DEFORMATIONS OF FOUR DIMENSIONAL LIE ALGEBRAS

ALICE FIALOWSKI AND MICHAEL PENKAVA

Abstract. We study the moduli space of four dimensional ordinary Lie algebras, and their versal deformations. Their classification is well known; our focus in this paper is on the deformations, which yield a picture of how the moduli space is assembled. Surprisingly, we get a nice geometric description of this moduli space essentially as an orbifold, with just a few exceptional points.

1. Introduction

Lie algebras of small dimension are still a central area of research, although their classification is basically known up to order 7 (for instance, see [11, 13, 14, 9, 12]). The reason for this is that they play a crucial role in physical applications (especially in dimension 4). Despite the classification of these algebras, the moduli space of Lie algebras in a given dimension is not well understood. We should mention [8], on the variety of n dimensional Lie algebra structures. Moreover, in the existing classifications there are often overlaps of families determined by parameters and the manner in which unique objects are singled out is somewhat artificial. Our solution of this problem is to consider the cohomology of the Lie algebras as well as their versal deformations, and use this information as a guide to their division into families. This is the additional information which provides us with a natural division of the moduli space of Lie algebras into families, as well giving us a geometric picture of the structure of the moduli space. We did a similar study for 3 dimensional Lie algebras in [6].

Date: July 4, 2005.

1991 *Mathematics Subject Classification.* 14D15,13D10,14B12,16S80,16E40, 17B55,17B70.

Key words and phrases. versal deformations, Lie algebras, moduli space .

The research of the authors was partially supported by grants from NSF-OTKA-MTA 38453, OTKA T043641 and T043034, and by grants from the University of Wisconsin-Eau Claire.

The goal of the present paper is to get an accurate picture of the moduli space of complex 4-dimensional Lie algebras. The key ingredient in our description will be the versal deformations of the elements in the moduli space; therefore cohomology will be a primary computational tool.

In this paper we will show that the moduli space of Lie algebras on $\mathbb{C}^4$ is essentially an orbifold given by the natural action of the symmetric group Σ_3 on the complex projective space $\mathbb{P}^2(\mathbb{C})$. In addition, there are two exceptional complex projective lines, one of which has an action of the symmetric group Σ_2. Finally, there are 6 exceptional points. The moduli space is glued together by the miniversal deformations, which determine the elements that one may deform to locally, so deformation theory determines the geometry of the space. The exceptional points play a role in refining the picture of how this space is glued together. By orbifold, we mean essentially a topological space quotiented out by the action of a group. In the case of $\mathbb{P}^n$, there is a natural action of Σ_{n+1} induced by the natural action of Σ_{n+1} on $\mathbb{C}^{n+1}$. An orbifold point is a point which is fixed by some element in the group. In the case of Σ_{n+1} acting on $\mathbb{P}^n$, points which have two or more coordinates with the same value are orbifold points, but there are some other ones, such as the point $(1:-1)=(-1:1)$.

In the classical theory of deformations, a deformation is called a jump deformation if there is a 1-parameter family of deformations of a Lie algebra structure such that every nonzero value of the parameter determines the same deformed Lie algebra, which is not the original one (see [7]). There are also deformations which move along a family, meaning that the Lie algebra structure is different for each value of the parameter. There can be multiple parameter families as well.

In the picture we will assemble, both of these phenomena arise. Some of the structures belong to families and their deformations simply move along the family to which they belong. If there is a jump deformation from an element to a member of a family, then there will always be deformations from that element along the family as well, although they will typically not be jump deformations. In addition, there are sometimes jump deformations either to or from the exceptional points, so these exceptional points play an interesting role in the picture of the moduli space.

The structure of this paper is as follows. After some preliminary definitions and explanation of notation, we will explain our classification of four dimensional Lie algebras, giving a comparison between our description of the isomorphism classes of Lie algebras and the ones in [2] and [1]. Our division of the algebras into families is based primarily on cohomological considerations; elements with the same cohomological description are placed into the same family in our decomposition. The correlation between our decomposition and the one in [1] is very close. The main differences arise out of our intention to divide up our families as projective spaces, a point of view which only partially occurs in [1].

After giving a description of the elements of the moduli space, we then study in detail miniversal deformations of each element, and determine how the local deformations behave. The main tool used in this paper is a constructive approach to the computation of miniversal deformations, which was first given in [4, 5]. We do not provide complete details about the method of construction, but try to provide enough information that the reader might be able to reconstruct miniversal deformations from the data we provide. Our goal here is to use the constructions to give a picture of the moduli space, rather than to demonstrate the constructions themselves.

Finally, we will assemble all the information we have collected to give a pictorial representation of the moduli space.

2. Preliminaries

In classical Lie algebra theory, the cohomology of a Lie algebra is studied by considering a differential on the dual space of the exterior algebra of the underlying vector space, considered as a cochain complex. If V is the underlying vector space on which the Lie algebra is defined, then its exterior algebra $\bigwedge V$ has a natural $\mathbb{Z}_2$-graded coalgebra structure as well. In this language, a Lie algebra is is simply a quadratic odd codifferential on the exterior coalgebra of a vector space. An odd codifferential is simply an odd coderivation whose square is zero. The space L of coderivations has a natural $\mathbb{Z}$-grading $L = \bigoplus L_n$, where L_n is the subspace of coderivations determined by linear maps $\phi : \bigwedge^n V \to V$. A Lie algebra is a codifferential in L_2, in other words, a quadratic codifferential. (L_∞ algebras are just arbitrary odd codifferentials.)

The space of coderivations has a natural structure of a $\mathbb{Z}_2$-graded Lie algebra. The condition that a coderivation d is a codifferential can be expressed in the form $[d, d] = 0$. The coboundary operator $D : L \to L$ is given simply by the rule $D(\varphi) = [d, \varphi]$ for $\varphi \in L$; the fact that $D^2 = 0$ is a direct consequence of the fact that d is an odd codifferential. Moreover, $D(L_n) \subseteq L_{n+1}$, which means that the cohomology $H(d) = \ker D / \operatorname{Im} D$ has a natural decomposition as a $\mathbb{Z}$-graded space: $H(d) = \prod H^n(d)$, where

$$H^n(d) = \ker(D : L_n \to L_{n+1}) / \operatorname{Im}(D : L_{n-1} \to L_n).$$

Recall that for an arbitrary vector space V of dimension n, the dimension of $\bigwedge^k V$ is just $\binom{n}{k}$. If $\{e_1,\dots,e_n\}$ is a basis of V, and $I=(i_1,\dots,i_k)$ is a multi-index with $i_1<\dots<i_k$, and we denote $e_I=e_{i_1}\cdots e_{i_k}$, then the e_I-s give a basis of $\bigwedge^k V$. Define $\varphi^I_j\in L_k$ by $\phi^I_j(e_J)=\delta^I_J e_j$, where δ^I_J is the Kronecker delta. The elements of L_k are all even if k is odd, and odd if k is even; to stress this difference, we will denote even elements as ϕ^I_j, but odd ones as ψ^I_j. Because we will be working with a four dimensional space, only L_0, L_1, L_2, L_3 and L_4 are nonzero, so 1 and 3 cochains are even, while 1, 2 and 4 cochains are odd. In general, the dimension of L_k is just $n\binom{n}{k}$, so for our case, $\dim L_0=4$, $\dim L_1=16$, $\dim L_2=24$, $\dim L_3=16$ and $\dim L_4=4$.

The Lie algebra structures are codifferentials in L_2. In order to represent a codifferential d as a matrix, we choose the following order for the increasing pairs $I=(i_1,i_2)$ of indices:

$$\{(1,2),(1,3),(2,3),(1,4),(2,4),(3,4)\},$$

and denote the ith element of this ordered set by $S(i)$. Using this order and the Einstein summation convention, we can express

$$d=a^i_j\varphi^{S(j)}_i.$$

Let $A=(a^i_j)$ be the matrix of coefficients of d. The first column represents $d(e_1e_2)$, the second $d(e_1e_3)$, etc. The Jacobi identity of the Lie algebra is given by the equation $[d,d]=0$, which can be expressed in matrix form as $AB=0$, where B is the matrix

$$B:=\begin{bmatrix} a^1_6 & -a^2_6 & -a^2_5-a^1_4 & -a^1_2-a^2_3\\ -a^1_5 & -a^3_6-a^1_4 & -a^3_5 & a^1_1-a^3_3\\ -a^3_6-a^2_5 & -a^2_4 & a^3_4 & a^3_2+a^2_1\\ a^1_3 & -a^4_6+a^1_2 & -a^4_5+a^1_1 & -a^4_3\\ -a^4_6+a^2_3 & a^2_2 & a^4_4+a^2_1 & a^4_2\\ a^4_5+a^3_3 & a^4_4+a^3_2 & a^3_1 & -a^4_1 \end{bmatrix}$$

Since AB is a 4×4 matrix, we obtain 16 quadratic relations among the coefficients that must be satisfied. In principle, it should be possible to use a computer algebra system to determine the solutions, but in our experience, this method has some drawbacks, unless one reduces the problem to some special cases, which we will do below.

In order to classify the solutions, we note that the dimension of the derived algebra is just the rank of A. We will show that the rank of A is never larger than 3. From this it follows that there is an ideal I of dimension 3 in the Lie algebra L, which gives an exact sequence of Lie algebras

$$0\to I\to L\to\mathbb{K}\to 0,$$

where $\mathbb{K}$ is the abelian Lie algebra of dimension 1. But then, the structure of L is completely determined by the structure of I as a Lie algebra, and an outer derivation δ of I. In [6], the moduli space of three dimensional Lie algebras was studied, and we will use the classification given there, because we will use in our classification the structure of the cohomology of these Lie algebras, which is given in detail in that paper.

3. Dimension of the Derived Algebra

We separate the types of Lie algebras into two distinct cases.

(1) Every independent pair of vectors spans a two dimensional subalgebra.
(2) There are independent vectors x, y and z so that $d(xy) = z$.

The first case is interesting, in that, up to isomorphism, over any field $\mathbb{K}$, there is exactly one nonabelian Lie algebra in each dimension greater than one satisfying this property, and it is given as an extension of a one dimensional Lie algebra by an abelian ideal. To see this, suppose that L has dimension at least two, is nonabelian, and satisfies the property that every independent pair of vectors spans a two dimensional subalgebra.

Let x'_1 and y' be two independent elements whose bracket $[x'_1, y'] = ax'_1 + by'$ does not vanish. We may assume that $a \neq 0$. If $x_1 = x'_1 + b/ay'$ and $y = 1/ay'$, then $[x_1, y] = x_1$. Next, suppose that x'_2 is independent of x_1 and y. Let $[x'_2, y] = ax'_2 + by$. Then $x_1 + ax'_2 + by = [x_1 + x'_2, y] = p(x_1 + x'_2) + qy$, for some p and q, so $a = 1$. Let $x_2 = x'_2 + by$. Then $[x_2, y] = x_2$. Now, express $[x_1, x_2] = ax_1 + bx_2$. Then $ax_1 + bx_2 - x_2 = [x_1 + y, x_2] = p(x_1 + y) + qx_2$, which implies that $a = 0$. Similarly, $x_1 + bx_2 = [x_1, y + x_2] = px_1 + q(y + x_2)$, so $b = 0$ and thus $[x_1, x_2] = 0$. The process can be repeated indefinitely, so we finally obtain a basis $\{x_1, \dots, x_n, y\}$ satisfying $[x_i, y] = x_i$, $[x_i, x_j] = 0$.

Finally, let us show that the bracket of any two elements is linearly dependent on them. Let $u = a^i x_i + by$ and $v = c^i x_i + dy$, then $[u, v] = a^i dx_i - c^i bx_i = du - bv$. Clearly, the x_i-s span an abelian ideal in the algebra, so L is an extension of the one dimensional Lie algebra spanned by y by this ideal. It follows that there is an abelian ideal of dimension n; moreover, this ideal coincides with the derived algebra, so the rank of the matrix A is precisely n, one less than the dimension of the vector space. In fact, the matrix A has precisely the form $A = \left[\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right]$, where I is the $n \times n$ identity matrix. This completes the description of the first case.

For the second case, suppose that there are linearly independent vectors such that $d(e_1e_2) = e_3$, so the matrix A of d satisfies $a_1^1 = a_1^2 = a_1^4 = 0$, $a_1^3 = 1$. One can easily check the possible solutions by considering subcases of this second case. For example, either e_1, e_2 and e_3 span a subalgebra, or we can assume that $d(e_1e_2) = e_4$. Since it is well known that the derived subalgebra of any 4 dimensional Lie algebra has dimension at most 3, we will not give a detailed analysis of this issue, and simply point out that the division into subcases can be carried out relatively easily. However, we note that even without breaking up the second case into subcases, we can solve the Jacobi identity using Maple, yielding around 40 solutions all of which have matrices of rank less than or equal to three. We note that the solutions are well defined over any field, so the fact that the derived algebra has dimension 3 is independent of the field $\mathbb{K}$ as well.

4. Extensions of $\mathbb{C}$ by a three dimensional ideal

From now on, in this paper, we shall assume that we are working over the base field $\mathbb{C}$. It is not difficult to classify the moduli space over $\mathbb{R}$ as well. Over fields of finite characteristic, and over other fields, even the classification of 3 dimensional Lie algebras is quite complicated.

Since the dimension of the derived algebra is never more than 3, every 4 dimensional Lie algebra is given as an extension of $\mathbb{C}$ by some three dimensional ideal. In [6], a complete classification of three dimensional algebras and their cohomology was given. We summarize the results about the cohomology in Table 1. Here we have realigned the family

Type	Codiff	H^1	H^2	H^3
$d_1 = \mathfrak{n}_3$	ψ_1^{23}	4	5	2
$d_2 = \mathfrak{r}_{3,1}(\mathbb{C})$	$\psi_1^{13} + \psi_2^{23}$	3	3	0
$d_2(1:1) = \mathfrak{r}_3(\mathbb{C})$	$\psi_1^{13} + \psi_1^{23} + \psi_2^{23}$	1	1	0
$d_2(\lambda:\mu) = \mathfrak{r}_{3,\mu/\lambda}(\mathbb{C})$	$\psi_1^{13}\lambda + \psi_1^{23} + \psi_2^{23}\mu$	1	1	0
$d_2(1:0) = \mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}$	$\psi_1^{13} + \psi_1^{23}$	2	1	0
$d_2(1:-1) = \mathfrak{r}_{3,-1}(\mathbb{C})$	$\psi_1^{13} + \psi_1^{23} - \psi_2^{23}$	1	2	1
$d_3 = \mathfrak{sl}_2(\mathbb{C})$	$\psi_3^{12} + \psi_2^{13} + \psi_1^{23}$	0	0	0

Table 1. Cohomology of Three Dimensional Algebras

of codifferentials as presented in [6] in order to identify elements which have the same cohomological type as belonging to the same family. The changes are actually modest: the family $d_2(\lambda:\mu)$ coincides with $d(\mu/\lambda)$

of that paper except that the new element d_2 was given as $d(1)$ in the paper, and the element $d(1:1)$ corresponds to the element d_2 in the previous paper. In addition, we have introduced projective notation for the family $d_2(\lambda:\mu)$. It should be noted that $d_2(\lambda:\mu)=d_2(\mu:\lambda)$, so the family can be identified with $\mathbb{P}^1(\mathbb{C})/\Sigma_2$, which makes it an orbifold with orbifold points at $d_2(1:1)$ and $d_2(1:-1)$, where there is some atypical phenomena in the moduli space. At the point $d_2(1:1)$, there is a doppelganger d_2, whose neighborhoods coincide with those of the point $d_2(1:1)$, and which also deforms infinitesimally into $d_2(1:1)$. At the point $d_2(1:-1)$, there is a deformation in the d_3 direction as well as a deformation in the direction of the family. Otherwise, members of the family $d_2(\lambda:\mu)$ deform only in the direction of the family itself. The codifferential d_1 has deformations into every other type of codifferential except d_2, which accounts for why it has such a large dimension of H^2.

In order to determine all the codifferentials of degree 4, it is only necessary to study the equivalence classes of codifferentials given by extending $\mathbb{C}$ by a 3 dimensional algebra, via an outer derivation. For this reason, in Table 1, we have denoted by H^1 the dimension of the outer derivations, unlike our convention in [6]. In most cases, an extension of $\mathbb{C}$ by a 3 dimensional algebra is equivalent to either an extension by the Heisenberg algebra d_1, or an extension by the zero algebra, that is, a three dimensional central extension of $\mathbb{C}$. For each of the types of 3 dimensional algebras in our classification in Table 1, we will analyze the extensions of $\mathbb{C}$, by studying the outer derivations.

Suppose that A is a matrix representing a codifferential d and A' is the matrix representing a codifferential d'. The codifferentials d and d' determine isomorphic Lie algebras, and we call them equivalent codifferentials, if there is a linear automorphism $g:V\to V$ such that $d'=g^{-1}d\tilde{g}$, where $\tilde{g}:\bigwedge^2 V\to\bigwedge^2 V$ is the induced isomorphism. If we represent g by the 4×4 matrix $G=(g^i_j)$, where $g(e_j)=g^i_je_i$, then $\tilde{g}$ is represented by the 6×6 matrix Q, in other words, $\tilde{g}(e_{S(j)})=Q^i_je_{S(i)}$, then the coefficients of Q are given by the formula

$$Q^i_j=g^m_kg^n_l-g^m_lg^n_k,\ \text{where } S(i)=(k,l) \text{ and } S(j)=(m,n).$$

It follows that d is equivalent to d' precisely when there is an invertible matrix G and a corresponding matrix Q such that $A'=G^{-1}AQ$. It is usually easier to check by computer whether there is a matrix G and corresponding Q so that $GA'=AQ$, but then one must be careful to check that $\det(G)\neq 0$.

4.1. **The simple Lie algebra $d_3=\mathfrak{sl}_2(\mathbb{C})$.** Since $\mathfrak{sl}_2(\mathbb{C})$ is simple, all derivations are inner. As a consequence, any extension of $\mathbb{C}$ by $\mathfrak{sl}_2(\mathbb{C})$

is just a direct sum $\mathfrak{sl}_2(\mathbb{C})\oplus\mathbb{C}$. This 4 dimensional algebra is given by the codifferential

$$d_3=\psi^{12}_3+\psi^{13}_2+\psi^{23}_1, \tag{1}$$

which represents the simple algebra $\mathfrak{sl}_2(\mathbb{C})\oplus\mathbb{C}$ in the BS list [2].

4.2. **The solvable Lie algebra $d_2=r_3(\mathbb{C})$.** This algebra is given by the codifferential

$$d_2=\psi^{13}_1+\psi^{23}_2.$$

$H^1(d_2)=\langle\varphi^2_2,\varphi^1_2,\varphi^2_1\rangle$. Thus a generic outer derivation of d_2 is given by $\delta=\varphi^2_2x+\varphi^1_2y+\varphi^2_1z$. An extension of $\mathbb{C}$ by δ is given by the rule $d(e_ie_4)=\delta(e_i)$. We compute

$$d(e_1e_4)=e_2y\quad d(e_2e_4)=e_1z+e_2x\quad d(e_3e_4)=0,$$

so that the general formula for an extension d of $\mathbb{C}$ by d_2 is

$$d=\psi^{13}_1+\psi^{23}_2+\psi^{14}_2y+\psi^{24}_1z+\psi^{24}_2x.$$

When $x^2+4yz\neq0$, d is equivalent to the codifferential

$$d^\sharp_2=\psi^{12}_1+\psi^{34}_3, \tag{2}$$

which represents the Lie algebra $\mathfrak{r}_2\oplus\mathfrak{r}_2$ in the BS list. When $x^2+4yz=0$ and the three parameters are not all 0, then the matrix can be transformed into the matrix of the codifferential

$$d_1(1:0)=\psi^{12}_3+\psi^{13}_3+\psi^{23}_4,+\psi^{14}_4,$$

which represents the Lie algebra $\mathfrak{g}_8(0)$ in the BS list.

4.3. **The solvable algebra $d_2(\lambda:\mu)$.** This algebra is given by the codifferential

$$d_2(\lambda:\mu)=\psi^{13}_1\lambda+\psi^{23}_1+\psi^{23}_2.$$

If we consider the trivial extension of $\mathbb{C}$ by $d_2(\lambda:\mu)$, then $\{e_1,e_2,e_4\}$ span an abelian ideal, so this case reduces to an extension of $\mathbb{C}$ by an abelian ideal. To analyze nontrivial extensions, first note that

$$\begin{aligned}H^1(d_2(1:0))&=\langle\varphi^1_1+\varphi^2_2,\varphi^3_2\rangle\\ H^1(d_2(\lambda:\mu))&=\langle\varphi^1_1+\varphi^2_2\rangle\quad\text{otherwise}\end{aligned}$$

If we extend our codifferential by the derivation $\delta=(\varphi^1_1+\varphi^2_2)x+\varphi^3_2y$, the extended codifferential is

$$d=d_2(\lambda:\mu)+(\varphi^{14}_1+\varphi^{24}_2)x+\varphi^{34}_2y.$$

When $x\neq0$ then if $\lambda=\mu$, the extended codifferential is equivalent to the codifferential $d_1(1:0)$, otherwise it is equivalent to the codifferential $d^\sharp_2$.

When $x = 0$ and $\mu \neq 0$, the codifferential is equivalent to the unextended codifferential which we will identify with the codifferential

$$d_3(\lambda : \mu : 0) = \psi_1^{14}\lambda + \psi_1^{24} + \psi_2^{24}\mu + \psi_2^{34},$$

which represents the Lie algebra $\mathfrak{r}_{3,\mu/\lambda}(\mathbb{C}) \oplus \mathbb{C}$ (unless $\lambda = \mu$, in which case it represents the Lie algebra $\mathfrak{r}_3(\mathbb{C}) \oplus \mathbb{C}$). When $\mu = 0$ and $x = 0$, then if $y \neq 0$, the extended codifferential is equivalent to $d_3(1 : 0 : 0)$, which represents the Lie algebra $\mathfrak{g}_2(0, 0)$, but when $y = 0$, the unextended codifferential is equivalent to the codifferential

$$d_3(0 : 1) = \psi_2^{34} + \psi_3^{34},$$

which represents the Lie algebra $\mathfrak{r}_2(\mathbb{C}) \oplus \mathbb{C}^2$.

4.4. **The Heisenberg Algebra $d_1 = n_3(\mathbb{C})$.** Let $d_1 = \psi_1^{23}$ be the three dimensional Heisenberg algebra. Then

$$H^1(d_1) = \langle \varphi_3^2, \varphi_2^3, \varphi_1^1 + \varphi_2^2, \varphi_1^1 + \varphi_3^3 \rangle$$

so $H^1(d_1)$ is four dimensional. If we consider a generic outer derivation

$$\delta = \varphi_3^2 a + \varphi_2^3 b + (\varphi_1^1 + \varphi_2^2)c + (\varphi_1^1 + \varphi_3^3)d,$$

the term $\psi_3^{24}a + \psi_2^{24}c + \psi_2^{34}b + \psi_3^{34}d + \psi_1^{14}(c + d)$ would be added to d_1 obtain the extended codifferential. If we set $a = a_5^3$, $b = a_6^2$, $c = a_5^2$ and $d = a_6^3$, then we get the extended codifferential

$$d = \psi_1^{23} + \psi_1^{14}(a_5^2 + a_6^3) + \psi_2^{24}a_5^2 + \psi_2^{34}a_6^2 + \psi_3^{24}a_5^3 + \psi_3^{34}a_6^3,$$

with matrix A given by $A = \begin{bmatrix} 0 & 0 & 1 & a_5^2+a_6^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5^2 & a_6^2 \\ 0 & 0 & 0 & 0 & a_5^3 & a_6^3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Let g be the linear transformation whose matrix is $G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & p & r & 0 \\ 0 & q & s & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Let $R = \begin{bmatrix} p & r \\ q & s \end{bmatrix}$, and assume $\det(R) = 1$. Now the matrix Q is given in block form by $Q = \begin{bmatrix} R & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & R \end{bmatrix}$. The matrix of $d' = g^{-1}d\tilde{g}$ is $A' = \begin{bmatrix} 0 & 0 & 1 & a_5^2+a_6^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_5'^2 & a_6'^2 \\ 0 & 0 & 0 & 0 & a_5'^3 & a_6'^3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, where

$$\begin{bmatrix} a_5'^2 & a_6'^2 \\ a_5'^3 & a_6'^3 \end{bmatrix} = R^{-1} \begin{bmatrix} a_5^2 & a_6^2 \\ a_5^3 & a_6^3 \end{bmatrix} R,$$

which means that if $V = \begin{bmatrix} a_5^2 & a_6^2 \\ a_5^3 & a_6^3 \end{bmatrix}$, then similar submatrices give equivalent codifferentials. Note that the a_4^1 coefficient $a_5^2 + a_6^3$ is just the trace of the matrix V, which is invariant under similarity transformations. Therefore, looking at the submatrix V alone, we have the following cases

- $V = \left[\begin{smallmatrix} \lambda & 1 \\ 0 & \mu \end{smallmatrix}\right]$, corresponding to the codifferential

$$d_1(\lambda:\mu) = \psi_1^{23} + \psi_1^{14}(\lambda+\mu) + \psi_2^{24}\lambda + \psi_2^{34} + \psi_3^{34}\mu. \tag{3}$$

This family of codifferentials should be thought of as a projective family, parameterizing $\mathbb{P}^1(\mathbb{C})$. There is an action of Σ_2 on this space which identifies $d_1(\lambda:\mu)$ with $d_1(\mu:\lambda)$. There are two orbifold points under this action: $d_1(1:1)$ and $d_1(1,-1)$. We can reasonably expect something unusual to happen at these orbifold points. In fact, $d_1(1:-1)$ represents the Lie algebra $\mathfrak{g}_7$ on the BS list while for all other values, *i.e.*, when $\lambda+\mu \neq 0$, $d_1(\lambda:\mu)$ represents the Lie algebra $\mathfrak{g}_8\left(\frac{\lambda\mu}{(\lambda+\mu)^2}\right)$.

The diagonal matrix $V = \operatorname{diag}(1,1)$. This is the only nonzero diagonalizable matrix which does not show up in the case above. Its associated codifferential is given by the formula

$$d_1^\sharp = \psi_1^{23} + 2\psi_1^{14} + \psi_2^{24} + \psi_3^{34}, \tag{4}$$

representing the Lie algebra $\mathfrak{g}_6$.

- $V = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Then the extended codifferential is equivalent to

$$d_2^\star = \psi_1^{24} + \psi_2^{34}, \tag{5}$$

representing the Lie algebra $\mathfrak{n}_4(\mathbb{C})$.

- $V = 0$. This is the original, unextended codifferential, which is equivalent to the codifferential

$$d_1 = \psi_1^{24}, \tag{6}$$

representing the Lie algebra $\mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}$.

4.5. **Extensions of $\mathbb{C}$ by an abelian ideal.** Since $H^1(0) = L_1(\mathbb{C}^3)$, the whole 9 dimensional cochain space, an extension of $\mathbb{C}$ by $\mathbb{C}^3$ is given by a matrix of the form $A = \left[\begin{smallmatrix} 0 & 0 & 0 & a_4^1 & a_5^1 & a_6^1 \\ 0 & 0 & 0 & a_4^2 & a_5^2 & a_6^2 \\ 0 & 0 & 0 & a_4^3 & a_5^3 & a_6^3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right]$. If we let $V = \left[\begin{smallmatrix} a_4^1 & a_5^1 & a_6^1 \\ a_4^2 & a_5^2 & a_6^2 \\ a_4^3 & a_5^3 & a_6^3 \end{smallmatrix}\right]$, then any matrix V' which is similar to V up to multiplication by a nonzero constant determines an equivalent codifferential. Since matrices which are constant multiples of each other determine the same codifferential, we can think of the nonequivalent codifferentials as being parameterized projectively. The decomposition of these matrices into distinct equivalence classes is as follows.

- The codifferential

$$d_3(\lambda:\mu:\nu) = \psi_1^{14}\lambda + \psi_1^{24} + \psi_2^{24}\mu + \psi_2^{34} + \psi_3^{34}\nu \tag{7}$$

for $(\lambda:\mu:\nu)\in\mathbb{P}^2(\mathbb{C})/\Sigma_3$, where the action of Σ_3 is given by permutation of the coordinates. These points determine an orbifold with orbifold points occurring along certain lines ($\mathbb{P}^1(\mathbb{C})$) where some of the parameters coincide. It might seem more natural to use diagonal matrices to represent this two parameter family; the choice here is based on cohomological considerations.

The codifferential

$$d_3(\lambda:\mu)=\psi^{14}_1\lambda+\psi^{24}_2\lambda+\psi^{34}_2+\psi^{34}_3\mu \tag{8}$$

for $(\lambda:\mu)\in\mathbb{P}^1(\mathbb{C})$. Here there is no action of the symmetric group.

- The Heisenberg algebra $d_1=\psi^{24}_1$. The only eigenvalue of the matrix is zero, and it has two Jordan blocks. We will see that every point in $d_3(\lambda,\mu)$ is infinitesimally close to this point.
- The solvable algebra d^*_2. The matrix has one Jordan block, with eigenvalue zero.
- The identity matrix determines the codifferential

$$d^*_3=\psi^{14}_1+\psi^{25}_2+\psi^{34}_2, \tag{9}$$

which represents the Lie algebra $\mathfrak{g}_1(1)$.
- The zero algebra $d=0$. Every point is infinitesimally close to this zero point.

We summarize these results and give the Lie bracket operations in standard terminology in the table below.

Type	Brackets
$d_1(\lambda:\mu)$	$[e_2,e_3]=e_3,[e_1,e_4]=(\lambda+\mu)e_1,$
	$[e_2,e_4]=\lambda e_2,[e_3,e_4]=e_2+\mu e_3$
$d_3(\lambda:\mu:\nu)$	$[e_1,e_4]=\lambda e_1,[e_2,e_4]=e_1+\mu e_2,[e_3,e_4]=e_2+\nu e_3$
$d_3(\lambda:\mu)$	$[e_1,e_4]=\lambda e_1,[e_2,e_4]=\lambda e_2,[e_3,e_4]=e_2+\mu e_3$
d_1	$[e_2,e_4]=e_1$
$d^\sharp_1$	$[e_2,e_3]=e_1,[e_1,e_4]=2e_1,[e_2,e_4]=e_2,[e_3,e_4]=e_3$
d^*_2	$[e_1,e_2]=e_1,[e_3,e_4]=e_2$
$d^\sharp_2$	$[e_1,e_2]=e_1,[e_3,e_4]=e_3$
d_3	$[e_1,e_2]=e_3,[e_1,e_3]=e_2,[e_2,e_3]=e_1$
d^*_3	$[e_1,e_4]=e_1,[e_2,e_4]=e_2,[e_3,e_4]=e_3$

TABLE 2. Table of Lie Bracket Operations

5. Comparison with the Burde-Steinhoff and Agaoka Lists

The comparison between the Burde-Steinhoff (BS) list and ours is slightly complicated. On the other hand, our decomposition is essentially the same as Agaoka's list, so we will just note the corresponding element, which is of the form $\mathbf{L}_i(\alpha)$ (see [1]).

5.1. $d_1(\lambda:\mu)=\mathbf{L}_8(\mu/\lambda)=\psi_1^{23}+\psi_1^{14}(\lambda+\mu)+\psi_2^{24}\lambda+\psi_2^{34}+\psi_3^{34}\mu$.

(1) When $\lambda+\mu\neq 0$, then
$$d_1(\lambda:\mu)=\mathfrak{g}_8\left(\tfrac{\lambda\mu}{(\lambda+\mu)^2}\right).$$

(2) When $\lambda+\mu=0$, then we have the codifferential $d_1(1:-1)$ and
$$d_1(1:-1)=\mathfrak{g}_7.$$

5.2. $d_3(\lambda:\mu:\nu)=\mathbf{L}_7(\lambda/\nu,\mu/\nu)=\psi_1^{14}\lambda+\psi_1^{24}+\psi_2^{24}\mu+\psi_2^{34}+\psi_3^{34}\nu$.

(1) When the trace $\lambda+\mu+\nu$ of the matrix V is nonzero and none of the parameters are equal to zero, then
$$d_3(\lambda:\mu:\nu)=\mathfrak{g}_2\left(\tfrac{\lambda\mu\nu}{(\lambda+\mu+\nu)^3},\tfrac{\lambda\mu+\lambda\nu+\mu\nu}{(\lambda+\mu+\nu)^2}\right).$$

(2) When exactly one of the parameters vanishes and the other two are not equal, then
$$d_3(\lambda:\mu:0)=\mathfrak{r}_{3,\mu/\lambda}(\mathbb{C})\oplus\mathbb{C}.$$

(3) When one of the parameters vanishes and the other two are equal we have the special point
$$d_3(1:1:0)=\mathfrak{r}_3(\mathbb{C})\oplus\mathbb{C}.$$

(4) When two of the parameters vanish, then we have the special point
$$d_3(1:0:0)=\mathfrak{g}_2(0,0).$$

(5) When the trace of V is zero, none of the parameters is equal to zero, and the parameters are not the three distinct roots of unity, then we have
$$d_3(\lambda:\mu:-\lambda-\mu)=\mathfrak{g}_3\left(\tfrac{(\lambda^2+\lambda\mu+\mu^2)^3}{(\lambda\mu(\lambda+\mu))^2}\right).$$

(6) When λ, μ and ν are the three distinct cube roots of unity, then
$$d_3(1:-1/2+1/2i\sqrt{3}:-1/2-1/2i\sqrt{3})=\mathfrak{g}_4.$$

5.3. $d_3(\lambda:\mu)=\mathbf{L}_4(\mu/\lambda)=\psi_1^{14}\lambda+\psi_2^{24}\lambda+\psi_2^{34}+\psi_3^{34}\mu$.

(1) When neither of the parameters vanish or are equal, then we have
$$d_3(\lambda:\mu)=\mathfrak{g}_1(\mu/\lambda).$$
(2) When $\mu=0$, then we have the special point
$$d_3(1:0)=\mathfrak{r}_{3,1}(\mathbb{C})\oplus\mathbb{C}.$$
(3) When $\lambda=0$ then we have the special point
$$d_3(0:1)=\mathbf{L}_4(\infty)=\mathfrak{r}_2(\mathbb{C})\oplus\mathbb{C}^2.$$
(4) When $\lambda=\mu$ then we have the special point
$$d(0)=d_3(1:1)=\mathfrak{g}_5.$$

5.4. **The special cases.**
$$\begin{aligned}
d_1&=\mathbf{L}_1=\mathfrak{n}_3(\mathbb{C})\oplus\mathbb{C}=\psi_1^{24}\\
d_1^\sharp&=\mathbf{L}_5=\mathfrak{g}_6=\psi_1^{23}+2\psi_1^{14}+\psi_2^{24}+\psi_3^{34}\\
d_2^*&=\mathbf{L}_2=\mathfrak{n}_4(\mathbb{C})=\psi_1^{24}+\psi_2^{34}\\
d_2^\sharp&=\mathbf{L}_9=\mathfrak{r}_2(\mathbb{C})\oplus\mathfrak{r}_2(\mathbb{C})=\psi_1^{12}+\psi_3^{34}\\
d_3&=\mathbf{L}_6=\mathfrak{sl}_2(\mathbb{C})\oplus\mathbb{C}=\psi_3^{12}+\psi_2^{13}+\psi_1^{23}\\
d_3^*&=\mathbf{L}_3=\mathfrak{g}_1(1)=\psi_1^{14}+\psi_2^{24}+\psi_3^{34}.
\end{aligned}$$

6. Deformations of the Lie Algebras

For the basic notion of deformations, we refer to [7, 10, 3, 4]. In some previous papers, we considered deformations of L_∞ algebras [5, 6]. In this paper, we will only consider Lie algebra deformations of our Lie algebras, which are determined by cocycles coming from H^2. We will not explore L_∞ deformations of the Lie algebras we study in this paper, but it would not be difficult to construct them from the cohomology computations we provide here.

In Table 3, we give a classification of the codifferentials according to their cohomology. Note that for the most part, elements from the same family have the same cohomology. In fact, the decomposition of the codifferentials into families was strongly influenced by the desire to associate elements with the same pattern of cohomology in the same family. This is why our family $d_3(\lambda:\mu:\nu)$ was not chosen to be the diagonal matrices. Similar considerations influenced our selection of the family $d_3(\lambda:\mu)$.

Type	H^1	H^2	H^3	H^4
d_3	1	0	1	1
$d_2^\sharp$	0	0	0	0
$d_1(1:-1)$	2	2	2	1
$d_1(1:0)$	1	2	1	0
$d_1(\lambda:\mu)$	1	1	0	0
$d_1^\sharp$	3	3	0	0
$d_3(1:-1:0)$	3	5	5	2
$d_3(\lambda:\mu:\lambda+\mu)$	2	3	1	0
$d_3(\lambda:\mu:0)$	3	3	1	0
$d_3(\lambda:\mu:-\lambda-\mu)$	2	2	1	1
$d_3(\lambda:\mu:\nu)$	2	2	0	0
$d_3(1:0)$	5	7	3	0
$d_3(0:1)$	6	6	2	0
$d_3(1:2)$	4	5	1	0
$d_3(1:-2)$	4	4	1	1
$d_3(\lambda:\mu)$	4	4	0	0
d_1	8	13	10	3
d_2^*	4	6	5	2
d_3^*	8	8	0	0

TABLE 3. Table of the Cohomology

6.1. **The codifferential $d_3=\mathfrak{sl}_2(\mathbb{C})\oplus\mathbb{C}$.** It is an easy calculation to show that

$$\begin{aligned}
H^1 &= \langle\varphi_4^4\rangle\\
H^2 &= 0\\
H^3 &= \langle\varphi_4^{123}\rangle\\
H^4 &= \langle\varphi_4^{1234}\rangle
\end{aligned}$$

Since H^2 vanishes, this algebra is rigid in terms of deformations in the Lie algebra sense.

6.2. **The codifferential $d_2^\sharp=\mathfrak{r}_2(\mathbb{C})\oplus\mathfrak{r}_2(\mathbb{C})$.** Since the cohomology vanishes entirely, this algebra has no interesting deformations or extensions. This algebra is the only four dimensional Lie algebra which is truly rigid in the L_∞ algebra sense, although d_3 is also rigid in the Lie algebra sense.

6.3. **The codifferential** $d_1(\lambda : \mu)$**.** In the generic case we have

$$H^1 = \langle 2\varphi_1^1 + \varphi_2^2 + \varphi_3^3 \rangle$$
$$H^2 = \langle \psi_1^{14} + \psi_2^{24} \rangle$$

and all higher cohomology vanishes. Thus, generically, the infinitesimal deformation is given by

$$d^{\,\mathrm{inf}} = d_1(\lambda + t, \mu). \tag{10}$$

Since $d^{\,\mathrm{inf}}$ is actually a member of the family $d_1(\lambda : \mu)$, it is immediate that $[d^{\,\mathrm{inf}}, d^{\,\mathrm{inf}}] = 0$, so the infinitesimal deformation is the miniversal deformation d^∞ and the base of the miniversal deformation is $\mathbb{C}[[t]]$. Moreover, it is transparent in this case that the deformations run in the direction of the family.

6.4. **The codifferential** $d_1(1 : -1)$**.** For this special case there are more cohomology classes than in the generic case. We have

$$H^1 = \langle 2\varphi_1^1 + \varphi_2^2 + \varphi_3^3, \varphi_1^4 \rangle$$
$$H^2 = \langle \psi_1 = \psi_1^{14} + \psi_2^{24}, \psi_2 = \psi_4^{23} \rangle$$
$$H^3 = \langle \varphi_4^{123}, \varphi_1^{123} + \varphi_4^{234} \rangle$$
$$H^4 = \langle \psi_4^{1234} \rangle.$$

Consider the universal infinitesimal deformation

$$d^{\,\mathrm{inf}} = d_1(1 : -1) + \psi_1 t^1 + \psi_2 t^2.$$

Then we have $\frac{1}{2}[d^{\,\mathrm{inf}}, d^{\,\mathrm{inf}}] = -(\varphi_1^{123} + \varphi_4^{234})t^1t^2$, which is a nontrivial cocycle. It follows that the infinitesimal deformation is miniversal, and the base of the miniversal deformation is $\mathcal{A} = \mathbb{C}[[t^1, t^2, t^3]]/(t^1t^2)$.

When $t^1t^2 \neq 0$, the miniversal deformation $d^\infty = d^{\,\mathrm{inf}}$ does not correspond to an actual deformation. The cohomology class of the cocycle $(\varphi_1^{123}+\varphi_4^{234})$ is called an obstruction to the extension of the infinitesimal deformation to higher order. In order to obtain an actual deformation out of the miniversal deformation, we need to restrict ourselves to the lines $t^1 = 0$ or $t^2 = 0$, along which the obstruction term vanishes. The cohomology H^2, which gives the tangent space to the moduli space, has dimension 2, but the deformations actually lie on two curves. Thus the dimension of the tangent space does not reveal the complete situation in terms of the deformations; one needs to construct the versal deformation to get the true picture.

A deformation along the line $t^2 = 0$ gives $d_1(1+t_1, : -1+t_1)$, the same pattern as we observed generically. On the other hand, a deformation along the line $t^1 = 0$ yields a coderivation which is equivalent to the codifferential d_3 . This is an example of a jump deformation, because

$d_1(1:-1)+\psi_2t^2 \sim d_3$ for all values of t^2. In the classical language of Lie brackets, we get the following bracket table:

$$[e_1,e_3]=e_1+t^2e_4,\quad [e_2,e_4]=e_2,\quad [e_3,e_4]=e_2-e_3$$

Both orbifold points in the family $d_1(\lambda:\mu)$ have some unusual features. The point $d_1(1:-1)$, because it has a jump deformation out of the family to d_3, and the point $d_1(1:1)$ because, as we will see shortly, there is a jump deformation to it from the element $d_1^\sharp$, which lies outside of the family. What is surprising is that the point $d_1(1:0)$, which is not an orbifold point, is also special.

6.5. **The codifferential** $d_1(1:0)$**.** The cohomology H^1 is the same as the generic case, while H^2 and H^3 are given by

$$H^2=\langle \psi_1=\psi_1^{14}+\psi_3^{34},\psi_2=\psi_2^{13}\rangle$$
$$H^3=\langle \phi=\varphi_2^{134}\rangle.$$

The universal infinitesimal deformation $d^{\,\mathrm{inf}}=d_1(1:0)+\psi_it^i$ satisfies $\frac{1}{2}[d^{\,\mathrm{inf}},d^{\,\mathrm{inf}}]=-2\phi t^1t^2$, so it is miniversal and the base of the versal deformation is $\mathbb{C}[[t^1,t^2]]/(t^1t^2)$. Along the line $t^2=0$, $d^\infty=d_1(1,t^1)$, so we deform along the family as in the generic case.

Along the line $t^1=0$, the deformation $d^{\,\mathrm{inf}}$ is equivalent to $d_2^\sharp$ for all values of t^2. Thus $d_1(1:0)$ has a jump deformation to the element $d_2^\sharp$. The classical form of the Lie brackets for the case $t^1=0$ is

$$[e_1,e_3]=t^2e_2,[e_2,e_3]=e_1,[e_1,e_4]=e_1,[e_2,e_4]=e_2,[e_3,e_4]=e_2$$

6.6. **The codifferential** $d_1^\sharp$**.** The cohomology is given by

$$H^1=\langle \varphi_1^1+\varphi_2^2,\varphi_3^2,\varphi_2^3\rangle$$
$$H^2=\langle \psi_1=\psi_3^{24},\psi_2=\psi_2^{34},\psi_3=\psi_1^{14}+\psi_3^{34}\rangle$$

The universal infinitesimal deformation $d^{\,\mathrm{inf}}=d_1^\sharp+\psi_it^i$ is miniversal since $[d^{\,\mathrm{inf}},d^{\,\mathrm{inf}}]=0$, so the base of the miniversal deformation is just $\mathcal{A}=\mathbb{C}[[t^1,t^2,t^3]]$.

Now let us consider which codifferential $d^\infty=d^{\,\mathrm{inf}}$ is equivalent to. Even though the deformation is defined for all values of the parameters, which element we deform to depends in a complicated manner on the parameters.

Except on the plane $t^1=0$, we have $d^\infty \sim d_1(\alpha:\beta)$ where

$$(\alpha,\beta)=2+t^3\pm\sqrt{(t^3)^2+4t^1t^2}.$$

On the plane $t^1=0$, we have $d^\infty\sim d_1(1+t^3:1)$. In particular, if $t^3=0$, we have $d^\infty\sim d(1:1)$. In fact, along the entire surface given by $(t^3)^2+4t^1t^2=0$, we have $d^\infty\sim d(1:1)$, so there is a two parameter family of jump deformations to $d_1(1:1)$.

Thus $d_1^\sharp$ has a jump deformation to $d_1(1:1)$ and deforms along the family $d_1(\alpha:\beta)$ as if it were the element $d_1(1:1)$ in this family. This is a pattern which will always emerge: *If a codifferential has a jump deformation to another codifferential, then it will deform also to every codifferential to which the element it jumps to deforms.*

We give the classical form of the Lie brackets for d^∞:

$$[e_2,e_3]=e_1,[e_1,e_4]=(2+t^3)e_1,[e_2,e_4]=e_2+t^1e_3,[e_3,e_4]=t^2e_2+(1+t^3)e_3.$$

Let us consider the picture including only the codifferentials d_3, $d_1^\sharp$ and $d_1(\lambda:\mu)$. The picture is very similar to that of the moduli space of three dimensional Lie algebras. The family $d_1(\lambda:\mu)$ consists of a $\mathbb{P}^1$ with an action of the symmetric group Σ_2, with orbifold points at $(1:1)$ and $(1:-1)$. The point $d_1(1:-1)$ has a jump deformation to d_3, the four dimensional simple Lie algebra, while there is a jump deformation from the point $d_1^\sharp$ to $d_1(1:1)$. The point $d_1(1:0)$ is not an orbifold point, but is still special, with a jump deformation to the point $d_2^\sharp$. We did not see this phenomenon in the 3 dimensional picture, but there was something special about the point $d_2(1:0)$ in the family of codifferentials $d_2(\lambda:\mu)$ (see Table 1), because $\dim(H^1(d_2(1:0)))=2$, instead of the generic value. Since H^1 influences the extensions of $\mathbb{C}$ by a Lie algebra, it is perhaps natural to expect that the 4 dimensional counterpart to $d_2(1:0)$ should not behave generically.

6.7. **The codifferential $d_3(\lambda:\mu:\nu)$.** Before examining the cohomology in the generic case, we want to make some general remarks about the family $d_3(\lambda:\mu:\nu)$, which we will call *the big family*, relating to the fact that the points correspond to $\mathbb{P}^2/\Sigma_3$, in contrast to $d_3(\lambda:\mu)$, which we will refer to as *the small family*. Let us refer to elements in the orbit of a point under the action of the symmetric group as conjugates. Most points in $\mathbb{P}^2$ have precisely 6 conjugates, and the stabilizer of the point is the trivial subgroup. The few exceptional cases are as follows.

(1) The points $(\lambda:\lambda:\mu)$, where $\lambda\neq\mu$ and their conjugates are stabilized by a subgroup of order 2, so they each have only 3 conjugates.
(2) The point $(1:-1:0)$ and its 3 conjugates are also stabilized by a subgroup of order 2.
(3) The point $(1:r:r^2)$, where r is a primitive cube root of unity, and its 2 conjugates, are stabilized by the alternating group A_3.
(4) The point $(1:1:1)$ is stabilized by the entire group Σ_3.

Next, consider the lines ($\mathbb{P}^1$) in $\mathbb{P}^2$ and the induced action of Σ_3 on the set of lines. For most lines, the stabilizer of the line is just the trivial subgroup, but again, there are a few exceptions.

(1) The line $(\lambda : \lambda : \mu)$ and its 3 conjugates are stabilized by subgroups of order 2.
(2) The lines $(\lambda : \mu : c(\lambda + \mu))$ and their conjugates are also stabilized by subgroups of order 2.

It turns out that when $c = 0$ or $c = \pm 1$, the cohomology of the codifferentials corresponding to points on the line $(\lambda : \mu : c(\lambda+\mu))$ does not follow the generic pattern. The cohomology of the codifferentials corresponding to points on the line $(\lambda : \lambda : \mu)$ is the same as the generic case with the exception of the points $(1 : 1 : 0)$, $(1 : 1 : 2)$ and $(1 : 1 : -2)$, which are the points of intersection of this line with the three other special lines. Note also that the lines $(\lambda : \mu : c(\lambda + \mu))$ all intersect in precisely the point $(1 : -1 : 0)$, which makes this point very special.

To determine the cohomology of a codifferential of type $d_3(\lambda : \mu : \nu)$, read Table 3 in descending order, and whichever is the first pattern it matches, that gives its cohomology. However, we will present the description of the cohomology in ascending order, because it is more natural to begin with the generic pattern, and then proceed to the more exotic cases.

6.8. **The codifferential** $d_3(\lambda : \mu : \nu)$**: the generic case.** Generically, we have

$$\begin{aligned} H^1 =&\langle\varphi^1_1(\lambda-\mu)+\varphi^2_1+\varphi^2_2(\mu-\nu)+\varphi^3_2,\\ &\varphi^1_1(-\lambda\mu+\lambda^2-\lambda\nu+\mu\nu)+\varphi^2_1(\lambda-\nu)+\varphi^3_1\rangle. \end{aligned}$$

For most generic values of $(\lambda : \mu : \nu)$ a natural basis to choose for H^2 would be $H^2 = \langle\psi^{24}_2, \psi^{34}_3\rangle$. Then $d^{\,\mathrm{inf}} = d_3(\lambda, \mu + t^1, \nu + t^2)$, so there is no difficulty in seeing what the deformations are equivalent to. However, for certain generic values of the parameters, the two cocycles above are not a basis of H^2, so we need to work with a more complex solution, which yields a basis of H^2 for all generic values. Let us take

$$H^2 = \langle\psi_1 = \psi^{24}_3, \psi_2 = \psi^{14}_3\rangle.$$

The universal infinitesimal deformation $d^{\,\mathrm{inf}} = d_3(\lambda : \mu : \nu) + \psi_i t^i$ is miniversal, with base $\mathcal{A} = \mathbb{C}[[t^1, t^2]]$. It is a bit more difficult to identify what the miniversal deformation $d^\infty = d^{\,\mathrm{inf}}$ is equivalent to when we take this more complicated basis of H^2. In fact, if we let x be a root of the polynomial

$$z^3 + (-\nu + 2\lambda - \mu)z^2 + (\mu\nu - \lambda\nu - \lambda\mu + \lambda^2 - t^1)z - t^2,$$

and y be a root of the polynomial

$$z^2 + (-x - \nu + \mu)z + x^2 + x(\lambda - \mu) - t^1,$$

then if g is given by the matrix $G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x(\lambda-\mu)+x^2 & y & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, we have

$$g^*(d^\infty) = d_3(\lambda + x : \mu + y - x : \nu - y).$$

Thus for generic values of $(\lambda : \mu : \nu)$, all deformations of $d_3(\lambda : \mu : \nu)$ simply move along this same big family.

6.9. **The codifferential** $d_3(\lambda : \mu : -\lambda - \mu)$**.** In this case H^1 is the same as the generic case, and for most values of μ, one can use the cocycles ψ_1^{14} and ψ_2^{24} as the basis of H^2. Since the brackets of these two cocycles vanish, the resulting infinitesimal deformation

$$d^{\,\mathrm{inf}} = d_3(\lambda : \mu : -\lambda - \mu) + \psi_1^{14} t^1 + \psi_2^{24} t^2$$

is miniversal, and in fact coincides with $d_3(\lambda + t^1, \mu + t^2, -\mu - \lambda)$. In this case, it is obvious that the deformations of our codifferential just lie along the big family. The values for which these two elements do not form a basis of H^2 are $(1 : -1 : 0)$, which we will cover separately, and $(1 : 1 : -2)$, for which $\psi_1 = \psi_3^{14}$ and $\psi_2 = \psi_2^{24}$ give a basis of H^2. It is still true that the brackets of these cocycles vanish, and the deformations lie along the big family, although the expression of the member of the family corresponding to the element d^∞ is more complicated in this case, and will be omitted.

Thus the family $d_3(\lambda : \mu : -\lambda - \mu)$ is not special in deformation theory. This is a bit surprising, since H^3 does not vanish for elements of this subfamily, so it would not have been unreasonable to expect that there would be some obstructions to the extension of an infinitesimal deformation.

6.10. **The codifferential** $d_3(\lambda : \mu : 0) = \mathfrak{r}_{3,\mu/\lambda}(\mathbb{C}) \oplus \mathbb{C}$**.** . The dimensions of H^1 and H^2 increase to 3, and H^3 is 1-dimensional as well. The two cocycles ψ_1 and ψ_2 chosen as basis elements for H^2 in the generic case remain nontrivial and one can find an independent nontrivial cocycle $\psi_1^{13} + \psi_2^{23}$. However, this choice of a basis turns out to be inconvenient, and a slight modification of the basis will make the presentation simpler. We have

$$\begin{aligned} H^1 =& \langle \varphi_1^1 \lambda(\lambda - \mu) + \varphi_1^2 \lambda + \varphi_1^3, \varphi_1^1 + \varphi_2^2 + \varphi_3^3, \varphi_3^4 \rangle \\ H^2 =& \langle \psi_1 = \psi_3^{24} + \psi_3^{14} \lambda, \psi_2 = \psi_3^{14}, \psi_3 = \psi_1^{13} + \psi_2^{23} \rangle. \end{aligned}$$

Let $d^{\,\mathrm{inf}} = d_3(\lambda:\mu:0) + \psi_i t^i$. We compute

$$[\psi_1, \psi_3] = -\varphi_1^{124} + \varphi_3^{234} + \varphi_3^{134}\lambda + \varphi_2^{124}\lambda = -D(\psi_2^{12} + \psi_3^{13})$$
$$[\psi_2, \psi_3] = \varphi_3^{134} + \varphi_2^{124},$$

so that

$$\tfrac{1}{2}[d^{\,\mathrm{inf}}, d^{\,\mathrm{inf}}] = -D(\zeta_1)t^1t^3 + (\varphi_3^{134} + \varphi_2^{124})t^2t^3,$$

where $\zeta_1 = \psi_2^{12} + \psi_3^{13}$.

Note that in the case $t^3 = 0$, since ψ_1 and ψ_2 span the same subspace as the ones we used in the generic case, a deformation with $t^3 = 0$ is equivalent to one in the family. Thus there is a two parameter family of deformations of $d_3(\lambda:\mu:0)$ along the big family $d_3(\alpha:\beta:\eta)$.

On the other hand, when t^3 does not vanish, we will have to consider how $d^{\,\mathrm{inf}}$ extends to a higher order deformation. It turns out that when $\lambda = \mu$, the codifferential $\varphi_3^{134} + \varphi_2^{124}$ is a coboundary, but otherwise, it can be taken as a basis of H^3, and so is an obstruction to the extension of $d^{\,\mathrm{inf}}$ to a higher order deformation. We will first consider this obstructed case.

6.10.1. $\mu \neq \lambda$. In this case we have

$$H^3 = \langle \phi = \varphi_3^{134} + \varphi_2^{124} \rangle.$$

We extend $d^{\,\mathrm{inf}}$ to the second order deformation

$$d^2 = d^{\,\mathrm{inf}} + \zeta_1 t^1 t^3.$$

Since $\frac{1}{2}[d^2, d^2] = \phi t^2 t^3$, the second order deformation is miniversal and the base of the miniversal deformation is $\mathcal{A} = \mathbb{C}[[t^1, t^2, t^3]]/(t^2t^3)$.

Thus any true deformation is given by taking $d^\infty = d^2$ with either $t^2 = 0$ or $t^3 = 0$. Since the case $t^3 = 0$ has already been examined, we consider the case $t^2 = 0$. In this case, the matrix A of the deformation d^∞ is given by $A = \begin{bmatrix} 0 & t^2 & 0 & \lambda & 1 & 0 \\ t^1t^3 & 0 & t^3 & 0 & \mu & 1 \\ 0 & t^1t^3 & 0 & \lambda t^1 & t^1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. When $t^1 = -\frac{(\lambda-\mu)^2}{4}$, then $d^\infty \sim d_1(1:0)$. Since $\lambda \neq \mu$, note that this deformation is not a jump deformation of $d_3(\lambda:\mu:0)$ but occurs some "distance" away from the original codifferential.

When $t^1 \neq -\frac{(\lambda-\mu)^2}{4}$, then $d^\infty = d_2^\sharp$. This is a jump deformation, since it is independent of the value of t^1, as long as it is small.

Thus we obtain that the deformations of $d_3(\lambda:\mu:0)$, for $\lambda \neq \mu$, live along two planes in the (t^1, t^2, t^3) space. One is the plane $t^3 = 0$ determining deformations along the family $d_3(\lambda:\mu:\nu)$, while those which lie in the plane $t^2 = 0$ are equivalent to $d_2^\sharp$, except along the line $t^1 = -\frac{(\lambda-\mu)^2}{4}$, which is not important to us, because this line does not include the origin. We say that a family of deformations is *not* *local* if the origin in the t-parameter space is not part of the family. Thus the deformations along the line $t^1 = -\frac{(\lambda-\mu)^2}{4}$ are not local, in this sense. Only local deformations play a role in determining how the moduli space is glued together.

6.10.2. $\mu = \lambda$. This is the codifferential $d_3(1:1:0) = \mathfrak{r}_3(\mathbb{C}) \oplus \mathbb{C}$. We have

$$[\psi_2, \psi_3] = \varphi_3^{134} + \varphi_2^{124} = -D(\zeta_2),$$

where $\zeta_2 = -\psi_3^{23} + \psi_3^{13} + \psi_4^{24} + \psi_2^{12}$. So the second order deformation is given by

$$d^2 = d^{\,\mathrm{inf}} + \zeta_1 t^1 t^3 + \zeta_2 t^2 t^3.$$

Since $\varphi = \varphi_3^{124}$ is a nontrivial 3-cocycle, we can take

$$H^3 = \langle \phi = \varphi_3^{124} \rangle.$$

Now

$$\tfrac{1}{2}[d^2, d^2] = -2\phi t^2 t^3 (t^1 + t^3) - D(\zeta_3) t^2 (t^3)^2 + (\varphi_4^{124} - \varphi_3^{123}) t^2 (t^3)^2 (t^1 + t^2),$$

where $\zeta_3 = \psi_4^{23}$. We did not obtain any second order relations, but because of the term involving φ in the bracket above, there is a third order relation $t^2 t^3 (t^1 + t^2)$. The last term in the bracket is of higher order, so can be ignored in computing the third order deformation. We can take

$$d^3 = d(1:1:0) + \psi_i t^i + \zeta_1 t^1 t^3 + \zeta_2 t^2 t^3 + \zeta_3 t^2 (t^3)^2.$$

One computes that

$$\begin{aligned}\tfrac{1}{2}[d^3, d^3] = &-2\phi t^2 t^3 (t^1 + t^3) + 2(\varphi_4^{124} - \varphi_3^{123}) t^2 (t^3)^2 (t^1 + t^2) \\ &+ 2\varphi_4^{123} t^2 (t^3)^3 (t^1 + t^2).\end{aligned}$$

But this term is equal to zero, using the third order relation. Thus the base of a versal deformation is $\mathcal{A} = \mathbb{C}[[t^1, t^2, t^3]]/(t^2 t^3 (t^1 + t^2))$, $d^\infty = d^3$, and the formal deformation corresponds to an actual deformation along the three planes $t^3 = 0$, $t^2 = 0$ and $t^2 = -t^1$.

The plane $t^3 = 0$ corresponds to the generic case, which gives a 2 parameter space of deformations along the family $d_3(\lambda : \mu : \nu)$.

Now consider the plane spanned by $t^2 = 0$. If neither t^1 nor t^3 vanish, we then have $d^\infty \sim d_2^\sharp$, so there is a jump deformation from $d_3(1:1:0)$ to $d_2^\sharp$, just as for the other points on the line $d_3(\lambda : \mu : 0)$.

When $t^2 = t^3 = 0$, we are on the plane $t^3 = 0$, which we discussed already. When $t^2 = t^1 = 0$, then we get a jump deformation to the point $d_1(1:0)$. This is not like the generic case.

Finally, let us consider the case when $t^2 = -t^1$ and $t^3 \neq 0$. Let us express $t^1 = \frac{\alpha\beta}{(\alpha+\beta)^2}$, and let $x = \frac{\alpha+\beta}{t^3}$. Then $g^*(d^3) = d_1(\alpha, \beta)$ if

g is given by the matrix $G = \begin{bmatrix} -x & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \\ 0 & \frac{\beta}{t^3} & -\frac{1}{t^3} & x \\ 0 & -\beta & 1 & 0 \end{bmatrix}$. Note that $(\alpha : \beta)$ is independent of t^3 as long as t^3 is nonzero. On the plane $t^2 = -t^1$, we see that for $t^1 = 0$, the deformation jumps to $d_1(1 : 0)$, but when $t^1 \neq 0$, we deform along the family $d_1(\alpha : \beta)$. It is as if $d_3(1 : 1 : 0)$ sits just above the point $d_1(1 : 0)$ and deforms as if it were that element. This is the usual pattern we have already discussed when there is a jump deformation to a point.

6.11. **The codifferential $d_3(\lambda : \mu : \lambda+\mu)$.** We exclude the codifferential $d_3(1 : 0 : 1)$ from our consideration here, because it coincides with the codifferential $d_3(1 : 1 : 0)$, which we treated previously. H^1 is the same as the generic case, and we have

$$\begin{aligned} H^2 = \langle \psi_1 =& 2\psi_1^{12}\lambda\mu(\lambda+\mu)^2 + \psi_2^{12}\lambda\mu^2(\lambda+\mu)(\lambda+2\mu) \\ &+ \psi_3^{12}\lambda^2\mu^2(\lambda+\mu)^2 - \psi_1^{13}\mu(\mu^2+2\lambda\mu+2\lambda^2) \\ &- \psi_2^{13}\mu^2(\lambda+\mu) + \psi_1^{23}(\mu^2+\lambda\mu+\mu^2) \\ &+ (\psi_1^{14}+\psi_2^{23})\lambda^2\mu^2(\lambda+\mu) \\ \psi_2 =& \psi_1^{14} \\ \psi_3 =& \psi_2^{14}\rangle. \end{aligned}$$

The brackets of ψ_1 with itself and ψ_3 are coboundaries, its bracket with ψ_2 is a nontrivial cocycle, and the rest of the brackets vanish. From this, one sees immediately that the second order relation is $t^1t^2 = 0$, but it is not so obvious what higher order terms might be necessary to add in order to obtain the relation on the base of the miniversal deformation. Since the space of 3-cocycles is 12 dimensional, we know that a miniversal deformation can be expressed in the form

$$d^\infty = d_3(\lambda, \mu, \lambda+\mu) + \psi_i t^i + \zeta_i x^i,$$

where $\zeta_1, \dots, \zeta_{11}$ is a pre-basis of the space of 3-coboundaries. In fact, we can give this pre-basis as

$$\{\zeta_i, i = 1, \dots 11\} = \{\psi_1^{12}, \psi_2^{12}, \psi_3^{12}, \psi_4^{12}, \psi_1^{13}, \psi_2^{13}, \psi_3^{13}, \psi_4^{13}, \psi_1^{23}, \psi_2^{23}, \psi_4^{23}\}.$$

Note that the first 10 of these vectors are just the first 10 elementary 2-cochains. Also

$$H^3 = \langle \phi = \varphi_3^{124} \rangle,$$

and we can complete the linearly independent set given by the $D(\zeta_i)$ and ϕ to a basis $\{D(\zeta_1), \dots, D(\zeta_1 1), \phi, \tau_1, \dots \tau_4\}$ of L_3. Then we must have

$$[d^\infty, d^\infty] = D(\zeta_1)s^1 + \cdots + D(\zeta_{11})s^{11} + \phi s^{12} + \tau_1 s^{13} + \cdots \tau_4 s^{16},$$

for some coefficients $s^1, \dots s^{16}$, where these coefficients are expressed as polynomials in the variables t^i and x^i. Now all of these coefficients must be equal to zero, once you take into account the relation on the base of the miniversal deformation, which is the coefficient s^{12}. The expression one obtains for s^{12} by direct computation from the form of d^∞ will have the variables x^i in it, but it should depend only on the variables t^i. The trick is to solve the first 11 equations for x^i as functions of the variables t^i, and then substitute these into the formula for s^{12} to obtain the relation on the base.

The relation on the base of the miniversal deformation is simply $t^1t^2 = 0$, which is exactly the second order relation. If you solve for the coefficients of $s^{13}, \dots, s^{16}$, then they turn out to be multiples of s^{12}, so they are equal to zero using the relation on the base.

Let us study the deformations of $d(\lambda : \mu : \lambda + \mu)$. Since the relation on the base of the miniversal deformation is $t^1t^2 = 0$, in any true deformation, we must have either $t^1 = 0$ or $t^2 = 0$.

When $t^1 = 0$, then $d^\infty = d_3(\lambda : \mu : \lambda + \mu) + \psi_2 t^2 + \psi_3 t^3$, so that for any values of t^2 and t^3 we have a deformation along the big family. In fact, $d^\infty \sim d_3(\alpha : \beta : \eta)$ where

$$\begin{aligned}
\alpha =& \tfrac{1}{2}(\lambda + \mu + t^2 + \sqrt{(t^2 + \lambda - \mu)^2 + 4t^3}) \\
\beta =& \tfrac{1}{2}(\lambda + \mu + t^2 - \sqrt{(t^2 + \lambda - \mu)^2 + 4t^3}) \\
\eta =& \lambda + \mu.
\end{aligned}$$

The interesting case is when $t^2 = 0$. The matrix of d^∞ is quite complicated, so we won't reproduce it here, but it should be noted that some terms have $t^3 - \lambda\mu$ in the denominator, so that $t^3 = \lambda\mu$ may not correspond to an actual deformation. When $t^1 \neq 0$, then $d^\infty \sim \mathfrak{g}_8\left(\frac{\lambda\mu - t^3}{(\lambda+\mu)^2}\right)$. In particular, if we set $t^3 = 0$, we see that there is a jump deformation to $d_1(\lambda : \mu)$, and that we also deform along the family $d_1(\lambda : \mu)$ when $t^3 \neq 0$.

6.12. **The codifferential** $d_3(1 : -1 : 0)$**.** For this codifferential, from the fact that H^2 and H^3 both have dimension 5, we expect to see some interesting phenomena, both because the tangent space to the space of deformations has dimension 5, and since H^3 has high dimension, the dimension of the variety of deformations would likely be lower than 5.

We can give bases for the cohomology as follows:

$$\begin{aligned}
H^1 &= \langle 2\varphi_1^1+\varphi_2^2+\varphi_1^3, \varphi_1^1+\varphi_2^2+\varphi_3^3, \varphi_3^4\rangle\\
H^2 &= \langle \psi_1=\psi^{24}_{,}\psi_2=\psi_3^{14}, \psi_3=\psi_3^{12}-\psi_3^{13}-\psi_3^{23}+\psi_4^{14}+\psi_4^{24}\\
&\qquad \psi_4=\psi_1^{23}-2\psi_2^{23}, \psi_5=\psi_4^{12}-\psi_4^{13}-\psi_4^{23}\rangle\\
H^3 &= \langle \phi_1=\varphi_1^{124}, \phi_2=\varphi_3^{124}, \phi_3=\varphi_2^{123}+\varphi_3^{123}-\varphi_4^{234}\\
&\qquad \phi_4=\varphi_4^{123}, \phi_5=\varphi_2^{123}+\varphi_4^{124}-\varphi_4^{234}\rangle.
\end{aligned}$$

A pre-basis of the 3-coboundaries is

$$\{\zeta_1,\dots,\zeta_9\}=\{\psi_1^{12},\psi_2^{12},\psi_3^{12},\psi_4^{12},\psi_1^{13},\psi_2^{13},\psi_3^{13},\psi_4^{13},\psi_4^{14}\}.$$

A miniversal deformation is given by

$$d^\infty = d_3(1:-1:0)+\psi_i t^i+\zeta_i x^i,$$

where the x^i are expressible as power series in the variables t^i. Since not all of the brackets of the ψ_i vanish, we do not expect that the coefficients x^i are all equal to zero, in general.

We can express

$$[d^\infty,d^\infty]=D(\zeta_i)s^i+\phi_i s^{9+i}+\tau_i s^{14+i},$$

where $D(\zeta_i)$, ϕ_i and τ_i form a basis of L_3. Solving $s^1=\cdots=s^9=0$ for $x^1,\dots,x^9$ in terms of $t^1,\dots,t^9$, and substituting these values of the x^i into the formulas for $s^{10},\dots,s^{14}$, we obtain 5 relations on the base of the versal deformation, the simplest of which is

$$\tfrac{(t^3(t^1)^2+4t^1t^2t^4+4t^2t^4)}{t^1-2}=0.$$

Some of these relations have t^1-1 or t^1-2 as a factor of the denominator, which means that there may not be a solution when t^1 takes on these values. There should be an actual, rather than just a formal power series solution for all values of t^i which make all 5 of the relations vanish. When we solved for the zeros of the relations, we obtained the following 5 solutions:

1) $t^1=t^2=t^4=0$
2) $t^3=t^4=t^5=0$
3) $t^2=t^3=t^5=0$
4) $t^3=t^5=0,\qquad t^1=-1$
5) $t^2=\frac{-t^1(t^1-2)^2}{8}, t^3=\frac{t^4(t^1-2)^2(t^1+1)}{2t^1}, t^5=\frac{(t^4)^2(t^1-2)^2(t^1+1)}{(t^1)^2}$.

Note that each of these solutions is only a 2-dimensional subvariety of the 5-dimensional tangent space.

For the first solution, the matrix of the corresponding d^∞ is

$$A=\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & -1 & 1\\ t^3 & -t^3 & -t^3 & 0 & 0 & 0\\ t^5 & -t^5 & -t^5 & t^3 & t^3 & 0\end{bmatrix}.$$

Along the curve $t^5=(t^3)^2$, $d^\infty\sim d_1(1:-1)$. For all other points on the (t^3,t^5)-plane, $d^\infty\sim d_3$. This fits with our prior observation that there is a jump deformation from $d_1(1:-1)$ to d_3. Thus we have jump deformations from $d_3(1:-1:0)$ to both $d_1(1:-1)$ and d_3.

For the second solution, the matrix of d^∞ is

$$A=\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & -1+t^1 & 1\\ 0 & 0 & 0 & t^2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

Deformations corresponding to this solution give a two parameter family of deformations along the big family $d_3(\lambda:\mu:\nu)$.

For the third solution, the matrix of d^∞ is

$$A=\begin{bmatrix} 0 & -t^4t^1 & t^4 & 1 & 1 & 0\\ 0 & 0 & -2t^4 & 0 & -1+t^1 & 1\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}.$$

When $t^4=0$, this is just a special case of the previous solution, and in fact, in this case $d^\infty=d_3(1:-1+t^1:0)$. Supposing that $t^4\neq 0$, then when $t^1\neq -1$, then $d^\infty\sim d_2^\sharp$ which is a jump deformation.

The fourth solution has $t^1=-1$, which means it is not local, so does not contribute to our picture of the moduli space. Although the solution is interesting, we will omit it here.

For the last solution, which is the most complicated of them all, the first three columns of the matrix of d^∞ are

$$\begin{bmatrix} 0 & t^1t^4 & t^4\\ \frac{t^4(t^1-2)^2(t^1+1)}{4} & 0 & -2t^4\\ \frac{t^4(t^1-2)^4(t^1+1)}{8t^1} & \frac{-t^4(t^1-2)^2(-4+3(t^1)^2)}{8t^1} & \frac{-t^4(t^1-2)^2(t^1+1)}{2t^1}\\ \frac{(t^4)^2(t^1+2)(t^1-2)^4(t^1+1)}{8(t^1)^2} & \frac{-(t^4)^2((t^1)^3-2(t^1)^2+4t^1+8)(t^1-2)^2}{8(t^1)^2} & \frac{-(t^4)^2(t^1+1)(t^1-2)^2}{(t^1)^2}\end{bmatrix}.$$

Note that t^1 appears in the denominator, so cannot vanish for this solution. If $t^4=0$, $t^1=-1$ or $t^1=2$, then the fifth solution coincides with one of the previous four, so we will not consider these cases here. The matrix of A is so complicated that in order to determine which standard form the codifferential is equivalent to three, we first had to transform A into a matrix of an equivalent codifferential which had a simpler matrix. We found that $d^\infty\sim d_1(\alpha:\beta)$ where

$$\alpha=\frac{t^1+\sqrt{5(t^1)^2-16t^1+16}}{2},\quad \beta=\frac{t^1-\sqrt{5(t^1)^2-16t^1+16}}{2}.$$

Note that if we were to set $t^1 = 0$ in the above, we would obtain the codifferential $d_1(1:-1)$, to which we already obtained a jump deformation in the first solution above.

The picture of the local deformations of $d_3(1:-1:0)$ is as follows. First, we can deform along the big family. Secondly, we can deform to $d_2^\sharp$, like any other member of the family $d_3(\lambda:\mu:0)$. Thirdly, like any other member of the family $d_3(\lambda:\mu:\lambda+\mu)$, we have a jump deformation to an element in the family $d_1(\lambda:\mu)$. Because the element we deform to is $d_1(1:-1)$, which has an extra deformation to the element d_3, we can also deform to this element, as well as deforming along the family $d_1(\lambda:\mu)$.

6.13. **The codifferential $d_3(\lambda:\mu)$.** This family does not have an action of the symmetric group, which is important to keep in mind. Generically, H^1 and H^2 are 4 dimensional. The generic basis of H^2 below consists of elements which are linearly independent nontrivial cocycles for generic values of λ and μ except in the special case $\lambda=\mu$, which we will treat separately. Of course, for those values of λ and μ for which $\dim H^2 > 4$, they do not span H^2. Generically, we have

$$H^1 = \langle \psi_2^1, \psi_2^2(\lambda-\mu)+\psi_2^3, \psi_2^2+\psi_3^3, \psi_1^2(\lambda-\mu)+\psi_1^3\rangle$$
$$H^2 = \langle \psi_1=\psi_3^{34}, \psi_2=\psi_2^{14}, \psi_3=\psi_1^{24}, \psi_4=\psi_2^{24}\rangle.$$

All of the brackets of these nontrivial cocycles vanish, so the miniversal deformation is just the first order deformation $d^\infty = d_3(\lambda:\mu)+\psi_i t^i$, and there are no relations on the base. The matrix of d^∞ is given by $A = \begin{bmatrix} 0&0&0&\lambda&t^3&0\\ 0&0&0&t^2&\lambda+t^4&1\\ 0&0&0&0&0&\mu+t^1\\ 0&0&0&0&0&0\end{bmatrix}$.

If $t^3 \neq 0$, then $d^\infty \sim d_3(\alpha,\beta,\eta)$, where

$$\alpha = \lambda + \frac{t^4+\sqrt{(t^4)^2+4t^2t^3}}{2}, \beta = \lambda + \frac{t^4-\sqrt{(t^4)^2+4t^2t^3}}{2}, \eta = \mu + t^1.$$

If $t^1 = t^2 = t^4 = 0$, then $d^\infty \sim d_3(\lambda:\lambda:\mu)$, so there is a jump deformation from $d_3(\lambda:\mu)$ to $d_3(\lambda:\lambda:\mu)$. Thus we see that the codifferential $d_3(\lambda:\mu)$ sits over the codifferential $d_3(\lambda:\lambda:\mu)$ and deforms along the big family as if it were that codifferential. All of the deformations of $d_3(\lambda:\mu)$ which do not lie along the hyperplane $t^3=0$ lie along the big family.

Now, consider the hyperplane $t^3 = 0$. The eigenvalues of the submatrix $B = \begin{bmatrix} \lambda&0&0\\ t^2&\lambda+t^4&1\\ 0&\mu+t^1 \end{bmatrix}$ are λ, $\lambda+t^4$ and $\mu+t^1$. If these eigenvalues are all distinct, then $d^\infty \sim d_3(\lambda:\lambda+t^4:\mu+t^1)$. Otherwise, one of the conditions $t^4=0$, $t^1=\lambda-\mu$, or $t^1-t^4=\lambda-\mu$ holds. Of these conditions, only the first one is local, so we will not consider the other two. Consider the plane $t^3 = t^4 = 0$. Unless $t^2 = 0$ or $t^1 = \lambda - \mu$, d^∞ is still equivalent to $d_3(\lambda : \lambda + t^4 : \mu + t^1)$. Again, the second condition is not local, so we will ignore it. On the line $t^2 = 0$, we have $d^\infty \sim d_3(\lambda : \mu + t^1)$, so we get a deformation along the $d_3(\lambda : \mu)$ family.

To summarize the generic deformation behavior of an element of the family $d_3(\lambda : \mu)$, we have the following picture. First, we can always deform along the family to which an element belongs, so there is a deformation along the family $d_3(\lambda : \mu)$. Secondly, there is a jump deformation to the element $d_3(\lambda : \lambda : \mu)$ in the big family. Whenever there is a jump deformation, then we can deform in any manner in which the element we jump to deforms, and thus there is a deformation along the big family as well. Note that the line $(\lambda : \lambda : \mu)$, which is one of the lines in $\mathbb{P}^2$ with nontrivial stabilizer, is the target of our jump deformations, so the elements $d_3(\lambda : \lambda : \mu)$ are special not in the sense that they have more deformations, but that there are extra deformations to them.

6.14. **The codifferential $d_3(1 : 1)$.** Even though the the dimension of H^2 for this element is the same as the generic case of $d_3(\lambda : \mu)$, we have to use a different basis for H^2 than in the generic case.

$$H^2 = \langle \psi_1 = \psi_3^{14}, \psi_2 = \psi_1^{14}, \psi_3 = \psi_1^{24}, \psi_4 = \psi_3^{24} \rangle.$$

As in the generic case, the brackets of these cocycles all vanish, so the universal infinitesimal deformation is the miniversal deformation d^∞, with matrix $A = \begin{bmatrix} 0&0&0&1+t^2&t^3&0\\ 0&0&0&0&1&1\\ 0&0&0&t^1&t^4&1\\ 0&0&0&0&0&0 \end{bmatrix}$. When $t^1 \neq 0$, we obtain a complicated deformation along the family $d_3(\alpha : \beta : \eta)$. To understand the solution a bit better, when we solve for a matrix transforming A into one representing a codifferential of the form $d_3(\alpha : \beta : \eta)$, we obtain a solution which satisfies

$$\begin{aligned} t^3 &= \frac{1/3(\alpha^3+\beta^3+\eta^3)(t^2+3)^3+(\alpha+\beta+\eta)(t^2+3)p(\alpha,\beta,\eta,t^2)+8(\alpha+\beta+\eta)^3}{t^1(\alpha+\beta+\eta)^3} \\ t^4 &= \frac{1/3(\alpha^2+\beta^2+\eta^2)(t^2+3)^2-1/2(\alpha+\beta+\eta)((t^2)^2+2t^2+3))}{(\alpha+\beta+\eta)^2}, \end{aligned}$$

where p is a polynomial which is homogeneous, quadratic and symmetric in α, β and η and quadratic in t^2. Consequently, when $t^2 \neq -3$, for any values of t^2, t^4 and t^3, we obtain exactly one solution up to the action of the symmetric group, and thus one member of the family $d_3(\alpha : \beta : \eta)$ is determined. This follows since the line $\alpha+\beta+\eta = 0$ intersects the quadric surface determined by the equation for t^4 above in exactly the orbifold points $(1 : \frac{-1+\sqrt{3}}{2} : \frac{-1-\sqrt{3}}{2})$ and $(1 : \frac{-1-\sqrt{3}}{2} : \frac{-1:\sqrt{3}}{2})$, which do not lie on the cubic surface determined by the equation for t^3.

When $t^1 \neq 0$ and $t^2 = t^3 = t^4 = 0$, then $d^\infty \sim d_3(1 : 1 : 1)$, so there is a jump deformation to this element, as we expect from the generic case.

When $t^1 = 0$, then the eigenvalues of the submatrix $B = \begin{bmatrix} 1+t^2 & t^3 & 0 \\ 0 & 1 & 1 \\ t^1 & t^4 & 1 \end{bmatrix}$ are $1 + t^2$ and $1 \pm \sqrt{t^4}$, so they are distinct unless $t^4 = 0$ or $t^4 = (t^2)^2$. Thus, except in these two cases we have $d^\infty \sim d_3(1+t^2, 1+\sqrt{t^4}, 1-\sqrt{t^4})$. On the plane $t^1 = t^4 = 0$ we have $d^\infty \sim d_3(1 + t^2, 1, 1)$.

On the surface $t^1 = 0$, $t^4 = (t^2)^2$ except on the curve $t^3 = 0$ we have $d^\infty \sim d_3(1 + t^2, 1 + t^2, 1 - t^2)$. Finally, on the curve $t^3 = 0$ on this surface we have $d^\infty \sim d_3(1 + t^2, 1 - t^2)$, so we obtain a deformation along the family $d_3(\lambda : \mu)$ on this curve.

Thus, just like any other generic value, there is one curve along which there is a jump deformation to the corresponding point $d_3(1 : 1 : 1)$ on the large family, another curve along which we deform along the $d_3(\lambda : \mu)$ family, and otherwise, all deformations are along the big family. In a way, it is surprising that the one point in $\mathbb{P}^2$ which is fixed by every permutation does not have any special properties in terms of deformation theory, but as we have seen, there just isn't anything particularly special about the deformations of this codifferential.

6.15. **The codifferential $d_3(1 : -2)$.** We have

$$\begin{aligned} H^2 =&\langle \psi_1 = \psi_2^{14}, \psi_2 = \psi_3^{24}, \psi_3 = \psi_3^{34}, \psi_4 = \psi_1^{24} \rangle \\ H^3 =&\langle \varphi_4^{123} \rangle. \end{aligned}$$

Even though $H^3 \neq 0$, it turns out that the brackets of all the ψ's with each other vanish, so the miniversal deformation d^∞ coincides with the infinitesimal deformation, and its matrix is given by $A = \begin{bmatrix} 0 & 0 & 0 & 1 & t^4 & 0 \\ 0 & 0 & 0 & t^1 & 1 & 1 \\ 0 & 0 & 0 & 0 & t^2 & -2+t^3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Because this matrix has no terms on the left hand side, it is natural to guess that the deformations are either along the family $d_3(\alpha : \beta : \eta)$ or the family $d_3(\lambda : \mu)$, with possibly a few exceptional codifferentials.

When $t^1 \neq 0$ we have a solution of the form

$$\begin{aligned} t^3 =& \tfrac{\alpha+\beta+\eta}{q} \\ t^2 =& -\tfrac{(\alpha+\eta-2q)(\alpha+\beta-2q)(\beta+\eta-2q)}{q^2(\alpha+\beta+\eta-3q)} \\ t^4 =& \tfrac{-(\alpha-q)(\beta-q)(\eta-q)}{t^1 q^2(\alpha+\beta+\eta-3q)}, \end{aligned}$$

where q is a nonzero free parameter. These equations are symmetric in α, β and η. If $t^3 \neq 0$, then $\alpha+\beta+\eta \neq 0$, and we can solve the first equation for q and get

$$\begin{aligned} t^2 =& \tfrac{-((\alpha+\beta)t^3-2(\alpha+\beta+\eta))((\beta+\eta)t^3-2(\alpha+\beta+\eta))((\alpha+\eta)t^3-2(\alpha+\beta+\eta)}{(\alpha+\beta+\eta)^3(t^3-3)}\\ t^4 =& \tfrac{-(\alpha t^3-(\alpha+\beta+\eta))(\beta t^3-(\alpha+\beta+\eta))(\eta t^3-(\alpha+\beta+\eta))}{t^1(t^3-3)(\alpha+\beta+\eta)^3}. \end{aligned}$$

We can express these equations in the form

$$\begin{aligned} t^2 =& \tfrac{\alpha\beta\eta(t^3)^3+(\alpha+\beta+\eta)(t^3-2)p(\alpha,\beta,\eta,t^3)}{(\alpha+\beta+\eta)^3(t^3-3)}\\ t^4 =& \tfrac{-\alpha\beta\eta(t^3)^3+(\alpha+\beta+\eta)r(\alpha,\beta,\eta,t^3)}{t^1(t^3-3)(\alpha+\beta+\eta)^3}, \end{aligned}$$

where p and q are homogeneous, quadratic and symmetric in α, β and η. The surfaces represented by these two equations are both cubic, so there are 9 points of intersection. Since every cubic which is given by a symmetric, homogeneous polynomial either contains the line $\alpha+\beta+\eta=0$ or intersects this line in precisely the points $(1:-1:0)$, $(1:0:-1)$ and $(0:1:-1)$, there are six points in the intersection of these two cubics not on this line, which uniquely determine the codifferential $d_3(\alpha:\beta:\eta)$ to which d^∞ is equivalent. The matrix representing the transformation can be chosen with nonzero determinant, as long as $t^3 \neq 3$. The condition $t^3 \neq 0$ can also be overcome, because if we substitute $t^3=0$ in the above, then the problem still has a solution.Thus, whenever $t^1 \neq 0$ and $t^3 \neq 3$, the deformation is equivalent to a member of the family $d_3(\alpha:\beta:\eta)$.

When $t^1=0$, then as long as $t^2 \neq 0$ and $t^3 \neq 1$, $d^\infty \sim d_3(\alpha:\beta:1)$, where $\alpha = \frac{t^3-1+\sqrt{(t^3-3)^2+4t^2}}{2}$ and $\beta = \frac{t^3-1-\sqrt{(t^3-3)^2+4t^2}}{2}$.

When $t^1=t^2=0$ and $t^4 \neq 0$ we have $d^\infty \sim d_3(1:1:t^3-2)$. As a consequence, if we set $t^3=0$, we have a jump deformation from $d_3(1:-2)$ to $d_3(1:1:-2)$. On the other hand, when $t^1=t^2=t^4=0$, then $d^\infty \sim d_3(1:t^3-2)$. When $t^1=0$ and $t^3=1$, then we also have a deformation along the big family. The upshot of all this analysis is that $d_3(1:-2)$ is really not special in terms of deformation theory. It deforms along its own family, jumps to $d_3(1:1:-2)$, and deforms along that family.

6.16. **The codifferential $d_3(1:2)$.** We have

$$\begin{aligned} H^2 &= \langle \psi_1 = \psi^{12}_2 - \psi^{12}_3 + \psi^{13}_2 + \psi^{13}_3, \psi_2 = \psi^{14}_1 + \psi^{34}_3,\\ &\qquad \psi_3 = \psi^{14}_2, \psi_4 = \psi^{34}_3, \psi_5 = \psi^{24}_1 \rangle\\ H^3 &= \langle \varphi^{124}_3 + \varphi^{134}_2 \rangle. \end{aligned}$$

This time, we do have some nonzero brackets, but only those brackets of ψ_1 with ψ_2, ψ_4 and ψ_5, with the first one being a nontrivial cocycle, so that the second order relation is $t^1t^2 = 0$. After some work, one obtains that there is one relation on the base of the versal deformation,

$$t^1t^2(-1 - t^4 + t^3t^5) = 0,$$

so that there are three distinct solutions for a true deformation, given by the three factors of the miniversal deformation. Notice that the third factor does not give rise to a local deformation.

Let us study the first solution, when $t^1 = 0$. This case is simplest. The matrix corresponding to d^∞ is $A = \begin{bmatrix} 0&0&0&1+t^4&t^5&0\\ 0&0&0&t^3&1&1\\ 0&0&0&0&0&2+t^2+t^4\\ 0&0&0&0&0&0 \end{bmatrix}$.

When $t^5 \neq 0$, then d^∞ is equivalent to $d_3(\alpha : \beta : 2 + t^2 + t^4)$, where $\{\alpha, \beta\} = \frac{2+t^4 \pm \sqrt{(t^4)^2+4t^3t5}}{2}$. If $t^2 = t^3 = t^4 = 0$, then the deformation is equivalent to $d_3(1 : 1 : 2)$ for all $t^5 \neq 0$, giving the expected jump deformation.

What happens if $t^5 = 0$? As long as $t^2 \neq -1$ and $t^2t^3 + t^4t^3 + t^3 + t^4 \neq 0$, then the deformation is still along the big family. If $t^2 \neq -1$, but $t^2t^3 + t^4t^3 + t^3 + t^4 = 0$, then as long as $t^3 \neq 0$ and $t^4 \neq 0$, the deformation is in the big family. If $t^4 = 0$, then $t^3 = 0$ or $t^2 = -1$, and in both cases we deform along the family $d_3(\alpha : \beta)$. Thus, the first solution to the relations on the base does not have any surprises.

The second solution to the relations on the base is $t^2 = 0$. We may as well assume that $t^1 \neq 0$ and that $-1 - t^4 + t^3t^5 \neq 0$ for this case. Then the matrix of d^∞ is $A = \begin{bmatrix} -t^1t^5 & \frac{-t^1t^5}{-1-t^4+t^3t^5} & \frac{t^1(t^5)^2}{-1-t^4+t^3t^5} & 1+t^4 & t^5 & 0\\ -t^1 & \frac{-t^1}{-1-t^4+t^3t^5} & \frac{t^1t^5}{-1-t^4+t^3t^5} & t^3 & 1 & 1\\ t^1(-1-t^4+t^3t^5) & t^1 & -t^1t^5 & 0 & 0 & 2+t^4\\ 0&0&0&0&0&0 \end{bmatrix}$.

The submatrix consisting of the first three columns of A has rank 1, so we can transform this matrix into a simpler matrix.

Recall that we assume that $t^1 \neq 0$. When $t^5 \neq 0$, then it turns out that $d^\infty \sim d_1(\alpha : \beta)$, where $\frac{\alpha\beta}{(\alpha+\beta)^2} = \frac{1-t^4-t^3t^5}{(2+t^4)^2}$. Also, if $t^5 = 0$ and $t^3 \neq 0$, then the deformation is equivalent to $d_1(1 + t^4 : 1)$. In particular, if $t^4 = 0$, we see that there is a jump deformation from our codifferential to the codifferential $d_1(1 : 1)$. On the other hand, if $t^3 = 0$ and $t^4 \neq 0$, we also deform to $d_1(1 + t^4 : 1)$. When $t^3 = t^4 = t^5 = 0$, there is a jump deformation of d^∞ to $d_1^\sharp$.

The picture for this element is more intriguing than for $d_3(1 : -2)$. In addition to the usual deformations along the family $d_3(\alpha, \beta)$, jump deformation to $d_3(1 : 1 : 2)$, and deformations along the big family, we see that $d_3(1 : 2)$ has a jump deformation to the codifferential $d_1^\sharp$.

Because $d_1^\sharp$ itself has a jump deformation to $d_1(1:1)$, we get a jump deformation to this element as well, and deformations along the family $d_1(\lambda:\mu)$. Thus we pick up far more deformations than we would expect considering that the dimension of H^2 is only one more than in the generic case. Again, the explanation for this "impossibility" has to do with the fact that the three dimensional tangent space to this element of the moduli space does not accurately reflect the nature of the deformations, which are all tangent to one of three planes in this space. The true picture is captured by the versal deformation.

6.17. **The codifferential $d_3(0:1)=\mathfrak{r}_2(\mathbb{C})\oplus\mathbb{C}^2$.** The cohomology is given by

$$\begin{aligned}
H^1 &= \langle \psi^1_2, \psi^2_1-\psi^3_1, \psi^2_2-\psi^3_2, \psi^4_1, \psi^4_2, \psi^1_1\rangle\\
H^2 &= \langle \psi_1=-\psi^{12}_2+\psi^{13}_2, \psi_2=\psi^{14}_1, \psi_3=\psi^{14}_2,\\
&\qquad \psi_4=\psi^{24}_1, \psi_5=\psi^{24}_2, \psi_6=\psi^{12}_1-\psi^{13}_1\rangle\\
H^3 &= \langle \varphi^{124}_1, \varphi^{124}_3\rangle.
\end{aligned}$$

Not all of the brackets of the nontrivial 2-cocycles vanish, so we obtain some relations on the base of the versal deformation. The second order relations are $t^1t^2+t^3t^6=0$ and $t^1t^4+t^5t^6=0$. The relations on the base are obtained by adding higher order terms to these second order relations. We will omit them for brevity, but instead will describe the solutions which may give actual deformations. There are 8 solutions, 4 of which not local. The local solutions are

$$\begin{aligned}
&1)\quad t^1=t^6=0\\
&2)\quad t^1=t^5=t^3=0\\
&3)\quad t^2=t^3=0, t^4=\tfrac{-t^5t^6}{t^1}\\
&4)\quad t^6=\tfrac{t^1t^2}{t^3}, t^4=\tfrac{t^2t^5}{t^3}.
\end{aligned}$$

The first solution corresponds to the matrix $A=\left[\begin{smallmatrix}0&0&0&t^2&t^4&0\\0&0&0&t^3&t^5&1\\0&0&0&0&0&1\\0&0&0&0&0&0\end{smallmatrix}\right]$. The codifferentials d^∞ associated to this matrix are easy to analyze. They usually lie in the big family, except for some special cases when they are in the small family. There is a jump deformation to $d_3(0:0:1)$.

The second solution corresponds to $A=\left[\begin{smallmatrix}t^6&-t^6&\frac{t^4t^6}{t^2-1}&t^2&t^4&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&0\end{smallmatrix}\right]$. If $t^6\neq 0$, then either $t^2\neq 1$ or $t^4=0$, and the deformation is equivalent to $d_2^\sharp$. As a consequence, there is a jump deformation to $d_2^\sharp$. When

$t^6 = 0$, then if $t^4 \neq 0$ or $t^2 \neq 1$, then the deformation is equivalent to $d_3(1 : t_2 : 0)$.

In the third solution, let us first assume t^6 does not vanish. Then the solution has matrix $A = \begin{bmatrix} -(t^5-1)t^6 & (t^5-1)t^6 & \frac{t^5t^6}{t^1} & t^5 & \frac{t^5}{t^1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. When $t^5 \neq 1$, then the deformation is equivalent to $d_2^\sharp$, so there is a jump deformation to $d_2^\sharp$.

Now let us assume that $t^6 = 0$ in the third solution. We can assume $t^1 \neq 1$, since that corresponds to the first solution. The matrix simplifies to $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ t^1(t^5-1) & t^1 & 0 & 0 & t^5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. If $t^5 \neq 1$, then the deformation is equivalent to $d_2^\sharp$ again.

Finally, let us consider the fourth solution, whose matrix is equivalent to $A = \begin{bmatrix} \frac{t^1(t^2+t^5-1)}{t^2-1} & \frac{t^1t^2(t^2+t^5-1)}{t^3(t^2-1)} & t^1 & t^2+t^5 & t^3 & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{-t^2}{t^3} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. There are two special cases that need to be considered, when $t^2 = 0$, in which case, the restriction $t^3 \neq 0$ does not apply, and the case when $t^5 = 0$. (The case $t^5 = 1 - t^2$ is not local.) In these cases, the restriction $t^2 \neq 1$ does not apply. Let us first address these special cases.

When $t^2 = 0$, if $t^1 \neq 0$ and $t^5 \neq 1$ then we get $d_2^\sharp$, but when $t^5 = 1$, we get $d_1(1 : 0)$.

From now on, we deal with the general case, assuming that $t^5 \neq 0$, $t^2 \neq 0$ $t^5 \neq 1 - t^2$. Then $t^2 \neq 1$ and $t^3 \neq 0$.

When $t^1 \neq 0$, the deformation is equivalent to $d_2^\sharp$; otherwise it is equivalent to $d_3(1 : t^2 + t^5 : 0)$.

The subcases are a bit tricky, but the same codifferentials keep showing up, so the final analysis of the deformations of this codifferential is not difficult. We either obtain a jump deformation to $d_3(1 : 0 : 0)$ or to $d_2^\sharp$, or we obtain a deformation along the big or small families.

6.18. **The codifferential** $d_3(1 : 0) = \mathfrak{r}_{3,1}(\mathbb{C}) \oplus \mathbb{C}$**.** The cohomology is given by

$$\begin{aligned} H^1 =& \langle \varphi_1^1, \varphi_2^1, \varphi_2^2 + \varphi_3^3, \varphi_3^4, \varphi_1^2 + \varphi_3^3 \rangle \\ H^2 =& \langle -\psi_1^{12} + \psi_2^{24} + \psi_4^{34}, \psi_2^{12} + \psi_4^{14}, \psi_1^{23}, \psi_3^{34}, \psi_1^{24}, \psi_2^{24}, \psi_3^{14} \rangle \\ H^3 =& \langle \varphi_1^{124}, \varphi_3^{124}, \varphi_1^{234} \rangle. \end{aligned}$$

Some of the brackets of the nontrivial 2-cocycles do not vanish, and we have the second order relations

$$t^1t^4 + 2t^3t^7 + 2t^2t^5 = 0, \quad t^2t^4 - t^2t^6 + t^1t^7 = 0, \quad t^1t^5 + t^3t^4 + t^3t^6 = 0.$$

We omit the long expressions for the seven relations on the base of the miniversal deformation, but remark that $1 + t^4 + t^6$ appears in the denominator of two of them, so there may be an obstruction to the extension of an infinitesimal deformation to a formal one.

The solution to the relations is quite complex; however, if we confine ourselves to solutions which are local, then we can reduce the problem to 9 relatively simple cases.

1) $t^1 = (t^4 + t^6)\sqrt{\frac{t^2t^3}{t^4(1+t^6)}}, t^7 = \frac{t^3(t^4+t^6)}{t^1}, t^5 = \frac{-t^3(t^4+t^6)}{t^1}$
2) $t^3 = \frac{(t^1)^2((t^6-t^4+2)^2+2(t^4-t^6))}{8t^2(t^4+t^6+2)}, t^7 = \frac{(t^4-t^6)(t^4-t^6-2)t^1}{8t^3}, t^5 = -\frac{t^3(t^4+t^6)}{t^1}$
3) $t^1 = t^2 = t^3 = 0$
4) $t^1 = t^2 = t^7 = 0, \quad t^6 = -t^4$
5) $t^1 = 0, t^4 = -t^6, t^7 = \frac{t^6(1+t^6)}{t^5}, t^2 = \frac{-t^3t^6(1+t^6)}{(t^5)^2)}$
6) $t^1 = t^3 = t^5 = 0$
7) $t^2 = t^4 = t^7 = 0, \quad t^5 = \frac{-t^3t^6}{t^1}$
8) $t^4 = t^5 = t^6 = t^7 = 0$
9) $t^3 = t^4 = t^5 = 0, \quad t^7 = \frac{t^2t^6}{t^1}$.

In the first solution, if $t^6 = 0$, or $4t^4 \neq (t^6 - t^4)^2$ then the deformation is equivalent to $d_2^\sharp$; otherwise, we get $d_1(1 : 0)$.

In the second solution the differential is equivalent to $d_1(\alpha : \beta)$, where

$$(\alpha, \beta) = t^4 + t^6 + 2 \pm \sqrt{5(t^4)^2 - 12t^4 - 6t^4t^6 + 4t^6 + 5(t^6)^2 + 4}.$$

(It may be more revealing to recognize this element as $\mathfrak{g}_8\left(\frac{4t^4-(t^6-t^4)^2}{(t^4+t^6+2)^2}\right)$). Note that since t^1 is any nonzero number, this means that there is a jump deformation from $d_3(1 : 0)$ to $d_1(1 : 0)$.

In the third solution, all deformations are either along the big family or the family $d_3(\alpha : \beta)$. If $t^4 = t^6 = t^7 = 0$ and $t^5 \neq 0$, then the deformation is equivalent to $d_3(1 : 1 : 0)$, so there is a jump deformation from $d_3(1 : 0)$ to this element.

In the fourth solution, we get $d_1(1 + t^6 : -t^6)$, so that if $t^6 = 0$, we see that there is a jump deformation to $d_1(1 : 0)$.

In the fifth solution, when $t^6 = 0$ then if $t^3 = 0$, we get $d_3(1 : 1 : 0)$; otherwise we obtain $d_1(1 : 0)$. If $t^3 = 0$ and $t^6 \neq 0$, then we get $d_3(1 + \sqrt{t^6(t^6 + 1)} : 1 - \sqrt{t^6(t^6 + 1)})$ (assuming $t^6 \neq -1$). When neither t^6 nor t^3 vanish, we get a jump deformation to $d_2^\sharp$.

In the sixth solution, if $t^2=0$, this reduces to a previous case. If $t^6=0$, then we get $d_2^\sharp$, a jump deformation. Otherwise, when $t^7\neq 0$ or $t^6\neq 1$, we get $d_1(1:t^6)$.

In the seventh solution, we always get $d_2^\sharp$.

In the eight solution, if $t^1=t^2=0$, this is a previous case. If $t^2=0$, but $t^1\neq 0$, or $t^1=0$ and $t^2\neq 0$, we get $d_2^\sharp$ unless $t^3=0$, in which case we get $d_1(1:0)$. When neither t^1 nor t^2 vanish, then we get $d_2^\sharp$; unless $(t^3)^2=4t^1t^2$, when we get $d_1(1:0)$.

In the ninth solution, we always get $d_2^\sharp$.

To summarize, we note that $d_3(1:0)$ jumps to $d_3(1:0:0)$ and $d_1(1:0)$ and it deforms along the the big and small families as usual. It also jumps to $d_2^\sharp$.

6.19. **The codifferential $d_3^\star$.** The cohomology is given by

$$\begin{aligned}H^1=&\langle\varphi^1_2,\varphi^1_3,\varphi^2_1,\varphi^2_2,\varphi^2_3,\varphi^3_1,\varphi^3_2,\varphi^3_3\rangle\\ H^2=&\langle\psi^{14}_2,\psi^{14}_3,\psi^{34}_3,\psi^{24}_3,\psi^{14}_1,\psi^{34}_2,\psi^{34}_1,\psi^{24}_1\rangle.\end{aligned}$$

The brackets of all 2-cocycles with each other vanish, so the infinitesimal deformation is miniversal. The matrix of d^∞ is given by $A=\begin{bmatrix}0&0&0&1+t^5&t^8&t^7\\0&0&0&t^1&1&t^6\\0&0&0&t^2&t^4&1+t^3\\0&0&0&0&0&0\end{bmatrix}$. The deformations are easy to analyze, because they are given by the equivalence classes of similar matrices of the 3×3 submatrix given by the parameters. It is easy to see that there are jump deformations to $d_3(1:1:1)$ and $d_3(1:1)$, as well as deformations along the families these two codifferentials belong to. There are no other possibilities for local deformations.

6.20. **The codifferential $d_2^\star=\mathfrak{n}_4(\mathbb{C})$.** The cohomology is given by

$$\begin{aligned}H^1=&\langle\varphi^4_3,2\varphi^1_1+\varphi^2_2+\varphi^4_4,\varphi^3_1,\varphi^1_1+\varphi^2_2+\varphi^3_3\rangle\\ H^2=&\langle\psi^{24}_2,\psi^{13}_1+\psi^{23}_2,\psi^{24}_3,\psi^{23}_4,\psi^{14}_3,\psi^{12}_1+\psi^{13}_2+\psi^{23}_3\rangle\\ H^3=&\langle\varphi^{124}_2,\varphi^{124}_3,\varphi^{123}_1,\varphi^{123}_4,\varphi^{124}_4-\varphi^{123}_3\rangle.\end{aligned}$$

With such a large H^3, it would be too much to imagine that the brackets of the cocycles vanish; in fact, there are 5 relations on the base of the miniversal deformation. Since they are fairly simple, we will give them:

$$
\begin{aligned}
&4t^2t^5 + (t^1)^2t^6 = 0\\
&2t^5t^6 - t^1t^2t^5 + t^1t^3t^6 = 0\\
&t^1t^4 + t^2t^6 = 0\\
&2t^2t^3t^4 - t^1t^4t^6 - t^1(t^2)^2t^6 = 0\\
&2t^4t^5 + 2t^1t^3t^4 - (t^1)^2t^2t^6 + t^1(t^6)^2 = 0.
\end{aligned}
$$

Note that the fourth relation has no second order term, and the fact that the relations have no denominators means that the miniversal deformation is constructed in a finite number of steps; in fact, since the highest degree term in a relation is of degree 4, the fourth order deformation is miniversal. The solution to the relations can be decomposed into 3 four dimensional subspaces and one more complex four dimensional piece as follows.

$$
\begin{aligned}
&1)\quad t^4 = t^5 = t^6 = 0\\
&2)\quad t^2 = t^4 = t^6 = 0\\
&3)\quad t^1 = t^2 = t^5 = 0\\
&4)\quad t^6 = \tfrac{-t^1t^4}{2t^2}, \quad t^3 = \tfrac{-(t^1)^2(t^4+(t^2)^2)}{4(t^2)^2}, \quad t^5 = \tfrac{t^4(t^1)^3}{8(t^2)^2}.
\end{aligned}
$$

For the first solution, the matrix of d^∞ is $A = \left[\begin{smallmatrix} 0 & t^2 & 0 & 0 & 1 & 0\\ t^2t^3 & 0 & t^2 & 0 & t^1 & 1\\ 0 & t^2t^3 & 0 & 0 & t^3 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right]$. When $t^2 = 0$, $d^\infty \sim d_3\left(0 : \frac{t^1+\sqrt{(t^1)^2+4t^3}}{2} : \frac{t^1-\sqrt{(t^1)^2+4t^3}}{2}\right)$. Assume $t^2 \neq 0$. Then, when $t^3 = 0$, if $t^1 \neq 0$, we have $d^\infty \sim d_2^\sharp$, and when $t^1 = 0$, we get $d_1(1 : 0)$. When $t^3 \neq 0$, if $(t^1)^2 + 4t^3 = 0$, then $d^\infty \sim d_1(1 : 0)$; otherwise it is equivalent to $d_2^\sharp$. Thus we get jump deformations to $d_2^\sharp$ and $d_1(1 : 0)$.

For the second solution, the matrix is given by $A = \left[\begin{smallmatrix} 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & t^1 & 1\\ 0 & 0 & 0 & t^5 & t^3 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right]$.

In this case $d^\infty \sim d_3(\alpha : \beta : \eta)$ where

$$\alpha + \beta + \eta = t^1q, \quad \alpha\beta\eta = t^5q^3, \quad \alpha\beta + \alpha\eta + \beta\eta = t^3q^2,$$

where q is an arbitrary nonzero parameter. As a consequence, there is a jump deformation from d_2^* to every member of the family $d_3(\alpha : \beta : \eta)$.

For the third solution, the matrix is given by $A = \left[\begin{smallmatrix} t^6 & 0 & 0 & 0 & 1 & 0\\ 0 & t^6 & 0 & 0 & 0 & 1\\ t^6t^3 & 0 & t^6 & 0 & t^3 & 0\\ 0 & 0 & t^4 & 0 & 0 & 0 \end{smallmatrix}\right]$.

When $t^6 = 0$, if $t^3 = 0$, then we get $d_1(1 : -1)$, while if $t^4 = 0$ we get $d_3(1 : -1 : 0)$, both jump deformations. When $t^6 = 0$ and neither t^3

nor t^4 vanishes, then the deformation is equivalent to d_3, another jump deformation.

Assume $t^6 \neq 0$. If $(t^6)^2 \neq -t^3t^4$ then we get d_3; otherwise we get $d_1(1:-1)$, both jump deformations.

For the fourth solution, the matrix is quite complicated, so we omit it. When $t^1 = 0$, then if $t^2 = 0$, we get $d_1(1:-1)$, and if $t^4 = 0$, then we get $d_1(1:0)$, both jump deformations; otherwise, we get a deformation along the family $d_1(\alpha:\beta)$.

When $t^1 \neq 0$ and $t^4 = 0$, then if $t^2 = 0$, we get the jump deformation $d_3(1:1:0)$; otherwise we get a jump deformation to $d_1(1:0)$.

When $t^1 \neq 0$ and $t^4 \neq 0$ and $t^4 = -(t^2)^4$, then we get a jump deformation to the element $d_1(1+\sqrt{5}, 1-\sqrt{5})$, which is just $\mathfrak{g}_8(-1)$ on the Burde-Steinhoff list.

When none of the three conditions above hold, then the deformation is equivalent to $d_1(t^2 + \sqrt{(t^2)^2 - 4t^4}, t^2 - \sqrt{(t^2)^2 - 4t^4}) = \mathfrak{g}_8\left(\frac{t^4}{(t^2)^2}\right)$. Since t^1 is an arbitrary nonzero number, these deformations are also jump deformations. Thus there is a jump deformation to any element of the family $d_1(\alpha:\beta)$.

To summarize, the deformations of d_2^* are as follows. There are jump deformations to every member of the big family and everything they deform to, which means we get jump deformations to the elements $d_2^\sharp$, d_3 and every element in the family $d_1(\lambda:\mu)$.

6.21. **The codifferential** $d_1 = \mathfrak{n}_3(\mathbb{C}) \oplus \mathbb{C}$**.** The cohomology is given by

$$\begin{aligned}
H^1 &= \langle \varphi_1^1 + \varphi_2^2, \varphi_1^1 + \varphi_4^4, \varphi_4^2, \varphi_1^3, \varphi_3^3, \varphi_2^4, \varphi_3^4 \rangle \\
H^2 &= \langle -\psi_3^{23}, \psi_4^{23}, \psi_1^{14}, \psi_2^{14}, \psi_3^{14}, \psi_3^{12}, \psi_3^{34}, \\
&\qquad \psi_4^{24}, \psi_2^{34}\psi_4^{12}, \psi_1^{13} + \psi_2^{23}, \psi_4^{14} - \psi_2^{12}, \psi_4^{34} - \psi_1^{13} \rangle \\
H^3 &= \langle \varphi_2^{234}, \varphi_4^{234}, \varphi_4^{124}, \psi_2^{134}, \psi_3^{134}, \psi_2^{123} + \psi_4^{134}, \psi_3^{123}, \psi_4^{123}, \psi_2^{124}, \psi_3^{124} \rangle.
\end{aligned}$$

There are 10 relations, none of which involve terms of higher order than 3, so in fact, the second order deformation is already miniversal. We will not give the relations here explicitly. Because the miniversal deformation is obtained in a finite number of steps, the relations are polynomial, not rational, in the parameters.

If all the parameters but t^{13} and t^{11} vanish, and $t^{13} = -t^{11}$ then the relations are satisfied, and we have a jump deformation to $d_1^\sharp$.

If all the parameters but t^{12}, t^5 and t^6 vanish, then if $t^{12} \neq 0$, the deformation is equivalent to d_3, so there is a jump deformation to d_3.

If we assume that

$$\begin{aligned} t^8 =& t^3 = 0, \quad t^1 = \tfrac{-t^6t^7}{t^5}, \quad t^{12} = \tfrac{-t^6t^9}{t^5}, \\ t^2 =& \tfrac{(t^6)^2t^9}{(t^5)^2}, \quad t^4 = \tfrac{-t^{12}t^5}{t^6}, \quad t^{11} = \tfrac{-t^6t^9}{t^5}, \quad t^{10} = \tfrac{t^6t^{12}}{t^5}, \end{aligned}$$

then if q is a free parameter and

$$\begin{aligned} &\alpha+\beta+\eta = \tfrac{t^7q}{t^5} \\ &(\alpha+\beta)(\alpha+\eta)(\beta+\eta) = \tfrac{-t^9q^3}{(t^5)^2} \\ &(\alpha\beta+\alpha\eta+\beta\eta) = \tfrac{t^{12}q^2}{t^5t^6}, \end{aligned}$$

we obtain a solution to the relations for which $d^\infty \sim d_3(\alpha:\beta:\eta)$. Whenever t^5 and t^6 don't vanish, there is a solution for every $(\alpha:\beta:\eta)$, which means that there is a jump deformation from d_1 to every element in the big family.

If all the parameters vanish except t^3, t^4 and t^7, then we obtain a solution for which $d^\infty \sim d_3(\alpha:\beta)$, where

$$\alpha = t^7q, \quad \alpha+\beta = t^3q, \quad \alpha\beta = -t^4q^2,$$

where again, q is a nonzero free parameter, so we also have jump deformations to every member of this family.

Similarly, if all the parameters but t^5, t^8, t^{10} and t^1 vanish, and $t^8 = t^1$, the relations are all satisfied, and if q is a nonzero parameter, then independently of the value of t^5 we have $d^\infty \sim d_1(\alpha:\beta)$, where q is a nonzero parameter and

$$\begin{aligned} \alpha+\beta =& -t^8q \\ \alpha\beta =& t^{10}q^2, \end{aligned}$$

so there is a jump deformation from d_1 to every member of the family $d_1(\alpha:\beta)$.

If t^5 and t^8 do not vanish, $t^1 = t^{12} = t^8$ and all the other parameters vanish, then the relations are satisfied and $d^\infty \sim d_2^\sharp$, which gives another jump deformation.

If all the parameters except t^5 vanish, the relations are satisfied, and we get a jump deformation to d_2^*.

Thus finally, we observe that d_1 has jump deformations to every codifferential except d_3^*. This pattern is completely analogous to the three dimensional Lie algebra case, where the corresponding element d_1 has jump deformations to every element except d_2, which is exactly the analog of the element d_3^*, one dimension lower.

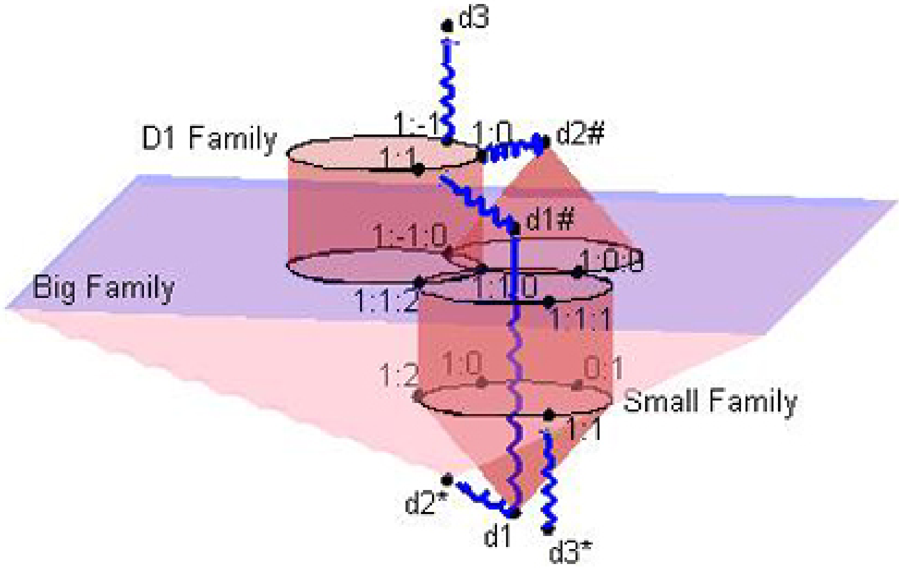

FIGURE 1. The Moduli Space of 4 dimensional Lie Algebras

7. Description of the Moduli Space

In Figure (1), we give a pictorial representation of the moduli space. The big family $d_3(\lambda:\mu:\nu)$ is represented as a plane, although in reality it is $\mathbb{P}^2/\Sigma_3$. The families $d_1(\lambda:\mu)$, $d_3(\lambda:\mu)$ and the three subfamilies $d_3(\lambda:\mu:0)$, $d_3(\lambda:\lambda:\mu)$ and $d_3(\lambda:\mu:\lambda+\mu)$ are represented by circles, mainly to reflect that the three subfamilies of the big family intersect in more than one point, because they each represent not a single $\mathbb{P}^1$, but several copies of $\mathbb{P}^1$ which are identified under the action of the symmetric group.

In the picture, jump deformations from special points are represented by curly arrows. The jump deformations from the small family $d_3(\lambda:\mu)$ to $d_3(\lambda:\lambda:\mu)$ and the jump deformations from $d_3(\lambda:\mu:\lambda+\mu)$ to $d_1(\lambda:\mu)$ are represented by cylinders. The jump deformations from the family $d_3(\lambda:\mu:0)$ to $d_2^\sharp$ and those from d_1 to the small family are represented by cones. Finally, the jump deformations from d_2^* to the big family are represented by an inverted pyramid shape. All jump deformations are either in an upwards or a horizontal direction.

The picture tries to capture the order of precedence of the deformations. For example, in the picture, you can trace a path of jump deformations from d_1 to $d_3(1:0)$ to $d_3(1:1:0)$ to $d_1(1:0)$ to $d_2^\sharp$.

8. Classifying a Particular Lie Algebra

In [1], it was shown that a four dimensional Lie algebra can be classified by computing certain invariants of the Lie algebra. Instead, our approach to classifying a Lie algebra, which we will outline here, used linear algebra.

Suppose that a codifferential d representing a Lie algebra structure has matrix A. Since the rank of the matrix is at most 3, it is easy to compute a new basis for which the matrix has the form $A = \left[\begin{smallmatrix} A' & \delta \\ 0 & 0 \end{smallmatrix}\right]$, where A' is a 3×3 matrix representing a 3 dimensional Lie algebra, and δ is a 3×3 matrix representing a derivation of this Lie algebra.

Next, consider the submatrix A'. If it has rank 3, then the codifferential is equivalent to d_3. Otherwise, we find a new basis in which the submatrix A' has been reduced to one with exactly as many rows as its rank. In fact, by using the classification methods for three dimensional Lie algebras, one can reduce the matrix A' to one of the standard forms.

At this point, the matrix δ representing the derivation on the three dimensional Lie algebra may not represent an outer derivation. However, by replacing the vector e_4 with a vector of the form $e_4' = ae_1 + be_2 + ce_3 + e_4$, one can replace the δ with one representing an outer derivation.

Once this has been accomplished, the classification scheme presented in this paper for determining the point in the moduli space corresponding to an extension of a three dimensional Lie algebra by an outer derivation can be applied. The precise identification scheme depends on which point in the moduli space of three dimensional Lie algebras occurs.

When computing versal deformations of the four dimensional Lie algebras, in most cases, we could identify the appropriate element by following a more simple scheme of solving for a matrix G such that the matrix $GA' = AQ$, where Q is the matrix representing the linear transformation g corresponding to G extended to $\bigwedge^2 V \to \bigwedge^2 V$ and A' is a matrix representing one of the nine types of elements in the moduli space.

However, because our matrices involved many parameters, it was sometimes too difficult for the computer to solve for the values of the parameters for which the A' and A matrices are equivalent. In those cases, we followed the more complicated scheme outlined above. In practice, we found that it was only necessary to follow the steps partially, because after transforming the matrix to eliminate some of the rows, we then were able to apply the simple scheme, and obtain a solution.

9. Acknowledgements

The authors would like to thank E. Vinberg for helpful discussions and the Max-Planck-Institut für Mathematik, Bonn for hosting both authors while they were finishing this paper.

References

1. Y. Agaoka, *An algorithm to determine the isomorphism classes of 4-dimensional complex Lie algebras*, Linear Algebra and its Applications **345** (2002), 85–118.
2. D. Burde and C. Steinhoff, *Classification of orbit closures of 4-dimensional complex Lie algebras*, Journal of Algebra **214** (1999), 729–739.
3. A. Fialowski, *Deformations of Lie algebras*, Mathematics of the USSR-Sbornik **55** (1986), no. 2, 467–473.
4. A. Fialowski and D. Fuchs, *Construction of miniversal deformations of Lie algebras*, Journal of Functional Analysis (1999), no. 161(1), 76–110.
5. A. Fialowski and M. Penkava, *Deformation theory of infinity algebras*, Journal of Algebra **255** (2002), no. 1, 59–88, math.RT/0101097.
6. ———, *Versal deformations of three dimensional Lie algebras as L_∞ algebras*, Communications in Contemporary Mathematics **7** (2005), no. 2, 145–165, math.RT/0303346.
7. M. Gerstenhaber, *On the deformations of rings and algebras I–IV*, Annals of Mathematics **79** (1964), 59–103; II, Annals of Mathematics **84** (1966), 1–19; III, Annals of Mathematics **88** (1968), 1–34; IV, Annals of Mathematics **99** (1974), 257–276.
8. A.A. Kirillov and Y.A. Neretin, *The variety A_n of n-dimensional Lie algebra structures*, Amer. Math. Soc. Transl. **137** (1987), no. 2, 21–30.
9. B. Komrakov and A. Tchourioumov, *Small dimensional and linear Lie algebras*, International Sophus Lie Centre Press, 2000.
10. A. Nijenhuis and R. Richardson, *Deformations of Lie algebra structures*, Jour. Math. Mech. **17** (1967), 89–105.
11. A.L. Onishik and and E.B. Vinberg (ed.) *Lie Groups and Lie Algebras III.(Structure of Lie groups and Lie algebras)*, Encyclopaedia of Math. Sci., 1994.
12. R.O. Popovich, V.M. Boyko, M.O. Nesterenko, and M.W. Lutfullin, *Realization of real low-dimensional Lie algebras*, Journal of Phys. A Math Gen. **36** (2003), 7337–7360.
13. M. Rhomdani, *Classification of real and complex nilpotent Lie algebras of dimension 7*, Linear and Multilinear Algebra **24** (1989), 167–189.
14. P. Turkowski, *Literature on the structure of low-dimensional nonsemisimple Lie algebras and its applications to cosmology*, Acta Cosmologica **20** (1994), 147–153.

Eötvös Loránd University, Budapest, Hungary
E-mail address: `fialowsk@cs.elte.hu`

University of Wisconsin, Eau Claire, WI 54702-4004
E-mail address: `penkavmr@uwec.edu`