MONODROMY OF PICARD-FUCHS DIFFERENTIAL EQUATIONS FOR CALABI-YAU THREEFOLDS

YAO-HAN CHEN, YIFAN YANG, AND NORIKO YUI

ABSTRACT. In this paper we are concerned with the monodromy of Picard-Fuchs differential equations associated with one-parameter families of Calabi-Yau threefolds. Our results show that in the hypergeometric cases the matrix representations of monodromy relative to the Frobenius bases can be expressed in terms of the geometric invariants of the underlying Calabi-Yau threefolds. This phenomenon is also verified numerically for other families of Calabi-Yau threefolds in the paper. Furthermore, we discover that under a suitable change of bases the monodromy groups are contained in certain congruence subgroups of $\mathrm{Sp}(4,\mathbb{Z})$ of finite index and whose levels are related to the geometric invariants of the Calabi-Yau threefolds.

1. INTRODUCTION

Let M_z be a family of Calabi-Yau n-folds parameterized by a complex variable $z \in \mathbb{P}^1(\mathbb{C})$, and ω_z be the unique holomorphic differential n-form on M_z (up to a scalar). Then the standard theory of Gauss-Manin connections asserts that the periods

$$\int_{\gamma_z} \omega_z$$

satisfy certain linear differential equations, called the *Picard-Fuchs differential equations*, where γ_z are r-cycles on M_z.

When $n = 1$, Calabi-Yau onefolds are just elliptic curves. A classical example of Picard-Fuchs differential equations is

$$(1-z)\theta^2 f - z\theta f - \frac{z}{4} f = 0, \qquad \theta = zd/dz, \tag{1}$$

satisfied by the periods

$$f(z) = \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-z)}}$$

of the family of elliptic curves $E_z : y^2 = x(x-1)(x-z)$.

When $n = 2$, Calabi-Yau manifolds are either 2-dimensional complex tori or $K3$ surfaces. When the Picard number of a one-parameter family of $K3$ surfaces is 19,

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the Picard-Fuchs differential equation has order 3. One of the simplest examples is

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 - z^{-1}x_1x_2x_3x_4 = 0 \subset \mathbb{P}^3,$$

whose Picard-Fuchs differential operator is

$$\theta^3 - 4z(4\theta+1)(4\theta+2)(4\theta+3). \tag{2}$$

Another well-known example is

$$(1-34z+z^2)\theta^3 + (3z^2-51z)\theta^2 + (3z^2-27z)\theta + (z^2-5z), \tag{3}$$

which is the Picard-Fuchs differential operator for the family of $K3$ surfaces

$$1-(1-XY)Z - zXYZ(1-X)(1-Y)(1-Z) = 0.$$

(See [7].) This differential equation appeared in Apéry's proof of irrationality of $\zeta(3)$. (See [5].)

When $n = 3$ and Calabi–Yau threefolds have the Hodge number $h^{2,1}$ equal to 1, the Picard-Fuchs differential equations have order 4. One of the most well-known examples of such Calabi–Yau threefolds is the quintic threefolds

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - z^{-1}x_1x_2x_3x_4x_5 = 0 \subset \mathbb{P}^4.$$

In [9], it is shown that the Picard-Fuchs differential operator for this family of Calabi–Yau threefolds is

$$\theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4). \tag{4}$$

(Actually, it is the mirror partner of the quintic Calabi–Yau threefolds that has Hodge number $h^{2,1} = 1$, but they share the same Picard-Fuchs differential equation.)

In this article we are concerned with the monodromy aspect of the Picard-Fuchs differential equations. Let

$$L: \; r_n(z)\theta^n + r_{n-1}(z)\theta^{n-1} + \cdots + r_0(z), \qquad r_i \in \mathbb{C}(z),$$

be a differential operator with regular singularities. Let z_0 be a singular point and S be the solution space of L at z_0. Then analytic continuation along a closed curve γ circling z_0 gives rise to an automorphism of S, called *monodromy.* If a basis $\{f_1, \ldots, f_n\}$ of S is chosen, then we have a matrix representation of the monodromy. Suppose that f_i becomes $a_{i1}f_1 + \cdots + a_{in}f_n$ after completing the loop γ, that is, if

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \longmapsto \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},$$

then the matrix representation of the monodromy along γ relative to the basis $\{f_i\}$ is the matrix (a_{ij}). The group of all such matrices are referred to as the *monodromy group* relative to the basis $\{f_i\}$ of the differential equation. Clearly, two different choices of bases may result in two different matrix representations for the same monodromy. However, it is easily seen that they are connected by conjugation by the matrix of basis change. Thus, the monodromy group is defined up to conjugation. In the subsequent discussions, for the ease of exposition, we may often drop the phrase "up to conjugation" about the monodromy groups, when there is no danger of ambiguities.

It is known that for one-parameter families of Calabi-Yau varieties of dimension one and two (i.e., elliptic curves and $K3$ surfaces, respectively), the monodromy groups are very often congruence subgroups of $SL(2,\mathbb{R})$. For instance, the monodromy group of (1) is $\Gamma(2)$, while those of (2) and (3) are $\Gamma_0(2)+\omega_2$ and $\Gamma_0(6)+\omega_6$, respectively, where ω_d denotes the Atkin-Lehner involution. (Technically speaking, the monodromy groups of (2) and (3) are subgroups of $SL(3,\mathbb{R})$ since the order of the differential equations is 3. But because (2) and (3) are symmetric squares of second-order differential equations, we may describe the monodromy in terms of the second-order ones.) Moreover, suppose that $y_0(z) = 1 + \cdots$ is the unique holomorphic solution at $z = 0$ and $y_1(z) = y_0(z)\log z + g(z)$ is the solution with logarithmic singularity. Set $\tau = cy_1(z)/y_0(z)$ for a suitable complex number c. Then z, as a function of τ, becomes a modular function, and $y_0(z(\tau))$ becomes a modular form of weight 1 for the order 2 cases and of weight 2 for the order 3 cases. For example, a classical result going back to Jacobi states that

$$\theta_3^2 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\theta_2^4}{\theta_3^4}\right),$$

where

$$\theta_2(\tau) = q^{1/8}\sum_{n\in\mathbb{Z}} q^{n(n+1)/2}, \qquad \theta_3(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2/2}, \qquad q = e^{2\pi i\tau},$$

or equivalently, that the modular form $y(\tau) = \theta_3^2$, as a function of $z(\tau) = \theta_2^4/\theta_3^4$, satisfies (1). Here ${}_2F_1$ denotes the Gauss hypergeometric function.

In this paper we will address the monodromy problem for Calabi-Yau threefolds. At first, given the experience with the elliptic curve and $K3$ surface cases, one may be tempted to guess that the monodromy group of such a differential equation will be the symmetric cube of some congruence subgroup of $SL(2,\mathbb{R})$. After all, there is a result by Stiller [23] (see also [26]) asserting that if $t(\tau)$ is a non-constant modular function and $F(\tau)$ is a modular form of weight k on a subgroup of $SL(2,\mathbb{R})$ commensurable with $SL(2,\mathbb{Z})$, then $F, \tau F, \ldots, \tau^k F$, as functions of t, are solutions of a $(k+1)$-st order linear differential equation with algebraic functions of t as coefficients. However, this is not the case in general. A quick way to see this is that the coefficients of the symmetric cube of a second order differential equation $y'' + r_1(t)y' + r_0(t)y = 0$ is completely determined by r_1 and r_0, but the coefficients of the Picard-Fuchs differential equations, including (4), do not satisfy the required relations. (The exact relations can be computed using Maple's command `symmetric_power`.) Nevertheless, in the subsequent discussion we will show that, with a suitable choice of bases, the monodromy groups for Calabi-Yau threefolds are contained in certain congruence subgroups of $\mathrm{Sp}(4,\mathbb{Z})$ whose levels are somehow described in terms of the geometric invariants of the manifolds in question. This is proved rigorously for the hypergeometric cases and verified numerically for other (e.g., non-hypergeometric) cases. Furthermore, our computation in the hypergeometric cases shows that the matrix representation of the monodromy around the finite singular point (different from the origin) relative to the Frobenius basis at the origin can be expressed completely using the geometric invariants of the associated Calabi-Yau threefolds. This phenomenon is also verified numerically in the non-hypergeometric cases. Although it is highly expected that geometric invariants will enter into the picture, in reality, geometry will dominate the entire picture in the sense that *every* entry of the matrix is expressed exclusively in terms of the geometric invariants.

The monodromy problem in general has been addressed by a number of authors. Papers relevant to our consideration include [6], [9], [11], [16], and [25], to name a few. In [6], Beukers and Heckman studied monodromy groups for the hypergeometric functions ${}_nF_{n-1}$. They showed that the Zariski closure of the monodromy groups of (4) is $\mathrm{Sp}(4,\mathbb{C})$. The same is true for other Picard-Fuchs differential equations for Calabi-Yau threefolds that are hypergeometric. In [9], Candelas et al. obtained precise matrix representations of monodromy for (4). Then Klemm and Theisen [16] applied the same method as that of Candelas et al. to deduce monodromy groups for three other hypergeometric cases. In [11] Doran and Morgan determined the monodromy groups for all the hypergeometric cases. Their matrix representations also involve geometric invariants of the Calabi-Yau threefolds. For Picard-Fuchs differential equations of non-hypergeometric type, there is not much known in literature. In [25] van Enckevort and van Straten computed the monodromy matrices numerically for a large class of differential equations. In many cases, they are able to find bases such that the monodromy matrices have rational entries. We will discuss the above results in more detail in Sections 3–5.

Our motivations of this paper may be formulated as follows. Modular functions and modular forms have been extensively investigated over the years, and there are great body of literatures on these subjects. As we illustrated above, the monodromy groups of Picard-Fuchs differential equations for families of elliptic curves and $K3$ surfaces are congruence subgroups of $SL(2,\mathbb{R})$. This modularity property can be used to study properties of the differential equations and the associated manifolds. For instance, in [18] Lian and Yau gave a uniform proof of the integrality of Fourier coefficients of the mirror maps for several families of $K3$ surfaces using the fact that the monodromy groups are congruence subgroups of $SL_2(\mathbb{R})$. For such an application, it is important to express monodromy groups in a proper way so that properties of the associated differential equations can be more easily discussed and obtained. Thus, the main motivation of our investigation is to find a good representation for monodromy groups from which further properties of Picard-Fuchs differential equations for Calabi-Yau threefolds can be derived.

The terminology "modularity" has been used for many different things. One aspect of the modularity that we would like to address is the modularity question of the Galois representations attached to Calabi–Yau threefolds, assuming that Calabi–Yau threefolds in question are defined over $\mathbb{Q}$. Let X be a Calabi–Yau threefold defined over $\mathbb{Q}$. We consider the L-series associated to the third étale cohomology group of X. It is expected that the L-series should be determined by some modular (automorphic) forms. The examples of Calabi–Yau threefolds we treat in this paper are those with the third Betti number equal to 4. It appears that Calabi–Yau threefolds with this property are rather scarce. Batyrev and Straten [4] considered 13 examples of Calabi–Yau threefolds with Picard number $h^{1,1} = 1$. Then their mirror partners will fulfill this requirement. (We note that more examples of such Calabi–Yau threefolds were found by Borcea [8].) All these 13 Calabi–Yau threefolds are defined as complete intersections of hypersurfaces in weighted projective spaces, and they have defining equations defined over $\mathbb{Q}$.

To address the modularity, we ought to have some "modular groups", and this paper offers candidates for appropriate modular groups via the monodromy group of the associated Picard–Fuchs differential equation (of order 4). In these cases, we expect that modular forms of more variables, e.g., Siegel modular forms associated to the modular groups for our congruence subgroups would enter the scene.

In general, the third Betti number of Calabi–Yau threefolds are rather large, and consequently, the dimension of the associated Galois representations would be rather high. To remedy this situation, we first decompose Calabi–Yau threefolds into motives, and then consider the motivic Galois representations and their modularity. Especially, when the principal motives (e.g., the motives that are invariant under the mirror maps) are of dimension 4, the modularity question for such motives should be accessible using the method developed for the examples discussed in this paper.

The modularity questions will be treated in subsequent papers.

2. Statements of results

To state our first result, let us recall that among all the Picard-Fuchs differential equations for Calabi-Yau threefolds, there are 14 equations that are hypergeometric of the form

$$\theta^4 - Cz(\theta + A)(\theta + 1 - A)(\theta + B)(\theta + 1 - B).$$

Their geometric descriptions and references are given in the following Table 1.

#	A	B	C	Description	H^3	$c_2 \cdot H$	c_3	Ref
1	1/5	2/5	3125	$X(5) \subset \mathbb{P}^4$	5	50	-200	[9]
2	1/10	3/10	$8 \cdot 10^5$	$X(10) \subset \mathbb{P}^4(1,1,1,2,5)$	1	34	-288	[20]
3	1/2	1/2	256	$X(2,2,2,2) \subset \mathbb{P}^7$	16	64	-128	[19]
4	1/3	1/3	729	$X(3,3) \subset \mathbb{P}^5$	9	54	-144	[19]
5	1/3	1/2	432	$X(2,2,3) \subset \mathbb{P}^6$	12	60	-144	[19]
6	1/4	1/2	1024	$X(2,4) \subset \mathbb{P}^5$	8	56	-176	[19]
7	1/8	3/8	65536	$X(8) \subset \mathbb{P}^4(1,1,1,1,4)$	2	44	-296	[20]
8	1/6	1/3	11664	$X(6) \subset \mathbb{P}^4(1,1,1,1,2)$	3	42	-204	[20]
9	1/12	5/12	12^6	$X(2,12) \subset \mathbb{P}^5(1,1,1,1,4,6)$	1	46	-484	[11]
10	1/4	1/4	4096	$X(4,4) \subset \mathbb{P}^5(1,1,1,1,2,2)$	4	40	-144	[17]
11	1/4	1/3	1728	$X(4,6) \subset \mathbb{P}^5(1,1,1,2,2,3)$	6	48	-156	[17]
12	1/6	1/4	27648	$X(3,4) \subset \mathbb{P}^5(1,1,1,1,1,2)$	2	32	-156	[17]
13	1/6	1/6	$2^8 \cdot 3^6$	$X(6,6) \subset \mathbb{P}^5(1,1,2,2,3,3)$	1	22	-120	[17]
14	1/6	1/2	6912	$X(2,6) \subset \mathbb{P}^5(1,1,1,1,1,3)$	4	52	-256	[17]

Some comments might be in order for the notations in the table. We employ the notations of van Enckevort and van Straten [25]. $X(d_1, d_2, \ldots, d_k) \subset \mathbb{P}^n(w_0, \ldots, w_n)$ stands for a complete intersection of k hypersurfaces of degree $d_1, \ldots, d_k$ in the weighted projective space with weight $(w_0, \cdots, w_n)$. For instance, $X(3,3) \subset \mathbb{P}^5$ is a complete intersection of two cubics in the ordinary projective 5-space $\mathbb{P}^5$ defined by

$$\{ Y_1^3 + Y_2^3 + Y_3^3 - 3\phi Y_4 Y_5 Y_6 = 0 \} \cap \{ -3\phi Y_1 Y_2 Y_3 + Y_4^3 + Y_5^3 + Y_6^3 = 0 \}.$$

Slightly more generally, $X(4,4) \subset \mathbb{P}^5(1,1,2,1,1,2)$ denotes a complete intersection of two quartics in the weighted projective 5-space $\mathbb{P}^5(1,1,2,1,1,2)$ and may be defined by the equations

$$\{ Y_1^4 + Y_2^4 + Y_3^2 - 4\phi Y_4 Y_5 Y_6 = 0 \} \cap \{ -4\phi Y_1 Y_2 Y_3 + Y_4^4 + Y_5^4 + Y_6^2 = 0 \}.$$

We note that all these examples of Calabi–Yau threefolds M have the Picard number $h^{1,1}(M) = 1$. Let $\mathcal{O}(H)$ be the ample generator of the Picard group $Pic(M) \simeq \mathbb{Z}$. The basic invariants for such a Calabi–Yau threefold M are the *degree* $d := H^3$, the *second Chern number* $c_2 \cdot H$ and the Euler number c_3 (the Euler characteristic of M). The equations are numbered in the same way as in [1].

In [9], using analytic properties of hypergeometric functions, Candelas et al. proved that with respect to a certain basis, the monodromy matrices around $z = 0$ and $z = 1/3125$ for the quintic threefold case (Equation 1 from Table 1) are

$$\begin{pmatrix} 51 & 90 & -25 & 0 \\ 0 & 1 & 0 & 0 \\ 100 & 175 & -49 & 0 \\ -75 & -125 & 35 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively. (Note that these two matrices are both in $\mathrm{Sp}(4, \mathbb{Z})$.) Applying the same method as that of Candelas et al., Klemm and Theisen [16] also obtained the monodromy of the one-parameter families of Calabi–Yau threefolds for Equations 2, 7, and 8. Presumably, their method should also work for several other hypergeometric cases. However, the method fails when the indicial equation of the singularity ∞ has repeated roots. To be more precise, it does not work for Equations 3–6, 10, 13 and 14. Moreover, the method uses the explicit knowledge that the singular point $z = 1/C$ is of conifold type. (Note that in geometric terms, a conical singularity is a regular singular point whose neighborhood looks like a cone with a certain base. For instance, a 3-dimensional conifold singularity is locally isomorphic to $XY - ZT = 0$ or equivalently, to $X^2 + Y^2 + Z^2 + T^2 = 0$. Reflecting to the Picard-Fuchs differential equations, this means that the local monodromy is unipotent of index 1.) Thus, it can not be applied immediately to study monodromy of general hypergeometric differential equations.

In [11] Doran and Morgan proved that if the characteristic polynomial of the monodromy around ∞ is

$$x^4 + (k-4)x^3 + (6 - 2k + d)x^2 + (k-4)x + 1,$$

then there is a basis such that the monodromy matrices around $z = 0$ and $z = 1/C$ are

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & d & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

respectively. It turns out that these numbers d and k both have geometric interpretation. Namely, the number $d = H^3$ is the degree of the associated threefolds and $k = c_2 \cdot H/12 + H^3/6$ is the dimension of the linear system $|H|$. Doran and Morgan's representation has the advantage that the geometric invariants can be read off from the matrices directly (although there is no way to extract the Euler number c_3 from the matrices), but has the disadvantage that the matrices are no longer in the symplectic group (in the strict sense).

Before we state our Theorem 1, let us recall the definition of Frobenius basis. Since the only solution of the indicial equation at $z = 0$ for each of the cases is 0 with multiplicity 4, the monodromy around $z = 0$ is maximally unipotent. (See [20] for more detail.) Then the standard method of Frobenius implies that at $z = 0$ there are four solutions y_j, $j = 0, \ldots,$, with the property that

$$y_0 = 1 + \cdots, \quad y_1 = y_0 \log z + g_1,$$
$$y_2 = \frac{1}{2} y_0 \log^2 z + g_1 \log z + g_2, \quad y_3 = \frac{1}{6} y_0 \log^3 z + \frac{1}{2} g_1 \log^2 z + g_2 \log z + g_3, \tag{6}$$

where g_i are all functions holomorphic and vanishing at $z = 0$. We remark that these solutions satisfy the relation

$$\begin{vmatrix} y_0 & y_3 \\ y_0' & y_3' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},$$

and therefore the monodromy matrices relative to the ordered basis $\{y_0, y_2, y_3, y_1\}$ are in $\mathrm{Sp}(4, \mathbb{C})$, as predicted by [6]. Now we can present our first theorem.

Theorem 1. *Let*

$$L : \theta^4 - Cz(\theta + A)(\theta + 1 - A)(\theta + B)(\theta + 1 - B)$$

be one of the 14 *hypergeometric equations, and* H^3*,* $c_2 \cdot H$*, and* c_3 *be geometric invariants of the associated Calabi-Yau threefolds given in the table above. Let* y_j*,* $j = 0, \ldots, 3$*, be the Frobenius basis specified by (6). Then with respect to the ordered basis* $\{y_3/(2\pi i)^3, y_2/(2\pi i)^2, y_1/(2\pi i), y_0\}$*, the monodromy matrices around* $z = 0$ *and* $z = 1/C$ *are*

$$\begin{pmatrix} 1 & 1 & 1/2 & 1/6 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1+a & 0 & ab/d & a^2/d \\ -b & 1 & -b^2/d & -ab/d \\ 0 & 0 & 1 & 0 \\ -d & 0 & -b & 1-a \end{pmatrix}, \tag{7}$$

respectively, where

$$a = \frac{c_3}{(2\pi i)^3} \zeta(3), \qquad b = c_2 \cdot H/24, \qquad d = H^3.$$

Remark 1. *We remark that by conjugating by the matrix*

$$\begin{pmatrix} d & 0 & b & a \\ 0 & d & d/2 & d/6 + b \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we do recover Doran and Morgan's representation (5).

The appearance of the geometric invariants c_2, c_3, H and d is not so surprising. In [9], it was shown that the conifold period, defined up to a constant as the holomorphic solution $f(z) = a_1(z - 1/C) + a_2(z - 1/C)^2 + \cdots$ at $z = 1/C$ that appears in the unique solution $f(z)\log(z - 1/C) + g(z)$ with logarithmic singularity at $z = 1/C$, is asymptotically

$$\frac{H^3}{6(2\pi i)^3} \log^3 z + \frac{c_2 \cdot H}{48\pi i} \log z + \frac{c_3}{(2\pi i)^3} \zeta(3) + \cdots \tag{8}$$

near $z = 0$. (See also [15].) Therefore, it is expected that the entries of the monodromy matrices should contain the invariants. However, it is still quite remarkable that the matrix is determined completely by the invariants alone. We have numerically verified the phenomenon for other families of Calabi-Yau threefolds, and also for general differential equations of Calabi-Yau type. (See [1] for the definition of a differential equation of Calabi-Yau type. See also Section 5 below.) It appears that if the differential equation has at least one singularity with exponents $0,1,1,2$, then there is always a singularity whose monodromy relative to the Frobenius basis is of the form stated in the theorem. Thus, this gives a numerical method to identify the possible geometric origin of a differential equation of Calabi-Yau type.

We emphasize that our proof of Theorem 1 is merely verification. That is, we can prove it, but unfortunately it does not give any geometric insight why the matrices are in this special form. It would be interesting to have a geometric interpretation of this fact.

Now conjugating the matrices by

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & d & d/2 & -b \\ -d & 0 & -b & -a \end{pmatrix}, \tag{9}$$

we can bring the matrices into the symplectic group $\mathrm{Sp}(4,\mathbb{Z})$. The results are

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ d & d & 1 & 0 \\ 0 & -k & -1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $z=0$ and $z=1/C$, respectively, where $k=2b+d/6$. Since the monodromy group is generated by these two matrices, we see that the group is contained in the congruence subgroup $\Gamma(d,\gcd(d,k))$, where the notation $\Gamma(d_1,d_2)$ with $d_2|d_1$ represents

$$\Gamma(d_1,d_2)=\left\{\gamma\in\mathrm{Sp}(4,\mathbb{Z}):\gamma\equiv\begin{pmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & * & * & * \end{pmatrix} \mod d_1\right\}$$
$$\bigcap\left\{\gamma\in\mathrm{Sp}(4,\mathbb{Z}):\gamma\equiv\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \mod d_2\right\}$$

We remark that the entries of the matrices in $\Gamma(d_1,d_2)$ satisfy certain congruence relations inferred from the symplecticity of the matrices. To be more explicit, let us recall that the symplectic group is characterized by the property that

$$\gamma=\begin{pmatrix} A & B \\ C & D \end{pmatrix}\in\mathrm{Sp}(2n,\mathbb{C}),$$

where A, B, C, and D are $n\times n$ blocks, if and only if

$$\gamma^{-1}=\begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}.$$

Thus, for

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

to be in $\Gamma(d_1, d_2)$, the integers a_{ij} should satisfy the implicit conditions

$$a_{22}a_{44} - a_{24}a_{42} \equiv 1, \quad a_{23} \equiv a_{14}a_{22} - a_{12}a_{24}, \quad a_{43} \equiv a_{14}a_{42} - a_{12}a_{44} \mod d_1,$$

and

$$a_{12} \equiv -a_{43} \mod d_2.$$

We now summarize our finding in the following theorem.

Theorem 2. *Let*

$$\theta^4 - Cz(\theta + A)(\theta + 1 - A)(\theta + B)(\theta + 1 - B)$$

be one of the 14 *hypergeometric equations. Let* y_j, $j = 0, \ldots, 3$ *be the Frobenius basis. Then relative to the ordered basis*

$$\frac{y_1}{2\pi i}, \quad y_0, \quad \frac{H^3}{2(2\pi i)^2}y_2 + \frac{H^3}{4\pi i}y_1 - \frac{c_2\cdot H}{24}y_0, \quad -\frac{H^3}{6(2\pi i)^3}y_3 - \frac{c_2\cdot H}{48\pi i}y_1 - \frac{c_3}{(2\pi i)^3}\zeta(3)y_0,$$

the monodromy matrices around $z = 0$ *and* $z = 1/C$ *are*

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ d & d & 1 & 0 \\ 0 & -k & -1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

with $d = H^3$, $k = H^3/6 + c_2\cdot H/12$, *respectively. They are contained in the congruence subgroups* $\Gamma(d_1, d_2)$ *for the* 14 *cases in the table below.*

#	A	B	d_1	d_2	#	A	B	d_1	d_2
1	1/5	2/5	5	5	8	1/6	1/3	3	1
2	1/10	3/10	1	1	9	1/12	5/12	1	1
3	1/2	1/2	16	8	10	1/4	1/4	4	4
4	1/3	1/3	9	3	11	1/4	1/3	6	1
5	1/3	1/2	12	1	12	1/6	1/4	2	1
6	1/4	1/2	8	2	13	1/6	1/6	1	1
7	1/8	3/8	2	2	14	1/6	1/2	4	1

Remark. We remark that what we show in Theorem 2 is merely the fact that the monodromy groups are contained in the congruence subgroups $\Gamma(d_1, d_2)$. Although the congruence subgroups $\Gamma(d_1, d_2)$ are of finite index in $\mathrm{Sp}(4, \mathbb{Z})$ (see the appendix by Cord Erdenberger for the index formula), the monodromy groups themselves may not be so. In fact, Zudilin has indicated to us a heuristic argument suggesting that the monodromy groups are too "thin" to be of finite index.

It would not be of much significance if the hypergeometric equations are the only cases where the monodromy groups are contained in congruence subgroups. Our numerical computation suggests that there are a number of further examples where the monodromy groups continue to be contained in congruence subgroups of $\mathrm{Sp}(4, \mathbb{Z})$. However, the general picture is not as simple as that for the hypergeometric cases.

As mentioned earlier, our numerical data suggest that the Picard-Fuchs differential equations for Calabi-Yau threefolds known in literature all have bases relative to which the monodromy matrices around the origin and some singular points of conifolds take the form (7) described in Theorem 1. Thus, with respect to the basis given in Theorem 2, the matrices around the origin and the conifold points again have the form (10). However, with this basis change, the monodromy matrices around other singularities may not be in $\mathrm{Sp}(4,\mathbb{Z})$, but in $\mathrm{Sp}(4,\mathbb{Q})$ instead, although the entries still satisfy certain congruence relations. Furthermore, in most cases, we can still realize the monodromy groups in congruence subgroups of $\mathrm{Sp}(4,\mathbb{Z})$, by a suitable conjugation.

Example 1. Consider the differential equation

$$\begin{aligned}
&25\theta^4 - 15z(51\theta^4 + 84\theta^3 + 72\theta^2 + 30\theta + 5) \\
&\quad + 6z^2(531\theta^4 + 828\theta^3 + 541\theta^2 + 155\theta + 15) \\
&\quad - 54z^3(423\theta^4 + 2160\theta^3 + 4399\theta^2 + 3795\theta + 1170) \\
&\quad + 243z^4(279\theta^4 + 1368\theta^3 + 2270\theta^2 + 1586\theta + 402) - 59049z^5(\theta+1)^4.
\end{aligned}$$

In [4] it is shown that this is the Picard-Fuchs differential equation for the Calabi-Yau threefolds defined as the complete intersection of three hypersurfaces of degree $(1,1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. The invariants are $H^3 = 90$, $c_2 \cdot H = 108$, and $c_3 = -90$. There are 6 singularities 0, $1/27$, $\pm i/\sqrt{27}$, $5/9$, and ∞ for the differential equation. Among them, the local exponents at $z = 5/9$ are $0, 1, 3, 4$ and we find that the monodromy around $z = 5/9$ is the identity. For others, our numerical computation shows that relative to the basis in Theorem 2 the monodromy matrices are

$$T_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 90 & 90 & 1 & 0 \\ 0 & -24 & -1 & 1 \end{pmatrix}, \qquad T_{1/27} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_{i/\sqrt{27}} = \begin{pmatrix} -17 & 3 & 1/3 & 1 \\ -54 & 10 & 1 & 3 \\ -972 & 162 & 19 & 54 \\ 162 & -27 & -3 & -8 \end{pmatrix}, \qquad T_{-i/\sqrt{27}} = \begin{pmatrix} -11 & 3 & 1/3 & -1 \\ 36 & -8 & -1 & 3 \\ -432 & 108 & 13 & -36 \\ 108 & -27 & -3 & 10 \end{pmatrix}.$$

¿From these, we see that the monodromy group is contained in the following group

$$\begin{aligned}
\Big\{(a_{ij}) \in \mathrm{Sp}(4,\mathbb{Q}) : a_{ij} \in \mathbb{Z}\ \forall (i,j) \neq (1,3),\ a_{13} \in \frac{1}{3}\mathbb{Z}, \\
a_{21}, a_{31}, a_{41}, a_{32}, a_{34} \equiv 0 \mod 18, \quad a_{11}, a_{33} \equiv 1 \mod 6, \\
a_{42} \equiv 0, \quad a_{22}, a_{44} \equiv 1 \mod 3\Big\}
\end{aligned}$$

Conjugating by

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we find that the monodromy group can be brought into the congruence subgroup $\Gamma(6,3)$.

Example 2. Consider the differential equation.

$$\begin{aligned}9\theta^4 &- 3z(173\theta^4 + 340\theta^3 + 272\theta^2 + 102\theta + 15)\\ &- 2z^2(1129\theta^4 + 5032\theta^3 + 7597\theta^2 + 4773\theta + 1083)\\ &+ 2z^3(843\theta^4 + 2628\theta^3 + 2353\theta^2 + 675\theta + 6)\\ &- z^4(295\theta^4 + 608\theta^3 + 478\theta^2 + 174\theta + 26) + z^5(\theta+1)^4\end{aligned}$$

This is the Picard-Fuchs differential equation for the complete intersection of 7 hyperplanes with the Grassmannian $G(2,7)$ with the invariants $H^3 = 42$, $c_2{\cdot}H = 84$, and $c_3 = -98$. (See [3].) The singularities are 0, 3, ∞, and the three roots $z_1 = 0.01621\ldots$, $z_2 = -0.2139\ldots$, and $z_3 = 289.197\ldots$ of $z^3 - 289z^2 - 57z + 1$. The monodromy around $z = 3$ is the identity. The others have the matrix representations

$$T_0 = \begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 42 & 42 & 1 & 0\\ 0 & -14 & -1 & 1\end{pmatrix}, \qquad T_{z_1} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

$$T_{z_2} = \begin{pmatrix} -13 & 7 & 1 & -2\\ 28 & -13 & -2 & 4\\ -196 & 98 & 15 & -28\\ 98 & -49 & -7 & 15\end{pmatrix}, \qquad T_{z_3} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 42 & 1 & 0 & 9\\ -196 & 0 & 1 & -42\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Thus, the monodromy group is contained in the subgroup $\Gamma(14,7)$.

3. A general approach

Let

$$y^{(n)} + r_{n-1}y^{(n-1)} + \cdots + r_1y' + r_0y = 0, \qquad r_i \in \mathbb{C}(z),$$

be a linear differential equation with regular singularities. Then the monodromy around a singular point z_0 with respect to the local Frobenius basis at z_0 is actually very easy to describe, as we shall see in the following discussion.

Consider the simplest cases where the indicial equation at z_0 has n distinct roots $\lambda_1, \ldots, \lambda_n$ such that $\lambda_i - \lambda_j \notin \mathbb{Z}$ for all $i \neq j$. In this case, the Frobenius basis consists of

$$y_j(z) = (z - z_0)^{\lambda_j} f_j(z), \qquad j = 1, \ldots, n,$$

where $f_j(z)$ are holomorphic near z_0 and have non-vanishing constant terms. It is easy to see that the matrix of the monodromy around z_0 with respect to $\{y_j\}$ is simply

$$\begin{pmatrix} e^{2\pi i\lambda_1} & 0 & \cdots & 0\\ 0 & e^{2\pi i\lambda_2} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & e^{2\pi i\lambda_n}\end{pmatrix}.$$

Now assume that the indicial equation at z_0 has $\lambda_1, \ldots, \lambda_k$, with multiplicities $e_1, \ldots, e_k$, as solutions, where $e_1 + \cdots + e_k = n$ and $\lambda_i - \lambda_j \notin \mathbb{Z}$ for all $i \neq j$. Then for each λ_j, there are e_j linearly independent solutions

$$\begin{aligned} y_{j,0} &= (z-z_0)^{\lambda_j} f_{j,0}, \\ y_{j,1} &= y_{j,0}\log(z-z_0) + (z-z_0)^{\lambda_j} f_{j,1}, \\ y_{j,2} &= \frac{1}{2} y_{j,0}\log^2(z-z_0) + (z-z_0)^{\lambda_j} f_{j,1}\log(z-z_j) + (z-z_0)^{\lambda_j} f_{j,2}, \\ \vdots \quad & \quad \vdots \\ y_{j,e_j-1} &= (z-z_0)^{\lambda_j} \sum_{h=0}^{e_j-1} \frac{1}{h!} f_{j,e_j-1-h}\log^h(z-z_0), \end{aligned}$$

where $f_{j,h}$ are holomorphic near $z = z_0$ and satisfy $f_{j,0}(z_0) = 1$ and $f_{j,h}(z_0) = 0$ for $h > 0$. Since $f_{j,h}$ are all holomorphic near z_0, the analytic continuation along a small closed curve circling z_0 does not change $f_{j,h}$. For other factors, circling z_0 once in the counterclockwise direction results in

$$(z-z_0)^{\lambda_j} \longmapsto e^{2\pi i\lambda_j}(z-z_0)^{\lambda_j}$$

and

$$\log(z-z_0) \longmapsto \log(z-z_0) + 2\pi i.$$

Thus, the behaviors of $y_{j,h}$ under the monodromy around z_0 are governed by

$$\begin{pmatrix} y_{j,0} \\ y_{j,1} \\ \vdots \\ y_{j,e_j-1} \end{pmatrix} \longmapsto \begin{pmatrix} \omega_j & 0 & \cdots & 0 \\ 2\pi i\omega_j & \omega_j & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \frac{(2\pi i)^{e_j-1}}{(e_j-1)!}\omega_j & \frac{(2\pi i)^{e_j-2}}{(e_j-2)!}\omega_j & \cdots & \omega_j \end{pmatrix} \begin{pmatrix} y_{j,0} \\ y_{j,1} \\ \vdots \\ y_{j,e_j-1} \end{pmatrix},$$

where $\omega_j = e^{2\pi i\lambda_j}$.

When the indicial equation of z_0 has distinct roots λ_i and λ_j such that $\lambda_i - \lambda_j \in \mathbb{Z}$, there are many possibilities for the monodromy matrix relative to the Frobenius basis, but in any case, the matrix still consists of blocks of entries that take the same form as above.

¿From the above discussion we see that monodromy matrices with respect to the local Frobenius bases are very easy to describe. Therefore, to find monodromy matrices uniformly with respect to a given fixed basis, it suffices to find the matrix of basis change between the fixed basis and the Frobenius basis at each singularity. When the differential equation is hypergeometric, this can be done using the (refined) standard analytic method, in which we first express the Frobenius basis at $z = 0$ as integrals of Barnes-Mellin type and then use contour integration to obtain the analytic continuation to a neighborhood of $z = \infty$. This gives us the monodromy matrices around $z = 0$ and $z = \infty$. Since the monodromy group is generated by these two matrices, the group is determined.

When the differential equation is not hypergeometric, we are unable to determine the matrices of basis change precisely. To obtain the matrices numerically we use the following idea. Let z_1 and z_2 be two singularities and $\{y_i\}$ and $\{\tilde{y}_j\}$, $i, j = 1, \ldots, n$, be their Frobenius bases. Observe that if $y_i = a_{i1}\tilde{y}_1 + \cdots + a_{in}\tilde{y}_n$, then we have

$$\begin{pmatrix} y_1 & y_1' & \cdots & y_1^{(n-1)} \\ y_2 & y_2' & \cdots & y_2^{(n-1)} \\ \vdots & \vdots & & \vdots \\ y_n & y_n' & \cdots & y_n^{(n-1)} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \tilde{y}_1 & \tilde{y}_1' & \cdots & \tilde{y}_1^{(n-1)} \\ \tilde{y}_2 & \tilde{y}_2' & \cdots & \tilde{y}_2^{(n-1)} \\ \vdots & \vdots & & \vdots \\ \tilde{y}_n & \tilde{y}_n' & \cdots & \tilde{y}_n^{(n-1)} \end{pmatrix}.$$

Thus, to determine the matrix (a_{ij}) it suffices to evaluate $y_i^{(k)}$ and $\tilde{y}_i^{(k)}$ at a common point. To do it numerically, we expand the Frobenius bases into power series and assume that the domains of convergence for the power series have a common point z_0. We then truncate and evaluate the series at z_0. This gives us approximation of the matrices of basis changes. We will discuss some practical issues of this method in Section 5.

4. The hypergeometric cases

Throughout this section, we fix the branch cut of $\log z$ to be $(-\infty, 0]$ so that the argument of a complex variable z is between $-\pi$ and π.

Recall that a hypergeometric function ${}_pF_{p-1}(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_{p-1}; z)$ is defined for $\beta_i \neq 0, -1, -2, \dots$ by

$${}_pF_{p-1}(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_{p-1}; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n}{(1)_n (\beta_1)_n \dots (\beta_{p-1})_n} z^n,$$

where

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)\dots(\alpha+n-1), & \text{if } n > 0, \\ 1, & \text{if } n = 0. \end{cases}$$

It satisfies the differential equation

$$[\theta(\theta+\beta_1-1)\dots(\theta+\beta_{p-1}-1) - z(\theta+\alpha_1)\dots(\theta+\alpha_p)]f = 0. \tag{11}$$

Moreover, it has an integral representation

$$\frac{1}{2\pi i} \frac{\Gamma(\beta_1)\dots\Gamma(\beta_{p-1})}{\Gamma(\alpha_1)\dots\Gamma(\alpha_p)} \int_{\mathcal{C}} \frac{\Gamma(s+\alpha_1)\dots\Gamma(s+\alpha_p)}{\Gamma(s+\beta_1)\dots\Gamma(s+\beta_{p-1})} \Gamma(-s)(-z)^s \, ds$$

for $|\arg(-z)| < \pi$, where $\mathcal{C}$ is any path from $-i\infty$ to $i\infty$ such that the poles of $\Gamma(-s)$ lie on the right of $\mathcal{C}$ and the poles of $\Gamma(s+a_k)$ lie on the left of $\mathcal{C}$. (See [21, Chapter 5].) Then one can obtain the analytic continuation of ${}_pF_{p-1}$ by moving the path of integration to the far left of the complex plane and counting the residues arising from the process. It turns out that this method can be generalized.

Lemma 1. *Let m be the number of 1's among β_k. Set*

$$F(h, z) = \sum_{n=0}^{\infty} \frac{(\alpha_1+h)_n \dots (\alpha_p+h)_n}{(1+h)_n (\beta_1+h)_n \dots (\beta_{p-1}+h)_n} z^{n+h}.$$

Then, for $j = 0, \dots, m$, the functions

$$\frac{\partial^j}{\partial h^j} F(h, z) \Big|_{h=0}$$

are solutions of (11). Moreover, if $|\arg(-z)| < \pi$ *and* h *is a small quantity such that* $\alpha_k + h$ *are not zero or negative integers, then* $F(h,z)$ *has the integral representation*

$$F(h,z) = -\frac{z^h}{2\pi i}\frac{\Gamma(\beta_1+h)\dots\Gamma(\beta_{p-1}+h)\Gamma(1+h)}{\Gamma(\alpha_1+h)\dots\Gamma(\alpha_p+h)}$$
$$\int_{\mathcal{C}} \frac{\Gamma(s+\alpha_1+h)\dots\Gamma(s+\alpha_p+h)}{\Gamma(s+\beta_1+h)\dots\Gamma(s+\beta_{p-1}+h)\Gamma(s+1+h)}\frac{\pi}{\sin\pi s}(-z)^s\,ds,$$

where $\mathcal{C}$ *is any path from* $-i\infty$ *to* $i\infty$ *such that the integers* $0,1,2,\dots$ *lies on the right of* $\mathcal{C}$ *and the poles of* $\Gamma(s+a_k+h)$ *lie on the left of* $\mathcal{C}$*.*

Proof. The first part of the lemma is just a specialization of the Frobenius method (see [14]) to the hypergeometric cases. We have

$$\theta(\theta+\beta_1-1)\dots(\theta+\beta_{p-1}-1)F(h,z) = h(h+\beta_1-1)\dots(h+\beta_{p-1}-1)z^h$$
$$+\sum_{n=1}^{\infty}\frac{(\alpha_1+h)_n\dots(\alpha_p+h)}{(1+h)_{n-1}(\beta_1+h)_{n-1}\dots(\beta_{p-1}+h)_{n-1}}z^{n+h}$$

and

$$z(\theta+\alpha_1)\dots(\theta+\alpha_p)F(h,z) = \sum_{n=0}^{\infty}\frac{(\alpha_1+h)_{n+1}\dots(\alpha_p+h)_{n+1}}{(1+h)_n(\beta_1+h)_n\dots(\beta_{p-1}+h)_n}z^{n+1+h}$$

It follows that

$$[\theta(\theta+\beta_1-1)\dots(\theta+\beta_{p-1}-1) - z(\theta+\alpha_1)\dots(\theta+\alpha_p)]F(h,z)$$
$$= h(h+\beta_1-1)\dots(h+\beta_{p-1}-1)z^h.$$

If the number of 1's among β_k is m, then the first non-vanishing term of the Taylor expansion in h of the last expression is h^{m+1}. Consequently, we see that

$$\frac{\partial^j}{\partial h^j}F(h,z)\Big|_{h=0}$$

are solutions of (11) for $j = 0,\dots,m$.

The proof of the second part about the integral representation is standard. We refer the reader to Chapter 5 of [21]. □

We now prove Theorem 1. Here we will only discuss the cases

$$(A,B) = (1/2,1/2),\ (1/3,1/3),\ (1/4,1/2),\ \text{and}\ (1/6,1/3),$$

representing the four classes whose indicial equations at $z=\infty$ have one root with multiplicity 4, two distinct roots, each of which has multiplicity 2, one repeated root and two other distinct roots, and four distinct roots, respectively. The other cases can be proved in the same fashion.

Proof of the case $(A,B) = (1/6,1/3)$. Let h denote a small real number, and let $F(h,z)$ be defined as in Lemma 1 with $p=4$, $\alpha_1 = 1/6$, $\alpha_2 = 1/3$, $\alpha_3 = 2/3$, $\alpha_4 = 5/6$, and $\beta_k = 1$ for all k. Then, by Lemma 1, the functions

$$y_j(z) = \frac{1}{j!}\frac{\partial^j}{\partial h^j}(C^{-h}F(h,Cz)), \qquad j = 0,\dots,3,$$

are solutions of

$$\theta^4 - 11664z(\theta+1/6)(\theta+1/3)(\theta+2/3)(\theta+5/6),$$

where $C = 11664$. In fact, by considering the contribution of the first term, we see that these four functions make up the Frobenius basis at $z = 0$.

We now express $C^{-h}F(h, Cz)$ using Lemma 1. By the Gauss multiplication theorem we have

$$\begin{aligned}\Gamma(s+1/6)&\Gamma(s+1/3)\Gamma(s+2/3)\Gamma(s+5/6)\\ &= \frac{\prod_{k=1}^{6}\Gamma(s+k/6)}{\Gamma(s+1/2)\Gamma(s+1)} = \frac{(2\pi)^{5/2}6^{-1/2-6s}\Gamma(6s+1)}{(2\pi)^{1/2}2^{-1/2-2s}\Gamma(2s+1)}.\end{aligned}$$

Thus, restricting z to the lower half-plane $-\pi < \arg z < 0$, by Lemma 1, we may write

$$\begin{aligned}C^{-h}F(h, Cz) = -\frac{z^h}{2\pi i}&\frac{\Gamma(1+h)^4\Gamma(1+2h)}{\Gamma(1+6h)}\\ &\times \int_{\mathcal{C}} \frac{\Gamma(6s+1+6h)}{\Gamma(s+1+h)^4\Gamma(2s+1+2h)}\frac{\pi}{\sin \pi s}e^{\pi i s}z^s\,ds,\end{aligned}$$

where $\mathcal{C}$ is the vertical line $\operatorname{Re} s = -1/12$. Now move the line of integration to $\operatorname{Re} s = -13/12$. This is justified by the fact that the integrand tends to 0 as $\operatorname{Im} s$ tends to infinity. The integrand has four simple poles $s = -n/6 - h$, $n = 1, 2, 4, 5$, between these two lines. The residues are

$$\frac{(-1)^{n-1}}{6\Gamma(n)}\frac{\pi e^{-\pi i(n/6+h)}}{\Gamma(1-n/6)^4\Gamma(1-n/3)\sin \pi(n/6+h)}z^{-n/6-h}.$$

Thus, we see that the analytic continuation of $C^{-h}F(h, z)$ to $|z| > 1$ with $-\pi < \arg z < 0$ is given by

$$C^{-h}F(h, z) = \sum_{n=1,2,4,5} a_n B_n(h) z^{-n/6} + (\text{higher order terms in } 1/z),$$

where

$$a_n = \frac{(-1)^n \pi e^{-\pi i n/6}}{6\Gamma(n)\Gamma(1-n/6)^4\Gamma(1-n/3)}, \qquad B_n(h) = \frac{\Gamma(1+h)^4\Gamma(1+2h)e^{-\pi i h}}{\Gamma(1+6h)\sin \pi(n/6+h)}.$$

On the other hand, since the local exponents at $z = \infty$ are 1/6, 1/3, 2/3, and 5/6, the Frobenius basis at $z = \infty$ consists of

$$\tilde{y}_n(z) = z^{-n/6} g_n(1/z), \qquad n = 1, 2, 4, 5,$$

where $g_n = 1 + \cdots$ are functions holomorphic at 0. It follows that for z with $-\pi < \arg z < 0$

$$y_j(z) = \frac{1}{j!}\sum_{n=1,2,4,5} a_n B_n^{(j)}(h)\tilde{y}_n(z).$$

Set $f_j(z) = y_j(z)/(2\pi i)^j$ for $j = 0, \dots, 3$ and $\tilde{f}_n = a_n \tilde{y}_n / \sin(n\pi/6)$ for $n = 1, 2, 4, 5$. Then using the evaluation

$$\Gamma'(1) = -\gamma, \qquad \Gamma''(1) = \gamma^2 + \zeta(2), \qquad \Gamma'''(1) = -\gamma^3 - 3\zeta(2)\gamma - 2\zeta(3),$$

we find

$$\begin{pmatrix} f_3 \\ f_2 \\ f_1 \\ f_0 \end{pmatrix} = M \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_4 \\ \tilde{f}_5 \end{pmatrix},$$

where

$$M=\begin{pmatrix} \eta-i\omega/4 & \eta+5\sqrt{3}i\omega^2/36 & \eta+5\sqrt{3}i\omega^4/36 & \eta-i\omega^5/4 \\ -5/12-i\omega/2 & 1/4-i\omega^2/2\sqrt{3} & 1/4-i\omega^4/2\sqrt{3} & -5/12-i\omega^5/2 \\ i\omega & i\omega^2/\sqrt{3} & i\omega^4/\sqrt{3} & i\omega^5 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

with

$$\omega=e^{\pi i/6}, \qquad \eta=\frac{68\zeta(3)}{(2\pi i)^3}.$$

Now let P be the path traveling along the real axis with $\arg z=-\pi+$ from $z=-2$ to $-\infty$ and then coming back along the real axis with $\arg z=\pi-$ to $z=-2$. The monodromy effect on $\tilde{y}_n(z)=z^{-n/6}g_n(1/z)$ is

$$\tilde{y}_n(z)\longmapsto \tilde{y}_n(e^{2\pi i}z)=e^{-2\pi i n/6}\tilde{y}_n(z).$$

Therefore, the matrix representation of the monodromy along P relative to the ordered basis $\{f_3,f_2,f_1,f_0\}$ is

$$T_\infty=M\begin{pmatrix}\omega^{-2} & 0 & 0 & 0\\ 0 & \omega^{-4} & 0 & 0\\ 0 & 0 & \omega^4 & 0\\ 0 & 0 & 0 & \omega^2\end{pmatrix}M^{-1}.$$

Now the path P is equivalent to that of circling once around $z=1/C$ and then once around $z=0$, both in the counterclockwise direction. Therefore, if we denote by T_0 and $T_{1/C}$ the monodromy matrices relative the basis $\{f_3,f_2,f_1,f_0\}$ around $z=0$ and $z=1/C$, respectively, then we have

$$T_\infty=T_{1/C}T_0.$$

Since T_0 is easily seen to be

$$T_0=\begin{pmatrix}1 & 1 & 1/2 & 1/6\\ 0 & 1 & 1 & 1/2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix},$$

we find

$$T_{1/C}=M\begin{pmatrix}\omega^{-2} & 0 & 0 & 0\\ 0 & \omega^{-4} & 0 & 0\\ 0 & 0 & \omega^4 & 0\\ 0 & 0 & 0 & \omega^2\end{pmatrix}M^{-1}T_0^{-1}=\begin{pmatrix}1+a & 0 & ab/d & a^2/d\\ -b & 1 & -b^2/d & -ab/d\\ 0 & 0 & 1 & 0\\ -d & 0 & -b & 1-a\end{pmatrix},$$

where

$$a=-\frac{204}{(2\pi i)^3}\zeta(3), \qquad b=\frac{7}{4}, \qquad d=3.$$

Comparing these numbers with the invariants, we find the matrix $T_{1/C}$ indeed takes the form (7) specified in the statement of Theorem 1. This proves the case $(A,B)=(1/6,1/3)$. □

Proof of the case $(A,B)=(1/4,1/2)$. Apply Lemma 1 with $p=4$, $\alpha_1=1/4$, $\alpha_2=3/4$, $\alpha_3=\alpha_4=1/2$, $\beta_k=1$ for all k, and set $C=1024$. Then

$$y_j(z)=\frac{1}{j!}\frac{\partial^j}{\partial h^j}(C^{-h}F(h,Cz)), \qquad j=0,\dots,3,$$

form the Frobenius basis for

$$\theta^4 - 1024z(\theta+1/4)(\theta+3/4)(\theta+1/2)^2.$$

Assuming that $-\pi < \arg z < 0$, we have

$$C^{-h}F(h,Cz) = -\frac{z^h}{2\pi i}\frac{\Gamma(1+h)^6}{\Gamma(1+2h)\Gamma(1+4h)} \times \int_{\mathcal{C}} \frac{\Gamma(4s+1+4h)\Gamma(2s+1+2h)}{\Gamma(s+1+h)^6}\frac{\pi}{\sin \pi s}e^{\pi i s}z^s\,ds,$$

where $\mathcal{C}$ is the vertical line $\operatorname{Re} s = -1/8$. The integrand has simple poles at $-k-h-1/4$ and $-k-h-3/4$, and double poles at $-k-h-1/2$ for $k = 0,1,2,\ldots$. The residues at $s = -h-n/4$, $n = 1,3$, are $a_nC_n(h)z^{-h-n/4}$, where

$$a_n = (-1)^{(n+1)/2}\frac{\pi\Gamma(1/2)e^{-\pi in/4}}{4\Gamma(1-n/4)^6}, \qquad C_n(h) = \frac{e^{-\pi ih}}{\sin\pi(h+n/4)}$$

At $s = -h-1/2$ we have

$$\frac{\Gamma(4s+1+4h)\Gamma(2s+1+2h)}{\Gamma(s+1+h)^6} = -\frac{1}{8\Gamma(1/2)^6}(s+h+1/2)^{-2} - \frac{3\log 2+1}{2\Gamma(1/2)^6}(s+h+1/2)^{-1}+\cdots,$$

$$\frac{\pi}{\sin\pi s} = -\frac{\pi}{\cos\pi h} + \pi^2\frac{\sin\pi h}{\cos^2\pi h}(s+h+1/2)+\cdots,$$

and

$$e^{\pi is}z^s = z^{-1/2-h}e^{-\pi i(h+1/2)}(1+(\pi i+\log z)(s+h+1/2)+\cdots).$$

Thus, the residue at $s=-h-1/2$ is

$$\frac{\pi e^{-\pi i(h+1/2)}}{8\Gamma(1/2)^6\cos\pi h}\left(\pi i+\log z+12\log 2+4-\pi\frac{\sin\pi h}{\cos\pi h}\right)z^{-h-1/2}.$$

Set

$$a_2 = -\frac{\pi e^{-\pi i/2}}{8\Gamma(1/2)^6}, \quad C_2(h) = \frac{e^{-\pi ih}}{\cos\pi h}, \quad C_2^*(h) = C_2(h)(\pi i+12\log 2+4-\pi\tan\pi h).$$

We find

$$C^{-h}F(h,Cz) = -a_1B_1(h)z^{-1/4} - a_2B_2(h)z^{-1/2}\log z - a_2B_2^*(h)z^{-1/2} - a_3B_3(h)z^{-3/4} + (\text{higher order terms in } 1/z),$$

where

$$B_n(h) = \frac{\Gamma(1+h)^6}{\Gamma(1+2h)\Gamma(1+4h)}C_n(h), \qquad B_2^*(h) = \frac{\Gamma(1+h)^6}{\Gamma(1+2h)\Gamma(1+4h)}C_2^*(h).$$

Let $y_j(z)$, $j=0,\ldots,3$, be the Frobenius basis at $z=0$, and

$$\tilde{y}_1(z) = z^{-1/4}(1+\cdots), \qquad \tilde{y}_3(z) = z^{-3/4}(1+\cdots),$$
$$\tilde{y}_2^*(z) = z^{-1/2}(1+\cdots), \qquad \tilde{y}_2(z) = (\log z+g(1/z))\tilde{y}_2^*(z)$$

be the Frobenius basis at ∞, where $g(t)$ is a function holomorphic and vanishing at $t=0$. Set $f_j(z)=y_j(z)/(2\pi i)^j$ for $j=0,\dots,3$, $\tilde f_n(z)=-a_n\tilde y_n(z)/\sin\pi(n/4)$ for $n=1,2,3$, and $\tilde f_2^*(z)=-a_2\tilde y_2^*(z)$. Using the fact that

$$y_j(z)=\frac{1}{j!}\frac{\partial^j}{\partial h^j}(C^{-h}F(h,Cz)),$$

we find

$$\begin{pmatrix} f_3\\ f_2\\ f_1\\ f_0\end{pmatrix}=\begin{pmatrix}\eta+(1-i)/48 & \eta-5/48 & -5\mu/12+\pi i\eta+4\mu\eta & \eta+(1+i)/48\\ (1-6i)/24 & 7/24 & \pi i/24+7\mu/6 & (1+6i)/24\\ (i-1)/2 & -1/2 & -2\mu & -(i+1)/2\\ 1 & 1 & \pi i+4\mu & 1\end{pmatrix}\begin{pmatrix}\tilde f_1\\ \tilde f_2\\ \tilde f_2^*\\ \tilde f_3\end{pmatrix}$$

where

$$\mu=3\log 2+1,\qquad \eta=\frac{22\zeta(3)}{(2\pi i)^3}.$$

Let P be the path from $z=-1$ with argument $-\pi$ to $-\infty$ and then back to $z=-1$ with argument π. The monodromy matrix for P relative to the ordered basis $\{\tilde f_1,\tilde f_2,\tilde f_2^*,\tilde f_3\}$ is

$$\begin{pmatrix}-i & 0 & 0 & 0\\ 0 & -1 & -2\pi i & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & i\end{pmatrix}.$$

Thus, the matrix with respect to the ordered basis $\{f_3,f_2,f_1,f_0\}$ is

$$T_\infty=\begin{pmatrix}1-8\eta & 1-8\eta & 1/2-19\eta/3 & 1/6-11\eta/3+8\eta^2\\ -7/3 & -4/3 & -61/72 & -41/72+7\eta/3\\ 0 & 0 & 1 & 1\\ -8 & -8 & -19/3 & -8/3+8\eta\end{pmatrix}.$$

Finally, it is easy to see that the monodromy around $z=0$ with respective to $\{f_3,f_2,f_1,f_0\}$ is

$$T_0=\begin{pmatrix}1 & 1 & 1/2 & 1/6\\ 0 & 1 & 1 & 1/2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 1\end{pmatrix},$$

and after a short computation we find that the monodromy $T_{1/C}=T_\infty T_0^{-1}$ around $z=1/C$ indeed takes the form claimed in the statement of Theorem 1. □

Proof of the case $(A,B)=(1/3,1/3)$. Let z be a complex number with $-\pi<\arg z<0$. By the same argument as before. We find that the Frobenius basis $\{y_j\}$ at $z=0$ can be expressed as

$$y_j(z)=\frac{1}{j!}\frac{\partial^j}{\partial h^j}(C^{-h}F(h,Cz)),$$

where $C=729$ and

$$C^{-h}F(h,Cz)=-\frac{z^h}{2\pi i}\frac{\Gamma(1+h)^6}{\Gamma(1+3h)^2}\int_{\mathcal C}\frac{\Gamma(3s+1+3h)^2}{\Gamma(s+1+h)^6}\frac{\pi}{\sin\pi s}e^{\pi i s}z^s\,ds.$$

Here h is assumed to be a real number and $\mathcal C$ denotes the vertical line $\operatorname{Re}s=-1/6$. Set

$$a_n=-\frac{\pi e^{-\pi i n/3}}{9\Gamma(1-n/3)^6},\qquad n=1,\ 2.$$

The residues at $z=-1/3-h$ and $z=-2/3-h$ are

$$a_1(\pi i+\log z+9\log 3-\pi\sqrt{3}+\cot\pi(1/3+h))z^{-1/3-h}e^{-\pi ih}$$

and

$$a_2(\pi i+\log z+9\log 3+\pi\sqrt{3}+6+\cot\pi(2/3+h))z^{-2/3-h}e^{-\pi ih},$$

respectively. Let

$$B_n(h)=\frac{\Gamma(1+h)^6}{\Gamma(1+3h)^2}\frac{e^{-\pi ih}}{\sin\pi(n/3+h)},\qquad n=1,\ 2,$$

and

$$B_1^*(h)=B_1(h)(\pi i+9\log 3-\pi\sqrt{3}+\pi\cot\pi(1/3+h)),$$
$$B_2^*(h)=B_2(h)(\pi i+9\log 3+\pi\sqrt{3}+6+\pi\cot\pi(2/3+h)).$$

Then we have

$$C^{-h}F(h,Cz)=-\sum_{n=1}^{2}a_n(B_n(h)z^{-n/3}\log z+B_n^*(h)z^{-n/3})+(\text{higher order terms}).$$

Now the Taylor expansions of $B_n(h)$ and $B_n^*(n)$ are

$$\sin\frac{\pi}{3}B_1\left(\frac{h}{2\pi i}\right)=1+\frac{i\omega}{\sqrt{3}}h-\left(\frac{i\omega}{2\sqrt{3}}+\frac{1}{12}\right)h^2+\left(\frac{i\omega}{12\sqrt{3}}+\eta\right)h^3+\cdots,$$

$$\sin\frac{\pi}{3}B_2\left(\frac{h}{2\pi i}\right)=1+\frac{i\omega^2}{\sqrt{3}}h-\left(\frac{i\omega^2}{2\sqrt{3}}+\frac{1}{12}\right)h^2+\left(\frac{i\omega^2}{12\sqrt{3}}+\eta\right)h^3+\cdots,$$

$$\begin{aligned}\sin\frac{\pi}{3}B_1^*\left(\frac{h}{2\pi i}\right)&=\left(\mu_1+\frac{2\pi\omega}{\sqrt{3}}\right)+\left(\frac{i\mu_1\omega}{\sqrt{3}}+\frac{2\pi i\omega}{3}\right)h-\left(\frac{i\mu_1\omega}{2\sqrt{3}}+\frac{\mu_1}{12}+\frac{\pi\omega}{2\sqrt{3}}\right)h^2\\&\quad+\left(\mu_1\eta+\frac{i\mu_1\omega}{12\sqrt{3}}+\frac{2\pi\eta\omega}{\sqrt{3}}-\frac{\pi i\omega}{6}\right)h^3+\cdots,\end{aligned}$$

$$\begin{aligned}\sin\frac{\pi}{3}B_2^*\left(\frac{h}{2\pi i}\right)&=\left(\mu_2+\frac{2\pi\omega^2}{\sqrt{3}}+6\right)+\left(\frac{i\mu_2\omega^2}{\sqrt{3}}-\frac{2\pi i\omega^2}{3}+2i\omega^2\sqrt{3}\right)h\\&\quad-\left(\frac{i\mu_2\omega^2}{2\sqrt{3}}+\frac{\mu_2}{12}+\frac{\pi\omega^2}{2\sqrt{3}}-\omega^2-\frac{3}{2}\right)h^2\\&\quad+\left(\mu_2\eta+\frac{i\mu_2\omega^2}{12\sqrt{3}}+\frac{2\pi\eta\omega^2}{\sqrt{3}}+\frac{\pi i\omega^2}{6}+6\eta+\frac{i\omega^2}{2\sqrt{3}}\right)h^3+\cdots,\end{aligned}$$

where

$$\omega=e^{\pi i/3},\qquad \eta=\frac{16\zeta(3)}{(2\pi i)^3},\qquad \mu_1=9\log 3-\pi\sqrt{3},\qquad \mu_2=9\log 3+\pi\sqrt{3}.$$

From these we can deduce the matrix of basis change between the Frobenius basis

$$f_j(z)=\frac{1}{(2\pi i)^j j!}\frac{\partial^j}{\partial h^j}C^{-h}F(h,Cz),\qquad j=0,\dots,3$$

and the basis

$$\tilde{f}_1^*(z)=-a_1z^{-1/3}(1+\cdots),\qquad \tilde{f}_2^*(z)=-a_2z^{-2/3}(1+\cdots),$$
$$\tilde{f}_1=(\log z+g_1(1/z))\tilde{f}_1^*(z),\qquad \tilde{f}_2=(\log z+g_2(1/z))\tilde{f}_2^*(z),$$

where $g_1(t)$ and $g_2(t)$ are functions holomorphic and vanishing at $t=0$. The monodromy matrix around ∞ with respect to the ordered basis $\{\tilde f_1, \tilde f_1^*, \tilde f_2, \tilde f_2^*\}$ is easily seen to be

$$\begin{pmatrix} e^{-2\pi i/3} & 2\pi i e^{-2\pi i/3} & 0 & 0 \\ 0 & e^{-2\pi i/3} & 0 & 0 \\ 0 & 0 & e^{2\pi i/3} & 2\pi i e^{2\pi i/3} \\ 0 & 0 & 0 & e^{2\pi i/3} \end{pmatrix}.$$

By the same argument as before, we find that the monodromy matrix with respect to the basis $\{f_3, f_2, f_1, f_0\}$ indeed takes the form claimed in the statement. This proves the case $(A,B)=(1/3,1/3)$. □

Proof of the case $(A,B)=(1/2,1/2)$. Let z be a complex number such that $-\pi < \arg z < 0$. We find that the Frobenius basis $\{y_j\}$ at $z=0$ can be expressed as

$$y_j(z) = \frac{1}{j!}\frac{\partial^j}{\partial h^j}(C^{-h}F(h,Cz)),$$

where $C=256$ and

$$C^{-h}F(h,Cz) = -\frac{z^h}{2\pi i}\frac{\Gamma(1+h)^8}{\Gamma(1+2h)^4}\int_{\mathcal C}\frac{\Gamma(2s+1+2h)^4}{\Gamma(s+1+h)^8}\frac{\pi}{\sin\pi s}e^{\pi i s}z^s\,ds.$$

Here h is assumed to be a small real number and $\mathcal C$ denotes the vertical line $\operatorname{Re} s = -1/4$. The integrand has quadruple poles at $s=-k-1/2-h$ for non-positive integers k. Moving the line of integration to $\operatorname{Re} s = -3/4$ and computing the residue at $s=-1/2-h$, we see that

$$C^{-h}F(h,Cz) = a_1\sum_{n=0}^3 \frac{B_n(h)}{n!}z^{-1/2}(\log z)^n + (\text{higher order terms in } 1/z),$$

where

$$a_1 = \frac{\pi e^{-\pi i/2}}{16\Gamma(1/2)^8}, \quad B_3(h) = \frac{\Gamma(1+h)^8 e^{-\pi i h}}{\Gamma(1+2h)^4\cos\pi h}, \quad B_2(h) = B_3(h)(\mu - \pi\tan\pi h),$$

$$B_1(h) = B_3(h)\left(-\frac{7}{6}\pi^2 + \frac{\mu^2}{2} - \pi\mu\tan\pi h + \pi^2\sec^2\pi h\right),$$

and

$$\begin{aligned} B_0(h) = B_3(h)\Big(&\frac{\mu^3}{6} - \frac{\pi}{2}\mu^2\tan\pi h + (\sec^2\pi h - 7/6)\pi^2\mu \\ &+ (5/6 - \sec^2\pi h)\pi^3\tan\pi h + 8\zeta(3)\Big), \end{aligned}$$

where $\mu = 16\log 2 + \pi i$. Let

$$\tilde f_0(z) = z^{-1/2}(1+\cdots), \qquad \tilde f_1(z) = \frac{1}{2\pi i}(\log z + g_1(1/z))\tilde f_0(z),$$

$$\tilde f_2(z) = \frac{1}{(2\pi i)^2}(\log^2 z/2 + g_1(1/z)\log z + g_2(1/z))\tilde f_0(z),$$

$$\tilde f_3(z) = \frac{1}{(2\pi i)^3}(\log^3 z/6 + g_1(1/z)\log^2 z/2 + g_2(1/z)\log z + g_3(1/z))\tilde f_0(z)$$

be the Frobenius basis at $z=\infty$ with $g_n(0)=0$. Using the evaluation

$$\Gamma'(1) = -\gamma, \qquad \Gamma''(1) = \frac{\pi^2}{12}+\frac{\gamma^2}{2}, \qquad \Gamma'''(1) = -\frac{1}{3}\zeta(3) - \frac{\pi^2\gamma}{12} - \frac{\gamma^3}{6},$$

we can find the analytic continuation of the Frobenius at $z=0$ in terms of $\tilde{f}_n(z)$. Now the monodromy around ∞ relative to the basis $\{\tilde{f}_3(z), \tilde{f}_2(z), \tilde{f}_1(z), \tilde{f}_0(z)\}$ is

$$\begin{pmatrix} -1 & -1 & -1/2 & -1/6 \\ 0 & -1 & -1 & -1/2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

From this we can determine the monodromy matrix around $z=1/C$ with respect to the Frobenius basis at $z=0$. We find that the result agrees with the general pattern depicted in Theorem 1, although the detailed computation is too complicated to be presented here. □

Of course, there is no reason why our approach should be applicable only to order 4 cases. Consider the hypergeometric differential equations of the form

$$L:\theta^5 - z(\theta+1/2)(\theta+A)(\theta+1-A)(\theta+B)(\theta+1-B). \tag{12}$$

The cases $(A,B)=(1/2,1/2)$, $(1/4,1/2)$, $(1/6,1/4)$, $(1/4,1/3)$, $(1/6,1/3)$, and $(1/8,3/8)$ have been used by Guillera [12, 13] to construct series representations for $1/\pi^2$. Applying the above method, we determine the monodromy of these differential equations in the following theorem whose proof will be omitted.

Theorem 3. *Let L be one of the differential equations in (12). Let y_i, $i=0,\ldots,4$, be the Frobenius basis at 0. Then the monodromy matrices around $z=0$ and $z=1/C$ with respect to the ordered basis $\{y_4/(2\pi i)^4, y_3/(2\pi i)^3, y_2/(2\pi i)^2, y_1/(2\pi i), y_0\}$ are*

$$\begin{pmatrix} 1 & 1 & 1/2 & 1/6 & 1/24 \\ 0 & 1 & 1 & 1/2 & 1/6 \\ 0 & 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} a^2 & 0 & -ab & (1-a^2)x & -b^2/2 \\ -c^2x/2 & 1 & -acx & c^2x^2/2 & -(1-a^2)x \\ -ac & 0 & 1-2a^2 & acx & -ab \\ 0 & 0 & 0 & 1 & 0 \\ -c^2/2 & 0 & -ac & c^2x/2 & a^2 \end{pmatrix},$$

respectively, where x is an integer multiple of $\zeta(3)/(2\pi i)^3$, a and c are positive real numbers such that a^2, ac, and c^2 are rational numbers, and b is a real number satisfying $a^2+bc=1$. The exact values of a, c, and $x'=(2\pi i)^3x/\zeta(3)$ are given in the following table.

A	B	a^2	c^2	x'
1/2	1/2	25/36	64	10
1/2	1/4	8/9	32	24
1/4	1/6	289/288	8	80
1/3	1/4	27/32	24	28
1/3	1/6	75/64	12	70
1/8	3/8	529/288	8	150

5. Differential equations of Calabi-Yau type

The Picard-Fuchs differential equations for families of Calabi-Yau threefolds known in literature have the common features that

(a) the singular points are all regular,
(b) the indicial equation at $z=0$ has 0 as its only solution,
(c) the indicial equation at one of the singularities has solutions $0,1,1,2$, corresponding to a conifold singularity,

(d) the unique holomorphic solution y around 0 with $y(0) = 1$ has integral coefficients in its power series expansion,
(e) the solutions $\lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_4$ of the indicial equation at $t = \infty$ are positive rational numbers and satisfy $\lambda_1 + \lambda_4 = \lambda_2 + \lambda_3 = r$ for some $r \in \mathbb{Q}$, and the characteristic polynomial of the monodromy around $t = \infty$ is a product of cyclotomic polynomials.
(f) the coefficients $r_i(z)$ of the differential equation satisfy

$$r_1 = \frac{1}{2} r_2 r_3 - \frac{1}{8} r_3^3 + r_2' - \frac{3}{4} r_3' r_3 - \frac{1}{2} r_3'',$$

(g) the instanton numbers are integers.

In [1] a fourth order linear differential equation satisfying all conditions except (c) is said to be of *Calabi-Yau type*. Using various techniques, Almkvist and etc. found more than 300 such equations. (See Section 5 of [2] for an overview of strategies of finding Calabi-Yau equations. The paper also contains a "superseeker" that tabulates the known Calabi-Yau equations, sorted according to the instanton numbers.) Among them, there are 178 equations that have singularities with exponents $0, 1, 1, 2$. It is speculated that all such equations should have geometric origin.

In [25] van Enckevort and van Straten numerically determined the monodromy for these 178 equations. They were able to find rational bases for 145 of them, among which there are 64 cases that are integral. Their method goes as follows. Let $z_1, \ldots, z_k$ be the singularities of a Calabi-Yau differential equation. They first chose a reference point p and piecewise linear loops each of which starts from p and encircles exactly one of z_i. Then the problem of determining analytic continuation becomes that of solving several initial value problems in sequences. This was done numerically using the `dsolve` function in `Maple`. Then they used the crucial observation that the Jordan form for the monodromy around a conifold singularity is unipotent of index one to find a rational basis. Finally, assuming that (5) and (8) hold for general differential equations of Calabi-Yau type, conjectural values of geometric invariants can be read off.

Here we present a different method of computing monodromy based on the approach described in Section 3. Let $0 = z_0, z_1, \ldots, z_n$ be the singular points of a Calabi-Yau differential equation, and assume that $f_{i,k}$, $i = 0, \ldots, n$, $k = 1, \ldots, 4$ form the Frobenius bases at z_i. According to Section 3, to find the matrix of basis change between $\{f_{i,k}\}$ and $\{f_{j,k}\}$, we only need to evaluate $f_{i,k}^{(m)}$ and $f_{j,k}^{(m)}$ at a common point ζ where the power series expansions of the functions involved all converge. In practice, the choice of ζ is important in order to achieve required precision in a reasonable amount of time.

Let R_i denote the radius of convergence of the power series expansions of the Frobenius basis at z_i. In general, R_i is equal to the distance from z_i to the nearest singularity $z_j \neq z_i$, meaning that if we truncate the power series expansion of $f_{i,k}$ at the nth term, the resulting error is

$$O_\epsilon \left((1+\epsilon)^n \frac{|\zeta - z_i|^n}{R_i^n} \right).$$

Of course, the O-constants depend on the differential equation and z_i. Since we do not have any control over them, in practice we just choose ζ in a way such that

$$\frac{|\zeta - z_i|}{R_i} = \frac{|\zeta - z_j|}{R_j}.$$

If this does not yield needed precision, we simply replace n by a larger integer and do the computation again.

Example. Consider

$$\theta^4 - 5(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4).$$

The singularities are $z_0 = 0$, $z_1 = 1/3125$, and $z_2 = \infty$. The radii of convergence for the Frobenius bases at 0 and 1/3125 are both 1/3125. Thus, to find the matrix of basis change, we expand the Frobenius bases, say, for 30 terms, and evaluate the Frobenius bases and their derivatives at $\zeta = 1/6250$. Then we use the idea in Section 3 to compute the monodromy matrix around z_1 with respective to the Frobenius basis at 0. We find that the computation agrees with (7) in Theorem 1 up to 7 digits.

The above method works quite well if the singularities of a differential equation are reasonably well spaced. However, it occurs quite often that a Calabi-Yau differential equation has a cluster of singular points near 0, and a couple of singular points that are far away. For example, consider Equation #19

$$\begin{aligned}529\theta^4 &- 23z(921\theta^4 + 2046\theta^3 + 1644\theta^2 + 621\theta + 92)\\ &- z^2(3808851\theta^4 + 13285584\theta^3 + 17726673\theta^2 + 10335228\theta + 221168)\\ &- 2z^3(4758861\theta^4 + 13101172\theta^3 + 10287791\theta^2 + 208932\theta - 27232)\\ &- 68z^4(8873\theta^4 + 14020\theta^3 + 5139\theta^2 - 1664\theta - 976)\\ &+ 6936z^5(\theta+1)^2(3\theta+2)(3\theta+4).\end{aligned}$$

The singularities are $z_0 = 0$, $z_1 = 1/54$, $z_2 = (11-5\sqrt{5})/2 = -0.090\ldots$, $z_3 = -23/34$, and $z_4 = (11+5\sqrt{5})/2 = 11.09\ldots$. In order to determine the monodromy matrix around z_4, we need to compute the matrix of basis change between the Frobenius basis at 1/54 and that at z_4. The radius of convergence for the Frobenius basis at 1/54 is 1/54, while that at z_4 is $z_4 - 1/54 = 11.07\ldots$. Even if we choose ζ optimally, we still need to expand the Frobenius bases for thousands of terms in order to achieve a precision of a few digits. In such situations, we can choose several points lying between the two singularities, compute bases for each of them, and then use the same idea as before to determine the matrices of basis change.

Take Equation 19 above as an example. We choose $w_k = (1+3^k)/54$ and $\zeta_k = (1+3^k/2)/54$ for $k = 0,\ldots,5$. The radius of convergence for the basis at w_k is $3^k/54$. Thus, evaluating the first n terms of the power series expansions at ζ_k and ζ_{k+1} will result in an error of

$$O_\epsilon((1/2+\epsilon)^n),$$

which is good enough in practice.

Using the above ideas we computed the monodromy groups of the differential equations of Calabi-Yau type that have at least one conifold singularity. Our result shows that if a differential equation comes from geometry, then the monodromy matrix around one of the conifold singularities with respect to the Frobenius basis at the origin takes the form (7). We then conjugate the monodromy matrices by the matrix (9) and find that the other matrices are also in $\mathrm{Sp}(4,\mathbb{Q})$. We now tabulate the results for the equations coming from geometry in the following table. Note that the notations $\Gamma(d_1,d_2)$ and $\Gamma(d_1,d_2,d_3)$, $d_2,d_3|d_1$, represent the congruence subgroups

$$\Gamma(d_1,d_2)=\left\{\gamma\in\mathrm{Sp}(4,\mathbb{Z}):\gamma\equiv\begin{pmatrix}1&*&*&*\\0&*&*&*\\0&0&1&0\\0&*&*&*\end{pmatrix}\mod d_1\right\}$$
$$\bigcap\left\{\gamma\in\mathrm{Sp}(4,\mathbb{Z}):\gamma\equiv\begin{pmatrix}1&*&*&*\\0&1&*&*\\0&0&1&0\\0&0&*&1\end{pmatrix}\mod d_2\right\}$$

and

$$\begin{aligned}\Gamma(d_1,d_2,d_3)=\Big\{(a_{ij})\in\mathrm{Sp}(4,\mathbb{Q}):\ &a_{ij}\in\mathbb{Z}\ \forall(i,j)\neq(1,3),\ a_{13}\in\frac{1}{d_3}\mathbb{Z},\\ &a_{21},a_{31},a_{41},a_{32},a_{34}\equiv 0\mod d_1,\\ &a_{42}\equiv 0,\quad a_{22},a_{44}\equiv 1\mod d_2,\\ &a_{11},a_{33}\equiv 1\mod\frac{d_1}{d_3}\Big\}.\end{aligned}$$

Note also that since the matrix

$$\begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

is always in the monodromy groups, it is not listed in the table. The reader should be mindful of this omission.

#	H^3	$c_2 \cdot H$	c_3	Generators	in	Ref
15	18	72	-162	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 18 & 18 & 1 & 0 \\ 0 & -9 & -1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 3 & 1/2 & -1 \\ 6 & -5 & -1 & 2 \\ -18 & 18 & 4 & -6 \\ 18 & -18 & -3 & 7 \end{pmatrix}$	$\Gamma(6,3,2)$	[4]
16	48	96	-128	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 48 & 48 & 1 & 0 \\ 0 & -16 & -1 & 1 \end{pmatrix} \begin{pmatrix} -5 & 2 & 1/4 & -1 \\ 24 & -7 & -1 & 4 \\ -144 & 48 & 7 & -24 \\ 48 & -16 & -2 & 9 \end{pmatrix}$	$\Gamma(24,8,4)$	[4]
17	90	108	-90	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 90 & 90 & 1 & 0 \\ 0 & -24 & -1 & 1 \end{pmatrix} \begin{pmatrix} -17 & 3 & 1/3 & 1 \\ -54 & 10 & 1 & 3 \\ -972 & 162 & 19 & 54 \\ 162 & -27 & -3 & -8 \end{pmatrix} \begin{pmatrix} -11 & 3 & 1/3 & -1 \\ 36 & -8 & -1 & 3 \\ -432 & 108 & 13 & -36 \\ 108 & -27 & -3 & 10 \end{pmatrix}$	$\Gamma(18,6,3)$	[4]
18	40	88	-128	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 40 & 40 & 1 & 0 \\ 0 & -14 & -1 & 1 \end{pmatrix} \begin{pmatrix} -5 & 4 & 1/2 & -1 \\ 12 & -7 & -1 & 2 \\ -72 & 48 & 7 & -12 \\ 48 & -32 & -4 & 9 \end{pmatrix}$	$\Gamma(4,2,2)$	[4]
19	46	88	-106	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 46 & 46 & 1 & 0 \\ 0 & -15 & -1 & 1 \end{pmatrix} \begin{pmatrix} -6 & 4 & 1/2 & -1 \\ 14 & -7 & -1 & 2 \\ -98 & 56 & 8 & -14 \\ 56 & -32 & -4 & 9 \end{pmatrix} \begin{pmatrix} -45 & 12 & 2 & -6 \\ 138 & -35 & -6 & 18 \\ -1058 & 276 & 47 & -138 \\ 276 & -72 & -12 & 37 \end{pmatrix}$	$\Gamma(2,2,2)$	[4]
20	54	72	-18	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 54 & 54 & 1 & 0 \\ 0 & -15 & -1 & 1 \end{pmatrix} \begin{pmatrix} 7 & -1 & -1/6 & 1 \\ -6 & 1 & 0 & -2 \\ 126 & -18 & -2 & 24 \\ -36 & 6 & 1 & -5 \end{pmatrix}$	$\Gamma(6,3,6)$	[4]
21	80	104	-88	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 80 & 80 & 1 & 0 \\ 0 & -22 & -1 & 1 \end{pmatrix} \begin{pmatrix} -11 & 5 & 1/2 & -1 \\ 24 & -9 & -1 & 2 \\ -288 & 120 & 13 & -24 \\ 120 & -50 & -5 & 11 \end{pmatrix} \begin{pmatrix} -19 & 4 & 1/2 & -2 \\ 80 & -15 & -2 & 8 \\ -800 & 160 & 21 & -80 \\ 160 & -32 & -4 & 17 \end{pmatrix}$	$\Gamma(8,2,2)$	[4]
22	70	100	-100	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 70 & 70 & 1 & 0 \\ 0 & -20 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 10 & 1 & 0 & 2 \\ -50 & 0 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -9 & 5 & 1/2 & -1 \\ 20 & -9 & -1 & 2 \\ -200 & 100 & 11 & -20 \\ 100 & -50 & -5 & 11 \end{pmatrix}$	$\Gamma(10,10,2)$	[4]
23	96	96	-32	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 96 & 96 & 1 & 0 \\ 0 & -24 & -1 & 1 \end{pmatrix} \begin{pmatrix} 9 & -1 & -1/8 & 1 \\ -8 & 1 & 0 & -2 \\ 288 & -32 & -3 & 40 \\ -64 & 8 & 1 & -7 \end{pmatrix}$	$\Gamma(8,8,8)$	[4]

24	15	66	−150	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\15&15&1&0\\0&-8&-1&1\end{pmatrix}\begin{pmatrix}-5&5&1&-2\\12&-9&-2&4\\-36&30&7&-12\\30&-25&-5&11\end{pmatrix}$	$\Gamma(3,1)$	[3]
25	20	68	−120	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\20&20&1&0\\0&-9&-1&1\end{pmatrix}\begin{pmatrix}-7&5&1&-2\\16&-9&-2&4\\-64&40&9&-16\\40&-25&-5&11\end{pmatrix}$	$\Gamma(4,1)$	[3]
26	28	76	−116	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\28&28&1&0\\0&-11&-1&1\end{pmatrix}\begin{pmatrix}-9&6&1&-2\\20&-11&-2&4\\-100&60&11&-20\\60&-36&-6&13\end{pmatrix}$	$\Gamma(4,1)$	[3]
27	42	84	98	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\42&42&1&0\\0&-14&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\42&1&0&9\\-196&0&1&-42\\0&0&0&1\end{pmatrix}$ $\begin{pmatrix}-13&7&1&-2\\28&-13&-2&4\\-196&98&15&-28\\98&-49&-7&15\end{pmatrix}$	$\Gamma(14,7)$	[3, 22]
28	42	84	−96	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\42&42&1&0\\0&-14&-1&1\end{pmatrix}\begin{pmatrix}-41&12&2&-6\\126&-35&-6&18\\-882&252&43&-126\\252&-72&-12&37\end{pmatrix}$	$\Gamma(42,2)$	[3]
186	57	90	−84	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\57&57&1&0\\0&-17&-1&1\end{pmatrix}\begin{pmatrix}-53&12&2&-6\\162&-35&-6&18\\-1458&324&55&-162\\324&-72&-12&37\end{pmatrix}$ $\begin{pmatrix}-17&8&1&-2\\36&-15&-2&4\\-324&144&19&-36\\144&-64&-8&17\end{pmatrix}$	$\Gamma(3,1)$	[24]

In the second table we list a few equations whose monodromy matrices with respect to our bases have integers as entries. Note that the numbers H^3, $c_2\cdot H$, and c_3 are all conjectural, obtained from evaluation of the monodromy around a singularity of conifold type. Note, again, that the matrix

$$\begin{pmatrix}1&0&0&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

is omitted from the table.

#	H^3	$c_2\cdot H$	c_3	Generators	in
29	24	72	−116	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\24&24&1&0\\0&-10&-1&1\end{pmatrix}\begin{pmatrix}-47&20&4&-10\\120&-49&-10&25\\-576&240&49&-120\\240&-100&-20&51\end{pmatrix}$	$\Gamma(24,2)$
33	6	36	−72	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\6&6&1&0\\0&-4&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\2&1&0&2\\-2&0&1&-2\\0&0&0&1\end{pmatrix}$	$\Gamma(2,2)$
42	32	80	−116	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\32&32&1&0\\0&-12&-1&1\end{pmatrix}\begin{pmatrix}-15&6&1&-3\\48&-17&-3&9\\-256&96&17&-48\\96&-36&-6&19\end{pmatrix}$	$\Gamma(16,4)$

51	10	64	-200	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\10&10&1&0\\0&-7&-1&1\end{pmatrix}\begin{pmatrix}-3&5&1&-2\\8&-9&-2&4\\-16&20&5&-8\\20&-25&-5&11\end{pmatrix}$	$\Gamma(2,1)$
63	5	62	-310	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\5&5&1&0\\0&-6&-1&1\end{pmatrix}\begin{pmatrix}-1&5&1&-2\\4&-9&-2&4\\-4&10&3&-4\\10&-25&-5&11\end{pmatrix}$	$\Gamma(1,1)$
73	9	30	12	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\9&9&1&0\\0&-4&-1&1\end{pmatrix}\begin{pmatrix}2&0&0&1\\3&-2&-1&0\\0&3&2&3\\-3&0&0&-2\end{pmatrix}$	$\Gamma(3,1)$
99	13	58	-120	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\13&13&1&0\\0&-7&-1&1\end{pmatrix}\begin{pmatrix}-5&4&1&-2\\12&-7&-2&4\\-36&24&7&-12\\24&-16&-4&9\end{pmatrix}$	$\Gamma(1,1)$
100	36	72	-72	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\36&36&1&0\\0&-12&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\12&1&0&4\\-36&0&1&-12\\0&0&0&1\end{pmatrix}\begin{pmatrix}-11&6&1&-2\\24&-11&-2&4\\-144&72&13&-24\\72&-36&-6&13\end{pmatrix}$	$\Gamma(12,12)$
101	25	70	-100	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\25&25&1&0\\0&-10&-1&1\end{pmatrix}\begin{pmatrix}-19&10&2&-4\\40&-19&-4&8\\-200&100&21&-40\\100&-50&-10&21\end{pmatrix}\begin{pmatrix}1&0&0&0\\60&1&0&16\\-225&0&1&-60\\0&0&0&1\end{pmatrix}$	$\Gamma(5,5)$
109	7	46	-120	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\7&7&1&0\\0&-5&-1&1\end{pmatrix}\begin{pmatrix}-3&3&1&-2\\8&-5&-2&4\\-16&12&5&-8\\12&-9&-3&7\end{pmatrix}$	$\Gamma(1,1)$
117	12	36	-32	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\12&12&1&0\\0&-5&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\4&1&0&4\\-4&0&1&-4\\0&0&0&1\end{pmatrix}\begin{pmatrix}-59&21&9&18\\-120&43&18&36\\-400&140&61&120\\140&-49&-21&-41\end{pmatrix}$	$\Gamma(4,1)$
118	10	40	-50	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\10&10&1&0\\0&-5&-1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\30&1&0&18\\-50&0&1&-30\\0&0&0&1\end{pmatrix}\begin{pmatrix}-19&10&4&-8\\40&-19&-8&16\\-100&50&21&-40\\50&-25&-10&21\end{pmatrix}$	$\Gamma(10,5)$
185	36	84	-120	$\begin{pmatrix}1&1&0&0\\0&1&0&0\\36&36&1&0\\0&-13&-1&1\end{pmatrix}\begin{pmatrix}-11&7&1&-2\\24&-13&-2&4\\-144&84&13&-24\\84&-49&-7&15\end{pmatrix}$	$\Gamma(12,1)$

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Appendix – The index of $\Gamma(d_1, d_2)$ in $\mathrm{Sp}(4, \mathbb{Z})$

by Cord Erdenberger

Institut für Mathematik, Universität Hannover
Welfengarten 1, 30167 Hannover, Germany
E-mail: erdenber@math.uni-hannover.de

In this appendix we will show that the groups $\Gamma(d_1, d_2)$ are indeed congruence subgroups in $\mathrm{Sp}(4, \mathbb{Z})$ and provide a formula for their index.

Recall that for $n \in \mathbb{N}$ the principal congruence subgroup of level n is defined by

$$\Gamma(n) := \{ M \in \mathrm{Sp}(4, \mathbb{Z}) \,|\, N \equiv \mathbf{I}_4 \pmod{n} \}.$$

It is the kernel of the map from $\mathrm{Sp}(4, \mathbb{Z})$ to $\mathrm{Sp}(4, \mathbb{Z}/n\mathbb{Z})$ given by reduction modulo n and thus a normal subgroup in $\mathrm{Sp}(4, \mathbb{Z})$. It is a well–known fact that this map is surjective and hence the sequence

$$\mathbf{I}_4 \to \Gamma(n) \hookrightarrow \mathrm{Sp}(4, \mathbb{Z}) \to \mathrm{Sp}(4, \mathbb{Z}/n\mathbb{Z}) \to \mathbf{I}_4$$

is exact. So the index of $\Gamma(n)$ in $\mathrm{Sp}(4, \mathbb{Z})$ is just the order of $\mathrm{Sp}(4, \mathbb{Z}/n\mathbb{Z})$ which is known to be

$$[\mathrm{Sp}(4, \mathbb{Z}) : \Gamma(n)] = |\mathrm{Sp}(4, \mathbb{Z}/n\mathbb{Z})| = n^{10} \prod (1 - p^{-2})(1 - p^{-4}),$$

where the product runs over all primes p such that $p|n$.

For $d_1, d_2 \in \mathbb{N}$, define

$$\widetilde{\Gamma}_1(d_1) := \left\{ M \in \mathrm{Sp}(4, \mathbb{Z}) \; : \; M \equiv \begin{pmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & * & * & * \end{pmatrix} \mod d_1 \right\},$$

$$\widetilde{\Gamma}_2(d_2) := \left\{ M \in \mathrm{Sp}(4, \mathbb{Z}) \; : \; M \equiv \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \mod d_2 \right\}$$

and set

$$\Gamma(d_1, d_2) := \widetilde{\Gamma}_1(d_1) \cap \widetilde{\Gamma}_2(d_2).$$

Note that

$$\Gamma(d_1) \subset \widetilde{\Gamma}_1(d_1) \text{ and } \Gamma(d_2) \subset \widetilde{\Gamma}_2(d_2).$$

Hence

$$\Gamma(d) = \Gamma(d_1) \cap \Gamma(d_2) \subset \widetilde{\Gamma}_1(d_1) \cap \widetilde{\Gamma}_2(d_2) = \Gamma(d_1, d_2),$$

where d is the least common multiple of d_1 and d_2. This shows that $\Gamma(d_1, d_2)$ is a congruence subgroup, i.e. it contains a principal congruence subgroup as a normal subgroup of finite index. Moreover, this implies that $\Gamma(d_1, d_2)$ has finite index in $\mathrm{Sp}(4, \mathbb{Z})$ and an upper bound is given by the index of $\Gamma(d)$ as given above.

We will from now on restrict to the case relevant to this paper, namely $d_2 | d_1$. Then $\Gamma(d_1, d_2)$ is in fact a subgroup of $\widetilde{\Gamma}_1(d_1)$, namely

$$\Gamma(d_1, d_2) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \in \widetilde{\Gamma}_1(d_1) : \begin{pmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod d_2 \right\}.$$

To obtain a formula for the index of this group in $\mathrm{Sp}(4,\mathbb{Z})$, we first calculate the index of $\widetilde{\Gamma}_1(d_1)$. Note that

$$\widetilde{\Gamma}_1(d_1)/\Gamma(d_1) < \mathrm{Sp}(4,\mathbb{Z})/\Gamma(d_1) \simeq \mathrm{Sp}(4,\mathbb{Z}/d_1\mathbb{Z})$$

and hence

$$[\mathrm{Sp}(4,\mathbb{Z}) : \widetilde{\Gamma}_1(d_1)] = [\mathrm{Sp}(4,\mathbb{Z}/d_1\mathbb{Z}) : \widetilde{\Gamma}_1(d_1)/\Gamma(d_1)].$$

The quotient $\widetilde{\Gamma}_1(d_1)/\Gamma(d_1)$ considered as a subgroup of $\mathrm{Sp}(4,\mathbb{Z}/d_1\mathbb{Z})$ via the above isomorphism is given by

$$\widetilde{\Gamma}_1(d_1)/\Gamma(d_1) \simeq \left\{ M \in \mathrm{Sp}(4,\mathbb{Z}/d_1\mathbb{Z}) \,:\, M = \begin{pmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & * & * & * \end{pmatrix} \right\}.$$

An element of this group has the following form

$$M = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & \alpha & a_{23} & \beta \\ 0 & 0 & 1 & 0 \\ 0 & \gamma & a_{43} & \delta \end{pmatrix}.$$

Let $\mathbb{J}_4 := \begin{pmatrix} 0 & -\mathbf{I}_2 \\ \mathbf{I_2} & 0 \end{pmatrix}$. The symplectic relation that ${}^tM\,\mathbb{J}_4\,M = \mathbb{J}_4$ then implies that $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/d_1\mathbb{Z})$. Furthermore it gives rise to the following linear system:

$$\begin{array}{ccccccc} a_{12} & + & \alpha\, a_{43} & - & \gamma\, a_{23} & = & 0 \\ a_{14} & + & \beta\, a_{43} & - & \delta\, a_{23} & = & 0 \end{array}$$

Writing this in matrix form, we have

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} -a_{43} \\ a_{23} \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{14} \end{pmatrix}.$$

If we choose a_{12}, a_{13}, a_{14} freely, the above linear system has a unique solution a_{23}, a_{43} as $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is in $\mathrm{SL}_2(\mathbb{Z}/d_1\mathbb{Z})$. This shows that

$$|\widetilde{\Gamma}_1(d_1)/\Gamma(d_1)| = d_1^3 \cdot |\mathrm{SL}_2(\mathbb{Z}/d_1\mathbb{Z})| = d_1^6 \prod (1 - p^{-2})$$

where the product runs over all primes p dividing d_1. So we have the index formula

$$[\mathrm{Sp}(4,\mathbb{Z}) : \widetilde{\Gamma}_1(d_1)] = [\mathrm{Sp}(4,\mathbb{Z}/d_1\mathbb{Z}) : \widetilde{\Gamma}_1(d_1)/\Gamma(d_1)] = d_1^4 \prod_{p|d_1} (1 - p^{-4}).$$

Now we are ready to calculate the index of $\Gamma(d_1, d_2)$ in $\mathrm{Sp}(4,\mathbb{Z})$. Since we assume that $d_2|d_1$, we have the following chain of subgroups:

$$\Gamma(d_1) < \Gamma(d_1, d_2) < \widetilde{\Gamma}_1(d_1) < \mathrm{Sp}(4,\mathbb{Z})$$

Note that

$$[\widetilde{\Gamma}_1(d_1) : \Gamma(d_1, d_2)] = [\widetilde{\Gamma}_1(d_1)/\Gamma(d_1) : \Gamma(d_1, d_2)/\Gamma(d_1)]$$

and by our above description this is just the index of the group

$$\left\{ M \in \mathrm{SL}_2(\mathbb{Z}/d_1\mathbb{Z}) \,:\, M \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod d_2 \right\}$$

in $\mathrm{SL}_2(\mathbb{Z}/d_1\mathbb{Z})$. An easy calculation shows that this index is equal to

$$d_2^2 \prod_{p|d_2}(1-p^{-2}).$$

Putting all these together, we get

$$\begin{aligned}[\mathrm{Sp}(4,\mathbb{Z}):\Gamma(d_1,d_2)] &= [\mathrm{Sp}(4,\mathbb{Z}):\widetilde{\Gamma}_1(d_1)]\cdot[\widetilde{\Gamma}_1(d_1):\Gamma(d_1,d_2)] \\ &= d_1^4\prod_{p|d_1}(1-p^{-4})\; d_2^2\prod_{p|d_2}(1-p^{-2}).\end{aligned}$$

We summarize the above calculation to obtain

Theorem. The group $\Gamma(d_1, d_2)$ is a congruence subgroup in $\mathrm{Sp}(4,\mathbb{Z})$ and its index is given by

$$|\mathrm{Sp}(4,\mathbb{Z}):\Gamma(d_1,d_2)| = d_1^4\prod_{p|d_1}(1-p^{-4})\; d_2^2\prod_{p|d_2}(1-p^{-2}).$$

In fact, we can do a similiar calculation without the assumption that $d_2|d_1$ and obtain the same formula as given above where one has to replace d_1 with the least common multiple of d_1 and d_2.

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Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, TAIWAN

E-mail address: peace.am92g@nctu.edu.tw

Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, TAIWAN

E-mail address: yfyang@math.nctu.edu.tw

Department of Mathematics and Statistics, Queen's University, Kingston, Ontario K3L 3N6, CANADA

E-mail address: yui@mast.queensu.ca