Zeta functions of groups: zeros and friendly ghosts.

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Abstract

We consider zeta functions defined as Euler products of $W(p, p^{-s})$ over all primes p where $W(X, Y)$ is a polynomial. Such zeta functions arise naturally in the context of zeta functions of algebraic groups and nilpotent groups. The analytic behaviour of such Euler products is controlled by something we call the ghost polynomial of $W(X, Y)$. The ghost is defined from the zeros of $W(p, Y)$ as p ranges over all primes. When the Euler product of the ghost is meromorphic on $\mathbb{C}$, an important property in relation to the analytic behaviour of the original zeta function, we say that $W(X, Y)$ has a friendly ghost. We give an explicit expression for the ghosts of classical reductive groups and prove that they are friendly. We also consider the ghosts of exceptional reductive groups and zeta functions of nilpotent groups. The concept of the ghost implies a new dichotomy for the representations of an algebraic group according to whether their associated zeta function has a friendly ghost or not.

1 Introduction

Given an infinite finitely generated group or finite dimensional algebra one of its important properties is the structure of the lattice of subgroups or subalgebras. Since Hey's thesis [15] in 1929, zeta functions have proved a powerful

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tool in capturing information about the structure of the associated lattice. The analytic properties of such zeta functions provide a deep understanding of the asymptotics of these lattices.

Hey's important contribution was to recognize that the zeta function describing the subalgebra structure of a central simple algebra could be described by a zeta function attached to the algebraic group GL_n defined as an integral with respect to the Haar measure on the group. Emboldened by the success of this original contribution the following definition of the zeta function of an algebraic group was proposed:

Definition 1.1 *(i) Let $\mathfrak{G}$ be a $\mathbb{Q}$-algebraic group with a choice of some $\mathbb{Q}$-rational representation $\rho : \mathfrak{G} \to \mathrm{GL}_n$. Define the local or p-adic zeta function of $\mathfrak{G}$ to be:*

$$Z_{\mathfrak{G},\rho,p}(s) = \int_{\mathfrak{G}_p^+} |\det(\rho(g))|_p^{-s} \mu_{\mathfrak{G}}(g)$$

where $\mathfrak{G}_p^+ = \rho^{-1}\rho\left(\mathfrak{G}\left(\mathbb{Q}_p\right)\right) \cap \mathrm{M}_n\left(\mathbb{Z}_p\right)$, the set of matrices whose entries are all p-adic integers, $|\cdot|_p$ denotes the p-adic valuation and $\mu_{\mathfrak{G}}$ denote the Haar measure on $\mathfrak{G}(\mathbb{Q}_p)$ normalized such that $\mu_{\mathfrak{G}}(\mathfrak{G}(\mathbb{Z}_p)) = 1$.

(ii) Define the global zeta function of $\mathfrak{G}$ to be

$$Z_{\mathfrak{G},\rho}(s) = \prod_{p \text{ prime}} Z_{\mathfrak{G},\rho,p}(s).$$

Such zeta functions were studied extensively in the 60's by Tamagawa [20], Satake [18] and Macdonald [17] but subsequently went somewhat out of favour. However recent work has breathed new life into these functions. The primary source of this resurrection comes from the important observation in [13] that these zeta functions are counting something. In particular they have an interpretation as a non-commutative zeta function of a Lie algebra or group, generalizing Hey's interpretation of the zeta function of GL_n as counting ideals in central simple algebras.

Definition 1.2 *(1) Let L be a finite dimensional Lie algebra over $\mathbb{Z}$ and let $\mathcal{X}(L)$ (respectively $\mathcal{X}_p(L)$) denote the lattice of subalgebras H of finite (respectively p-power) index with the property that $H \otimes \mathbb{Z}_p \cong L \otimes \mathbb{Z}_p$ for each prime p. Define the zeta function $\zeta_L(s)$ of L to be*

$$\zeta_L(s) = \sum_{H \in \mathcal{X}(L)} |L : H|^{-s}.$$

The function $\zeta_L(s)$ *has an Euler product over all primes* p *of the local zeta functions*

$$\zeta_{L,p}(s) = \sum_{H \in \mathcal{X}_p(L)} |L : H|^{-s}.$$

(2) Let G *be a finitely generated nilpotent group and define* $\mathcal{X}^\wedge(G)$ *(respectively* $\mathcal{X}_p^\wedge(G)$*) to be the lattice of subgroups* H *of finite (respectively* p*-power) index with the property that the profinite completions* $\widehat{H} \cong \widehat{G}$ *are isomorphic. Define the zeta function* $\zeta_G^\wedge(s)$ *of* G *to be*

$$\zeta_G^\wedge(s) = \sum_{H \in \mathcal{X}^\wedge(G)} |G : H|^{-s}.$$

The function $\zeta_G^\wedge(s)$ *has an Euler product over all primes* p *of the local zeta functions*

$$\zeta_{G,p}^\wedge(s) = \sum_{H \in \mathcal{X}_p^\wedge(G)} |G : H|^{-s}.$$

Proposition 1.1 *(1) Let* L *be a finite dimensional Lie algebra over* $\mathbb{Z}$ *and* $\mathfrak{G}_L = \mathrm{Aut}(L \otimes \mathbb{Q})$*, the* $\mathbb{Q}$*-algebraic group of automorphisms with a representation* $\rho : \mathfrak{G}_L \to \mathrm{GL}_n$ *determined by a choice of basis for* L*. Then for almost all primes* p

$$\zeta_{L,p}(s) = Z_{\mathfrak{G}_L,\rho,p}(s).$$

(2) Let G *be a finitely generated nilpotent group. Then there exists an associated Lie algebra* L *over* $\mathbb{Z}$ *such that for almost all primes* p

$$\zeta_{G,p}^\wedge(s) = \zeta_{L,p}(s) = Z_{\mathfrak{G}_L,\rho,p}(s).$$

Work of Bryant and Groves [2] shows that it is possible to realise any algebraic group as a quotient of the automorphism group $\mathfrak{G}_L$ associated to a nilpotent group or Lie algebra. (Note there is always some unipotent part coming from the so-called IA-automorphisms, automorphisms which act trivially modulo the derived group.) In section 3.3 of [9], an explicit construction of a Lie algebra $L(X_l)$ is provided for each classical group

$\mathfrak{G}(X_l) = \mathrm{GL}_n = \mathrm{GL}_{l+1}, \mathrm{GO}_n = \mathrm{GO}_{2l+1}, \mathrm{GSp}_n = \mathrm{GSp}_{2l}$ or $\mathrm{GO}_n^+ = \mathrm{GO}_{2l}^+$ of type $X_l = A_l, B_l, C_l$ and D_l respectively such that

$$\zeta_{L(X_l),p}(s) = Z_{\mathfrak{G}(X_l),\rho,p}(s(1+2/n)-n).$$

Example [9] If we take

$$L = L(C_3) = \langle x_1, x_2, x_3, y_1, y_2, y_3, z : [x_i, y_j] = \delta_{ij} z, [x_i, x_j] = [y_i, y_j] = [x_i, z] = [y_i, z] = 0\rangle$$

where δ_{ij} is the Kronecker delta function then

$$\zeta_{L,p}(s) = Z_{\mathrm{GSp}_6,p}(4s/3-6).$$

The second important reason for the resurrection of these zeta functions of algebraic groups comes from explicit formulae due to Igusa which now exist for the split algebraic groups which details a rational function $W(X,Y) \in \mathbb{Q}(X,Y)$ defined explicitly in terms of combinatorial data associated to the building of $\mathfrak{G}$ such that for each p, $Z_{\mathfrak{G},\rho,p}(s) = W(p, p^{-s})$. This combinatorial expression calculated in [16] by Igusa and extended in [9] provides new invariants associated to these split algebraic groups which are important in their own right.

Example [16] and [9] *Let* $\mathfrak{G} = \mathrm{GSp}_6$ and ρ be the natural representation. Then

$$\zeta_{\mathrm{GSp}_6}(s/3) = \zeta(s)\zeta(s-3)\zeta(s-5)\zeta(s-6)\prod_{p \text{ prime}} W(p, p^{-s})$$

where $\zeta(s)$ *is the Riemann zeta function and*

$$W(X,Y) = \left(1 + (X + X^2 + X^3 + X^4)Y + X^5Y^2\right).$$

The Encyclopedia of Mathematics [10] includes as one of the important properties of a 'zeta function' the possibility of meromorphic continuation beyond the region of convergence of the function. Such meromorphic continuation leads to the possibility of determining the asymptotics of the arithmetic invariants which the zeta function encodes. In general, the further one can meromorphically continue a zeta function, the better asymptotic results one can expect. It is important therefore to know what are the limits of such continuation. In the best case one might hope that the zeta function has meromorphic continuation to the whole of the complex plane. Another important question is the location of the zeros of a zeta function.

This paper grew then out of these related questions for the zeta functions detailed above which capture important information about the lattice of subgroups or subalgebras. The explicit formulae led us to the consideration therefore of the analytic properties of zeta functions defined as Euler products of a two variable polynomial $W(X,Y) \in \mathbb{C}[X,Y]$ (with $W(X,0) = 1$):

$$Z(s) = \prod_{p \text{ prime}} W(p, p^{-s}).$$

The best introduction to the ideas we have developed in this paper is perhaps to explain an example which motivated our discovery of what we call *the ghost zeta function* attached to the functions above. This ghost turns out to be fundamental to the question of the existence of natural boundaries for the meromorphic continuation of these zeta functions, but is also interesting in its own right as natural invariant of these zeta functions. Consider the explicit expression for $\zeta_{\mathrm{GSp}_6}(s/3)$ detailed above. It is the zeros of this function which lie at the heart of this paper. The zeros of $\zeta(s)$ we leave to those experts of non-commutative geometry and quantum chaos. Rather it is the zeros of

$$W(p,p^{-s}) = \left(1 + p^{1-s} + p^{2-s} + p^{3-s} + p^{4-s} + p^{5-2s}\right)$$

which piqued our interest and in particular hold the key to whether the corresponding Euler product has meromorphic continuation to the whole of $\mathbb{C}$.

In the above situation, as p tends to infinity, the zeros of $W(p,Y)$ can be approximated by the zeros of an associated polynomial

$$\widetilde{W}(p,Y) = 1 + pY + p^4Y + p^5Y^2 = (1+pY)(1+p^4Y),$$

in the sense that the two roots of $W(p,Y)$ can be arranged as $\alpha_p(1)$ and $\alpha_p(2)$ so that

$$\begin{aligned} \alpha_p(1) \cdot p &\to -1 \\ \alpha_p(2) \cdot p^4 &\to -1. \end{aligned}$$

It turns out that it is a general phenomenon for two variable polynomials $W(X,Y)$ that the zeros of $W(p,Y)$ on multiplication by some rational power of p always have a non-zero limit. The ghost polynomial is built then out of these limits and captures in some sense what the polynomial $W(X,Y)$ is trying to be in the limit as X tends to infinity. More precisely to any two variable polynomial $W(X,Y)$ there exists an associated polynomial $\widetilde{W}(X,Y)$ which we call *the ghost polynomial of* $W(X,Y)$ with the property that

(1) $\widetilde{W}(X,Y) = g_1(X^{u_1}Y^{v_1}) \cdots g_r(X^{u_r}Y^{v_r})$ where $g_i(Z)$ is a polynomial in one variable with constant coefficient 1 and (non-zero) zeros c_{ij}; and

(2) we can order the zeros of $W(p,Y)$ as $\alpha_{p,i,j}$ so that

$$\alpha_{p,i,j} \cdot p^{u_i/v_j} \to c_{ij}.$$

The explanation for the existence of this ghost is provided by the Puiseux power series associated to $W(X,Y)$, as we shall explain in section 2. There is an easy combinatorial way to construct the ghost where each polynomial $g_i(Z)$ corresponds to a line in the associated Newton polygon of the polynomial whose gradient is v_i/u_i.

The reason why these zeros and the ghosts are particularly interesting appears when we consider the question of meromorphic continuation. Let us return to our motivating example. In [4] the following analytic behaviour of $\zeta_{\mathrm{GSp}_6}(s)$ is established:

Example (continued) [4] $\zeta_{\mathrm{GSp}_6}(s/3)$ *converges on* $\Re(s) > 7$ *and has meromorphic continuation to* $\Re(s) > 4$. *However* $\Re(s) = 4$ *is a natural boundary beyond which* $\zeta_{\mathrm{GSp}_6}(s/3)$ *cannot be meromorphically continued.*

The natural boundary exists because every point on the boundary $\Re(s) = 4$ can be realised as a limit point of zeros in $\Re(s) > 4$. In particular for p large enough, choosing the zero $\alpha_p(2)$ of $W(p,Y)$, $W(p,p^{-s})$ has a zero for s satisfying

$$p^{-s} = \alpha_p(2) = (-1+\delta_p)\,p^{-4}$$

where δ_p is something small and positive, i.e. for all $n \in \mathbb{Z}$, $W(p,p^{-s})$ has a zero at

$$s = 4 - \frac{\log(1-\delta_p)}{\log p} + \frac{(2n-1)\pi i}{\log p}.$$

For these zeros to be of any use, it is important that they lie on the right hand side of $\Re(s) = 4$, i.e. in the region into which $Z(s)$ has been successfully meromorphically continued. This is the case since

$$\epsilon_p = \frac{\log\,(1-\delta_p)}{\log p} < 0$$

for p large enough. It is rather striking that in all the examples documented in [4], the zeros always end up on the right hand side of the candidate natural boundary. This is by no means guaranteed for an arbitrary two variable polynomial.

Note that $\epsilon_p \to 0$ as $p \to \infty$. If $A = 4 + ai$ is any point on $\Re(s) = 4$ one can choose $n_p \in \mathbb{Z}$ with the property that

$$\frac{(2n_p - 1)\pi}{\log p} \to a.$$

Hence every point on $\Re(s) = 4$ is a limit point of zeros of $Z(s)$ lying on $\Re(s) > 4$. Therefore $Z(s)$ cannot be meromorphically continued beyond $\Re(s) > 4$.

The natural boundary occurs then because the zeros of $W(p, p^{-s})$ are shifted away to the right of the natural boundary. However the ghost function has been defined as the limit of these zeros. In other words, the zeros of $\widetilde{W}(p, p^{-s})$ are located exactly on the lines $\Re(s) = 4$ and $\Re(s) = 1$.

The primary significance of the ghost is that it is likely to hold the key to understanding in general whether $Z(s) = \prod_p W(p, p^{-s})$ has a natural boundary or not. The following conjecture has been partially confirmed in [4] and generalizes an old result of Estermann:

Conjecture 1.2 *$Z(s) = \prod_p W(p, p^{-s})$ can be meromorphically continued to the whole complex plane if and only if there exist cyclotomic polynomials $g_i(U)$ and integers u_i, v_i for $i = 1, ..., r$ such that $W(X, Y) = g_1(X^{u_1}Y^{v_1})...g_r(X^{u_r}Y^{v_r})$.*

By a cyclotomic polynomial we mean any factor of $(1 - U^n)$. Estermann's Theorem is the case when $W(X, Y) = W(Y)$, a polynomial in one variable.

The following would then be a corollary of this conjecture:

Corollary 1.3 *Suppose that Conjecture 1.2 is true. Then $Z(s)$ has meromorphic continuation to the whole complex plane if and only if $W(X, Y)$ is its own ghost, (i.e. $W(X, Y) = \widetilde{W}(X, Y) = g_1(X^{u_1}Y^{v_1}) \cdots g_r(X^{u_r}Y^{v_r})$) and, for each $i = 1, .., r$, the polynomial $g_i(U)$ is a product of cyclotomic polynomials.*

Because of the straightforward combinatorial procedure to calculate the ghost polynomial from its Newton polygon, the ghost would give a good criterion for checking for meromorphic continuation.

The current state of Conjecture 1.2 as documented in [4] is as follows. Suppose $\widetilde{W}(X, Y) = g_1(X^{u_1}Y^{v_1}) \cdots g_r(X^{u_r}Y^{v_r})$. If $W(X, Y) = \widetilde{W}(X, Y)$ and for each $i = 1, .., r$ the polynomial $g_i(U)$ is a product of cyclotomic polynomials then it is clear that $Z(s)$ has meromorphic continuation. This is the easy part of Conjecture 1.2. Otherwise there exists

$$\beta = \max \left\{ \begin{array}{c} u_i/v_i : g_i(U) \text{ is not a product of cyclotomic polynomials} \\ \text{or } g_i(X^{u_i}Y^{v_i}) \text{ is not a factor of } W(X,Y) \end{array} \right\}.$$

In [4] it is shown how to continue $Z(s)$ meromorphically up to $\Re(s) = \beta$. This is done by using an identity of the form $W(X,Y) = \prod_{(n,m)\in\mathbb{N}^2}(1-X^nY^m)^{c_{nm}}$ where $c_{nm} \in \mathbb{Z}$. The line $\Re(s) = \beta$ is then proved to be a natural boundary except for one class of polynomials which remain elusive from current analysis. These polynomials have the property that all the zeros of $W(p^{u_i}, p^{-v_i s})$ close to $\Re(s) = \beta$ are on the left of $\Re(s) = \beta$ (which means we can't get the sort of argument working as in our example above) and there are infinitely many pairs (n, m) with $c_{nm} \neq 0$ and $(n + 1/2)/m \leq \beta < (n + 1)/m$. Note that in general $n/m < \beta$ except for finitely many pairs. These infinitely many pairs imply that multiplication by an infinite number of zeta functions is required to continue $Z(s)$ to $\Re(s) = \beta$, (since $\beta < (n+1)/m$) but we can't see the zeros of these zeta functions because they lie to the left of $\Re(s) = \beta$ (since $(n + 1/2)/m \leq \beta$). An example of such a polynomial is $1 + Y + XY^2$. For more details see [4]. We would conjecture that in this outstanding case, the natural boundary will still be located at $\Re(s) = \beta$. Hence the ghost would also provide an easy way to read off the natural boundary if it exists:

Conjecture 1.4 *If $Z(s) = \prod_p W(p, p^{-s})$ cannot be meromorphically continued to the whole complex plane then $\Re(s) = \beta$ is a natural boundary where*

$$\beta = \max \left\{ \begin{array}{c} u_i/v_i : g_i(U) \textit{ is not a product of cyclotomic polynomials} \\ \textit{or } g_i(X^{u_i}Y^{v_i}) \textit{ is not a factor of } W(X,Y) \end{array} \right\}$$

and $\widetilde{W}(X,Y) = g_1(X^{u_1}Y^{v_1}) \cdots g_r(X^{u_r}Y^{v_r})$ is the ghost of $W(X,Y)$.

By Corollary 1.3 it is important to know whether the ghost is a product of cyclotomic polynomials. We therefore make the following:

Definition 1.3 *The zeta function $Z(s) = \prod_p W(p, p^{-s})$ has a* friendly ghost *$\widetilde{W}(X,Y) = g_1(X^{u_1}Y^{v_1}) \cdots g_r(X^{u_r}Y^{v_r})$ if for each $i = 1, .., r$ the polynomial $g_i(U)$ is a product of cyclotomic polynomials.*

Equivalently we have by Estermann's Theorem:

Definition 1.4 *The zeta function $Z(s) = \prod_p W(p, p^{-s})$ has a* friendly ghost *if the* ghost zeta function *defined as $\widetilde{Z}(s) = \prod_p \widetilde{W}(p, p^{-s})$ has meromorphic continuation to the whole of $\mathbb{C}$.*

Even if the original zeta function is not meromorphic, it is still of interest to investigate whether the associated ghost zeta function is friendly. After all, the ghost has the effect of smoothing out the first barrier to the possibility of meromorphic continuation of $\prod_p W(p, p^{-s})$ by moving the zeros onto the natural boundary.

Returning to our example, although we have shown that $Z(s)$ has a natural boundary, nonetheless the ghost zeta function

$$\widetilde{Z}(s) = \prod_p (1 + p^{1-s})(1 + p^{4-s}) = \frac{\zeta(s-1)\zeta(s-4)}{\zeta(2s-2)\zeta(2s-8)}$$

does have meromorphic continuation to the whole of $\mathbb{C}$.

The main results of this paper concern the propensity of ghosts of zeta functions of groups to be friendly, although not always.

Our first results concern the zeta functions $Z_{\mathfrak{G},\rho}(s)$ of the classical groups $\mathrm{GL}_{l+1}, \mathrm{GO}_{2l+1}, \mathrm{GSp}_{2l}$ and GO_{2l}^{+}, i.e. the $\mathbb{Q}$-split reductive algebraic group $\mathfrak{G}$ of type A_l, B_l, C_l and D_l with their natural representations ρ. In section 3 we explain Igusa's explicit expression mentioned above calculated in [16] and [9] for the zeta function $Z_{\mathfrak{G},\rho,p}(s)$ in terms of the associated root system. (Igusa's definition of the zeta function is slightly different to our definition. [9] explains these explicit formulae in our context.) In particular there exist polynomials $P(X, Y)$ and $Q(X, Y)$ such that for almost all primes p,

$$Z_{\mathfrak{G},\rho,p}(s) = \frac{P(p, p^{-s})}{Q(p, p^{-s})}.$$

Let $\Phi(X_l)$ denote the root system associated to X_l and $\{\alpha_1, ..., \alpha_l\}$ a basis for the root system. Define positive integers $a_1, ..., a_l$ by

$$\prod_{\alpha \in \Phi(X_l)^+} \alpha = \prod_{i=1}^{l} \alpha_i^{a_i}.$$

The denominator has the special form

$$Q(X,Y)=(1-Y^m)\prod_{i=1}^{l}(1-X^{a_i}Y^{b_i})$$

where m and $b_1,...,b_l$ are integers arising from the choice of representation ρ.

We prove the following theorem announced in [6]:

Theorem 1.5 *The zeta function $Z_{\mathfrak{G},\rho}(s)$ of a $\mathbb{Q}$-split reductive algebraic group $\mathfrak{G}$ of type A_l, B_l, C_l and D_l and their natural representations ρ have friendly ghosts.*

Note that the special shape of the denominator means that $\prod_p Q(p,p^{-s})$ is meromorphic on $\mathbb{C}$ (and hence is its own ghost) and so we can concentrate on the products of the numerator. Explicit expressions are proved for the ghost zeta functions $\widetilde{P}(X,Y)$ in terms of the combinatorial data of the associated root system:

Theorem 1.6 *(1) If $\mathfrak{G}=\mathrm{GL}_{l+1}$ is of type A_l, then*

$$Z_{\mathfrak{G}}(s)=\zeta(s)...\zeta(s-l)=\widetilde{Z}_{\mathfrak{G}}(s).$$

(2) If $\mathfrak{G}=\mathrm{GO}_{2l+1}$ is of type B_l, then

$$\frac{\widetilde{P}(X,Y)}{Q(X,Y)}=\frac{(1+Y)\prod_{i=1}^{l-1}(1+X^{a_i}Y)}{(1-Y)\prod_{i=1}^{l-1}(1-X^{a_i}Y)}.$$

(For the natural representation $m=b_1=...=b_l=1$.)

(3) If $\mathfrak{G}=\mathrm{GSp}_{2l}$ is of type C_l, then

$$\frac{\widetilde{P}(X,Y)}{Q(X,Y)}=\frac{(1+XY)\prod_{i=1}^{l-2}(1+X^{(a_i/2)+1}Y)}{(1-Y)(1-X^{a_l}Y)\prod_{i=1}^{l-1}(1-X^{a_i/2}Y)}.$$

(For the natural representation $m=b_1=...=b_{l-1}=2$ and $b_l=1$.)

(4) If $\mathfrak{G}=\mathrm{GO}_{2l}^{+}$ is of type D_l, then

$$\frac{\widetilde{P}(X,Y)}{Q(X,Y)}=\frac{(1+Y)\prod_{i=1}^{l-2}(1+X^{a_i/2}Y)}{(1-Y)(1-X^{a_{l-1}}Y)(1-X^{a_l}Y)\prod_{i=1}^{l-2}(1-X^{a_i/2}Y)}.$$

(For the natural representation $m=b_1=...=b_{l-2}=2$ and $b_{l-1}=b_l=1$.)

Note that in (3) and (4) some cancelling has been made between common factors of $\widetilde{P}(X,Y)$ and $Q(X,Y)$ of the form $(1+Y)\prod_{i=1}^{l-1}(1+X^{a_i/2}Y)$ and $(1+Y)\prod_{i=1}^{l-2}(1+X^{a_i/2}Y)$ respectively. In fact the first author has shown by using certain symmetries of the root system that these common factors are in fact factors of the original polynomial $P(X,Y)$.

The classical work of Hey [15] and Tamagawa [20] for the case of GL_{l+1} proved that the associated zeta function, and hence the zeta functions of central simple algebras, have meromorphic continuation to the whole complex plane. In contrast the first author in [4] has used the proven part of Conjecture 1.2 and the ghosts calculated above to prove that $Z_{\mathfrak{G},\rho}(s)$ has a natural boundary for each $\mathbb{Q}$-split reductive algebraic group $\mathfrak{G}$ of type B_l, C_l and D_l with their natural representations ρ. The precise location of the natural boundary can be determined in terms of the combinatorial data of the root system (see the end of section 3 for a precise statement). As far as we are aware, this is the first time that this analytic character of these zeta functions has been determined.

In general, the irreducible representations of the $\mathbb{Q}$-split reductive algebraic groups of type X_l are parameterized by the dominant integral weight lattice $\Lambda^+ = \{m_1\varpi_1 + \cdots + m_l\varpi_l : m_i \in \mathbb{N}\}$ spanned by the fundamental dominant weights $\varpi_1, ..., \varpi_l$.

The ghost zeta function implies an interesting dichotomy between those weights ϖ whose associated representation gives rise to a friendly ghost and those with unfriendly ghosts. We therefore make the following:

Definition 1.5 *Call an irreducible representation ρ of a $\mathbb{Q}$-split reductive algebraic group $\mathfrak{G}$ of type X_l* friendly *if the associated zeta function $Z_{\mathfrak{G},\rho}(s)$ has a friendly ghost. Otherwise call the representation* unfriendly.

It seems an interesting question to understand how this dichotomy divides the associated weight lattice Λ^+. In section 5 we prove the following:

Theorem 1.7 *(1) $(A_2, n_1\varpi_1 + n_2\varpi_2)$ is friendly if and only if $n_1 = 0$ or $n_2 = 0$.*

(2) In $C_2(= B_2)$ let $\varpi_1 = \alpha_1 + (1/2)\alpha_2$ and $\varpi_2 = \alpha_1 + \alpha_2$ where α_1 is the short root. Then $n_1\varpi_1 + n_2\varpi_2$ is unfriendly if and only if $n_1 = 2n_2 \neq 0$.

(3) In G_2 let $\varpi_1 = 2\alpha_1 + \alpha_2$ and $\varpi_2 = 3\alpha_1 + 2\alpha_2$ where α_1 is the short root. Then $n_1\varpi_1 + n_2\varpi_2$ is unfriendly if and only if $n_1 = 3n_2 \neq 0$.

We have documented in section 4 for the $\mathbb{Q}$-split simply connected algebraic groups of exceptional type (i.e. E_6, E_7, E_8, F_4 and G_2) which of the fundamental weights are friendly and a description of the corresponding ghost zeta function. Unlike the case of Theorem 1.5, we only have a finite number of cases to check and hence we can use a computer to calculate $Z_{\mathfrak{G},\rho}(s)$ and their ghosts. To evaluate Igusa's explicit expression for $Z_{\mathfrak{G},\rho}(s)$ in the case of E_8 requires quantifying over all elements of the associated Weyl group $W(E_8)$, something which is computational too heavy in general. However we have written a very short computer program in the language GAP which calculates very quickly enough of $Z_{\mathfrak{G},\rho}(s)$ to be able to deduce an expression for the ghost $\widetilde{Z}_{\mathfrak{G},\rho}(s)$ using the Newton polygon. There are some quite striking cyclotomic polynomials which appear in the ghosts of these exceptional groups. For example, consider the case of the first fundamental weight of E_6 which has the following expression in terms of the root lattice

$$\varpi_1 = 1/3(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6)$$

(where we are using the notation of Chapter VI section 4 of [1]). Then the numerator $P(X,Y)$ of $Z_{\mathfrak{G},\rho}(s)$ has a friendly ghost with the following description as a product of cyclotomic polynomials:

$$\begin{aligned}&(1+Z_1)\left(1+Z_2^2+Z_2^3+Z_2^4+Z_2^5+Z_2^7\right)\left(1+Z_3^2+Z_3^3+Z_3^4+Z_3^6\right)\\&\left(1+Z_4^2+Z_4^3+Z_4^5\right)\left(1+Z_5^2+Z_5^4\right)\end{aligned}$$

where

$$Z_1 = X^{15}Y, Z_2 = X^7Y, Z_3 = X^6Y, Z_4 = X^4Y, Z_5 = Y.$$

(Note that $1 + Z^2 + \cdots + Z^{n-2} + Z^n = (1 - Z + Z^2)(1 + Z + \cdots + Z^{n-2})$ and $(1 - Z + Z^2)$ is a factor of $(1 - Z^6)$.)

A summary of the results is as follows where the fundamental weights are indexed as Chapter VI section 4 of [1]:

Theorem 1.8 *(1) In E_6 the weights ϖ_1 and ϖ_6 are friendly whilst $\varpi_2, \varpi_3, \varpi_4$ and ϖ_5 are unfriendly.*

(2) In E_7 the weights $\varpi_1, \varpi_2, \varpi_6$ and ϖ_7 are friendly whilst ϖ_3, ϖ_4 and ϖ_5 are unfriendly.

(3) In E_8 the weights $\varpi_1, \varpi_2, \varpi_6, \varpi_7$ and ϖ_8 are friendly whilst ϖ_3, ϖ_4 and ϖ_5 are unfriendly.

(4) In F_4 and G_2 all the fundamental weights are friendly.

The ghost provides therefore both a new invariant associated to a root system and choice of weight, but it also provides a powerful tool in determining the analytic properties of the original zeta function of the algebraic group.

Another important source of such zeta functions in our setting comes from other zeta functions that can be defined for an arbitrary residually finite, finitely generated group G. In Definition 1.2 (2) zeta functions captured a special lattice of subgroups. In [13] zeta functions are defined which capture the whole lattice of subgroups $\mathcal{X}^{\leq}$ or the lattice of normal subgroups $\mathcal{X}^{\triangleleft}$:

$$\begin{aligned} \zeta_G(s) &= \zeta_G^{\leq}(s) = \sum_{H \in \mathcal{X}^{\leq}} |G:H|^{-s} \\ \zeta_G^{\triangleleft}(s) &= \sum_{H \in \mathcal{X}^{\triangleleft}} |G:H|^{-s}. \end{aligned}$$

When G is virtually soluble of finite rank then these functions converge for s in some right half of the complex plane. They were first introduced in a paper [13] by the second author, Segal and Smith and studied in the particular case that G is nilpotent. In this case the zeta functions have a natural Euler product: for $* \in \{\leq, \triangleleft\}$

$$\zeta_G^*(s) = \prod_{p \text{ prime}} \zeta_{G,p}^*(s).$$

In section 6 we document the ghosts of zeta functions of various nilpotent groups. The examples constitute a list of all explicit calculations that have so far been made of the zeta functions $\zeta_G(s)$ and $\zeta_G^{\triangleleft}(s)$ (see [13] and [12]). All the examples are finitely universal in the following sense:

Definition 1.6 *(1) We call* $Z(s)$ universal *if there exist polynomials* $W(X,Y)$ *and* $V(X,Y)$ *such that for almost all primes* p,

$$Z_p(s) = \frac{W(p,p^{-s})}{V(p,p^{-s})}$$

and the denominator has the special form

$$V(X,Y) = \prod_{i=1}^{l} (1 - X^{a_i} Y^{b_i}).$$

(2) We call $Z(s)$ finitely universal *if there exist finitely many polynomials $W_1(X,Y), ..., W_r(X,Y)$ and a polynomial $V(X,Y)$ such that for each prime p there exists $1 \leq i \leq r$ and*

$$Z_p(s) = \frac{W_i(p, p^{-s})}{V(p, p^{-s})}$$

and $V(X,Y)$ again has the special form detailed in (1). We shall put $i_p = i$ if $Z_p(s) = \frac{W_i(p,p^{-s})}{V(p,p^{-s})}$.

Since all the examples for which we have an explicit computation are finitely universal, we can therefore talk about their associated ghost functions. If $Z(s)$ is finitely universal then put $i_p = i$ if $Z_p(s) = \frac{W_i(p,p^{-s})}{V(p,p^{-s})}$ and define the ghost to be

$$\widetilde{Z}(s) = \prod_{p \text{ prime}} \frac{\widetilde{W_{i_p}}(p, p^{-s})}{V(p, p^{-s})}.$$

As above we call $\widetilde{Z}(s)$ a friendly ghost if it has meromorphic continuation to $\mathbb{C}$. These examples arise when we consider zeta functions attached to matrix groups over number fields, for example see section 6.2. However we do not have such a nice criterion for being friendly now since even if $\widetilde{W_{i_p}}(X,Y)$ is a product of cyclotomic polynomials, the corresponding partition of the primes according to whether $i_p = i$ might cause $\widetilde{Z}(s)$ to have a natural boundary.

For example we would conjecture the following which may already be known:

Conjecture 1.9 *Let $Z(s) = \prod_{p \equiv 1 \mod 4}(1 - p^{-s})$. Then $Z(s)$ has a natural boundary beyond which the function cannot be meromorphically continued.*

The question was asked in [13] whether it might be true in general for an arbitrary nilpotent group that their zeta functions are finitely universal. However an example due to the first author of a class two nilpotent group whose local zeta functions depend on counting points $\mod p$ on the elliptic curve $Y^2 = X^3 - X$ shows that this is not the case (see [8]). The example arose out of theoretical work contained in [7] which proves that $\zeta_G(s)$ and $\zeta_G^{\triangleleft}(s)$ always have some meromorphic continuation beyond the region of convergence. The paper shows that in general the local factors of a nilpotent group depend on counting points $\mod p$ on varieties. Explicit formulae are given for the local factors in terms of the resolution of singularities for an associated polynomial.

It seems that the phenomenon discovered in this paper, that the zeros of the local factors have a limit as p tends to infinity, does not extend to zeta functions which are not universal or finitely universal. For example, the behaviour of the zeros of the Hasse-Weil zeta function of an elliptic curve is conjectured to be in general wild.

Take for example the Hasse-Weil zeta function $L(E,s)$ of the elliptic curve $E := Y^2 - X^3 + X$, where this wild behaviour was demonstrated by Hecke [14]. Its local factor at a prime $p \neq 2$ is

$$1 - a_p p^{-s} + p^{1-2s} = \left(1 - \pi p^{-s}\right)\left(1 - \overline{\pi} p^{-s}\right)$$

where $p - a_p$ is the number of points $\mod p$ in E, a formula which appears as the last entry in the diaries of Gauss. It is known that $|\pi| = |\overline{\pi}| = p^{1/2}$ for all p but the arguments of the $\pi, \overline{\pi}$ for $p \equiv 1 \mod 4$ lie densely on the unit circle (see [14]). In particular, $\pi \cdot p^{-1/2}$ will not have a limit as p tends to infinity, although the absolute values are all 1. This dense behaviour of the arguments is conjectured to be a general phenomenon for the local factors of the Hasse-Weil zeta function of a smooth curve, related to the Sato-Tate Conjectures. The absolute values are known due to the Riemann hypothesis for the associated local zeta functions.

More generally the Riemann hypothesis for a smooth projective variety V implies that the zeros of the local factors of the Hasse-Weil zeta function of V can be ordered as $\omega_{p,1}, ..., \omega_{p,n}$ so that for each i some normalization exists with the property that $\omega_{p,i} \cdot p^{n_i}$ has constant absolute value.

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2 Definition of the ghost

Let $W(X,Y) \in \mathbb{C}[X,Y]$. We are interested in the behaviour of the zeros of the polynomial $W(p,Y)$ as the prime p tends to infinity. We shall suppose $W(X,0) = 1$. This is a sensible assumption if we are going to be considering

$W(X,Y)$ which give rise to an analytic function $\prod_p W(p,p^{-s})$ on some right half complex plane. Let l be the degree of Y in $W(X,Y)$.

It turns out that the zeros of $W(p,Y)$ have the property that they can be indexed $\alpha_{p,i}$ with $1 \le i \le l$, such that for each i there exists some fractional power $p^{d_i/q}$ with the property that $\alpha_{p,i}p^{d_i/q}$ has a non-zero limit c_i as p tends to infinity. The *ghost polynomial* that we define in this section will be built from the limit points of these zeros. The reason why these non-zero limits exist is provided by the Puiseux power series expression for zeros as we shall now explain.

It is easier to consider the behaviour of the Puiseux power series at 0. Therefore put $R(X,Y)$ equal to the polynomial

$$\begin{aligned} R(X,Y) &= X^{\deg_X W} W(X^{-1},Y) \\ &= a_l(X)Y^l + \cdots a_1(X)Y + a_0(X) \end{aligned}$$

where $a_i(X) \in \mathbb{C}[X]$ and has degree n_i. Note that since $W(X,Y)$ is not divisible by X, $\deg_X R = \deg_X W = d_X$, say.

There exist $q \in \mathbb{N}$ and finite tailed Laurent series $\Omega_1(X), ..., \Omega_l(X)$ in fractional powers $X^{1/q}$ of X with coefficients in the algebraic closure of $\mathbb{Q}$ with the property that in some neighbourhood of $X = 0$, the zeros of $R(X,Y) = 0$, thought of as a polynomial in Y, are given by the power series $\Omega_1(X), ..., \Omega_l(X)$. These power series are known as the *Puiseux power series* and generalize the implicit function theorem by giving an expression for all the zeros in a neighbourhood of a point.

We shall be interested in the first term of the Puiseux power series:

$$\Omega_i(X) = c_i \left(X^{1/q}\right)^{e_i} + O(X^{(1/q)(e_i+1)})$$

where the symbol $O(X^{(1/q)(e_i+1)})$ represents a power series in $X^{1/q}$ starting with $X^{(1/q)(e_i+1)}$ or some higher term and $e_i \in \mathbb{Z}$.

Definition 2.7 *The* ghost polynomial $\widetilde{W}(X,Y)$ *of* $W(X,Y)$ *is defined to be*

$$\widetilde{W}(X,Y) = X^{d_X - n_l} \prod_{i=1}^{l} (YX^{e_i/q} - c_i).$$

(Note that since the Puiseux series are in general Laurent series d_i may be negative. We shall see in the combinatorial analysis below that multiplying by $X^{d_X - n_l}$ ensures that the ghost is a polynomial.) The zeros of $W(p,Y)$ are given by $\Omega_i(p^{-1})$. Note that, as stated on page 102 of [21], if the coefficients $a_i(X)$ converge in some region then each root $\Omega_i(X)$ also converges in the same region. In our case the $a_i(X)$ are polynomials so $\Omega_i(X)$ converges for all p. By the power series expression for Ω_i, we see that these zeros do indeed have the property that $\Omega_i(p^{-1}) \cdot p^{e_i/q}$ tends to c_i as p tends to infinity. The ghost polynomial $\widetilde{W}(X,Y)$ may therefore equivalently be defined as the polynomial with the property that $\widetilde{W}(p,Y)$ has zeros $c_i \cdot p^{-e_i/q}$ for all primes p.

The nature of the zeros of the original polynomial $W(p,Y)$ is what gave rise to natural boundaries in the associated zeta function

$$Z(s) = \prod_{p \text{ prime}} W(p, p^{-s}).$$

The reason as explained in [4] is that the zeros of $W(p,Y)$ are not exactly at $c_i \cdot p^{-e_i/q}$ but, owing to the $O(X^{(1/q)(e_i+1)})$ term, are shifted away from this limiting zero. As p tends to infinity, as we have seen, these zeros tend to the limiting zeros $c_i \cdot p^{-e_i/q}$. If the $O(X^{(1/q)(e_i+1)})$ term places the zeros in a region of analytic continuation of $Z(s)$, as seems to always happen in zeta functions arising from groups, then we get a natural boundary at $\Re(s) = e_i/q$ for some i since every point on this boundary can be realised as limit points of zeros of $\prod_{p \text{ prime}} W(p,p^{-s})$.

What then if we try to remove this natural boundary by taking the zeta function whose local factors all have zeros at these limiting zeros rather than shifted by some error term? Namely what if we take the Euler product of the ghost polynomials? Is it possible that this will no longer have a natural boundary since we have removed the barrier to analytic continuation coming from the zeros being shifted away from $\Re(s) = e_i/q$.

We therefore make the following:

Definition 2.8 *The* ghost zeta function $\widetilde{Z}(s)$ *of* $Z(s) = \prod_{p\ prime} W(p,p^{-s})$ *is defined to be the Euler product constructed from the associated ghost polynomial* $\widetilde{W}(X,Y)$ *:*

$$\widetilde{Z}(s) = \prod_{p\ prime} \widetilde{W}(p, p^{-s}).$$

We shall say that the zeta function $Z(s)$ *has a* friendly ghost *if* $\widetilde{Z}(s)$ *has meromorphic continuation to the whole of the complex plane.*

Estermann's Theorem (which in its most general form can be found in [3]) tells us when the ghost will be friendly:

Proposition 2.10 *Let* $h(Y) = 1 + a_1Y + \cdots a_dY^d = \prod(1-\alpha_iY) \in \mathbb{C}[Y]$. *Set* $Z(s) = \prod_p h(p^{-s})$, *which converges for* $\Re(s) > 1$. *Then*

(1) $Z(s)$ *can be meromorphically continued to* $\Re(s) > 0$.

(2) If $|\alpha_i| = 1$ *for* $i = 1, ..., d$ *(in which case* $h(Y)$ *is a product of cyclotomic polynomials) then* $Z(s)$ *can be meromorphically continued to the whole complex plane. Otherwise* $\Re(s) = 0$ *is a natural boundary.*

Note that the ghost is a polynomial with coefficients in $\mathbb{C}$ but need not have constant coefficient 1. This follows from Proposition 2.12 below which constructs the ghost from the Newton Polygon. It is clear that if the constant coefficient is not 1 or -1 then the associated Euler product will not be meromorphic anywhere. So actually the ghost could make the analytic behaviour worse than before! Estermann's Theorem gives us then the following criterion for the ghost zeta function to be friendly:

Proposition 2.11 *The ghost zeta function* $\widetilde{Z}(s)$ *is friendly if and only if, for each* $i = 1, ..., l$, c_i *is on the unit circle and hence a root of unity.*

By Proposition 2.11 it is far from guaranteed that the ghost will be friendly. The rest of the paper concerns the frequent occurrence of friendly ghosts in the context of zeta functions associated to groups.

We conclude this section with an easy geometric way to calculate the ghost zeta function which will be our principal tool in the ensuing calculations.

Let $\mathcal{L}_W$ denote the set of lattice points (n, m) which correspond to the occurrence of a non-zero term X^nY^m in $W(X, Y)$. For each $1 \leq m \leq l$, $n_m = \max\{n : (n, m) \in \mathcal{L}_W\}$. The Newton polygon consists of the lines marking out the convex hull of these lattice points. We shall be interested in lines starting from the point $(n_0, 0)$ and ascending to the line with end point (n_l, l) marking out the right hand convex hull which we call $\mathcal{N}$. Let $\mathfrak{l}_1, ..., \mathfrak{l}_r$ be the lines making up $\mathcal{N}$.

For each line $\mathfrak{l}_i$, let $m_i = \min\{m : (n, m) \in \mathfrak{l}_i \cap \mathcal{L}_W\}$. Let u_i and v_i be coprime with the property that

$$v_i/u_i = \frac{(m_{i+1} - m_i)}{(n_{m_{i+1}} - n_{m_i})},$$

the gradient of the line $\mathfrak{l}_i$. Consider the quasi-homogeneous polynomial made up of all the homogeneous components of $W(X,Y)$ sitting on the line $\mathfrak{l}_i$:

$$\sum_{(n,m)\in\mathfrak{l}_i\cap\mathcal{L}_W} a_{n,m}X^nY^m = X^{n_{m_i}}Y^{m_i}\widetilde{W_i}(X^{u_i}Y^{v_i})$$

where $\widetilde{W_i}(Z)$ is a polynomial of degree $(m_{i+1}-m_i)/v_i$ in one variable with non-zero constant coefficient $a_{n_{m_i},m_i}$.

Theorem 2.12 *The ghost polynomial of* $W(X,Y)$ *is*

$$\widetilde{W}(X,Y) = X^{d_X-n_l}\prod_{i=1}^{r}\widetilde{W_i}(X^{u_i}Y^{v_i}).$$

Proof The Puiseux series are formal solutions to a polynomial in Y with coefficients in $\mathbb{Q}[X]$. Appendix B of [19] gives a constructive procedure to construct these solutions, the first stage of which gives an expression for the leading term from which we are defining our ghosts.

One begins by substituting into $R(X,Y)$ an undetermined series

$$\overline{Y} = X^w(c+c^{'}X^{w^{'}}+\cdots).$$

Let us suppose that

$$R(X,Y) = a_{r_k}(X)Y^{r_k}+\cdots a_{r_1}(X)Y^{r_1}+a_0(X)$$

where $a_{r_j}(X) = a_{n_{r_j},r_j}X^{d_X-n_{r_j}}+O(X^{d_X-n_{r_j}+1})$ is non-zero. We get after this substitution:

$$\begin{aligned} R(X,\overline{Y}) &= (a_{n_0,0}X^{d_X-n_0}+\cdots) \\ &\quad +(a_{n_{r_1},r_1}X^{d_X-n_{r_1}}+\cdots)X^{r_1w}(c+\cdots)^{r_1} \\ &\quad +\cdots \\ &\quad +(a_{n_{r_k},r_k}X^{d_X-n_{r_k}}+\cdots)X^{r_kw}(c+\cdots)^{r_k}. \end{aligned} \tag{1}$$

Here the expressions shown are those of minimal degree and $+\cdots$ represents a power series or polynomial with terms of higher degree. If we now expand the right hand side and rearrange as a fractional power series in X, a necessary condition for this expression $R(X,\overline{Y})$ to be zero is that the terms of each separate order that is represented must cancel, in particular the terms of the lowest order. These are to be found amongst the terms of lowest order in each line of (1), i.e. among the terms

$$a_{n_0,0}X^{d_X-n_0}, a_{n_{r_1},r_1}c^{r_1}X^{d_X-n_{r_1}+r_1w}, \dots, a_{n_{r_k},r_k}c^{r_k}X^{d_X-n_{r_k}+r_kw}$$

although not all these terms will appear.

Recall that we are trying to find at least necessary conditions on c and w_1 to make $R(X,\overline{Y})$ vanish. It is necessary for the lowest term to vanish that there be at least two terms occurring in this list with lowest exponent and that the sum of the coefficients of all the terms must be zero. This puts the condition on w that at least two of the numbers

$$d_X - n_0, d_X - n_{r_1} + r_1 w, ..., d_X - n_{r_k} + r_k w$$

must be equal and less than the rest. The key to this is the Newton polygon. Plot points P_{r_j} with coordinates $(d_X - n_{r_j}, r_j)$ and draw lines starting from the point P_0, round to P_{r_k} marking out the left hand convex hull which we shall call $\overline{\mathcal{N}}$. The condition that we have to satisfy, namely that there exist r_a and r_b such that for all $i = 0, ..., k$

$$d_X - n_{r_a} + r_a w = d_X - n_{r_b} + r_b w \leq d_X - n_{r_j} + r_j w,$$

means precisely that the line joining P_{r_a} and P_{r_b} is a segment of $\overline{\mathcal{N}}$. It follows that the possible values of w are

$$\frac{n_{r_a} - n_{r_b}}{r_a - r_b}$$

namely the negative of the inverse of the gradients of the lines making up $\overline{\mathcal{N}}$. When one of these lines $\overline{\mathfrak{l}_i}$ say, has been selected, with $w(\overline{\mathfrak{l}_i})$ the corresponding value of w, there are only finitely many possible values of c which will make the sum of all the coefficients of all the terms of lowest exponent vanish, namely those given by the equation

$$\sum_{P_{r_j} \in \overline{\mathfrak{l}_i}} a_{n_{r_j},r_j} c^{r_j} = 0.$$

Note that we are assuming that c is a non-zero solution. Therefore c is a solution to

$$\sum_{P_{r_j} \in \overline{\mathfrak{l}_i}} a_{n_{r_j},r_j} c^{r_j - r(\overline{\mathfrak{l}_i})} = 0$$

where $r(\overline{\mathfrak{l}_i}) = \min\{r_j : P_{r_j} \in \overline{\mathfrak{l}_i}\}$. Put $C(\overline{\mathfrak{l}_i})$ to be the values of c corresponding to $\overline{\mathfrak{l}_i}$. This gives then the first approximation to the l solutions for Y of $R(X,Y) = 0$. One goes on to use further Newton polygons to successively approximate the Puiseux power series. However for our ghost we are happy to know that the first term is given by the expressions

$$cX^{w(\overline{\mathfrak{l}_i})}$$

for $c \in C(\overline{\mathfrak{l}_i})$ and $\overline{\mathfrak{l}_i} \in \overline{\mathcal{N}}$. The ghost polynomial is therefore given by:

$$X^{d_X - n_l} \prod_{\overline{\mathfrak{l}_i} \in \overline{\mathcal{N}}} \prod_{c \in C(\overline{\mathfrak{l}_i})} \left(Y X^{w(\overline{\mathfrak{l}_i})} - c \right).$$

We now show how to relate this to the convex hull $\mathcal{N}$ defined from $W(X,Y)$. $\overline{\mathcal{N}}$ is just the reflection of $\mathcal{N}$ in the vertical axis with an additional translation to the right by d_X. The lines $\overline{\mathfrak{l}_i}$ making up the left convex hull $\overline{\mathcal{N}}$ then correspond to the lines $\mathfrak{l}_i$ of $\mathcal{N}$ under this transformation and $\mathfrak{l}_i \cap \mathcal{L}_W = \left\{ (n_{r_j}, r_j) : P_{r_j} = (d_X - n_{r_j}, r_j) \in \overline{\mathfrak{l}_i} \right\}$. Hence $w(\overline{\mathfrak{l}_i})$ is the inverse of the gradient of the corresponding line $\mathfrak{l}_i$ of $\mathcal{N}$, namely v_i/u_i. Now

$$\begin{aligned} \prod_{c \in C(\overline{\mathfrak{l}_i})} \left(Y X^{w(\overline{\mathfrak{l}_i})} - c \right) &= \sum_{P_{r_j} \in \overline{\mathfrak{l}_i}} a_{n_{r_j}, r_j} \left(Y X^{w(\overline{\mathfrak{l}_i})} \right)^{r_j - r(\overline{\mathfrak{l}_i})} \\ &= X^{-n_{m_i}} Y^{-m_i} \sum_{(n,m) \in \mathfrak{l}_i \cap \mathcal{L}_W} a_{n,m} X^n Y^m. \end{aligned}$$

Hence the ghost polynomial is

$$X^{d_X - n_l} \prod_{\mathfrak{l}_i \in \mathcal{N}} \left(X^{-n_{m_i}} Y^{-m_i} \sum_{(n,m) \in \mathfrak{l}_i \cap \mathcal{L}_W} a_{n,m} X^n Y^m \right) = X^{d_X - n_l} \prod_{i=1}^{r} \widetilde{W}_i(X^{u_i} Y^{v_i}).$$

We point out that to ensure that this is in fact a polynomial we need to check that

$$d_X - n_l = - \sum_{w(\overline{\mathfrak{l}_i}) < 0} w(\overline{\mathfrak{l}_i}) (r(\overline{\mathfrak{l}_{i+1}}) - r(\overline{\mathfrak{l}_i})).$$

This follows since the left hand side is the horizontal distance between the point $(0, r(\overline{\mathfrak{l}_i}))$ at which the lines of $\overline{\mathcal{N}}$ turn to the right and ascend to the last point $(d_X - n_{r_k}, r_k) = (d_X - n_l, l)$.

3 Zeta functions of algebraic groups

We begin in this section by explaining an explicit formula for the zeta function of an algebraic group as defined in the introduction. This explicit formula was first developed in [16] for split algebraic groups and irreducible representations and extended in [9] and [5] by the first author and Lubotzky to more general settings. We begin by broadening the definition from the introduction to a algebraic groups defined over an arbitrary field:

Definition 3.9 *Let G be a linear algebraic group over a field k. Let $\rho : G \to \mathrm{GL}_n$ be a k-rational representation.*

(1) If k is a finite extension of $\mathbb{Q}_p$ we define the 'zeta function of the algebraic group G at the representation ρ' to be

$$Z_{G(k),\rho}(s) = \int_{G^+} |\det \rho(g)|^s \, \mu_G(g)$$

where $G^+ = \rho^{-1}\left(\rho\left(G\left(k\right)\right) \cap \mathrm{M}_n\left(\vartheta_k\right)\right)$, ϑ_k is the ring of integers of k and μ_G denotes the right Haar measure on $G(k)$ normalized such that $\mu_G\left(G\left(\vartheta_k\right)\right) = 1$.

(2) If k is a finite extension of $\mathbb{Q}$ and $\mathfrak{p}$ is a prime of k we define the 'local zeta function of the algebraic group G at the representation ρ and the prime $\mathfrak{p}$ ' to be

$$Z_{G(k),\rho,\mathfrak{p}}(s) = Z_{G(k_{\mathfrak{p}}),\rho}(s)$$

We define the 'global zeta function of G at the representation ρ' to be Euler product

$$Z_{G(k),\rho}(s) = \prod_{\mathfrak{p}} Z_{G(k),\rho,\mathfrak{p}}(s)$$

In [9] and [16] formulae were given for the evaluation of these integrals for certain algebraic groups and representations. Some care should be exercised as the definition of our zeta function here differs slightly from that in Igusa's paper [16]. He uses the canonical measure μ_c which is identical to the Haar measure on $G\left(\vartheta_k\right)$ but is approximated by some function of the Haar measure on the whole group. Also his normalisation is at the Iwahori subgroup rather than at $G\left(\vartheta_k\right)$. He also chooses a generator f for the group $\mathrm{Hom}(G, \mathbb{G}_m)$ as his integrand rather than the determinant of the representation. The above definitions are the most natural in our context since they are precisely those that define the zeta functions of nilpotent groups. We therefore take our formulae from [9]. Most of the differences between the two definitions are cosmetic though.

However we shall be restricting ourselves to irreducible representations of the split reductive algebraic groups (or Chevalley groups) which were precisely the cases considered by Igusa.

The formulae are defined in terms of combinatorial data associated with the Dynkin diagram corresponding to the Chevalley group and information about the weights of the irreducible representation. We shall identify $G(k)$ with its image as an irreducible subgroup of $\mathrm{GL}_n(k)$. We can then write $G(k)$ as an almost direct product of a one dimensional central torus $\mathbb{G}_m 1_n$ (which we shall assume is precisely the scalar matrices in $\mathrm{GL}_n(k)$) and the derived group G'. The derived group is assumed to be a split semisimple algebraic group or Chevalley group whose Dynkin diagram we shall denote by X_l where l is the rank of the Chevalley group. Let T denote the maximal split torus which we shall assume consists of all the diagonal matrices in $G(k)$. Assume $G(k)$ has very good reduction. In the case of a global field k, $G(k_\mathfrak{p})$ has good reduction for almost all primes $\mathfrak{p}$. Choose a basis $\Pi = \{\alpha_1, ..., \alpha_l\}$ for the root system Φ of G and by ϖ the dominant weight of the contragredient (irreducible) representation $\rho^* =^t \rho^{-1}$. Let m denote the order of the centre of G' and by μ_m the group of mth roots of unity so that $\mathbb{G}_m 1_n \cap G' = \mu_m 1_n$. (Warning: these are still as algebraic groups so the centre of $G'(k)$ may be trivial since $\mu_m(k)$ could be trivial.) Let f be the generator for $\mathrm{Hom}(G, \mathbb{G}_m)$ which satisfies $f^{n/m} = \det$. Let α_0 denote the restriction of f to T. Then there exist integers $b_i > 0$ for $1 \leq i \leq l$ such that

$$\varpi^m = \alpha_0^{-1} \cdot \prod_{i=1}^{l} \alpha_i^{b_i}.$$

The second set of numerical data we need for a formula are the positive integers $a_1, ..., a_l$ which express the sum of the positive roots in terms of the primitive roots:

$$\prod_{\alpha \in \Phi^+} \alpha = \prod_{i=1}^{l} \alpha_i^{a_i}.$$

We can now write down our formula for the zeta function. Let q denote the size of the residue field of k and W the finite Weyl group of Φ and $\lambda(w)$

the length of an element w of the Weyl group in terms of the fundamental reflections in the hyperplanes defined by the primitive roots, then

$$Z_{G(k),\rho}(s) = \frac{\left(\sum_{w\in W} q^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} q^{a_j - b_j(n/m)s}\right)}{\left(1 - q^{-m(n/m)s}\right) \prod_{j=1}^{l} \left(1 - q^{a_j - b_j(n/m)s}\right)}. \tag{2}$$

Define two polynomials $P(X,Y)$, $Q(X,Y) \in \mathbb{Z}[X,Y]$ by

$$\begin{aligned} P(X,Y) &= \left(\sum_{w\in W} X^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} X^{a_j} Y^{b_j}\right) \\ Q(X,Y) &= (1 - Y^m) \prod_{j=1}^{l} \left(1 - X^{a_j} Y^{b_j}\right) \end{aligned}$$

so that $Z_{G,\rho}(s) = \frac{P(p,p^{-(n/m)s})}{Q(p,p^{-(n/m)s})}$. Then it was proved in [16] and [9] that these polynomials satisfy a functional equation

$$\frac{P(X^{-1},Y^{-1})}{Q(X^{-1},Y^{-1})} = (-1)^{l+1} X^{\mathrm{card}(\Phi^+)} Y^m \frac{P(X,Y)}{Q(X,Y)}.$$

We are concerned with calculating the ghost zeta function. Since $Q(X,Y)$ is already its own ghost, we concentrate on the polynomial $P(X,Y)$. Note that this polynomial is a natural invariant now for the root system and some choice of weight in the weight space of the root system. It will be of interest to understand more generally the algebraic geometry of this curve. We concentrate in this paper on the ghost which indicates the nature of the curve as $X \to \infty$. By the last section, to calculate the ghost we need to construct the associated Newton Polygon marking out the right-hand side of the convex hull of the lattice points (n,m) corresponding to homogeneous components $X^n Y^m$.

The following proposition makes the calculation of the ghost both theoretically and computationally much easier:

Proposition 3.13 *The polynomials $P(X,Y)$ and*

$$\overline{P}(X,Y) = \left(\sum_{\Pi_0 \subset \Pi} X^{-\mathrm{card}(\Phi_0^+)} \prod_{\alpha_j \in \Pi_0} X^{a_j} Y^{b_j}\right)$$

have the same ghost polynomial where Φ_0 is the subroot system of Φ generated by Π_0.

Proof In general, a polynomial $\sum_{(n,m)\in\mathcal{L}} a_{n,m}X^nY^m$ has the same ghost as $\sum_m a_{n_m,m}X^{n_m}Y^m$ where $n_m = \max\{n : (n,m) \in \mathcal{L}\}$ since the Newton polygon only depends on the extremal points of the lattice $\mathcal{L}$. For each subset Π_0 of the simple roots Π, put

$$l_{\Pi_0} = \min\left\{l(w) : w(\Pi_0) \subset \Phi^- \text{ but } w(\Pi \setminus \Pi_0) \subset \Phi^+\right\}$$

and let a_{Π_0} be the number of w attaining this minimum amongst those w satisfying $w(\Pi_0) \subset \Phi^-$ but $w(\Pi \setminus \Pi_0) \subset \Phi^+$. Then $P(X,Y)$ has the same ghost as

$$\overline{P}(X,Y) = \left(\sum_{\Pi_0\subset\Pi} a_{\Pi_0}X^{-l_{\Pi_0}} \prod_{\alpha_j\in\Pi_0} X^{a_j}Y^{b_j}\right).$$

The following lemma implies then that this is precisely the polynomial $\overline{P}(X,Y)$ described in the statement of the proposition.

Lemma 3.14 *Let Π_0 be a subset of the simple roots Π and let Φ_0 be the subroot system of Φ generated by Π_0. Then there is a unique element $w \in W$ of shortest length with the property that $w(\Pi_0) \subset \Phi^-$ but $w(\Pi \setminus \Pi_0) \subset \Phi^+$. The length of this element is* $\mathrm{card}(\Phi_0^+)$.

Proof Let w_0 be the element which maps Φ_0^+ to Φ_0^-. Certainly then $w_0(\Pi_0) \subset \Phi^-$. The length of this element is $\mathrm{card}(\Phi_0^+)$ in terms of the natural generators w_α where $\alpha \in \Pi_0$. We can't get any shorter than this using all generators w_α where $\alpha \in \Pi$ since the length is still the number of positive roots sent to negative roots in Φ which is at least $\mathrm{card}(\Phi_0^+)$. But notice that we have now shown that it is exactly that number hence $\Pi \setminus \Pi_0$ must be sent to positive roots since $(\Pi \setminus \Pi_0) \cap \Phi_0 = \emptyset$. But now the length of any element sending Π_0 to negative roots must be at least $\mathrm{card}(\Phi_0^+)$ since the length is the number of positive roots in Φ^+ sent to negative roots and if Π_0 gets sent to negative roots then so does Φ_0^+. Why is this element w_0 unique? If there is another element w with length $\mathrm{card}(\Phi_0^+)$ which maps $w(\Pi_0) \subset \Phi^-$ then $w(\Phi_0^+) \subset \Phi^-$. This identifies the $\mathrm{card}(\Phi_0^+)$ positive roots sent to negative roots. Hence $w(\Phi^+\backslash\Phi_0^+) \subset \Phi^+$. Consider the element ww_0. This maps $ww_0(\Pi_0) \subset \Phi^+$ and hence $ww_0(\Phi_0^+) \subset \Phi^+$. But $w_0(\Phi^+\backslash\Phi_0^+) \subset \Phi^+\backslash\Phi_0^+$ hence $ww_0(\Phi^+\backslash\Phi_0^+) \subset \Phi^+$. But this means ww_0 sends no positive roots to negative roots, i.e. $ww_0 = 1$. Since w_0 is self inverse we get the uniqueness that we required. This completes the proof of the lemma and also the proposition.

The calculation of the ghosts for each root system have their own special features according to the nature of the root system and the corresponding dominant weight. However there are a number of features which the calculations have in common which we record. Let us begin though with a table recording the relevant values of a_i, b_i and m for each classical group:

G of type X_l	m	a_i	b_i	$\mathrm{card}(\Phi^+)$
GL_{l+1} of type A_l	$l+1$	$i(l-i+1)$	i	$l(l+1)/2$
GO_{2l+1} of type B_l	1	$i(2l-i)$	1	l^2
GSp_{2l} of type C_l	2	$\begin{cases} i(2l-i+1) \text{ if } i<l \\ l(l+1)/2 \text{ if } i=l \end{cases}$	$\begin{cases} 2 \text{ if } i<l \\ 1 \text{ if } i=l \end{cases}$	l^2
GO_{2l}^+ of type D_l	2	$\begin{cases} i(2l-i-1) \text{ if } i<l-1 \\ l(l-1)/2 \text{ if } i\geq l-1 \end{cases}$	$\begin{cases} 2 \text{ if } i<l-1 \\ 1 \text{ if } i\geq l-1 \end{cases}$	$l(l-1)$

We record some useful combinatorial facts about these invariants (where no proof is given the lemma is regarded as an obvious calculation):

Lemma 3.15 *(a) For $X_l = B_l$, $i \leq l$,*

$$\begin{aligned} a_i &= \sum_{j=l-i+1}^{l} (2j-1); \\ a_i - a_{i-1} &= 2(l-i+1) - 1. \end{aligned}$$

(b) For $X_l = C_l$, $i \leq l-1$

$$\begin{aligned} a_i &= \sum_{j=l-i+1}^{l} 2j; \\ a_i - a_{i-1} &= 2(l-i+1). \end{aligned}$$

(c) For $X_l = D_l$, $i \leq l-2$

$$\begin{aligned} a_i &= \sum_{j=l-i}^{l-1} 2j; \\ a_i - a_{i-1} &= 2(l-i). \end{aligned}$$

In particular, all sequences a_i are strictly increasing in the corresponding range for i.

Lemma 3.16

$$\mathrm{card}B_k^+ = \mathrm{card}C_k^+ = \sum_{j=1}^{k}(2i-1)$$

$$\begin{aligned} \mathrm{card}B_{k+1}^+ - \mathrm{card}B_k^+ &= \mathrm{card}C_{k+1}^+ - \mathrm{card}C_k^+ = 2k+1 \\ \mathrm{card}D_{k+1}^+ - \mathrm{card}D_k^+ &= 2k. \end{aligned}$$

Lemma 3.17 *For $X_l = B_l$ or C_l then*

$$\begin{aligned} m_k &= \max\left\{a_{j_1} + \cdots + a_{j_k} - \mathrm{card}\left(\langle \alpha_{j_1}, ..., \alpha_{j_k}\rangle^+\right) : \{j_1, ..., j_k\} \subset \{1, ..., l-1\}\right\} \\ &= a_{l-k} + \cdots + a_{l-1} - \mathrm{card}A_k^+. \end{aligned}$$

The maximum is only achieved once.
If $X_l = D_l$ then

$$\begin{aligned} m_k &= \max\left\{ \begin{array}{c} a_{j_1} + \cdots + a_{j_k} - \mathrm{card}\left(\langle \alpha_{j_1}, ..., \alpha_{j_k}\rangle^+\right) : \\ \{j_1, ..., j_{k-1}\} \subset \{1, ..., l-2\}, j_k \in \{l-1, l\} \end{array} \right\} \\ &= a_{l-k} + \cdots + a_{l-1} - \mathrm{card}A_k^+. \end{aligned}$$

The maximum is achieved twice.

Proof In the case of D_l it is clear that we get two occurrences of the maximum value since $a_{l-1} = a_l$. Hence we shall assume henceforth that in this case $j_k = l-1$. We prove the lemma by induction. For $k=1$ the result follows for $X_l = B_l$ and C_l since a_i is strictly increasing for $i \leq l-1$. For D_l we have insisted that $j_1 = l-1$.

Suppose we have proved that $a_{l-k} + \cdots + a_{l-1} - \mathrm{card}A_k^+ > a_{j_1} + \cdots + a_{j_k} - \mathrm{card}\left(\langle \alpha_{j_1}, ..., \alpha_{j_k}\rangle^+\right)$ for all other choices of $\{j_1, ..., j_k\} \subset \{1, ..., l-1\}$ (where, in the case of D_l, we have the added restriction that $j_k = l-1$). Then for $\{j_1, ..., j_{k+1}\} \subset \{1, ..., l-1\}$ where we order the j_i (and $j_{k+1} = l-1$ for $X_l = D_l$)

$$a_{l-k-1} + ... + a_{l-1} - \mathrm{card}(A_{k+1}^+) = a_{l-k} + ... + a_{l-1} - \mathrm{card}(A_k^+) + a_{l-k-1} - (k+1)$$

whilst

$$a_{j_1} + \cdots + a_{j_{k+1}} - \mathrm{card}\left(\left\langle \alpha_{j_1}, ..., \alpha_{j_{k+1}}\right\rangle^+\right) \leq a_{j_2} + \cdots + a_{j_{k+1}} - \mathrm{card}\left(\left\langle \alpha_{j_2}, ..., \alpha_{j_{k+1}}\right\rangle^+\right) + a_{j_1} - 1$$

since the worst that can happen is that $\left\langle \alpha_{j_1}, ..., \alpha_{j_{k+1}} \right\rangle = A_1 + \left\langle \alpha_{j_2}, ..., \alpha_{j_{k+1}} \right\rangle$. Now we apply induction and the fact that $a_{l-k-1} - a_{j_1} \geq a_{l-k-1} - a_{l-k-2} \geq 2(l - (l-k-1)) = 2k+2 > k$. This completes the proof of the lemma.

In fact we won't need to consider the value of m_k in the case of B_l as we shall see in this case since $b_i = 1$ for all i. However it cost nothing to include it as a case. For B_l, the next lemma calculates the relevant value for computing the ghost:

Lemma 3.18 *(i) For $X_l = B_l$,and $k \geq 0$*

$$\overline{m_{k+1}} = \max \left\{ \begin{array}{c} a_{j_1} + \cdots + a_{j_k} + a_{j_{k+1}} - \text{card}\left(\left\langle \alpha_{j_1}, ..., \alpha_{j_k}, \alpha_{j_{k+1}} \right\rangle^+ \right) : \\ \{j_1, ..., j_k, j_{k+1}\} \subset \{1, ..., l\} \end{array} \right\}$$
$$= a_{l-k} + \cdots + a_{l-1} + a_l - \text{card} X_{k+1}^+.$$

(ii) For C_l and $k \geq 0$

$$\begin{array}{rcl} \overline{m_{k+1}} & = & \max \left\{ \begin{array}{c} a_{j_1} + \cdots + a_{j_k} + a_l - \text{card}\left(\langle \alpha_{j_1}, ..., \alpha_{j_k}, \alpha_l \rangle^+ \right) : \\ \{j_1, ..., j_k\} \subset \{1, ..., l-1\} \end{array} \right\} \\ & = & a_{l-k} + \cdots + a_{l-1} + a_l - \text{card} X_{k+1}^+. \end{array}$$

(iii) For D_l and $k \geq 1$ (where we interpret $D_2 = A_1 \times A_1$)

$$\begin{array}{rcl} \overline{m_{k+1}} & = & \max \left\{ \begin{array}{c} a_{j_1} + \cdots + a_{j_{k-1}} + a_{l-1} + a_l - \text{card}\left(\left\langle \alpha_{j_1}, ..., \alpha_{j_{k-1}}, \alpha_{l-1}, \alpha_l \right\rangle^+ \right) : \\ \{j_1, ..., j_{k-1}\} \subset \{1, ..., l-2\} \end{array} \right\} \\ & = & a_{l-k} + \cdots + a_{l-1} + a_l - \text{card} X_{k+1}^+. \end{array}$$

The maximum is only achieved once in each case.

Proof As in the previous lemma we proceed by induction. We consider first the initial value of k. The only case with any choice is $X_l = B_l$ and the maximum value follows because $a_l > a_i$.

Suppose now that we have shown that $a_{l-k} + \cdots + a_{l-1} + a_l - \text{card} X_{k+1}^+ > a_{j_1} + \cdots + a_{j_k} + a_{j_{k+1}} - \text{card}\left(\left\langle \alpha_{j_1}, ..., \alpha_{j_k}, \alpha_{j_{k+1}} \right\rangle^+ \right)$ for all other choices of $\{j_1, ..., j_{k+1}\} \subset \{1, ..., l\}$ (where, in the case of C_l, we have the added restriction that $j_{k+1} = l$ and, in the case of D_l, we have that $j_{k+1} = l$ and

$j_k = l-1$). Then for $\{j_1, ..., j_{k+2}\} \subset \{1, ..., l\}$ where we order the j_i (with the restriction $j_{k+2} = l$ if $X_l = C_l$ and $j_{k+2} = l$ and $j_{k+1} = l-1$ if $X_l = D_l$)

$$\begin{aligned} & a_{l-k-1} + \cdots + a_{l-1} + a_l - \mathrm{card} X_{k+2}^+ \\ = & \left\{ \begin{array}{c} a_{l-k} + \cdots + a_{l-1} + a_l - \mathrm{card} X_{k+1}^+ + a_{l-k-1} - (2k+3) \text{ if } X_l = B_l \text{ or } C_l \\ a_{l-k} + \cdots + a_{l-1} + a_l - \mathrm{card} X_{k+1}^+ + a_{l-k-1} - (2k+2) \text{ if } X_l = D_l \end{array} \right. \end{aligned}$$

whilst

$$\begin{aligned} & a_{j_1} + \cdots + a_{j_k} + a_{j_{k+1}} + a_{j_{k+2}} - \mathrm{card}\left(\left\langle \alpha_{j_1}, ..., \alpha_{j_{k+1}}, \alpha_{j_{k+2}} \right\rangle^+\right) \\ \leq \; & a_{j_2} + \cdots + a_{j_{k+1}} + a_{j_{k+2}} - \mathrm{card}\left(\left\langle \alpha_{j_2}, ..., \alpha_{j_{k+1}}, \alpha_{j_{k+2}} \right\rangle^+\right) + a_{j_1} - 1 \end{aligned}$$

since, as above, the worst that can happen is that $\left\langle \alpha_{j_1}, ..., \alpha_{j_{k+1}}, \alpha_{j_{k+2}} \right\rangle = A_1 + \left\langle \alpha_{j_2}, ..., \alpha_{j_{k+1}}, \alpha_{j_{k+2}} \right\rangle$. Now we apply induction and the fact that

$$\begin{aligned} & a_{l-k-1} - a_{j_1} \geq a_{l-k-1} - a_{l-k-2} \\ = & \left\{ \begin{array}{l} 2k+4 \text{ if } X_l = B_l \\ 2k+3 \text{ if } X_l = C_l \\ 2k+2 \text{ if } X_l = D_l \end{array} \right. \end{aligned}$$

Hence we (just) get our inductive step. This completes the proof of the lemma.

Lemma 3.19 *Suppose $X_l = B_l$. Put $a_0 = 0$ and $\overline{m_0} = 0$. Then for $k = 1, ..., l$*

$$\overline{m_k} - \overline{m_{k-1}} = a_{l-k}.$$

In particular, $\overline{m_k} - \overline{m_{k-1}}$ is a strictly decreasing sequence.

Proof

$$\begin{aligned} \overline{m_k} - \overline{m_{k-1}} &= a_{l-(k-1)} - \mathrm{card}(B_k^+) + \mathrm{card}(B_{k-1}^+) \\ &= \left(\sum_{j=k}^{l} (2j-1) \right) - (2k-1) \\ &= a_{l-k}. \end{aligned}$$

Theorem 3.20 *The ghost polynomial* $\widetilde{P}(X,Y)$ *associated to* GO_{2l+1} *or* B_l *and its natural representation is*

$$\prod_{i=0}^{l-1}(YX^{a_i}+1).$$

Hence $Z_{\mathrm{GO}_{2l+1}}(s)$ *has a friendly ghost.*

Proof This is the easiest case since $b_i = 1$ for $i = 1, ..., l$. The degree of $P(X,Y)$ and $\overline{P}(X,Y)$ is l. By Proposition 3.13 it suffices to calculate the ghost of $\overline{P}(X,Y)$. For each $0 \le i \le l$, we put $n_i = \max\{n : (n,i) \in \mathcal{L}_{\overline{P}}\}$. We are interested in constructing the part of the Newton Polygon consisting of lines starting from the point $(n_0, 0)$ and ascending to the line with end point (n_l, l) marking out the right hand convex hull $\mathcal{N}$ of the lattice points corresponding to $\overline{P}(X,Y)$.

The fact that $b_i = 1$ for $i = 1, ..., l$ implies that $n_i = \overline{m_i}$ as defined and calculated in Lemma 3.18. For $i = 1, ..., l$ let $\mathfrak{l}_i$ be the line joining $(\overline{m_{i-1}}, i-1)$ and $(\overline{m_i}, i)$. By Lemma 3.19 the lines have strictly increasing gradients with value $v_i/u_i = (\overline{m_i} - \overline{m_{i-1}})^{-1}$ hence mark out the convex hull $\mathcal{N}$. To construct the ghost we right down for each line $\mathfrak{l}_i$ $(i = 1, ..., l)$ the corresponding quasi homogeneous polynomial:

$$\begin{aligned}
\widetilde{P}_i(X^{u_i}Y^{v_i}) &= X^{-\overline{m_{i-1}}}Y^{-(i-1)} \sum_{(n,j)\in \mathfrak{l}_i \cap \mathcal{L}_{\overline{P}}} a_{n,j}X^nY^j \\
&= 1 + X^{\overline{m_i}-\overline{m_{i-1}}}Y \\
&= 1 + X^{a_i}Y.
\end{aligned}$$

Note that $a_{\overline{m_i},i} = 1$ since the maximum value $\overline{m_i}$ is attained only once. Therefore the ghost polynomial of $P(X,Y)$ is the product of these terms as detailed in the statement of the Theorem.

Lemma 3.21 *Suppose* $X_l = C_l$. *Put* $a_0 = 0$. *Then*

(i) for $k = 0, ..., l-1$

$$\overline{m_{k+1}} - m_k = a_{l-(k+1)}/2;$$

(ii) for $k = 1, ..., l-1$

$$m_k - \overline{m_k} = a_{l-(k+1)}/2 + 1.$$

Each sequence is strictly decreasing.

Proof (i)

$$\begin{aligned}
\overline{m_{k+1}} - m_k &= a_l - \mathrm{card}(C_{k+1}^+) + \mathrm{card}(A_k^+) \\
&= \left(\sum_{j=1}^{l} j\right) - \left(\sum_{j=1}^{k+1}(2j-1)\right) + \left(\sum_{j=1}^{k} j\right) \\
&= \left(\sum_{j=k+2}^{l} j\right) \\
&= a_{l-(k+1)}/2.
\end{aligned}$$

(ii)

$$\begin{aligned}
m_k - \overline{m_k} &= a_{l-k} - a_l - \mathrm{card}(A_k^+) + \mathrm{card}(C_k^+) \\
&= \left(\sum_{j=k+1}^{l} 2j\right) - \left(\sum_{j=1}^{l} j\right) - \left(\sum_{j=1}^{k} j\right) + \left(\sum_{j=1}^{k}(2j-1)\right) \\
&= \left(\sum_{j=k+2}^{l} j\right) + 1 \\
&= a_{l-(k+1)}/2 + 1.
\end{aligned}$$

Theorem 3.22 *The ghost polynomial $\widetilde{P}(X,Y)$ associated to GSp_{2l} or C_l and its natural representation is*

$$\prod_{i=0}^{l-1}(YX^{a_i/2}+1)\prod_{i=0}^{l-2}(YX^{a_i/2+1}+1).$$

Hence $Z_{\mathrm{GSp}_{2l}}(s)$ has a friendly ghost.

Proof This case is slightly more complicated than for B_l since $b_i = 2$ for $i = 1, ..., l-1$ and $b_l = 1$. So we are going to get a case analysis according to whether the degree of Y is odd or even. The degree of $P(X,Y)$ and $\overline{P}(X,Y)$ in this case is $2l-1$. By Proposition 3.13 it suffices to calculate the ghost of $\overline{P}(X,Y)$. For each $0 \le i \le 2l-1$, we put $n_i = \max\{n : (n,i) \in \mathcal{L}_{\overline{P}}\}$. We are interested in constructing the part of the Newton Polygon consisting of lines starting from the point $(n_0, 0)$ and ascending to the line with end point $(n_{2l-1}, 2l-1)$ marking out the right hand convex hull $\mathcal{N}$ of the lattice points corresponding to $\overline{P}(X,Y)$.

Now

$$\begin{aligned} n_i &= \max\{n : (n,i) \in \mathcal{L}_{\overline{P}}\} \\ &= \max\left\{a_{j_1} + \cdots + a_{j_r} - \operatorname{card}\left(\langle \alpha_{j_1}, ..., \alpha_{j_r}\rangle^+\right) : b_{j_1} + \cdots + b_{j_r} = i\right\}. \end{aligned}$$

If $i = 2k$ is even then this means that $r = k$ and $l \notin \{j_1, ..., j_r\}$ since b_l is the only odd value of b_i. Hence $n_{2k} = m_k$ as defined and calculated in Lemma 3.17. If $i = 2k+1$ then $r = k+1$ and $l \in \{j_1, ..., j_r\}$ for the same reason that this is the only way to achieve an odd exponent given that $b_i = 2$ for $i = 1, ..., l-1$. Hence $n_{2k+1} = \overline{m_{k+1}}$ as defined and calculated in Lemma 3.18.

For $i = 1, ..., 2l-1$ let $\mathfrak{l}_i$ be the line joining $(n_{i-1}, i-1)$ and (n_i, i). By Lemma 3.21 the lines have strictly increasing gradients with value

$$\begin{aligned} v_i/u_i &= \left(m_k - \overline{m_k}\right)^{-1} \text{ if } i = 2k \\ v_i/u_i &= \left(\overline{m_{k+1}} - m_k\right)^{-1} \text{ if } i = 2k+1 \end{aligned}$$

hence mark out the convex hull $\mathcal{N}$. To construct the ghost we write down for each line $\mathfrak{l}_i$ $(i = 1, ..., 2l-1)$ the corresponding quasi homogeneous polynomial:

$$\begin{aligned} \widetilde{P}_i(X^{u_i}Y^{v_i}) &= X^{-n_{i-1}}Y^{-(i-1)} \sum_{(n,j)\in\mathfrak{l}_i\cap\mathcal{L}_{\overline{P}}} a_{n,j}X^nY^j \\ &= \begin{cases} 1 + X^{m_k - \overline{m_k}}Y \text{ if } i = 2k \\ 1 + X^{\overline{m_{k+1}} - m_k}Y \text{ if } i = 2k+1 \end{cases} \\ &= \begin{cases} 1 + X^{a_{l-(k+1)}/2+1}Y \text{ if } i = 2k \\ 1 + X^{a_{l-(k+1)}/2}Y \text{ if } i = 2k-1. \end{cases} \end{aligned}$$

Note that $a_{n_i,i} = 1$ since the maximum values of m_i and $\overline{m_i}$ are attained only once. Therefore the ghost polynomial of $P(X,Y)$ is the product of these terms as detailed in the statement of the Theorem.

Lemma 3.23 *Suppose $X_l = D_l$. Put $a_0 = 0$ and $\overline{m_1} = 0$. Then for $k = 1, ..., l-1$*

$$\overline{m_{k+1}} - m_k = m_k - \overline{m_k} = a_{l-k-1}/2.$$

These values are strictly decreasing.

Proof Firstly

$$\begin{aligned}
\overline{m_{k+1}} - m_k &= a_l - \mathrm{card}(D_{k+1}^+) + \mathrm{card}(A_k^+) \\
&= \sum_{j=1}^{l-1} j - \sum_{j=1}^{k} 2j + \sum_{j=1}^{k} j \\
&= \sum_{j=k+1}^{l-1} j \\
&= a_{l-(k+1)}/2.
\end{aligned}$$

For $k \geq 2$ the second identity is as follows:

$$\begin{aligned}
m_k - \overline{m_k} &= a_{l-k} - a_l - \mathrm{card}(A_k^+) + \mathrm{card}(D_k^+) \\
&= \sum_{j=k}^{l-1} 2j - \sum_{j=1}^{l-1} j - \sum_{j=1}^{k} j + \sum_{j=1}^{k-1} 2j \\
&= a_{l-(k+1)}/2.
\end{aligned}$$

Note that we have used the fact that the formula $\mathrm{card}(D_k^+) = k(k-1)$ holds when $D_2 = A_1 \times A_1$ and $D_3 = A_3$.

The case $k = 1$ needs special treatment since we have a different formula for a_{l-1} and we have defined $\overline{m_1} = 0$:

$$\begin{aligned}
m_1 &= a_{l-1} - \mathrm{card}(A_1^+) \\
&= \sum_{j=1}^{l-1} j - 1 \\
&= a_{l-2}/2.
\end{aligned}$$

Theorem 3.24 *The ghost polynomial $\widetilde{P}(X, Y)$ associated to GO_{2l}^+ or D_l and its natural representation is*

$$\prod_{i=0}^{l-2} (YX^{a_i/2} + 1)^2.$$

Hence $Z_{\mathrm{GO}_{2l}^+}(s)$ has a friendly ghost.

Proof This is the trickiest of the root systems. Again we have a case distinction between odd and even powers of Y. But we also get in this case that the lines of the convex hull include three lattice points, the middle one with multiplicity two. This gives us the squared terms in the ghost.

Recall that for D_l we have $b_i = 2$ for $i = 1, ..., l-2$ and $b_{l-1} = b_l = 1$. There are going to several ways now to realise a fixed power of Y so we shall need a little more care. The degree of $P(X,Y)$ and $\overline{P}(X,Y)$ in this case is $2l-2$. As above it suffices to calculate the ghost of $\overline{P}(X,Y)$. For each $0 \le i \le 2l-2$, we put $n_i = \max\{n : (n,i) \in \mathcal{L}_{\overline{P}}\}$ and $\mathcal{N}$ the part of the Newton Polygon of $\overline{P}(X,Y)$ consisting of lines starting from the point $(n_0, 0)$ and ascending to the line with end point $(n_{2l-2}, 2l-2)$ marking out the right hand convex hull.

Now

$$\begin{aligned} n_i &= \max\{n : (n,i) \in \mathcal{L}_{\overline{P}}\} \\ &= \max\left\{a_{j_1} + \cdots + a_{j_r} - \operatorname{card}\left(\langle \alpha_{j_1}, ..., \alpha_{j_r}\rangle^+\right) : b_{j_1} + \cdots + b_{j_r} = i\right\}. \end{aligned}$$

Start with the case that $i = 2k+1$ is odd. Then as above this means that one and only one of the $j_1, ..., j_r$ must be $l-1$ or l and the rest can range over elements of $\{1, ..., l-2\}$ and that $r = k+1$. Hence $n_{2k+1} = m_{k+1}$ as defined and calculated in Lemma 3.17 and this value is realised twice, giving multiplicity two to the lattice point (n_i, i), i.e. $a_{n_i,i} = 2$.

The case of $i = 2k$ needs a little more care. There are two sorts of ways to achieve an even power of Y. One is to take $r = k$ and $\{j_1, ..., j_k\} \subset \{1, ..., l-2\}$. The other is to take $r = k+1$, $\{j_1, ..., j_{k-1}\} \subset \{1, ..., l-2\}$ and $j_k = l-1$ and $j_{k+1} = l$. Using the same argument as Lemma 3.17 one can show in the first case

$$\begin{aligned} &\max\left\{a_{j_1} + \cdots + a_{j_r} - \operatorname{card}\left(\langle \alpha_{j_1}, ..., \alpha_{j_r}\rangle^+\right) : \{j_1, ..., j_k\} \subset \{1, ..., l-2\}\right\} \\ = \ & a_{l-(k+1)} + \cdots + a_{l-2} - \operatorname{card} A_k^+. \end{aligned}$$

In the second case

$$\begin{aligned} &\max\left\{\begin{array}{c} a_{j_1} + \cdots + a_{j_{k-1}} + a_{l-1} + a_l - \operatorname{card}\left(\left\langle \alpha_{j_1}, ..., \alpha_{j_{k-1}}, \alpha_{l-1}, \alpha_l\right\rangle^+\right) : \\ \{j_1, ..., j_{k-1}\} \subset \{1, ..., l-2\} \end{array}\right\} \\ = \ & a_{l-k} + \cdots + a_{l-1} + a_l - \operatorname{card} D_{k+1}^+ \\ = \ & \overline{m_{k+1}}. \end{aligned}$$

Since the second of these cases is strictly bigger than the first,

$$\begin{aligned} & a_{l-1}+a_l-\mathrm{card}D_{k+1}^{+}-a_{l-(k+1)}+\mathrm{card}A_k^{+} \\ = \; & 2\sum_{j=1}^{l-1} j-2\sum_{j=1}^{k} j-2\sum_{j=k+1}^{l-1} j+\sum_{j=1}^{k} j \\ = \; & \sum_{j=1}^{k} j>0, \end{aligned}$$

we get that $n_{2k}=\overline{m_{k+1}}$. Note that this also includes our definition of $\overline{m_1}=0$, which is certainly the value of n_0, as a special case separate from the above analysis. This maximum is only achieved once hence $a_{n_i,i}=1$.

For $i=1,...,l-1$ let $\mathfrak{l}_i$ be the line joining $(n_{2i-2},2i-2)$ and $(n_{2i},2i)$. By Lemma 3.23 the lines have strictly increasing gradients with value

$$v_i/u_i=\left(\overline{m_{i+1}}-m_i\right)^{-1}=\left(m_i-\overline{m_i}\right)^{-1}$$

hence mark out the convex hull $\mathcal{N}$. To construct the ghost we write down for each line $\mathfrak{l}_i$ $(i=1,...,l-1)$ the corresponding quasi homogeneous polynomial:

$$\begin{aligned} \widetilde{P}_i(X^{u_i}Y^{v_i}) &= X^{-n_{i-1}}Y^{-(i-1)}\sum_{(n,j)\in\mathfrak{l}_i\cap\mathcal{L}_{\overline{P}}} a_{n,j}X^nY^j \\ &= 1+2X^{a_{l-i-1}/2}Y+X^{a_{l-k-1}}Y^2 \\ &= \left(1+X^{a_{l-i-1}/2}Y\right)^2. \end{aligned}$$

Therefore the ghost polynomial of $P(X,Y)$ is the product of these terms for $i=1,...,l-1$, as detailed in the statement of the Theorem.

Note that we have conspicuously missed out the case of the analysis of A_l and its natural representation. We appeal to the explicit calculation of $Z_{\mathrm{GL}_{l+1}}(s)$ in this case first calculated in [15]:

$$Z_{\mathrm{GL}_{l+1}}(s)=\zeta(s)...\zeta(s-l).$$

Hence we get:

Theorem 3.25 *$Z_{\mathrm{GL}_{l+1}}(s)$ is its own friendly ghost.*

Note that without appealing to the explicit formula but proceeding by the same analysis as above for the other root systems turns out to be rather tricky.

Since $b_i = i$ for $i = 1, ..., l$ it is not such a straightforward combinatorial procedure to realise a given power Y and then to maximize the corresponding powers of X.

We can use the explicit formula however to give an expression for the polynomial $P(X, Y)$ corresponding to A_l:

$$\frac{P(X,Y)}{\prod_{i=1}^{l+1}\left(1-(X^{(l-i+1)}Y)^i\right)} = \frac{1}{\prod_{i=1}^{l+1}(1-X^{(l-i+1)}Y)}.$$

Hence we get:

Theorem 3.26 *The polynomial $P(X, Y)$ associated to GL_{l+1} or A_l and its natural representation is*

$$\prod_{i=1}^{l+1}\left(\sum_{j=0}^{i-1}(X^{l-i+1}Y)^j\right)$$

which is a product of cyclotomic polynomials and hence $P(X, Y) = \widetilde{P}(X, Y)$ is its own friendly ghost as we would expect from Theorem 3.25.

Note that although the ghosts of the classical groups $\mathrm{GO}_{2l+1}, \mathrm{GSp}_{2l}$ and GO_{2l}^{+} are all friendly, the zeta functions have been proved in [4] to have the following natural boundaries in terms of the combinatorial data of the Dynkin diagram:

Theorem 3.27 *(1) If $G = \mathrm{GO}_{2l+1}$ of type B_l then (a) the abscissa of convergence of $Z_G(s)$ is $\alpha_G = a_{l-1}+1 = l^2$; and (b) $Z_G(s)$ has a natural boundary at $\beta_G = a_{l-1} = l^2 - 1$;*

(2) If $G = \mathrm{GSp}_{2l}$ of type C_l then (a) the abscissa of convergence of $Z_G(s)$ is $\alpha_G = a_{l-2}/2 + 2 = l(l + 1)/2 - 1$; and (b) $Z_G(s)$ has a natural boundary at $\beta_G = a_{l-2}/2 + 1 = l(l + 1)/2 - 2$;

(3) If $G = \mathrm{GO}_{2l}^{+}$ of type D_l then (a) the abscissa of convergence of $Z_G(s)$ is $\alpha_G = a_{l-2}/2 + 1 = l(l - 1)/2$; and (b) $Z_G(s)$ has a natural boundary at $\beta_G = a_{l-2}/2 = l(l - 1)/2 - 1$.

Note that the ghost also serves to offer certain polynomials which might be factors of the original polynomial. For example, exploiting symmetries in the root system, the first author has proved that part of the ghost in the case of GSp_{2l} and GO_{2l}^{+} is in fact a factor of the original polynomial:

Theorem 3.28 *(a) If $G = \mathrm{GSp}_{2l}$ of type C_l then $\prod_{i=1}^{l-1}(YX^{a_i/2}+1)$ is an actual factor of $P(X,Y)$ and cancels with a similar factor in the denominator $Q(X,Y)$.*

(b) If $G = \mathrm{GO}_{2l}^{+}$ of type D_l then $\prod_{i=1}^{l-2}(YX^{a_i/2}+1)$ is an actual factor of $P(X,Y)$ and cancels with a similar factor in the denominator $Q(X,Y)$.

4 Ghosts of zeta functions of groups of exceptional type

In this section we consider the zeta functions corresponding to exceptional groups of the form $\rho_i(\mathfrak{G}(X_l)) \cdot \mathbb{G}_m 1_n \leq \mathrm{GL}_n$ where $\mathfrak{G}(X_l)$ is the Chevalley group of type $X_l \in \{E_6, E_7, E_8, F_4, G_2\}$ or the simply connected split semisimple group of type X_l and ρ_i is the representation of degree n corresponding to the fundamental weight ϖ_i $(i = 1, ..., l)$. Note that the dominant weight ϖ of ${}^t\rho_i^{-1}$ when restricted to $\mathfrak{G}(X_l)$ is still ϖ_i. Recall that the b_i are defined by

$$\varpi^m = \alpha_0^{-1} \cdot \prod_{i=1}^{l} \alpha_i^{b_i}$$

where m is defined as the order of $\rho_i(\mathfrak{G}(X_l)) \cap \mathbb{G}_m 1_n$. This is the same as the multiple of ϖ_i required to put it in the root lattice.

We have used a computer program to calculate the functions in this case. In the cases G_2, F_4 and E_6 we have calculated the full zeta functions, details of which may be obtained from the authors. The following program in GAP, version 3.4.4, will output the appropriate numerator where the corresponding values of l, root system $R \in \{A, B, C, D, E, F, G\}$, b_i must be inputted (marked in the programme in bold)

```
l:=l;
x := X(Rationals);
x.name := ''x'';
r := LaurentPolynomialRing(Rationals);
y := X(r);
s := LaurentPolynomialRing(r);
y.name := ''y'';
x := x*y^0;
RequirePackage(''chevie'');
```

```
W := CoxeterGroup( ''R'', l );
a:=Sum(Sublist(W.roots,[1..W.N]));
b:=[];
b[1]:=b_1;
...
b[l]:=b_l;
otherfac:=function(w)
     local j, result;
         result:=1;
         for j in [1..l] do
             if j^w > W.N then
                         result:=result*x^ (a[j])*y^(b[j]);
                 fi;
         od;
     return result;
end;
P:=0;
for w in Elements(W) do
 P:=P+x^(-CoxeterLength( W, w ))*otherfac(w);
 od;
P;
```

As this requires quantifying over the whole of the Weyl group, it is computationally quite heavy. For the cases of E_7 and E_8 we used the following GAP program which computes the polynomial $\overline{P}(X,Y)$ defined in Proposition 3.13. By that Proposition recall that $\overline{P}(X,Y)$ has the same ghost as $P(X,Y)$.

```
l:=l;
x := X(Rationals);
x.name := ''x'';
r := LaurentPolynomialRing(Rationals);
y := X(r);
s := LaurentPolynomialRing(r);
y.name := ''y'';
x := x*y^0;
RequirePackage(''chevie'');
W := CoxeterGroup( ''R'', l );
a:=Sum(Sublist(W.roots,[1..W.N]));
b:=[];
```

```
b[1]:=b_1;
...
b[l]:=b_l;
otherfac:=function(w)
     local j, result;
         result:=1;
         for j in [1..l] do
              if j in w then
                   result:=result*x^(a[j])*y^(b[j]);
              fi;
         od;
     return result;
end;
P:=0;
for w in Combinations([1..l]) do
     P:=P+x^(-ReflectionSubgroup(W,w).N)*otherfac(w);
     od;
P;
```

The Newton polygon corresponding to these polynomials and the associated ghosts were calculated by a (rather clumsy) program we wrote in Mathematica, which again is obtainable from the authors but not sufficiently elegant to reproduce here.

4.1 $E6$

(1) $\varpi_1 = 1/3(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6)$

$$\begin{aligned}\widetilde{P}_1(X,Y) &= (1+Z_1)\left(1+Z_2^2+Z_2^3+Z_2^4+Z_2^5+Z_2^7\right)\left(1+Z_3^2+Z_3^3+Z_3^4+Z_3^6\right)\\ &\quad \left(1+Z_4^2+Z_4^3+Z_4^5\right)\left(1+Z_5^2+Z_5^4\right)\end{aligned}$$

where

$$Z_1 = X^{15}Y, Z_2 = X^7Y, Z_3 = X^6Y, Z_4 = X^4Y, Z_5 = Y.$$

$(E6, \varpi_1)$ has a friendly ghost.

(2) $\varpi_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$

$$\begin{aligned}\widetilde{P}_2(X,Y) &= \left(1+2Z_1+Z_1^2\right)\left(1+2Z_2+3Z_2^2+2Z_2^3+Z_2^4\right)\\ &\quad \left(1+2Z_3+2Z_3^2+Z_3^3\right)(1+2Z_4)(2+Z_5)\end{aligned}$$

where

$$Z_1 = X^{15}Y, Z_2 = X^{14}Y, Z_3 = X^{11}Y, Z_4 = XY, Z_5 = Y.$$

These are all products of cyclotomic polynomials except for the last two factors $(1+2Z_4)(2+Z_5)$. So $(E6, \varpi_2)$ has an unfriendly ghost (just).

(3) $\varpi_3 = 1/3(5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6)$

$$\begin{aligned}\widetilde{P}_3(X,Y) &= (1+Z_1)\left(1+Z_2^2+Z_2^3+Z_2^4+Z_2^5+Z_2^7\right)\\ &\quad \left(1+Z_3^4+Z_3^5+Z_3^6+Z_3^8+Z_3^9+Z_3^{10}+Z_3^{11}+Z_3^{12}+Z_3^{13}+Z_3^{17}\right)\\ &\quad \left(1+Z_4^4+Z_4^5\right)\left(1+Z_5+Z_5^5\right)\end{aligned}$$

where

$$Z_1 = X^{15}Y^4, Z_2 = X^7Y^2, Z_3 = X^3Y, Z_4 = XY, Z_5 = Y.$$

Again these are all products of cyclotomic polynomials except for the last two factors $(1+Z_4^4+Z_4^5)(1+Z_5+Z_5^5)$. For example

$$\begin{aligned}&\left(1+Z_3^4+Z_3^5+Z_3^6+Z_3^8+Z_3^9+Z_3^{10}+Z_3^{11}+Z_3^{12}+Z_3^{13}+Z_3^{17}\right)\\ =\ &(1+Z_3)\left(1-Z_3^2+Z_3^4\right)\left(1-Z_3+Z_3^2-Z_3^3+Z_3^4\right)^2\left(1+Z_3+Z_3^2+Z_3^3+Z_3^4\right).\end{aligned}$$

So $(E6, \varpi_3)$ has an unfriendly ghost (just).

(4) $\varpi_4 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6$

$$\begin{aligned}\widetilde{P}_4(X,Y) &= \left(1+2Z_1+Z_1^2\right)\left(1+2Z_2^2+Z_2^3+3Z_2^4+2Z_2^5+2Z_2^6+3Z_2^7+Z_2^8+2Z_2^9+Z_{11}\right)\\ &\quad \left(1+2Z_3+Z_3^2\right)(1+2Z_4)\left(2+Z_5^2\right)\end{aligned}$$

where

$$Z_1 = X^{15}Y, Z_2 = X^7Y, Z_3 = X^4Y, Z_4 = XY, Z_5 = Y.$$

Again these are all products of cyclotomic polynomials except for the last two factors $(1+2Z_4)(2+Z_5)$. So $(E6, \varpi_4)$ has an unfriendly ghost (just).

(5) $\varpi_5 = 1/3(4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 5\alpha_6)$

It turns out in this case that the zeta function (and hence its ghost) is the same as that corresponding to the representation ϖ_3. In particular the ghost is again unfriendly.

(6) $\varpi_6 = 1/3(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6)$

This case too turns out to have the same zeta function (and hence ghost) as that corresponding to the representation ϖ_1. In particular the ghost is friendly.

Note that both these identities are reflected by a corresponding symmetry in the root system

$$\begin{array}{ccccc} \alpha_1 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 \\ \circ - & - \circ - & - \circ - & - \circ - & - \circ \\ & & | & & \\ & & \circ & & \\ & & \alpha_2 & & \end{array}$$

where ϖ_i is dual to α_i.

4.2 $E7$

(1) $\varpi_1 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$

$$\begin{aligned} \widetilde{P}_1(X,Y) &= (1+Z_1)\left(1+Z_2+Z_2^2\right)\left(1+Z_3+2Z_3^2+2Z_3^3+Z_3^4+Z_3^5\right) \\ &\quad \left(1+Z_4+2Z_4^2+Z_4^3+Z_4^4\right)\left(1+Z_5+Z_5^2+Z_5^3\right)(1+Z_6)(1+Z_7) \end{aligned}$$

where

$$\begin{aligned} Z_1 &= X^{26}Y, Z_2 = X^{25}Y, Z_3 = X^{24}Y, \\ Z_4 &= X^{22}Y, Z_5 = X^{17}Y, Z_6 = XY, Z_7 = Y. \end{aligned}$$

$(E7, \varpi_1)$ has a friendly ghost.

(2) $\varpi_2 = 1/2(4\alpha_1 + 7\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 8\alpha_6 + 3\alpha_7)$

$$\begin{aligned} \widetilde{P}_2(X,Y) &= (1+Z_1)(1+Z_2)(1+Z_3)\left(1+Z_4^4+Z_4^8\right)\left(1+Z_5^2+Z_5^4+Z_5^6\right) \\ &\quad (1+Z_6)(1+Z_7)\left(1+Z_8^3\right)(1+Z_9)(1+Z_{10}^3) \end{aligned}$$

where

$$\begin{aligned} Z_1 &= X^{26}Y^3, Z_2 = X^{33}Y^4, Z_3 = X^{74}Y^9, Z_4 = X^8Y, Z_5 = X^{15}Y^2, \\ Z_6 &= X^{39}Y^7, Z_7 = X^3Y, Z_8 = X^2Y, Z_9 = XY, Z_{10} = Y. \end{aligned}$$

$(E7, \varpi_2)$ has a friendly ghost.

(3) $\varpi_3 = 3\alpha_1 + 4\alpha_2 + 6\alpha_3 + 8\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7$

$$\begin{aligned}\widetilde{P}_3(X,Y) &= \left(1+Z_1^2\right)\left(1+Z_2+Z_2^2\right)\left(1+Z_3^2+2Z_3^4+2Z_3^6+Z_3^8+Z_3^{10}\right)\\ &\quad \left(1+Z_4^2+Z_4^3+2Z_4^4+Z_4^5+Z_4^6+2Z_4^7+Z_4^8+Z_4^9+Z_4^{11}\right)\\ &\quad \left(1+Z_5^2+Z_5^3\right)\left(1+Z_6\right)\left(1+Z_7^2\right)\end{aligned}$$

where

$$\begin{aligned} Z_1 &= X^{13}Y, Z_2 = X^{25}Y^2, Z_3 = X^{12}Y, Z_4 = X^{11}Y,\\ Z_5 &= X^6Y, Z_6 = XY, Z_7 = Y.\end{aligned}$$

All factors are products of cyclotomic polynomials (including the polynomial of degree 11) except for the factor $(1+Z_5^2+Z_5^3)$. Hence $(E7, \varpi_3)$ has an unfriendly ghost.

(4) $\varpi_4 = 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7$

$$\begin{aligned}\widetilde{P}_4(X,Y) &= \left(1+Z_1\right)\left(1+Z_2+Z_2^2\right)\left(1+Z_3\right)\left(1+Z_4^3+Z_4^4+2Z_4^6+Z_4^7+Z_4^8+2Z_4^9+2Z_4^{10}\right.\\ &\quad \left.+Z_4^{11}+Z_4^{12}+2Z_4^{13}+2Z_4^{14}+Z_4^{15}+Z_4^{16}+2Z_4^{17}+Z_4^{19}+Z_4^{20}+Z_4^{23}\right)\\ &\quad \left(1+Z_5^3+Z_5^4+Z_5^6\right)\left(1+Z_6^2\right)\left(1+Z_7\right)\left(1+Z_8^3\right)\end{aligned}$$

where

$$Z_1 = X^{26}Y^3, Z_2 = X^{25}Y^3, Z_3 = X^{33}Y^4, Z_4 = X^8Y, Z_5 = X^6Y, Z_6 = X^3Y, Z_7 = XY, Z_8 = Y$$

All factors are products of cyclotomic polynomials (including the polynomial of degree 23) except for the factor $(1+Z_5^3+Z_5^4+Z_5^6)$. Hence $(E7, \varpi_4)$ has an unfriendly ghost.

(5) $\varpi_5 = 1/2\left(6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 10\alpha_6 + 5\alpha_7\right)$

$$\begin{aligned}\widetilde{P}_5(X,Y) &= \left(1+Z_1^3\right)\left(1+Z_2^2+Z_2^3+Z_2^4+Z_2^5+Z_2^7\right)\left(1+Z_3\right)\left(1+Z_4^5+Z_4^6+Z_4^9\right.\\ &\quad \left.+Z_4^{10}+Z_4^{11}+Z_4^{12}+Z_4^{14}+Z_4^{16}+Z_4^{17}+Z_4^{18}+Z_4^{19}+Z_4^{22}+Z_4^{23}+Z_4^{28}\right)\\ &\quad \left(1+Z_5^5+Z_5^6\right)\left(1+Z_6^3\right)\left(1+Z_7\right)\left(1+Z_8^5\right)\end{aligned}$$

where

$$Z_1 = X^{11}Y^2, Z_2 = X^{16}Y^3, Z_3 = X^{26}Y^5, Z_4 = X^5Y, Z_5 = X^3Y, Z_6 = X^2Y, Z_7 = XY, Z_8 = Y.$$

Again all factors are products of cyclotomic polynomials (including the polynomial of degree 28) except for $(1+Z_5^5+Z_5^6)$. Hence $(E7, \varpi_5)$ has an unfriendly ghost.

(6) $\varpi_6 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7$

$$\begin{aligned}\widetilde{P}_6(X,Y) &= (1+Z_1)\left(1+Z_2^2+Z_2^3+Z_2^4+Z_2^5+Z_2^7\right)\left(1+Z_3^2+Z_3^3+Z_3^4+Z_3^6\right)\\ &\quad \left(1+2Z_4^2+Z_4^3+Z_4^4+2Z_4^5+Z_4^7\right)\left(1+Z_5^2\right)\left(1+Z_6^2\right)\end{aligned}$$

where

$$Z_1 = X^{33}Y^2, Z_2 = X^{16}Y, Z_3 = X^{15}Y, Z_4 = X^{13}Y, Z_5 = X^5Y, Z_6 = Y.$$

$(E7, \varpi_6)$ has a friendly ghost.

(7) $\varpi_7 = 1/2\,(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7)$

$$\begin{aligned}\widetilde{P}_7(X,Y) &= (1+Z_1)\left(1+Z_2^2+Z_2^3+Z_2^4+Z_2^5+Z_2^7\right)\left(1+Z_3^2+Z_2^3+Z_3^4+Z_3^6\right)\\ &\quad \left(1+Z_4^2+Z_4^3+Z_4^5\right)\left(1+Z_5^2+Z_5^4\right)\left(1+Z_6^3\right)\end{aligned}$$

where

$$Z_1 = X^{33}Y^2, Z_2 = X^{16}Y, Z_3 = X^{15}Y, Z_4 = X^{13}Y, Z_5 = X^9Y, Z_6 = Y.$$

$(E7, \varpi_7)$ has a friendly ghost.

4.3 $E8$

(1) $\varpi_1 = 4\alpha_1 + 5\alpha_2 + 7\alpha_3 + 10\alpha_4 + 8\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8$

$$\begin{aligned}\widetilde{P}_1(X,Y) &= (1+Z_1)\left(1+Z_2^2+Z_2^4\right)\left(1+Z_3+Z_3^2+Z_3^3\right)\\ &\quad \left(1+Z_4^2+Z_4^4+Z_4^5+Z_4^6+Z_4^7+Z_4^8+Z_4^9+Z_4^{11}+Z_4^{13}\right)\\ &\quad \left(1+Z_5^2+Z_5^4+Z_5^5+Z_5^6+Z_5^8+Z_5^{10}\right)\left(1+Z_6^2+Z_6^5+Z_6^7\right)(1+Z_7)(1+Z_8)\end{aligned}$$

where

$$\begin{aligned}Z_1 &= X^{57}Y^2, Z_2 = X^{28}Y, Z_3 = X^{55}Y^2, Z_4 = X^{27}Y,\\ Z_5 &= X^{26}Y, Z_6 = X^{23}Y, Z_7 = X^{13}Y^2, Z_8 = XY^2.\end{aligned}$$

$(E8, \varpi_1)$ has a friendly ghost.

(2) $\varpi_2 = 5\alpha_1 + 8\alpha_2 + 10\alpha_3 + 15\alpha_4 + 12\alpha_5 + 9\alpha_6 + 6\alpha_7 + 3\alpha_8$

$$\begin{aligned}\widetilde{P}_2(X,Y) &= \left(1+Z_1^3\right)\left(1+Z_2+Z_2^2\right)\left(1+Z_3+Z_3^2+Z_3^3\right)(1+Z_4)\left(1+Z_5^3+Z_5^5+Z_5^6\right.\\ &\quad \left.+Z_5^8+Z_5^8+Z_5^9+Z_5^{10}+Z_5^{11}+Z_5^{12}+Z_5^{13}+Z_5^{14}+Z_5^{16}+Z_5^{17}+Z_5^{19}+Z_5^{22}\right)\\ &\quad \left(1+Z_6^3+Z_6^5+Z_6^6+Z_6^9+Z_6^{10}+Z_6^{12}+Z_6^{15}\right)\left(1+Z_7^3\right)(1+Z_8)(1+Z_9)\end{aligned}$$

where

$$\begin{aligned} Z_1 &= X^{19}Y, Z_2 = X^{56}Y^3, Z_3 = X^{55}Y^3, Z_4 = X^{91}Y^5, \\ Z_5 &= X^{18}Y, Z_6 = X^{17}Y, Z_7 = X^{10}Y, Z_8 = X^{13}Y^2, Z_9 = XY^3. \end{aligned}$$

$(E8, \varpi_2)$ has a friendly ghost.

(3) $\varpi_3 = 7\alpha_1 + 10\alpha_2 + 14\alpha_3 + 20\alpha_4 + 16\alpha_5 + 12\alpha_6 + 8\alpha_7 + 4\alpha_8$

$$\begin{aligned} \widetilde{P}_3(X,Y) &= \left(1+Z_1^4\right)\left(1+Z_2^4+Z_2^8\right)\left(1+Z_3+Z_3^2+Z_3^3\right) \\ &\quad \left(1+Z_4^2+Z_4^4+Z_4^5+Z_4^6+Z_4^7+Z_4^8+Z_4^9+Z_4^{11}+Z_4^{13}\right) \\ &\quad \left(1+Z_5^4+Z_5^7+Z_5^8+Z_5^{10}+Z_5^{11}+Z_5^{12}+Z_5^{15}+Z_5^{16}+Z_5^{17}\right. \\ &\quad \left.+Z_5^{19}+Z_5^{20}+Z_5^{23}+Z_5^{27}\right)\left(1+Z_6^4+Z_6^7\right)\left(1+Z_7\right)\left(1+Z_8\right) \end{aligned}$$

where

$$\begin{aligned} Z_1 &= X^{57}Y^4, Z_2 = X^{14}Y, Z_3 = X^{55}Y^4, Z_4 = X^{27}Y^2 \\ Z_5 &= X^{13}Y, Z_6 = X^{10}Y, Z_7 = X^{13}Y^3, Z_8 = XY^4. \end{aligned}$$

All factors are products of cyclotomic polynomials (including the polynomial of degree 27) except for one rogue factor $\left(1+Z_6^4+Z_6^7\right)$. Hence $(E8, \varpi_3)$ has an unfriendly ghost.

(4) $\varpi_4 = 10\alpha_1 + 15\alpha_2 + 20\alpha_3 + 30\alpha_4 + 24\alpha_5 + 18\alpha_6 + 12\alpha_7 + 6\alpha_8$

$$\begin{aligned} \widetilde{P}_4(X,Y) &= \left(1+Z_1^3\right)\left(1+Z_2^2+Z_2^4\right)\left(1+Z_3+Z_3^2+Z_3^3\right)\left(1+Z_4\right) \\ &\quad \left(\begin{array}{c} 1+Z_5^6+Z_5^{10}+Z_5^{12}+Z_5^{15}+Z_5^{16}+Z_5^{18}+Z_5^{20}+Z_5^{21}+Z_5^{22}+Z_5^{24} \\ +Z_5^{25}+Z_5^{26}+Z_5^{27}+Z_5^{28}+Z_5^{31}+Z_5^{32}+Z_5^{33}+Z_5^{34}+Z_5^{35}+Z_5^{37} \\ +Z_5^{38}+Z_5^{39}+Z_5^{41}+Z_5^{43}+Z_5^{44}+Z_5^{47}+Z_5^{49}+Z_5^{53}+Z_5^{59} \end{array}\right) \\ &\quad \left(1+Z_6^6+Z_6^{10}+Z_6^{15}\right)\left(1+Z_7^5\right)\left(1+Z_8\right)\left(1+Z_9\right) \end{aligned}$$

where

$$\begin{aligned} Z_1 &= X^{19}Y^2, Z_2 = X^{28}Y^3, Z_3 = X^{55}Y^6, Z_4 = X^{91}Y^{10}, \\ Z_5 &= X^9Y, Z_6 = X^8Y, Z_7 = X^6Y, Z_8 = X^{13}Y^4, Z_9 = XY^6. \end{aligned}$$

All factors are products of cyclotomic polynomials (including the polynomial of degree 59) except for one rogue factor $\left(1+Z_6^6+Z_6^{10}+Z_6^{15}\right)$. Hence $(E8, \varpi_4)$ has an unfriendly ghost.

(5) $\varpi_5 = 8\alpha_1 + 12\alpha_2 + 16\alpha_3 + 24\alpha_4 + 20\alpha_5 + 15\alpha_6 + 10\alpha_7 + 5\alpha_8$

$$\begin{aligned}\widetilde{P}_5(X,Y) &= (1+Z_1)(1+Z_2)\left(1+Z_3^2+Z_3^3+Z_3^4+Z_3^7\right)\left(1+Z_4+Z_4^2\right)\\ &\quad \left(\begin{array}{c} 1+Z_5^5+Z_5^8+Z_5^{10}+Z_5^{12}+Z_5^{13}+Z_5^{15}+Z_5^{16}+Z_5^{17}+Z_5^{18}+Z_5^{21}\\ +Z_5^{22}+Z_5^{23}+Z_5^{24}+Z_5^{26}+Z_5^{27}+Z_5^{29}+Z_5^{31}+Z_5^{34}+Z_5^{39}\end{array}\right)\\ &\quad \left(1+Z_6^5+Z_6^8\right)\left(1+Z_7^2\right)(1+Z_8)(1+Z_9)\end{aligned}$$

where

$$\begin{aligned}Z_1 &= X^{57}Y^5, Z_2 = X^{91}Y^8, Z_3 = X^{45}Y^4, Z_4 = X^{56}Y^5,\\ Z_5 &= X^{11}Y, Z_6 = X^9Y, Z_7 = X^{15}Y^2, Z_8 = X^{13}Y^3, Z_9 = XY^5.\end{aligned}$$

All factors are products of cyclotomic polynomials (including the polynomial of degree 39) except for one rogue factor $\left(1+Z_6^5+Z_6^8\right)$. Hence $(E8, \varpi_5)$ has an unfriendly ghost.

(6) $\varpi_6 = 6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 12\alpha_6 + 8\alpha_7 + 4\alpha_8$

$$\begin{aligned}\widetilde{P}_6(X,Y) &= (1+Z_1)\left(1+Z_2^6+Z_2^9+Z_2^{12}+Z_2^{15}+Z_2^{21}\right)\left(1+Z_3^2+Z_3^3+Z_3^4+Z_3^6\right)(1+Z_4)\\ &\quad \left(1+Z_5^4+Z_5^6+Z_5^8+Z_5^9+Z_5^{10}+Z_5^{13}+Z_5^{14}+Z_5^{15}+Z_5^{17}+Z_5^{19}+Z_5^{23}\right)\\ &\quad \left(1+Z_6^4\right)(1+Z_7)(1+Z_8)(1+Z_9)\end{aligned}$$

where

$$\begin{aligned}Z_1 &= X^{91}Y^6, Z_2 = X^{15}Y, Z_3 = X^{44}Y^3, Z_4 = X^{57}Y^4\\ Z_5 &= X^{14}Y, Z_6 = X^{10}Y, Z_7 = X^{17}Y^2, Z_8 = X^{13}Y^2, Z_9 = XY^4.\end{aligned}$$

$(E8, \varpi_6)$ has a friendly ghost.

(7) $\varpi_7 = 4\alpha_1 + 6\alpha_2 + 18\alpha_3 + 12\alpha_4 + 10\alpha_5 + 8\alpha_6 + 6\alpha_7 + 3\alpha_8$

$$\begin{aligned}\widetilde{P}_7(X,Y) &= (1+Z_1)\left(1+Z_2^2+Z_2^3+Z_2^4+Z_2^5+Z_2^7\right)\left(1+Z_3^4+Z_3^6+Z_3^8+Z_3^{12}\right)\\ &\quad \left(1+Z_4^4+Z_4^6+Z_4^{10}\right)\left(1+Z_5^3+Z_5^4+Z_5^7+Z_5^8+Z_5^{11}\right)\left(1+Z_6^3\right)(1+Z_7)\end{aligned}$$

where

$$\begin{aligned}Z_1 &= X^{91}Y^4, Z_2 = X^{45}Y^2, Z_3 = X^{22}Y, Z_4 = X^{21}Y,\\ Z_5 &= X^{19}Y, Z_6 = X^{10}Y, Z_7 = XY^3.\end{aligned}$$

$(E8, \varpi_7)$ has a friendly ghost.

(8) $\varpi_8 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$

$$\begin{aligned}\widetilde{P}_8(X,Y) &= (1+Z_1)\left(1+Z_2^2+Z_2^3+Z_2^4+Z_2^5+Z_2^7\right)\left(1+Z_3^2+Z_3^3+Z_3^4+Z_3^6\right)\\ &\quad \left(1+Z_4^2+Z_4^3+Z_4^5\right)\left(1+Z_5^2+Z_5^4\right)\left(1+Z_6^3\right)(1+Z_7)\end{aligned}$$

where

$$\begin{aligned}Z_1 &= X^{91}Y^2, Z_2 = X^{45}Y, Z_3 = X^{44}Y,\\ Z_4 &= X^{42}Y, Z_5 = X^{38}Y, Z_6 = X^{29}Y, Z_7 = XY^2.\end{aligned}$$

$(E8, \varpi_8)$ has a friendly ghost.

4.4 $F4$

(1) $\varpi_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$

$$\widetilde{P}_1(X,Y) = (1+Z_1)\left(1+Z_2^2+Z_2^4\right)\left(1+Z_3^3\right)(1+Z_4)$$

where

$$Z_1 = X^{21}Y^2, Z_2 = X^{10}Y, Z_3 = X^8Y, Z_4 = XY^2.$$

$(F4, \varpi_1)$ has a friendly ghost.
(2) $\varpi_2 = 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4$

$$\widetilde{P}_2(X,Y) = (1+Z_1)\left(1+Z_2^3+Z_2^4+Z_2^7+Z_2^8+Z_2^{11}\right)\left(1+Z_3^3\right)(1+Z_4)$$

where

$$Z_1 = X^{21}Y^4, Z_2 = X^5Y, Z_3 = X^3Y, Z_4 = XY^3.$$

$(F4, \varpi_2)$ has a friendly ghost.
(3) $\varpi_3 = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4$

$$\widetilde{P}_3(X,Y) = (1+Z_1)\left(1+Z_2^2+Z_2^3+Z_2^4+Z_2^5+Z_2^7\right)\left(1+Z_3^2\right)(1+Z_4)(1+Z_5)$$

where

$$Z_1 = X^{15}Y^2, Z_2 = X^7Y, Z_3 = X^6Y, Z_4 = X^9Y^2, Z_5 = XY^2.$$

$(F4, \varpi_3)$ has a friendly ghost.

(4) $\varpi_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$

$$\widetilde{P}_4(X, Y) = (1 + Z_1)\left(1 + Z_2 + Z_2^2\right)(1 + Z_3)(1 + Z_4)(1 + Z_5)(1 + Z_6)(1 + Z_7)$$

where

$$Z_1 = X^{15}Y, Z_2 = X^{14}Y, Z_3 = X^{13}Y, Z_4 = X^{12}Y, Z_5 = X^{11}Y, Z_6 = X^6Y, Z_7 = XY.$$

$(F4, \varpi_4)$ has a friendly ghost.

5 Two-dimensional root systems

In this section we give a description of the partition of the weight space of the two dimensional root systems according to whether the corresponding ghosts are friendly or unfriendly.

5.1 A_2

The irreducible representations are parameterized by the weight lattice. For A_2 this is spanned by the two fundamental weights

$$\begin{aligned} \varpi_1 &= 1/3\,(2\alpha_1 + \alpha_2) \\ \varpi_2 &= 1/3\,(\alpha_1 + 2\alpha_2)\,. \end{aligned}$$

Each pair $(n_1, n_2) \in \mathbb{N}^2$ defines an irreducible representation with fundamental weight $n_1\varpi_1 + n_2\varpi_2$. The case that $(n_1, n_2) = (0, 0)$ corresponds to the trivial representation. Hence we get just the zeta function of the one dimensional torus corresponding to $\mathbb{G}_m$ which is meromorphic. We assume now that $(n_1, n_2) \neq (0, 0)$. If n_1 and n_2 are both divisible by 3 then $m = 1$; otherwise $m = 3$. The corresponding values of b_1 and b_2 are then given by

$$\begin{aligned} b_1 &= m/3\,(2n_1 + n_2) \\ b_2 &= m/3\,(n_1 + 2n_2)\,. \end{aligned}$$

The polynomial $P_{(n_1,n_2)}(X, Y)$ corresponding to this representation then takes the following form

$$1 + Z^{2n_1+n_2} + Z^{n_1+2n_2} + XZ^{2n_1+n_2} + XZ^{n_1+2n_2} + XZ^{3n_1+3n_2}$$

where $Z = Y^{m/3}$. The ghost polynomial has the following description

$$\widetilde{P}_{(n_1,n_2)}(X,Y) = \begin{cases} \left(1 + XZ^{2n_1+n_2}\right)\left(1 + Z^{n_2-n_1} + Z^{n_1+2n_2}\right) \text{ if } n_1 < n_2 \\ \left(1 + 2XZ^{3n}\right)\left(2 + Z^{3n}\right) \text{ if } n_1 = n_2 = n \\ \left(1 + XZ^{n_1+2n_2}\right)\left(1 + Z^{n_1-n_2} + Z^{2n_1+n_2}\right) \text{ if } n_1 > n_2. \end{cases}$$

We therefore get the following:

Theorem 5.29 *$(A_2, n_1\varpi_1 + n_2\varpi_2)$ has a friendly ghost if and only if $n_1 = 0$ or $n_2 = 0$.*

Proof $\left(1 + Z^{n_2-n_1} + Z^{n_1+2n_2}\right)$ is cyclotomic if and only if $2\,(n_2 - n_1) = n_1 + 2n_2$. Similarly for $\left(1 + Z^{n_1-n_2} + Z^{2n_1+n_2}\right)$. The factors $\left(1 + 2XZ^{3n}\right)$ and $\left(2 + Z^{3n}\right)$ are never cyclotomic.

5.2 $B_2 = C_2$

In this case the weight lattice is spanned by the two fundamental weights

$$\begin{aligned} \varpi_1 &= \alpha_1 + 1/2\alpha_2 \\ \varpi_2 &= \alpha_1 + \alpha_2 \end{aligned}$$

where α_1 is the short root.

Each pair $(n_1, n_2) \in \mathbb{N}^2 \backslash (0,0)$ defines an irreducible representation with fundamental weight $n_1\varpi_1 + n_2\varpi_2$. If n_1 is divisible by 2 then $m = 1$; otherwise $m = 2$. The corresponding values of b_1 and b_2 are then given by

$$\begin{aligned} b_1 &= m/2\,(2n_1 + 2n_2) \\ b_2 &= m/2\,(n_1 + 2n_2)\,. \end{aligned}$$

The polynomial $P_{(n_1,n_2)}(X,Y)$ corresponding to this representation then takes the following form

$$1 + Z^{n_1+2n_2} + XZ^{n_1+2n_2} + X^2Z^{n_1+2n_2} + XZ^{2n_1+2n_2} + X^2Z^{2n_1+2n_2} + X^3Z^{2n_1+2n_2} + X^3Z^{3n_1+4n_2}$$

where $Z = Y^{m/2}$. The ghost polynomial has the following description

$$\widetilde{P}_{(n_1,n_2)}(X,Y) = \begin{cases} \left(1 + X^2Z^{n_1+2n_2}\right)\left(1 + XZ^{n_1}\right)\left(1 + Z^{n_1+2n_2}\right) \text{ if } n_1 > 2n_2 \\ \left(1 + X^2Z^{2n} + X^3Z^{3n}\right)\left(1 + Z^{2n}\right) \text{ if } n_1 = 2n_2 = n \\ \left(1 + X^3Z^{2n_1+2n_2}\right)\left(1 + Z^{n_1+2n_2}\right) \text{ if } n_1 < 2n_2. \end{cases}$$

We therefore get:

Theorem 5.30 *$(C_2, n_1\varpi_1 + n_2\varpi_2)$ has an unfriendly ghost if and only if $n_1 = 2n_2 \neq 0$.*

Proof $\left(1 + X^2Z^{2n} + X^3Z^{3n}\right)$ is not cyclotomic.

5.3 G_2

In this case the weight lattice is spanned by the two fundamental weights

$$\begin{aligned} \varpi_1 &= 2\alpha_1 + \alpha_2 \\ \varpi_2 &= 3\alpha_1 + 2\alpha_2 \end{aligned}$$

where α_1 is the short root.

Each pair $(n_1, n_2) \in \mathbb{N}^2 \backslash (0,0)$ defines an irreducible representation with fundamental weight $n_1\varpi_1 + n_2\varpi_2$. In this case $m = 1$. The corresponding values of b_1 and b_2 are then given by

$$\begin{aligned} b_1 &= 2n_1 + 3n_2 \\ b_2 &= n_1 + 2n_2. \end{aligned}$$

The polynomial $P_{(n_1,n_2)}(X,Y)$ corresponding to this representation then takes the following form

$$1 + \left(X + X^2 + X^3 + X^4 + X^5\right) Y^{n_1+2n_2} + \left(X^5 + X^6 + X^7 + X^8 + X^9\right) Y^{2n_1+3n_2} + X^{10}Y^{3n_1+5n_2}.$$

The ghost polynomial has the following description

$$\widetilde{P}_{(n_1,n_2)}(X,Y) = \begin{cases} (1 + X^5Y^{n_1+2n_2})(1 + X^4Y^{n_1+n_2})(1 + XY^{n_1+2n_2}) \text{ if } n_1 > 3n_2 \\ (1 + X^5Y^{5n} + X^9Y^{9n})(1 + XY^{5n}) \text{ if } n_1 = 3n_2 = 3n \\ (1 + X^9Y^{2n_1+3n_2})(1 + XY^{n_1+2n_2}) \text{ if } n_1 < 3n_2. \end{cases}$$

We therefore get:

Theorem 5.31 *$(G_2, n_1\varpi_1 + n_2\varpi_2)$ has an unfriendly ghost if and only if $n_1 = 3n_2 \neq 0$.*

Proof $(1 + X^5Y^{5n} + X^9Y^{9n})$ is not cyclotomic.

6 Ghosts of zeta functions of nilpotent groups

In this section we record some ghost zeta functions attached to zeta functions counting subgroups in nilpotent groups. We consider only examples of nilpotent groups whose zeta functions have been explicitly calculated and have natural boundaries. Currently there is not such an extensive list of such explicit calculations. The natural boundaries for all the following have been established in [4]. We refer to [13] for other examples which are meromorphic on the whole of $\mathbb{C}$ and hence are the same as their ghost.

6.1 Free nilpotent class two group on three generators

Let F_2^3 denote the free nilpotent group of class two on three generators. Then

$$\zeta^{\triangleleft}_{F_{2,3}}(s) = \zeta(s)\zeta(s-1)\zeta(s-2)\zeta(5s-8)\zeta(6s-9)\prod_{p \text{ prime}} W(p,p^{-s})$$

where $\zeta(s)$ is the Riemann zeta function and

$$W(X,Y) = \left(1 + X^3Y^3 + X^4Y^3 + X^6Y^5 + X^7Y^5 + X^{10}Y^8\right).$$

The polynomial $W(X,Y)$ has a friendly ghost $\widetilde{W}(X,Y)$ with the following description:

$$\widetilde{W}(X,Y) = (1 + X^3Y^3)(1 + X^7Y^5).$$

6.2 The Heisenberg group over a number field

Let $U_3(R_n)$ denote the group of 3×3 upper triangular matrices (with 1's on the diagonal) over a ring of integers R_n of an algebraic number field K_n of degree n over $\mathbb{Q}$. For $n = 1$, the zeta functions $\zeta_{U_3(\mathbb{Z})}(s)$ and $\zeta^{\triangleleft}_{U_3(\mathbb{Z})}(s)$ have both been calculated and are meromorphic functions on $\mathbb{C}$ (see [13]). For $n = 2$, the following expression for $\zeta^{\triangleleft}_{U_3(R_2)}(s)$ was calculated in [13]:

$$\zeta^{\triangleleft}_{U_3(R_2)}(s) = \zeta(s)\zeta(s-1)\zeta(s-2)\zeta(s-3)\zeta(5s-4)\zeta(5s-5)\zeta_{K_2}(3s-4)/\zeta_{K_2}(5s-4)$$

where $\zeta_K(s)$ is the Dedekind zeta function of the number field K. So $\zeta^{\triangleleft}_{U_3(R_2)}(s)$ is in fact meromorphic on the whole of $\mathbb{C}$. Note that this is an example where the local factors are finitely universal (rather than universal) in the sense of Definition 1.6. Since these local factors are built from cyclotomic polynomials, they are their own ghosts. Hence $\zeta^{\triangleleft}_{U_3(R_2)}(s)$ is its own ghost.

The calculation for a cubic number field K_3 is at present incomplete. No formula has been given for the split primes (of which there are infinitely many). Formulae have been given for those primes p which ramify in K_3 (of which there are finitely many). For those primes p which remain prime in K_3, so-called inert primes (of which there are possibly infinitely many depending on the number field K_3) we have the following:

$$\zeta^{\triangleleft}_{G,p}(s) = \left(\prod_{i=0}^{5} \frac{1}{(1-p^{i-s})}\right) \frac{W(p,p^{-s})}{(1-p^{8-7s})(1-p^{14-8s})(1-p^{18-9s})}$$

where

$$W(X,Y)=\left(1+X^6Y^7+X^7Y^7+X^{12}Y^8+X^{13}Y^8+X^{19}Y^{15}\right).$$

The associated ghost polynomial for these inert primes is

$$\widetilde{W}(X,Y)=\left(1+X^6Y^7\right)\left(1+X^{13}Y^8\right).$$

This is then a product of cyclotomic polynomials. It would be interesting now to calculate the ghosts for the split primes since now there is no guarantee even if they too are products of cyclotomic polynomials, that the ghost zeta function of $\zeta^{\triangleleft}_{U_3(R_3)}(s)$ will be friendly. The expectation might be that the ghosts pick out the local factors of the associated Dedekind zeta function of R_3 and hence the ghost will be friendly because $\zeta_{K_3}(s)$ is meromorphic. These musings certainly raise the following question generalizing Estermann's Theorem:

Question *Suppose there are finitely many polynomials $h_j(Y)=\prod(1-\alpha_{ij}Y)\in\mathbb{C}[Y]$ $(j=1,...,r)$ and a partition of the primes into subsets $\mathcal{P}_j$ with $j=1,...,r$. Under what conditions does*

$$\prod_{j=1}^{r}\prod_{p\in\mathcal{P}_j}h_j(p^{-s})$$

have meromorphic continuation to the whole of $\mathbb{C}$?

6.3 Grenham's Examples

The next examples are calculations made by Dermott Grenham in his D.Phil. thesis [12] of zeta functions of groups G_n with the following presentation:

$$G_n=\langle x_1,...,x_n,u_1,...,u_{n-1}|[x_i,x_n]=u_i,[x_i,x_j]=[x_k,u_j]=1\ \ (\,i,j\le n-1,k\le n)\rangle$$

These groups are class two nilpotent groups. G_2 is the Heisenberg group of 3×3 upper triangular matrices with integer entries. More generally, G_n can be thought of as $n-1$ copies of this Heisenberg group with one off diagonal entry identified in each copy. G_3 is also the group of 4×4 upper triangular matrices with integer entries quotiented out by the one-dimensional centre consisting of matrices with entries just in the top right hand corner.

Grenham calculated $\zeta_{G_n,p}(s)$ and $\zeta^{\triangleleft}_{G_n,p}(s)$ for $n\le5$. They are all universal built from products of local Riemann zeta functions together with one polynomial which we shall denote in each case by $W_n(X,Y)$ and $W_n^{\triangleleft}(X,Y)$. The complete formulae can be found in [4] which is the first place that these functions have appeared in print. In [4] it is documented which examples are meromorphic and which have natural boundaries. We record here the $W_n(X,Y)$ and $W_n^{\triangleleft}(X,Y)$ corresponding to the groups with natural boundaries and their ghosts. The example $W_4(X,Y)$ is especially interesting as it gives us our first example for nilpotent groups of an unfriendly ghost.

6.3.1 $W_3(X,Y)$

$$W_3(X,Y) = 1 + X^3Y^2 + X^4Y^2 - X^4Y^3 - X^5Y^3 - X^8Y^5$$

has a friendly ghost

$$\widetilde{W_3}(X,Y) = \left(1 + X^4Y^2\right)\left(1 - X^4Y^3\right).$$

6.3.2 $W_4(X,Y)$

$$\begin{aligned}W_4(X,Y) = {} & 1 + X^4Y^2 + X^5Y^2 + X^6Y^2 - X^5Y^3 - X^6Y^3 - X^7Y^3 + X^8Y^3 + X^9Y^3 \\ & - X^9Y^4 - X^{10}Y^4 - X^{11}Y^4 - X^{14}Y^6 - X^{15}Y^6 - X^{16}Y^6 + X^{16}Y^7 + X^{17}Y^7 \\ & - X^{18}Y^7 - X^{19}Y^7 - X^{20}Y^7 + X^{19}Y^8 + X^{20}Y^8 + X^{21}Y^8 + X^{25}Y^{10}\end{aligned}$$

has an unfriendly ghost

$$\widetilde{W_4}(X,Y) = \left(1 + X^6Y^2 + X^9Y^3\right)\left(1 - X^{11}Y^4\right)\left(X^5Y^3 - 1\right)$$

since $1 + Z^2 + Z^3$ is not a product of cyclotomic polynomials.

6.3.3 $W_4^{\triangleleft}(X,Y)$

$$W_4^{\triangleleft}(X,Y) = 1 + X^4Y^3 + X^5Y^3 + X^8Y^5 + X^9Y^5 + X^{13}Y^8$$

has a friendly ghost

$$\widetilde{W_4^{\triangleleft}}(X,Y) = \left(1 + X^9Y^5\right)\left(1 + X^4Y^3\right).$$

6.3.4 $W_5(X,Y)$

$W_5(X,Y) =$

$$\begin{aligned}
&1+YX^2+Y^2X^4+Y^2X^5+Y^2X^6+2Y^2X^7+Y^2X^8+Y^3X^9+2Y^3X^{10}+Y^3X^{11}\\
&+2Y^3X^{12}+Y^3X^{13}+Y^4X^{12}+2Y^4X^{14}+2Y^4X^{15}+Y^4X^{16}+Y^4X^{17}+2Y^5X^{17}\\
&+Y^5X^{18}+2Y^5X^{19}+Y^5X^{20}-Y^6X^{18}-Y^6X^{20}+Y^6X^{21}+2Y^6X^{22}+2Y^6X^{23}\\
&+2Y^6X^{24}+Y^6X^{25}-Y^7X^{22}-2Y^7X^{23}-2Y^7X^{24}-2Y^7X^{25}-Y^7X^{26}+Y^7X^{27}\\
&+Y^7X^{29}-Y^8X^{27}-2Y^8X^{28}-Y^8X^{29}-2Y^8X^{30}-Y^9X^{30}-Y^9X^{31}-2Y^9X^{32}\\
&-2Y^9X^{33}-Y^9X^{35}-Y^{10}X^{34}-2Y^{10}X^{35}-Y^{10}X^{36}-2Y^{10}X^{37}-Y^{10}X^{38}-Y^{11}X^{39}\\
&-2Y^{11}X^{40}-Y^{11}X^{41}-Y^{11}X^{42}-Y^{11}X^{43}-Y^{12}X^{45}-Y^{13}X^{47}
\end{aligned}$$

has a friendly ghost

$$\begin{aligned}
\widetilde{W_5}(X,Y) &= \left(1+Y^3X^{13}\right)\left(1+YX^4+\left(YX^4\right)^3+\left(YX^4\right)^4\right)\\
&\quad\left(1-Y^4X^{14}\right)\left(-1-X^2Y-X^4Y^2\right).
\end{aligned}$$

6.3.5 $W_5^{\triangleleft}(X,Y)$

$$\begin{aligned}
W_5^{\triangleleft}(X,Y) &= 1+Y^3X^5+Y^3X^6+Y^3X^7+Y^5X^{10}+Y^5X^{11}+2Y^5X^{12}+Y^5X^{13}\\
&+Y^7X^{15}+Y^7X^{16}+Y^7X^{17}+Y^8X^{17}+Y^8X^{18}+Y^8X^{19}+Y^{10}X^{21}+2Y^{10}X^{22}\\
&+Y^{10}X^{23}+Y^{10}X^{24}+Y^{12}X^{27}+Y^{12}X^{28}+Y^{12}X^{29}+Y^{15}X^{34}
\end{aligned}$$

has a friendly ghost

$$\widetilde{W_5^{\triangleleft}}(X,Y) = \left(1+Y^5X^{13}\right)\left(1+Y^7X^{16}\right)\left(1+Y^3X^5\right).$$

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