-\widetilde{P}_v)(H)|_{\mathbb{P}(E_p)} = \mathcal{O}_{\mathbb{P}_1}(-2) \otimes \mathcal{O}_{\mathbb{P}_1}(1) = \mathcal{O}_{\mathbb{P}_1}(-1) .$$

Hence, applying Rf_{v*} to (3.6), we get an isomorphism

$$0 \to E \to R^0 f_{v*}(\mathcal{O}_{\widetilde{P}_v}(H)) \to 0$$

which coincides with (3.2).

Now let us switch on the gluing procedure prescribed by the framing data $a : \vec{E}(\Gamma) \to SL(2,\mathbb{C})$ to combine the spectral curves $\widetilde{P}_v$ together. Consider each $\widetilde{P}_v$ as projective line equipped with an involution $i_v : \widetilde{P}_v \to \widetilde{P}_v$ such that $\widetilde{P}_v \stackrel{w}{\to} P_v = \widetilde{P}_v / i_v$ is the double covering (3.4). Then the ramification points (3.5) become the fixed points of i_v and 3 picked points $p_{e_i} \in P_v$ turn to 6 points $p^{\pm}_{e_i} \in \widetilde{P}_v$ forming a triple of i_v-conjugated pairs. Gluing $\widetilde{P}_{v_1}$ with $\widetilde{P}_{v_2}$ in 2 points $p^{\pm}_e$ for each $e = (v_1, v_2) \in S(v_1) \cap S(v_2) \subset E(\Gamma)$ and all $v_1, v_2 \in V(\Gamma)$, we get a reducible curve $\widetilde{P}_\Gamma(\phi)$ of arithmetical genus $4g-3$ with the involution

$$i_\phi : \widetilde{P}_\Gamma(\phi) \to \widetilde{P}_\Gamma(\phi)$$

such that the quotient $\widetilde{P}_\Gamma(\phi)/i_\phi = P_\Gamma$ is the original ll-curve we have started with. Under this procedure, the local line bundles L_ϕ on $\widetilde{P}_v$ are glued to the global line bundle

$$L_\phi \in \operatorname{Pic} \widetilde{P}_\Gamma(\phi) ,$$

which restricts onto each $\widetilde{P}_v$ as $L_\phi|_{\widetilde{P}_v} = \mathcal{O}_{\widetilde{P}_v}(1)$.

Now, using the geometric interpretation of the Hitchin systems via Prym varieties developed in [T4], [T5] one can easily prove the following

Proposition 3.1 *The initial Higgs field* (3.2) *on* P_Γ *is uniquely recovered from the triple* $(\widetilde{P}_\Gamma(\phi), i_\phi, L_\phi)$*;*

and the fiber of the Hitchin map (3.3)*:*

$$\pi \; : \; H^0(P_\Gamma, \mathrm{ad}\, \widetilde{E}_a \otimes K_\Gamma) \to H^0(P_\Gamma, \mathcal{O}_{P_\Gamma}(2\, K_\Gamma))$$

over a regular quadratic differential $\Omega \in H^0(P_\Gamma, \mathcal{O}_{P_\Gamma}(2K_\Gamma))$ *coincides with the Prym variety:*

$$\pi^{-1}(\Omega) = \mathrm{Prym}_w = \mathrm{Pic}_{1,\dots,1}\Big(\widetilde{P}_\Gamma(\phi)/w^*(\mathrm{Pic}(P_\Gamma)\Big) \; .$$

Andrey Tyurin died in Bonn at 27th of October going by bus to the Institute. The death was instantaneous and unexpected. The text looked unfinished but it was decided that really it contains the result and the argument to prove it; one just needs to remember two classical papers listed below and combine these ones with [T1]. Thus it should be strongly recommended to use these sources for understanding of the present text, just slightly edited by Alexei Gorodentsev and Nikolai Tyurin.

References

[A] I. V. Artamkin. *Algebraic geometry of ll-curves.* in preparation

[H] N. Hitchin. *Stable bundles and integral systems.* Duke Math. J. v.54 n.1 (1987), 91–114.

[HT] A. Hatcher, W. Thurston. *A presentation for the mapping class group of closed oriented surface.* Topology **19** (3) (1980), 221–237.

[JW] L. C. Jeffrey, J. Weitsman. *Bohr - Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula.* Commun. Math. Phys. **150** (1992), 593–630.

[T1] *A. Tyurin. Quantization, Classical and Quantum Field Theory and Theta-Functions.* Les publication du CRM, a paraitre 2002; arXiv: math.AG/0210466

[T2] *A. Tyurin. On the Mumford – Narasimhan problem.* Preprint MPI n.129 (2002), available at `math.AG/0210135`, 1–27.

[T3] *A. Tyurin. Three mathematical facets of $SU(2)$-spin networksm.* Preprint available at `math.DG/0011035`, 1–20.

[T4] *A. Tyurin. Periods and principal parts on a Riemann surface with a flat structure.* Izv. Akad. Nauk SSSR, Ser. Mat. 41:6 (1977), 1425–1443; translated in: Math. USSR-Izv. 11 (1979).

[T5] *A. Tyurin. On periods of quadratic differentials.* Uspekhi Mat. Nauk 33:6 (1978), 59–88; translated in: Russian Math. Surveys, 33:6 (1979), 169–221.