STABLE 2–BUNDLES WITH $(c_1,c_2) = (0,2)$ OVER A HOMOTOPY K3 SURFACE

by

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Introduction

Let S be a compact simply–connected elliptic surface and $\mathcal{E} \longrightarrow S$ be a holomorphic 2–bundle with $c_1(\mathcal{E}) = 0$. In light of Donaldson's work in [D1] it has become clear that one can choose some special polarization ω on S and have a good understanding on the nature of ω–stable 2–bundles $\mathcal{E} \longrightarrow S$ with $c_2(\mathcal{E}) = 1$. See for instance [FM], [LO] or [OV]. Recently this idea has been extended to the cases when $c_2(\mathcal{E}) > 1$. Working with certain carefully chosen polarizations, Friedman has had an extensive investigation in [F] on such stable bundles and obtained some nice qualitative information on their moduli spaces provided $c_2(\mathcal{E})$ is larger than some specified constant. The purpose of this note is to depict another nice aspect of the theory through a particular example concerning $c_2(\mathcal{E}) = 2$. We shall be more attentive to the problem of multiplicity two structures on some surface and our discussion here should be viewed as a slight supplement to that of [F].

To be more precise, let S_3 be a homotopy K3 surface having precisely a multiple fibre F_3 of multiplicity three. We study here holomorphic 2–bundles $\mathcal{E} \longrightarrow S_3$ with $(c_1(\mathcal{E}),c_2(\mathcal{E})) = (0,2)$ which are stable in certain polarization ω of S_3. The first issue of our discussion is to give a complete description of the moduli space $M_2(\omega)$ of these bundles. A peculiar feature of $M_2(\omega)$ is that due to the appearance of $H^2(sl(\mathcal{E})) \simeq \mathbb{C}$ in the deformation complex its dimension is higher than the "virtual" one. Nevertheless this moduli space is smooth. In order to describe $M_2(\omega)$ recall a known fact that as an elliptic surface S_3 is a fibration over the complex projective line $\mathbb{P}_1$. We denote the projection map from S_3 onto $\mathbb{P}_1$ by Ψ.

Theorem 1 The moduli space $M_2(\omega)$ is a smooth complex 3–dimensional manifold modelled on the proper transformation $\tilde{Y}$ of

$$Y = \{(z_1,z_2) \in (S_3\backslash F_3) \times (S_3\backslash F_3) : \Psi(z_1) = \Psi(z_2);\ z_1,z_2 \text{ distinct}\}$$

in the blow–up of $(S_3\backslash F_3) \times (S_3\backslash F_3)$ along the diagonal after dividing by the involution $\mathbb{Z}_2$ on $(S_3\backslash F_3) \times (S_3\backslash F_3)$ of interchanging factors.

This theorem in essence asserts $M_2(\omega) \simeq \tilde{Y}/\mathbb{Z}_2$. Over the moduli space $M_2(\omega)$ there is a complex line bundle $\zeta \longrightarrow M_2(\omega)$ of interest, arising from the assignment $\mathcal{E} \longrightarrow H^2(\mathrm{sl}(\mathcal{E})) \simeq \mathbb{C}$. The second issue of our discussion is to identify this bundle, or rather its square $\zeta^{\otimes 2}$, over $M_2(\omega)$. Let $[\Delta_{\tilde{Y}/\mathbb{Z}_2}]$ denote the line bundle over $M_2(\omega)$ associated to the diagonal divisor $\Delta_{\tilde{Y}/\mathbb{Z}_2}$ on $M_2(\omega)$.

Theorem 2 $\zeta^{\otimes 2} \simeq [\Delta_{\tilde{Y}/\mathbb{Z}_2}]$ as line bundles on $M_2(\omega)$.

We shall show this result by constructing explicitly a universal bundle $\mathcal{F}$ over $\tilde{Y}/\mathbb{Z}_2 \times S_3$, despite the existence of such a bundle does not follow from general consideration.

The motive of this work is to provide certain necessary material in a calculation related to the work of Donaldson in [D1] and [D2] while the interest of Theorem 2 lies in that after identifying $M_2(\omega)$ with the corresponding moduli space of anti–self–dual (ASD) connections (c.f. [D1]) determination of the "cokernel bundle" $\zeta \longrightarrow M_2(\omega)$ gives vital information on how to recover certain transversal intersections cutting moduli spaces of ASD connections sufficiently close to $M_2(\omega)$. This will be discussed elsewhere.

In the course of establishing these two results it requires a lot of checkings just applying the same technique to many different situations in question. For this reason we do not find it too enlightening to give all the details here and shall gross over those points which can be settled by straightforward arguments once the idea of the proof for one or two cases has been given. Those interested are referred to [Mo] for a more detailed discussion explaining these results.

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§1 Construction of stable 2–bundles using the residue map

Let $S_3 \xrightarrow{\Psi} \mathbb{P}_1$ be a homotopy K3 surface as mentioned in the introduction and suppose always $\mathcal{E} \longrightarrow S_3$ is a holomorphic 2–bundle with $(c_1(\mathcal{E}), c_2(\mathcal{E})) = (0,2)$. We shall assume the surface S_3 has the following generic properties:

(P1) the multiple fibre $F_3 \subset S_3$ is smooth, and

(P2) any other (simple) singular fibre $F_{a_\nu} = \Psi^{-1}(a_\nu)$ of S_3 contains only, and precisely one, node $n_\nu \in F_{a_\nu}$ as singularity.

Here in (P2) a point $n_\nu \in F_{a_\nu}$ is called a node if there are local coordinates (u_ν, v_ν) on S_3 centred at n_ν such that $\Psi = u_\nu^2 + v_\nu^2$. The goal of this section is to determine and construct all stable 2–bundles $\mathcal{E} \longrightarrow S_3$ relative to Kähler forms ω obtained by adding to arbitrary ones a large multiple of $\Psi^* \omega_{\mathbb{P}_1}$, where $\omega_{\mathbb{P}_1}$ denotes the Fubini–Study form on $\mathbb{P}_1$.

We begin with an observation that $h^0(\mathcal{E} \otimes [F_3]) > 0$ regardless $\mathcal{E}$ is stable or not. This is a consequence of the Riemann–Roch formula applied to $\mathcal{E} \otimes [F_3]$ incorporate with the (Serre duality) isomorphism

$$H^2(\mathcal{E} \otimes [F_3]) \simeq H^0(\mathcal{E} \otimes [F_3])^*$$

as $K_{S_3} \simeq [F_3]^{\otimes 2}$ on S_3. Now by following the line of arguments given for instance in [LO] one deduces that if $\mathcal{E}$ is to be ω–stable then every section s of $\mathcal{E} \otimes [F_3]$ has to vanish on a codimension two subset $Z = Z(s)$ of S_3. Consequently $\mathcal{E} \otimes [F_3]$ is an extension

$$(1.1) \qquad 0 \longrightarrow \mathcal{O} \xrightarrow{\;s\;} \mathcal{E} \otimes [F_3] \longrightarrow [F_3]^{\otimes 2} \otimes I \longrightarrow 0$$

for some ideal sheaf I of isolated zero(s) on S_3 and such locally free extensions are that we have interest to construct.

Despite having codimension two zero set Z, the section s of $\mathcal{E} \otimes [F_3]$ does not necessarily induce an ideal sheaf I of simple zeros on S_3 as $c_2(\mathcal{E} \otimes [F_3]) = 2$ and this is where our discussion diverges from previous ones concerning questions of similiar kind. Indeed one finds in (1.1) the ideal sheaf I is either that of

(a) two isolated simple zeros z_1, z_2 on S_3, or in degenerate cases,

(b) a single zero $z \in S_3$ with a multiplicity two structure of some kind.

To be more precise in case (b), we put $\mathcal{O}_Z = \mathcal{O}/I$. Then we say I is an ideal sheaf of a zero $z \in S_3$ with a multiplicity two structure if $\dim_{\mathbb{C}} \mathcal{O}_{Z,z} = 2$.

Having specified the nature of the ideal sheaf I we proceed to study, separately for the cases (a) and (b), if there are locally free extensions $\mathcal{E} \otimes [F_3]$ in (1.1). The results we obtain for case (a) can be summarized as follows.

(1.2) Proposition There is a 1–1 correspondence between

(i) equivalence classes of ω–stable bundles $\mathcal{E} \longrightarrow S_3$ fitting into an exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{s} \mathcal{E} \otimes [F_3] \longrightarrow [F_3]^{\otimes 2} \otimes I_{z_1, z_2} \longrightarrow 0$$

for some ideal sheaf I_{z_1, z_2} of two simple zeros z_1, z_2 on S_3, and

(ii) pairs of distinct points z_1, z_2 on a common fibre of S_3 other than F_3.

This proposition follows from the residue theorem for vector bundles (c.f. [GH] p. 731) asserting in this particular case the condition for a locally free extension $\mathcal{E} \otimes [F_3]$ to exist in (1.2) is equivalent to that the points z_1, z_2 in $\{s = 0\}$ are to satisfy the Cayley–Bacharach property relative to the linear system $|K_{S_3} \otimes [F_3]^{\otimes 2}|$; that is, any section of the bundle $K_{S_3} \otimes [F_3]^{\otimes 2}$ vanishing at either of the two points has to vanish at the other. As $[F_3]^{\otimes 3} \simeq \Psi^* \mathcal{O}_{\mathbb{P}_1}(1)$, one infers readily this is the case only if the points z_1, z_2 lie on a common fibre of S_3. Assuming $\mathcal{E}$ is stable, we may rule out the possibility that this common fibre is F_3 since in which case one finds $h^0(\mathcal{E}) = h^0([F_3] \otimes I) = 1$, an obvious violation to the stability condition. It is a routine matter then to check 2–bundles $\mathcal{E}$ otherwise obtained in (1.2) are ω–stable and moreover determines uniquely a pair of points z_1, z_2 on S_3 as $h^0(\mathcal{E} \otimes [F_3]) = 1$. However the converse that such an ω–stable bundle is uniquely determined up to isomorphism by the associated pair of points on S_3 requires a study of certain spectral sequence that we are not to discuss here; similar problems will come up later and it will be clear how the argument should go.

Now we consider case (b). In order to state the result we need a more vivid interpretation of an ideal sheaf I of a zero $z \in S$ with a multiplicity two structure. For simplicity write I_z for such an ideal sheaf. In principle a multiplicity two structure defined by I_z is a tangential direction in which the point z is approached. This is indicated in that after making suitable choices one can find generators s_1, s_2 of I_z taking the following form locally at z:

$$s_1 = (au + bv)^2, \qquad s_2 = cu + dv$$

where a,d,c,d are complex numbers satisfying $ad - bc \neq 0$. We say in this situation I_z has a multiplicity two structure along $a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}$ at z and identify I_z with the projectivized element $[a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}] \in \mathbb{P}(T_z S_3)$. Now we can state the result for case (b) as follows.

(1.3) Proposition. There is a 1–1 correspondence between

(i) equivalence classes of ω–stable bundle $\mathcal{E} \to S_3$ fitting into an exact sequence

$$0 \to \mathcal{O} \to \mathcal{E} \otimes [F_3] \to [F_3]^{\otimes 2} \otimes I_z \to 0$$

for some ideal sheaf I_z on S_3 with a multiplicity two structure, and

(ii) the points in $\{d\phi = 0\} \subset \mathbb{P}(T(S_3 \backslash F_3))$.

It is easy to check topologically $\{d\phi = 0\}$ in $\mathbb{P}(S_3 \backslash F_3)$ is just the blow–up $\widehat{S_3 \backslash F_3}$ of $S_3 \backslash F_3$ at all the nodes n_ν on singular fibres of S_3. In this topological model, a point z of $\widehat{S_3 \backslash F_3}$ not lying in the exceptional curves E_{n_ν} is to be identified with the projectivized tangent vector $[T_z F_{\Psi(z)}] \in \mathbb{P}(T_z S_3)$ where $F_{\Psi(z)}$ denotes the fibre to which z belongs. Note that points in the exceptional curves $E_{n_\nu} \subset \widehat{S_3 \backslash F_3}$ are already elements of $\mathbb{P}(T_{n_\nu} S_3) \subset \mathbb{P}(T(S_3 \backslash F_3))$ in a natural way.

In order to show this proposition we need a few facts of the residue theorem for vector bundles applying to

$$(1.4) \qquad 0 \to [F_3]^{-2} \xrightarrow{s} \mathcal{E} \otimes [F_3]^{-1} \to I_z \to 0$$

where $[F_3]^{-2}$ denotes $([F_3])^{-1})^{\otimes 2}$, the square of the dual bundle $[F_3]^{-1}$ for $[F_3]$. As $H^1([F_3]^{-2}) = 0$, the spectral sequence relating global and local Ext groups takes the form

$$0 \to \mathrm{Ext}^1(I_z,[F_3]^{-2}) \to \underline{\mathrm{Ext}}^1(I_z,[F_3]^{-2}) \to H^2([F_3]^{-2}) \to \ldots$$

with both $\underline{\mathrm{Ext}}^1(I_z[F_3]^{-2}) \simeq \mathcal{O}_z$ and $H^2([F_3]^{-2})$ are isomorphic to $\mathbb{C}^2$. For our purpose it is enough to consider I_z is locally generated by $s_1 = u$, $s_2 = v^2$ in some local coordinates (u,v) of z. In such cases $\mathcal{O}_z \simeq \mathbb{C}[1,v]$ and an element $e_z \in \underline{\mathrm{Ext}}^1(I_z,[F_3]^{-2})$ corresponding to $\lambda_z + \mu_z v \in \mathcal{O}_z$ lifts to $\mathrm{Ext}^1(I_z,[F_3]^{-2})$ provided that the residue pairing

$$\mathrm{Res}_{\{0\}} \frac{(\lambda_z+\mu_z v)\,\phi(u,v)du\wedge dv}{uv^2}$$

vanishes for all $\phi \in H^0(K_{S_3}\otimes[F_3]^{\otimes 2})$. Finally, such a lifting gives locally free extension precisely when e_z is a unit in $\underline{\mathrm{Ext}}^1(I_z,[F_3]^{-2})$.

We apply this framework to many different situations when showing proposition (1.3). Consider first $z \in S_3\backslash F_3$ is not a node. Let $\Psi = u$ so that the multiplicity two structure is along the fibre direction $T_{\Psi(z)}F$. In this case we have

$$\mathrm{Res}_{\{0\}} \frac{(\lambda_z+\mu_z v)\,\phi(u,v)du\wedge dv}{uv^2} = \mu_z\phi(0)$$

for all $\phi \in H^0(K_{S_3}\otimes[F_3]^{\otimes 2})$ and so precisely the unit λ_z lifts to $\mathrm{Ext}^1(I_z,[F_3]^{-2})$. Now we can check in (1.3) corresponding to I_z there is an ω-stable bundle $\mathcal{E}$ which is unique up to isomorphism as the proposition asserts. Suppose now $\Psi = v$ so that the multiplicity two structure is transversal to the fibre direction. We get this time

$$\mathrm{Res}_{\{0\}} \frac{(\lambda_z+\mu_z v)\,\phi(u,v)du\wedge dv}{uv^2} = \mu_z\phi(0) + \lambda_z \frac{\partial\phi}{\partial v}(0)\,.$$

Note that ϕ is a section of $K_{S_3} \otimes [F_3]^{\otimes 2} \simeq [F_3] \otimes \Psi^* \mathcal{O}_{\mathbb{P}_1}(1)$ and hence we have $\frac{\partial\phi}{\partial v}(0) \neq 0$ whenever $\phi(0) = 0$. Thus the above expression vanishes for all ϕ only if $\lambda_z = 0$ but then the same requirement forces $\mu_z = 0$ as well. Thus there is no locally free extension in (1.4) for such I_z. To complete the proof of proposition (1.3) we are to study the cases when z is a point of F_3 or a node on a singular fibre. A straightforward investigation as in previous cases yields the desired result.

Remark This line of arguments in fact applies equally well to study ω–stable 2–bundles with $(c_1,c_2) = (0,1)$ over S_3 but shows that such bundles do not exist at all (c.f. [LO]).

In §3 we shall be discussing a cokernel bundle $\zeta \longrightarrow M_2(\omega)$ and the following proposition ensures the existence of such a bundle.

(1.5) Proposition For every ω–stable 2–bundle $\mathcal{E} \longrightarrow S_3$ with $(c_1(\mathcal{E}),c_2(\mathcal{E})) = (0,2)$ we have $h^2(\text{End } \mathcal{E}) = 2$, where End $\mathcal{E}$ denotes the endomorphism bundle of $\mathcal{E}$.

Proof. By previous discussions, $\mathcal{E}$ comes from an extension

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \otimes [F_3] \longrightarrow [F_3]^{\otimes 2} \otimes I \longrightarrow 0$$

and it is easy to deduce from this

$$h^2(\text{End } \mathcal{E}) = 1 + h^0(\mathcal{E} \otimes [F_3]^{\otimes 3} \otimes I) \text{ and } h^0(\mathcal{E} \otimes [F_3]^{\otimes 3}) = 2\,.$$

For our purpose it suffices to check in the exact sequence

(1.6) $$0 \longrightarrow [F_3]^{\otimes 2} \longrightarrow \mathcal{E}\otimes [F_3]^{\otimes 3} \longrightarrow [F_3]^{\otimes 4}\otimes I \longrightarrow 0$$

the lift $\tilde{s} \in H^0(\mathcal{E}\otimes [F_3]^{\otimes 3})$ of a non–trivial section $s \in H^0([F_3]^{\otimes 4}\otimes I)$ is not an element of $H^0(\mathcal{E} \otimes [F_3]^{\otimes 3}\otimes I)$. This can be accomplished by considering the Koszul complex associated to the exact sequence (1.6) just as in [FM] and therefore we omit the argument here.

§2. Construction of the moduli space

The purpose of this section is to construct the moduli space $M_2(\omega)$ of ω–stable 2–bundles $\mathcal{E} \longrightarrow S_3$ with $(c_1(\mathcal{E}),c_2(\mathcal{E})) = (0,2)$ and then show such a moduli space is smooth. We first define a variety $\tilde{Y}$ which "doubly" parametrizes all such stable bundles. To begin with, let

$$Y = \{(z_1,z_2) \in ((S_3\backslash F_3) \times (S_3\backslash F_3))\backslash \Delta_{S_3\backslash F_3} : \Psi(z_1) = \Psi(z_2)\} .$$

Then we blow up $(S_3\backslash F_3)\times(S_3\backslash F_3)$ along the diagonal $\Delta_{S_3\backslash F_3}$ and obtain a new manifold $\widehat{(S_3\backslash F_3)\times}(S_3\backslash F_3)$ on where there is a projection π mapping onto $((S_3\backslash F_3)\times(S_3\backslash F_3))\backslash \Delta_{S_3\backslash F_3}$. We define $\tilde{Y}$ to be the closure of $\pi^{-1}(Y)$ in $\widehat{(S_3\backslash F_3)\times}(S_3\backslash F_3)$.

(2.1) Proposition $\tilde{Y}$ is a smooth complex 3–dimensional manifold.

Proof: It is not difficult to see firstly $Y \subset \hat{Y}$ is always a smooth manifold. Indeed, as long as neither of the points $z_1, z_2 \in S_3 \setminus F_3$ is a node on a singular fibre of $S_3 \setminus F_3$, we can easily find local coordinates for the pair $(z_1, z_2) \in Y$. It is just a simple application of the fact that the symmetric product of a Riemann surface is a smooth manifold. In the case when the point, say, z_1 is a node on a singular fibre of $S_3 \setminus F_3$, we choose local coordinates (u_i, v_i) for the points $z_i \in S_3 \setminus F_3$ so that

$$\Psi = u_1^2 + v_1^2 \qquad \text{near } z_1, \qquad \text{and}$$
$$\Psi = u_2 \qquad \text{near } z_2.$$

A neighbourhood of $(z_1, z_2) \in Y$ is then given by $\{f = 0\}$ where $f = u_2 - u_1^2 - v_1^2$. As $df \neq 0$, it follows Y is smooth.

To prove $\hat{Y}$ is also smooth, we let $U \subset S_3 \setminus F_3$ be a neighbourhood of a point $z \in S_3 \setminus F_3$ and (u,v) be local coordinates on U. Denote by (u_i, v_i) the local coordinates on the i–th U factor in $U \times U$. Let

$$w_1 = u_1 - u_2, \qquad w_2 = v_1 - v_2,$$
$$w_3 = u_1 + u_2, \qquad w_4 = v_1 + v_2$$

be a change of coordinates on $U \times U$. Clearly, the diagonal Δ_U of $U \times U$ is given by $\{w_1 = w_2 = 0\}$ and the blow–up $\widehat{U \times U}$ of $U \times U$ along Δ_U is the manifold

$$\{(w_1, w_2, w_3, w_4, [\ell_1, \ell_2]) \in U \times U \times \mathbb{P}_1 : w_1 \ell_2 = w_2 \ell_1\}.$$

Now if $z \in S_3 \setminus F_3$ is not a node on a singular fibre of $S_3 \setminus F_3$, we can take $\Psi = u$ on U. It follows $Y \cap (U \times U) = \{w_1 = 0\}$ and from which one infers

(2.2) $$\tilde{Y} \cap (U\widehat{\times}U) = \{\ell_1 = 0\}.$$

On the other hand, if z is a node on a singular fibre, we let $\Psi = u^2 + v^2$ on U. The condition $\Psi(z_1) = \Psi(z_2)$ on $U \times U \setminus \Delta_U$ reads $u_1^2 + v_1^2 = u_2^2 + v_2^2$, or $(u_1-u_2)(u_1+u_2) + (v_1-v_2)(v_1+v_2) = 0$. In terms of coordinates (w_1,w_2,w_3,w_4), it becomes $w_1w_3 + w_2w_4 = 0$. By the fact that $[\ell_1,\ell_2] = [w_1,w_2]$ for $(w_1,w_2) \neq (0,0)$, one finds then

(2.3) $$\tilde{Y} \cap (U\widehat{\times}U) = \{\ell_1 w_2 = \ell_2 w_1;\ \ell_1 w_3 + \ell_2 w_4 = 0\}.$$

It is clear both in (2.2) and (2.3) $\tilde{Y} \cap (U\widehat{\times}U)$ is smooth and so the proposition follows.

Now we wish to explain what the relation between the manifold $\tilde{Y}$ and the moduli space $M_2(\omega)$ is.

(2.4) Proposition The manifold $\tilde{Y}$ naturally parametrizes all ω–stable 2–bundles $\mathcal{E} \longrightarrow S_3$ with $(c_1(\mathcal{E}),c_2(\mathcal{E})) = (0,2)$. On the submanifold Y of $\tilde{Y}$, the parametrization is two to one.

Proof Understandably this parametrization is induced from the nature of the zero(s) associated to sections $s \in H^0(\mathcal{E} \otimes [F_3])$ by previous discussion in §1. Hence the second assertion is obvious. We are to show points z in $\tilde{Y} \setminus Y$ correspond to ideal sheaves I_z in a natural way. It is enough to work with local models $U\widehat{\times}U$ on where points in $\tilde{Y} \setminus Y$ are characterized by the conditions $w_1 = w_2 = 0$. One observes in such situations $w_3 = 2u$ and $w_4 = 2v_1$. Thus w_3 and w_4 are essentially local coordinates on U. Meanwhile, along $w_1 = w_2 = 0$ there are natural identification (cf. [GH] p. 603) as follows:

$$(0,0,w_3,w_4,[\ell_1,\ell_2]) \in \widehat{U\times U} \longrightarrow [\ell_1 \frac{\partial}{\partial w_1} + \ell_2 \frac{\partial}{\partial w_2}] \in \mathbb{P}((N_{\Delta_U})_{(0,0,w_3,w_4)}) .$$

The latter is identified with $[\ell_1 \frac{\partial}{\partial w_3} + \ell_2 \frac{\partial}{\partial w_4}] \in \mathbb{P}((T_{\Delta_U})_{(0,0,w_3,w_4)})$ since in the present context the normal bundle N_{Δ_U} of Δ_U in $U \times U$ is in essence the tangent bundle T_{Δ_U} of Δ_U. If $z \in U$ is not a node on a singular fibre of $S_3 \backslash F_3$, we may take $\Psi = u$ and obtain as before

$$(\tilde{Y}\backslash Y) \cap (\widehat{U\times U}) = \{(0,0,w_3,w_4[\ell_1,\ell_2]) : \ell_1 = 0\} .$$

This corresponds under the identification the element $[\frac{\partial}{\partial v}] \in \mathbb{P}(T_{(2u,2v)}S_3)$ which is precisely the tangential direction $T_{\Psi(2u,2v)}F$ of the fibre. Therefore at the point $z \in U$ the element $\{\ell_1 = 0\}$ in $\mathbb{P}(T_zS_3)$ corresponds to the ideal sheaf $I_z = (u,v^2)$ as wished. On the other hand, if $z \in U$ is a node on a singular fibre, we assume $\Psi = u^2 + v^2$ as before and find for any point $y \in U$ that

$$((\tilde{Y}\backslash Y) \cap (\widehat{U\times U})) \cap \mathbb{P}(T_y,S_3)$$
$$= \{[\ell_1,\ell_2] \in \mathbb{P}(T_yS_3) : \ell_1 w_2 = \ell_2 w_1;\ \ell_1 w_3 + \ell_2 w_4 = 0\} .$$

Since the point $z \in U$ is given by $w_1 = w_2 = w_3 = w_4 = 0$, it follows $((\tilde{Y}\backslash Y) \cap (\widehat{U\times U})) \cap \mathbb{P}(T_zS_3)$ is the whole of $\mathbb{P}(T_zS_3)$. We conclude therefore $\tilde{Y} \cap \mathbb{P}(T(S_3\backslash F_3))$ parametrizes all stable bundles $\mathcal{E}$ in our consideration having a section s of $\mathcal{E}\otimes [F_3]$ with a zero of multiplicity two. The proposition follows.

Now we are in a position to describe the moduli space $M_2(\omega)$. On the manifold Y there is an $\mathbb{Z}_2$–action interchanging the order of the pair $(z_1,z_2) \in (S_3\backslash F_3)\times(S_3\backslash F_3)$. In terms of coordinates (w_1,w_2,w_3,w_4) the $\mathbb{Z}_2$–action is given by

$$(w_1,w_2,w_3,w_4) \longrightarrow (-w_1,-w_2,w_3,w_4)\,.$$

This action extends to the blow–up model $\widehat{U\times U}$ in the obvious way and it is clear $M_2(\omega)$ models $\hat{Y}/\mathbb{Z}_2$.

(2.5) <u>Proposition</u> $\hat{Y}/\mathbb{Z}_2$ is smooth.

<u>Proof</u>. It is enough to check local models $\hat{Y} \cap (\widehat{U\times U})/\mathbb{Z}_2$ are smooth. In general, the manifold

$$\widehat{U\times U} = \{(w_1,w_2,w_3,w_4,[\ell_1,\ell_2]) \in U\times U\times\mathbb{P}_1 : w_1\ell_2 = w_2\ell_1\}$$

is covered by the patches $\{\ell_1\neq 0\}$ and $\{\ell_2\neq 0\}$. On $\{\ell_1\neq 0\}$ there are local coordinates $(w_1,w_3,w_4,\frac{\ell_2}{\ell_1})$ and we define on $\{\ell_1\neq 0\}/\mathbb{Z}_2$ local coordinates

$$(z_1,w_3,w_4,\frac{\ell_2}{\ell_1}) = (w_1^2,w_3,w_4,\frac{\ell_2}{\ell_1})\,.$$

Clearly $(z_1,w_3,w_4,\frac{\ell_2}{\ell_1})$ corresponds precisely to the pair of points

$$(w_1,w_1\,\frac{\ell_2}{\ell_1},w_3,w_4,[\ell_1,\ell_2]),\ (-w_1,-w_1\,\frac{\ell_2}{\ell_1},w_3,w_4,[\ell_1,\ell_2])$$

on $U\widehat{\times}U$ as $w_2 = w_1 \frac{\ell_2}{\ell_1}$. Likewise we write

$$(z_2, w_3, w_4, \frac{\ell_1}{\ell_2}) = (w_2^2, w_3, w_4, \frac{\ell_1}{\ell_2})$$

for local coordinates on $\{\ell_2 \neq 0\}/\mathbb{Z}_2$. Now in (2.2) we define for the local neighbourhood $\hat{Y} \cap (U\widehat{\times}U)/\mathbb{Z}_2$, where

$$\hat{Y} \cap (U\widehat{\times}U) = \{(w_1, w_2, w_3, w_4, [\ell_1, \ell_2]) : \ell_1 w_2 = \ell_2 w_1;\ \ell_1 = 0\},$$

local coordinates $(z_2, w_3, w_4) = (w_2^2, w_3, w_4)$. Similarly in (2.3) we write for $\tilde{Y} \cap (U\widehat{\times}U)/\mathbb{Z}_2$, where

$$\hat{Y} \cap (U\widehat{\times}U) = \{(w_1, w_2, w_3, w_4, [\ell_1, \ell_2]) : \ell_1 w_2 = \ell_2 w_1;\ \ell_1 w_3 + \ell_2 w_4 = 0\},$$

local coordinates

$$(z_1, w_4, \frac{\ell_2}{\ell_1}) = (w_1^2, w_4, \frac{\ell_2}{\ell_1}) \qquad \text{on } \{\ell_1 \neq 0\}/\mathbb{Z}_2 \text{ and}$$
$$(z_2, w_3, \frac{\ell_1}{\ell_2}) = (w_2^2, w_3, \frac{\ell_1}{\ell_2}) \qquad \text{on } \{\ell_2 \neq 0\}/\mathbb{Z}_2.$$

This shows $\hat{Y}/\mathbb{Z}_2$ admits a smooth manifold structure and proves the proposition.

Before closing this section we wish to discuss two more things required later. The first one is that when constructing a universal bundle $\not{s}$ over $\hat{Y}/\mathbb{Z}_2 \times S_3$ in the next section we shall come across two line bundles on $\hat{Y}/\mathbb{Z}_2$ which are more convenient to describe here. Using previous notations the local functions $z_i = 0$ on $\{\ell_i \neq 0\}/\mathbb{Z}_2$ define a diagonal divisor $\Delta_{\tilde{Y}/\mathbb{Z}_2}$ on $\tilde{Y}/\mathbb{Z}_2$ and we write $[\Delta_{\tilde{Y}/\mathbb{Z}_2}]$ for the associated line bundle. Likewise one can introduce a line bundle ζ^{-1} over $\tilde{Y}/\mathbb{Z}_2$ using defining functions

$$\begin{aligned} \ell_2 &= 0 \qquad \text{on} \qquad \{\ell_1 \neq 0\}/\mathbb{Z}_2 \text{ and} \\ \ell_1 &= 0 \qquad \text{on} \qquad \{\ell_2 \neq 0\}/\mathbb{Z}_2 . \end{aligned}$$

Note that the dual bundle ζ of ζ^{-1} when restricted to the diagonal $\Delta_{\tilde{Y}/\mathbb{Z}_2} \subset \tilde{Y}/\mathbb{Z}_2$ is in fact the tautological bundle over $\Delta_{\tilde{Y}/\mathbb{Z}_2} \hookrightarrow \mathbb{P}(T(S_3 \setminus F_3))$. From the relation $\ell_1 w_2 = \ell_2 w_1$ one infers $\ell_1^2 z_2 = \ell_2^2 z_1$ which gives

$$\frac{z_1}{z_2} = \frac{\ell_1^2}{\ell_2^2} \text{ on } \{\ell_1 \neq 0\}/\mathbb{Z}_2 \cap \{\ell_2 \neq 0\}/\mathbb{Z}_2 .$$

As a consequence the bundle ζ over $\tilde{Y}/\mathbb{Z}_2$ is a root of $[\Delta_{\tilde{Y}/\mathbb{Z}_2}]$ or that $\zeta^{\otimes 2} \simeq [\Delta_{\tilde{Y}/\mathbb{Z}}]$. The second thing we shall need is that this root is unique up to equivalence. The following proposition ensures this is the case.

(2.6) Proposition The moduli space $M_2(\omega) \simeq \tilde{Y}/\mathbb{Z}_2$ is simply-connected.

Proof Observe first for a smooth fibre F of $S_3 \xrightarrow{\Psi} \mathbb{P}_1$ the symmetric product $S^2F = (F \times F)/\mathbb{Z}_2$ is a smooth manifold admitting a fibration structure

$$(2.7) \qquad \begin{array}{ccc} \mathbb{P}_1 & \longrightarrow & S^2F \\ & & \big\downarrow \text{pr} \\ & & F \simeq \Delta_F \subset S^2F \end{array}$$

via a projection map pr sending a pair of points on F to their sum. By applying this framework to all smooth fibres F of S_3 we obtain naturally a fibration in the following way:

$$\begin{array}{ccc} \mathbb{P}_1 \longrightarrow & (\tilde{Y}/\mathbb{Z}_2)\backslash\tilde{\Psi}^{-1}(A) \\ & \big\downarrow \\ & (S_3\backslash F_3)\backslash\Psi^{-1}(A) \end{array} \tag{2.8}$$

where $A = \{a \in \mathbb{P}_1 :$ the fibre $F_a = \Psi^{-1}(a)$ of S_3 is singular$\}$ while the map $\tilde{\Psi}$ when restricted to smooth copies of $S^2F \hookrightarrow \tilde{Y}/\mathbb{Z}_2$ is simply the projection map pr in (2.7). As $\pi_1(\mathbb{P}_1) = 0$, we deduce from (2.8) that every essential loop of $(\tilde{Y}/\mathbb{Z}_2)\backslash\tilde{\Psi}^{-1}(A)$ is realized by one in $(S_3\backslash F_3)\backslash\Psi^{-1}(A)$.

Now to show $\pi_1(\tilde{Y}/\mathbb{Z}_2) = 0$ observe first there is an obvious surjective map

$$\pi_1((\tilde{Y}/\mathbb{Z}_2)\backslash\tilde{\Psi}^{-1}(A)) \longrightarrow \pi_1(\tilde{Y}/\mathbb{Z}_2) .$$

For our purpose we argue every loop in $(\tilde{Y}/\mathbb{Z}_2)\backslash\tilde{\Psi}^{-1}(A)$ deforms to the trivial one in $\tilde{Y}/\mathbb{Z}_2$. As explained above such a loop can be chosen to be in $(S_3\backslash F_3)\backslash\Psi^{-1}(A)$ and hence must be trivial in $\tilde{Y}/\mathbb{Z}_2$ using the facts $\pi_1(S_3/F_3) = 0$ (c.f. [K] lemma 4) and $S_3\backslash F_3 \simeq \Delta_{S_3\backslash F_3} \hookrightarrow \tilde{Y}/\mathbb{Z}_2$. The proposition follows.

§3. Determination of the cokernel bundle

As explained in (1.5) for every element $[\mathcal{E}] \in M_2(\omega)$ we have $H^2(\mathrm{End}\ \mathcal{E}) \simeq \mathbb{C}^2$. If we denote $\mathrm{sl}(\mathcal{E}) \subset \mathrm{End}\ \mathcal{E}$ the bundle of trace–free endomorphisms, then one finds $H^2(\mathrm{sl}(\mathcal{E})) \simeq \mathbb{C}$ since $H^2(\mathcal{O}) \simeq \mathbb{C}$ on S_3. Thus the assignment $\mathcal{E} \longrightarrow H^2(\mathrm{sl}(\mathcal{E}))$ defines a line bundle over $M_2(\omega) \simeq \tilde{Y}/\mathbb{Z}_2$. We shall show in this section the square of such a bundle identifies with the diagonal divisor $\Delta_{\tilde{Y}/\mathbb{Z}_2}$ on $\tilde{Y}/\mathbb{Z}_2$.

Note first on the product space $\tilde{Y}/\mathbb{Z}_2 \times S_3$ there exists tautologically a "universal" ideal sheaf $\mathcal{J}$ whose defining functions when restricted to $\{\tilde{y}\} \times S_3$ is precisely the one on S_3 parametrized by $\tilde{y} \in \tilde{Y}/\mathbb{Z}_2$. For our purpose, it is enough to construct a universal bundle $\mathcal{F} \longrightarrow M_2(\omega) \times S_3$ in an exact sequence

$$(3.1) \qquad 0 \longrightarrow pr_1^*L \longrightarrow \mathcal{F} \otimes pr_2^*[F_3] \longrightarrow pr_2^*[F_3]^{\otimes 2} \otimes \mathcal{J} \longrightarrow 0.$$

Here pr_i denotes the obvious projection map from $\tilde{Y}/\mathbb{Z}_2 \times S_3$ to the i–th factor while L is some line bundle over $\tilde{Y}/\mathbb{Z}_2$. The reason for this is as follows. Granted the existence of $\mathcal{F}$ in (3.1) we observe the $\mathbb{C}^2$–bundle over $\tilde{Y}/\mathbb{Z}_2$ given by $\mathcal{E} \longrightarrow H^2(\mathrm{End}\ \mathcal{E})$ is just the second direct image sheaf $(pr_1)_{*2}(\mathrm{End}\ \mathcal{F})$. By Serre duality this $\mathbb{C}^2$–bundle is dual to

$$(pr_1)_*((\mathrm{End}\ \mathcal{F}) \otimes pr_2^*K_{S_3}) \simeq (pr_1)_*(\mathrm{End}\ \mathcal{F}) \otimes pr_2^*[F_2]^{\otimes 2})$$

and it is easy to deduce from this

$$((pr_1)_{*2}(\mathrm{sl}(\mathcal{F})))^* \simeq L^{-1} \otimes ((pr_1)_*(\mathcal{F} \otimes pr_2^*[F_3]))^{\otimes 2}$$

as $\mathrm{End}\ \mathcal{F} \simeq \mathcal{F}^* \otimes \mathcal{F}$ and

$$\mathscr{E}^* \simeq \Lambda^2(\mathscr{E}^*) \otimes \mathscr{E} \simeq (pr_1^* L^{-1}) \otimes \mathscr{E} .$$

Using $(pr_1)_*(pr_2^*[F_3]^{\otimes 2} \otimes \mathscr{I}) = 0$ in (3.1) we obtain

$$(pr_1)_*(\mathscr{E} \otimes pr_2^*[F_3]) \simeq (pr_1)_*(pr_1^* L) \simeq L$$

and hence that

$$(pr_1)_{*2}(sl(\mathscr{E})) \simeq (L^{-1} \otimes L^{\otimes 2})^{-1} \simeq L^{-1} .$$

We shall see in the explicit construction of $\mathscr{E}$ the bundle L will be taken to be the dual of $\zeta \longrightarrow M_2(\omega) \simeq Y/\mathbb{Z}_2$ defined in §2. Therefore we have

$$(pr_1)_*(sl(\mathscr{E})) \simeq \zeta$$

and moreover that

$$(pr_1)_*(sl(\mathscr{E}))^{\otimes 2} \simeq \zeta^{\otimes 2} \simeq [\Delta_{\tilde{Y}/\mathbb{Z}_2}]$$

as asserted in Theorem 2.

A main problem of constructing the universal bundle $\mathscr{E} \longrightarrow \tilde{Y}/\mathbb{Z}_2 \times S_3$ in (3.1) is to find a root for the diagonal divisor $\Delta_{\tilde{Y}/\mathbb{Z}_2}$ on $\tilde{Y}/\mathbb{Z}_2$ but we have already known in §2 there is indeed one, namely, the bundle $\zeta \longrightarrow \tilde{Y}/\mathbb{Z}_2$. Nevertheless it might not be entirely obvious why such a root should exist in the first place. It is undoubtedly the case however if we restrict our attention to a subfamily $\mathscr{E}_{S^2F}$ of bundles parametrized by the symmetric product $S^2F \hookrightarrow \tilde{Y}/\mathbb{Z}_2$ of a smooth fibre F on S_3 since in which case we may explicitly determine the second Stiefel–Whitney class $w_2([\Delta_{S^2F}])$ for the line bundle $[\Delta_{S^2F}]$ associated to the diagonal divisor Δ_{S^2F} on S^2F and conclude whereby $[\Delta_{S^2F}]$ always has a root as $w_2([\Delta_{S^2F}]) = 0$. In fact we may further deduce in such cases the manifold S^2F is free of torsion and so the topological obstruction to the existence of $\not{\mathcal{S}}_{S^2F}$ imposed by the Brauer class on $S^2F \times S_3$ automatically vanishes. All these nice properties of S^2F have been discussed by Macdonald in [Ma]. As a matter of fact the construction of $\not{\mathcal{S}}_{S^2F}$ is fundamental to the existence of the universal bundle $\not{\mathcal{S}}$ in (3.1) as we shall see in a moment. To avoid repeating material however we play low the role of $\not{\mathcal{S}}_{S^2F}$ here in this discussion.

A standard procedure of finding whether or not a universal bundle $\not{\mathcal{S}} \longrightarrow \tilde{Y}/\mathbb{Z}_2 \times S_3$ exists in (3.1) is to study the spectral sequence

$$
\begin{aligned}
(3.2) \qquad 0 &\longrightarrow H^1(\tilde{Y}/\mathbb{Z}_2 \times S_3; pr_1^* \mathcal{P} \otimes pr_2^*[F_3]^{-2}) \\
&\longrightarrow Ext^1(\tilde{Y}/\mathbb{Z}_2 \times S_3; \not{\mathcal{S}}, pr_1^* L \otimes pr_2^*[F_3]^{-2}) \\
&\longrightarrow H^0(\tilde{Y}/\mathbb{Z}_2 \times S_3; \underline{Ext}^1(\not{\mathcal{S}} pr_1^* L \otimes pr_2^*[F_3]^{-2})) \\
&\longrightarrow H^2(\tilde{Y}/\mathbb{Z}_2 \times S_3; pr_2^*[F_3]^{-2}) \longrightarrow \ldots
\end{aligned}
$$

which we shall examine very carefully. Observe first

$$H^1(\tilde{Y}/\mathbb{Z}_2 \times S_3; pr_1^* L \otimes pr_2^*[F_3]^{-2}) = 0$$

by the Künneth formula as $H^0(S_3;[F_3]^{-2}) = H^1(S_3;[F_3]^{-2}) = 0$. To proceed on, let $X = \mathrm{supp}(\mathcal{O}_{\tilde{Y}/\mathbb{Z}_2 \times S_3}/\not{\mathcal{S}}) \subset \tilde{Y}/\mathbb{Z}_2 \times S_3$. Note that X is a manifold and the restricted projection $pr_1|_X : X \longrightarrow \tilde{Y}/\mathbb{Z}$ is a two–to–one covering map branched along the diagonal $\Delta_{\tilde{Y}/\mathbb{Z}_2}$ of $\tilde{Y}/\mathbb{Z}_2$. Via the local duality theorem (c.f. [GH]) we have an isomorphism

$$(3.3) \qquad \underline{\mathrm{Ext}}^1(\mathcal{J}, pr_1^*L \otimes pr_2^*[F_3]^{-2} \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Lambda^2(\mathcal{J}/\mathcal{J}^2), pr_1^*L \otimes pr_2^*[F_3]^{-2} \otimes \mathcal{O}_X)$$

and our task is to find a line bundle $L \longrightarrow \tilde{Y}/\mathbb{Z}_2$ to make the sheavf above a copy of $\mathcal{O}_X$. This suggests taking $L \simeq \zeta^{-1}$ as we are going to explain.

Note first $\Lambda^2(\mathcal{J}/\mathcal{J}^2)$ is the determinant of the conormal bundle N_X^* of X in $\tilde{Y}/\mathbb{Z}_2 \times S_3$ and so we have

$$(3.4) \qquad \begin{aligned} \Lambda^2(\mathcal{J}/\mathcal{J}^2) &\simeq \Lambda^5((T^*_{\tilde{Y}/\mathbb{Z}_2 \times S_2}|_X) \otimes \Lambda^3 T_X \\ &\simeq (pr_1^* K_{Y/\mathbb{Z}_2} \otimes pr_2^* K_{S_3})|_X \otimes \Lambda^3 T_X \\ &\simeq pr_1^* K_{\tilde{Y}/\mathbb{Z}_2}|_X \otimes \Lambda^3 T_X \end{aligned}$$

since K_{S_3} is trivial on $S_3 \backslash F_3$. On the other hand, as X is a branched covering of $\tilde{Y}/\mathbb{Z}_2$, there is a ramification divisor on X which we denote by $pr_1^{-1}(\Delta_{\tilde{Y}/\mathbb{Z}_2})$. Substituting the Hurwicz formula

$$K_X = pr_1^* K_{\tilde{Y}/\mathbb{Z}_2} \otimes \mathcal{O}_X(pr_1^{-1}(\Delta_{\tilde{Y}/\mathbb{Z}_2}))$$

in (3.4) we obtain

(3.5) $$\Lambda^2(\mathcal{I}\mathcal{J}) \simeq \mathcal{O}_X(\mathrm{pr}_1^{-1}(\Delta_{\tilde{Y}/\mathbb{Z}_2}))^{-1} \simeq (\mathrm{pr}_1^*\zeta^{-1})|_X .$$

Note that the second isomorphism comes from the facts $\zeta^{\otimes 2} \simeq [\Delta_{\tilde{Y}/\mathbb{Z}_2}]$ on $\tilde{Y}/\mathbb{Z}_2$ and

$\mathrm{pr}_*^1([\Delta_{\tilde{Y}/\mathbb{Z}_2}])|_X \simeq (\mathrm{pr}_1^{-1}(\Delta_{\tilde{Y}/\mathbb{Z}_2})|_X)^{\otimes 2}$ on X. These two identifications combine to give

$$(\mathrm{pr}_1^*\zeta)^{\otimes 2}|_X \simeq (\mathrm{pr}_1^{-1}(\Delta_{\tilde{Y}/\mathbb{Z}_2})|_X)^{\otimes 2}$$

and hence the asserted isomorphism as the manifold X is simply–connected (c.f. proposition (2.6)).

It is now clear after taking $L \simeq \zeta^{-1}$ in (3.2) one deduces from (3.3) and (3.5) that

$$H^0(\tilde{Y}/\mathbb{Z}_2\times S_3;\underline{\mathrm{Ext}}^1(\mathcal{I}\mathrm{pr}_1^*\zeta^{-1}\otimes \mathrm{pr}_2^*[F_3]^{-2}) \simeq H^0(\mathcal{O}_X)$$

as $\mathrm{pr}_2^*[F_3]^{-2}|_X$ is trivial. Since $H^0(\mathcal{O}_X)$ contains the constant function 1 as an element, the universal bundle $\mathcal{J}$ in (3.1) exists if the obstruction

$$H^2(\tilde{Y}/\mathbb{Z}_2\times S_3;\mathrm{pr}_1^*\zeta^{-1}\otimes \mathrm{pr}_2^*[F_3]^{-2}) \simeq H^0(\tilde{Y}/\mathbb{Z}_2,\zeta^{-1})\otimes H^2(S_3;[F_3]^{-2})$$

vanishes or equivalently that $H^0(\tilde{Y}/\mathbb{Z}_2,\zeta^{-1}) = 0$ as $(H^2(S_3;[F_3]^{-2}) \simeq \mathbb{C}^2$. To see it is indeed the case we note first the bundle ζ^{-1} when restricted to copies $S^2F \hookrightarrow \tilde{Y}/\mathbb{Z}_2$ of symmetric products of smooth fibre F on S_3 has no holomorphic section. This is a consequence of the fact that $\zeta^{\otimes 2}|_{S^2F} \simeq [\Delta_{\tilde{Y}/\mathbb{Z}_2}]|_{S^2F}$ is represented by an effective divisor on S^2F. As the union of such copies of S^2F is a dense open set in $\tilde{Y}/\mathbb{Z}$, we divisor on S^2F. As the union of such copies of S^2F is a dense open set in $\check{Y}/\mathbb{Z}$, we conclude $H^0(\check{Y}/\mathbb{Z}_2;\zeta^{-1}) = 0$ and therefore the existence of the universal bundle $\not{\mathcal{E}}$ in (3.1). This completes our discussions here.

REFERENCES

[D1] Donaldson, S.K. "Irrationality and the h–cobordism conjecture" J. Diff. Geom. 26 (1987) 141–168.

[D2] Donaldson, S.K. "Polynomial invariants for smooth four–manifolds" Preprint.

[F] Friedman, R. "Rank two vector bundles over regular elliptic surfaces" Invent. Math. 96, 283–332 (1989).

[FM] Friedman, R. and Morgan, J.W. "On the diffeomorphism types of certain algebraic surfaces I,II" J. Diff. Geom 21 (1988) 297–398.

[GH] Griffiths, P. and Harris, J. "Principal of algebraic geometry" John Wiley, New York (1978).

[LO] Lübke, M. and Okonek, C. "Stable bundles on regular elliptic surfaces " J. Reine Angew. Math. 378 (1987) 32–45.

[Ma] Macdonald, I.G. "Symmetric products of an algebraic curve" Topology, Vol. 1, pp. 319–343 (1962).

[Mo] Mong, K.C. "Some differential invariants of 4–manifolds" Oxford D. Phil. thesis (1988).

[OV] Okonek, C. and Van de Ven, A. "Stable bundles and differentiable structures on certain elliptic surfaces" Invent. Math. 86 (1986) 357–370.