THE HYPEROSCULATING SPACES OF HYPERSURFACES

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MPI/89 – 61

Almost-Periodic Attractors for a Class of Nonautonomous Reaction-Diffusion Equations on $\mathbb{R}^N$. +

by

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+ Work supported in part by the "Gruppo Nazionale per la Fisica Matematica" of the Italian Research Council (CNR) and by the Swiss National Science Foundation under Grant .

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§ 1. Introduction

At present, it seems that we still do not have an effective definition of Weierstrass points for varieties with higher dimensions. There are attempts including [9], [3] by Mount–Villamayor and Iitaka.

Now we try to analyse the problem in a very simple case i.e. hypersurfaces. We wish to know what will happen to the "special points" on a hypersurface for the sheaf of hyperplane sections. It may shed a light on the problem.

On the other hand, studying these special points is connected much with polar loci, singularities of mappings ([4] – [8], [10], [11]). But generally there they have a strong tool namely some kind of "theorem of genericness" (e.g. [6], [10]) to facilitate studying.

Intuitively, let $X \subset \mathbb{P}^n$ be a smooth hypersurface. We consider all of its tangent hyperplanes in $\mathbb{P}^n$. Then there exists an integer $b_2 \geq 2$ such that almost all of them have exactly contact of order b_2 with X and the others have that of higher order. We called such a b_2 coordinate gap number and the contact point with higher order a

b_2–inflection . The set of inflections whith certain natural structure is called hyperosculating space.

In [14] – [16] we discussed the hyperosculating space of surfaces in $\mathbb{P}^3$, now we intend to generalize that to arbitrary hypersurfaces.

Our main results are the following.

(a) $b_2 = 2$ or p^m for some $m \geq 1$, where $p \neq 2$ is the characteristic of ground field, and $b_2 = p^m$ if and only if the defining polynomial for X can be written as

$$\sum_{i=0}^{n} X_i F_i(X_0, \dots, X_m)^{p^m},$$

where p^m is the largest exponential for such an expression.

(b) If $p = 0$ or $p \nmid (\deg X - 1)$, then X has $b_2 = 2$ and a finite number of inflections. Furthermore, as a 0–cycle it is

$$\sum_{i=0}^{n-2} \sum_{k=0}^{i} 3^{n-1-i}(2-n)^{i-k} C_k(\Lambda^{n-2}\Omega_x^{\vee}) C_1(L)^{i-k}(C_1(\Omega_x) + (n-1)C_1(L))^{n-1-i} \cap [X],$$

where L is the sheaf of hyperplane sections of X in $\mathbb{P}^n$.

(c) For X with $b_2(X) = p^m$, we have $\deg X = 1 + kp^m$, $k \geq 1$. Then for generic such a X it has only a finite number of p^m–inflections, and as a 0–cycle it is

$$\sum_{k=0}^{n-1}\sum_{i=0}^{k}\begin{bmatrix} n-1-k+i \\ i \end{bmatrix} p^{(k-i)m}C_{k-i}(\Omega_x)C_1(L)^i(C_1(\Omega_x)+3C_1(L))^{n-1-k}\cap[X] .$$

For one who wishes to generalize the concept of Weierstrass point from curves to varieties of higher dimensions and if one wishes that one's definition would also include the simplest case as shown in this paper, then either one would permit the appearance of "continuous parts" of Weierstrass points or one would give more restriction until the "Weierstrass points" were finite. Of course, at the same time that "Weierstrass points" were asked to be expressed effectively.

§ 2. Notations and generalities

First set up our notations.

$X \subset \mathbb{P}^n_k$, a smooth hypersurface in $\mathbb{P}^n$.

K , algebraically closed field of arbitrary characteristic p but $p \neq 2$.

G , the polynomial for defining X and we always assume that the coordinates hyperplanes $X_i = 0$, $i = 0, \dots, n$ form a transversal sequence to X .

Point means closed point.

$L = \mathcal{O}_X(1)$, the sheaf of hyperplane sections of X in $\mathbb{P}^n$.

$P_x^m(L)$, the sheaf of m–principal part of L on X , i.e., $P_x^m(L) = p_*(\mathcal{O}_{\Delta^m} \otimes q^* L)$, where p , q are the first and the second projections from $X \times X$ to X ; where, if letting q_Δ be the ideal of definition for the diagonal Δ of $X \times_K X$, $\mathcal{O}_{\Delta^m} = \mathcal{O}_{X \times X}/q_\Delta^{m+1}$.

$V = H^0(X,L)$.

$V_X = V \otimes_K \mathcal{O}_X$.

$a_m : V_X \to P_X^m(L)$, the canonical morphism of taking m–truncated Taylor series [9], which is defined from the short exact sequence

$$0 \longrightarrow q_\Delta^{m+1} \otimes q^* L \longrightarrow \mathcal{O}_{X \times X} \otimes q^* L \longrightarrow \mathcal{O}_{\Delta^m} \otimes q^* L \longrightarrow 0$$

by taking the long exact sequence of their direct image ([7], [8]); additionally, we have some diagrams about a_ms' with exact rows and colomns:

(A1)

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \Omega_X(L) & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & V_X & \xrightarrow{a_1} & P^1_X(L) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E} & \longrightarrow & V_X & \xrightarrow{a_0} & L & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \Omega_X(L) & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

.
.
.

(Am)

$$
\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
 & & S^m\Omega_X(L) \\
 & & \downarrow \\
V_X & \xrightarrow{\quad a_m \quad} & P^m_X(L) \\
\| & & \downarrow \\
V_X & \xrightarrow{\quad a_{m-1} \quad} & P^{m-1}_X(L) \\
 & & \downarrow \\
 & & 0
\end{array}
$$

where S^m denotes the mth symmetric operator.

b_2, the coordinate gap number, is the least number m such that a_m is injective.

I, the hyperosculation space, defined as a scheme with ideal of definition $F^{\nu_{b_2}-(n+1)}(\mathrm{im}\, a_{b_2})$, where ν_{b_2} is the rank of $P_X^{b_2}(L)$ and $F^k(M)$ is the kth Fitting ideal of a sheaf M ([10]).

<u>Proposition 2.1</u>. The necessary and sufficient condition for $b_2(X) > 2$ are $G_{ij} = 0$ for all $0 \le i,j \le n$, where $G(X_0, \dots, X_n)$ is the polynomial for defining X and $G_{ij} = \partial^2 G/\partial X_i \partial X_j$.

<u>Corollary</u>. If $b_2(X) > 2$, then p divides $\deg X - 1$.

<u>Proof</u>. Theorem 3.1 and its Corollary in [14].

<u>Theorem 2.2</u>. $b_2(X) = 2$ or p^m; and $b_2 = p^m$ if and only if

$$G = \sum_{i=0}^{n} X_i F_i(X_0, \dots, X_m)^{p^m},$$ where p^m is the largest number in such a form of G.

The proof of the theorem is much like that in the case of $n = 3$ ([16]), but some expression appearing in the proof are needed in the sequel, so we shall give a sketch of proof.

<u>Proof</u>.

(a) Suppose $b_2(X) > 2$, then by Corollary p divides $\deg X - 1$ and we can write G into the form as in Theorem.

Without loss of generality, we may assume any two of the divisors $[F_0], \dots, [F_n]$ on

X have no common component and $F_n \neq 0$.

We take a point Q in $U_{0n} = \{F_n \neq 0\} \cap \{X_0 \neq 0\}$ and let B be the completion of the local ring at Q . In B we develope those coordinate functions into truncated Taylor series, then, if letting $x_1 = X_1/X_0, \dots, x_{n-1} = X_{n-1}/X_0$, $z = X_n/X_0$, we obtain

$$a_{S,Q}(z) = Z + R_1 dx_1 + \dots + R_{n-1} dx_{n-1} + \sum_{i_1 + \dots + i_{n-1} \geq 2} R_{i_1, \dots, i_{n-1}} dx_1^{i_1} \dots dx_{n-1}^{i_{n-1}}$$

for arbitrary integer $S \geq 2$, where all R_i and $R_{i_1, \dots, i_{n-1}}$ are elements of B .

(b) Substituting the expression of $a_S(z)$ into $G = 0$ at Q , and then comparing the coefficients of various independent differentials, we have

(i) $R_i = f_i^{p^m}(x_1, \dots, z)/f_n^{p^m}(x_1, \dots, z)$,

where $f_k(x_1, \dots, z) = F_k\left[1, \frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right]$, so $R_i \in \mathcal{O}_{Q,X}$;

(ii) $R_{i_1 \dots i_{n-1}} = 0$ for $i \leq i_1 + \dots + i_{n-1} < p^m$;

(iii) $R_{i_1 \dots i_{n-1}} = 0$ for $i_1 + \dots + i_{n-1} = p^m$ but at least two of $i_1, \dots, i_{n-1}$ are not zero;

(iv) $R_{p^m 0 \dots 0}, \dots, R_{0 \dots 0 p^m}$ are in $\mathcal{O}_{Q,X}$ and satisfy the following relations:

$$R_i^{p^m}\left[f_{0n}^{p^m} + x_1 f_{in}^{p^m} + \dots + x_{n-1} f_{n-1n}^{p^m} + z f_{nn}^{p^m}\right] +$$

$$+\left[f_{0i}^{p^m} + x_1 f_{1i}^{p^m} + \dots + x_{n-1} f_{n-1i}^{p^m} + z f_{ni}^{p^m}\right] +$$

$$+ f_n^{p^m} R_{0\dots 0\, p^m\, 0 \dots 0} = 0$$

for $i = 1, \dots, n-1$; where, in the subscript of $R_{0\dots 0p^m0\dots 0}$, p^m is at the i–th position and $f_{ij} = \partial f_i/\partial x_j$, $f_{in} = \partial f_i/\partial z$.

(c) From (i) – (iv) in (b) we see that $b_2(X) \geq p^m$. We conclude that $b_2 = p^m$. Otherwise, from $R_{0\dots 0\, p^m\, 0\dots 0} = 0$ we have

$$(***)_i \qquad F_i^{p^{2m}}\left[\sum_{j=0}^{n} X_j F_{jn}^{p^m}\right] - F_n^{p^{2m}}\left[\sum_{j=0}^{n} X_j F_{ji}^{p^m}\right] = 0$$

for $i = 0, \dots n$.

And furthermore from $(***)_i$ we deduce that

$$(****) \qquad \sum_{j=0}^{n} X_j F_{ji}^{p^m} = 0$$

are valid for $i = 0, \dots, n$ since $[F_i]$ have no common components on X and $\deg F_i^{p^{2m}} > \deg\left[\sum_{j=0}^{n} X_j F_{ji}^{p^m}\right]$ (for details see [16]).

Finally, differentiating (****) with respect to X_k respectively we obtain $F_{ji} = 0$ for all i , j , which contradicts to the property of F_i : which cannot be written as $H(X_0^{p^m}, \ldots, X_n^{p^m})$, $m \geq 1$ anymore.

§ 3. Case $b_2 = p^m$, $m \geq 1$.

From diagrams (A1) – (Ap^m) and (iv) in the proof of Theorem 2.2, we have

$$a_{p^m}(x_i) = dx_i + \ldots , \quad i = 1, \ldots , n-1 ,$$

$$a_{p^m}(z) = \sum_{i=1}^{n-1} R_{0, \ldots , p^m, \ldots , 0} (dx_i)^{p^m} ,$$

and that a_m is injective. Then by composing from (A1) to (Ap^m) we have an injective homomorphism

$$0 \longrightarrow R \xrightarrow{i} S^{p^m} \Omega_X(L) .$$

Therefore, I is defined by Fitting ideal $F^{\binom{p^m + n - 2}{n - 2} - 1}$ (coker i) .

For expressing the Fitting ideal explicitly we try to factor i through a locally free (n – 1)–subsheaf of $S^{p^m} \Omega_X(L)$, and which is also a direct factor of $S^{p^m} \Omega_X(L)$.

Lemma 3.1. There is a $(n-1)$–subsheaf $\mathcal{F}$ of $S^{p^m}\Omega_X$, which is locally generated by $\{(dx_1)^{p^m}, \dots, (dx_{n-1})^{p^m}\}$.

Proof. We only need to check that in a fixed coordinate neighborhood, every coordinate neighborhoods and their corresponding modules form a sheaf. But it is obvious, since, if letting (g_{ij}) be the matrix of coordinate transformation, then we have

$$\left[g_{ij}^{p^m}\right]\left[g_{jk}^{p^m}\right] = \left[g_{ik}^{p^m}\right].$$

Lemma 3.2.

$$C(\mathcal{F}) = \sum_{\ell=0}^{n-1} p^{\ell m} C_\ell(\Omega_X),$$

where $C(\mathcal{F})$ is the total Chern class of $\mathcal{F}$, $C_\ell(\Omega_X)$ is the ℓ–th Chern class of Ω_X.

Proof. We define a homomorphism

$$\varphi_1 : \mathcal{F}^\vee \longrightarrow \Omega_X^\vee$$

by $\mu_i \longmapsto \lambda_i^{p^m}$ locally, where $\{\mu_i\}$, $\{\lambda_i\}$ are local basis of $\mathcal{F}^\vee$, $\Omega_X^\vee$ which are dual to $(dx_1)^{p^m}, \dots, (dx_{n-1})^{p^m}$ and $dx_1, \dots, dx_{n-1}$ respectively. It is well–defined φ_1 induces a graded homomorphism

$$\psi_1 : \bigoplus_{\ell \geq 0} S^{\ell} \mathcal{F}^{\vee} \longrightarrow \bigoplus_{\ell \geq 0} S^{\ell} \Omega_X^{\vee} .$$

Because that any homogeneous prime ideal in $\oplus S^{\ell} \Omega_X^{\vee}$ (of course, we are arguing locally) containing im φ_1 must contain $\bigoplus_{\ell \geq 1} S^{\ell} \Omega_X^{\vee}$, ψ_1, determines a morphism

$$\begin{array}{ccc} \Phi_1 : P_1 = \mathbb{P}(\Omega_X^{\vee}) & \longrightarrow & S_1 = \mathbb{P}(\mathcal{F}^{\vee}) \\ & \pi_1 \searrow \quad \swarrow \gamma_1 & \\ & X & \end{array} .$$

Locally, Φ_1 is essentially a Frobenius morphism and hence flat.

We have the tautological exact sequences on P_1, S_1 respectively:

(1) $$0 \longrightarrow \mathcal{O}_{P_1}(-1) \longrightarrow \pi_1^* \Omega_X \longrightarrow Q_{P_1} \longrightarrow 0$$

(2) $$0 \longrightarrow \mathcal{O}_{S_1}(-1) \longrightarrow \gamma_1^* \mathcal{F} \longrightarrow Q_{S_1} \longrightarrow 0 .$$

From the flatness of Φ_1 and noting that $\Phi_1^* \mathcal{O}_{S_1}(-1) = \mathcal{O}_{P_1}(-p^m)$, we have a commutative diagram on P_1:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{P_1}(-1) & \longrightarrow & \pi_1^* \Omega_X & \longrightarrow & Q_{P_1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \varphi_2 & & \\ 0 & \longrightarrow & \mathcal{O}_{P_1}(-p^m) & \longrightarrow & \pi_1^* \mathcal{F} & \longrightarrow & \Phi_1^* Q_{S_1} & \longrightarrow & 0 . \end{array}$$

Now replacing X, Ω_X, $\mathcal{F}$, φ_1 with P_1, $\Phi_1^* Q_S$, Q_{P_1}, $\varphi_2^\vee$ respectively and doing what we did just above, we obtain on $P_2 = \mathbb{P}(Q_{P_1}^\vee)$ a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{P_2}(-1) & \longrightarrow & \pi_1^* Q_{P_1} & \longrightarrow & Q_{P_2} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \varphi_2 & & \\ 0 & \longrightarrow & \mathcal{O}_{P_2}(-p^m) & \longrightarrow & \pi_2^* Q_{S_1} & \longrightarrow & \Phi_2^* Q_{S_2} & \longrightarrow & 0\,. \end{array}$$

The composition $\pi_1 \circ \pi_2 : P_2 \longrightarrow X$ is flat and hence gives an injective homomorphism

$$(\pi_1 \circ \pi_2)^* : A^*(X) \longrightarrow A^*(P_2)\,,$$

where A^* is the symbol of Chow ring.

Continuing this process until we obtain the splitting space for both Ω_x and $\mathcal{F}$ ([1]). This means, there exists a scheme P and a morphism $\pi : P \longrightarrow X$ such that

(i) π is flat,

(ii) π induces an injective homomorphism

$$A^* X \longrightarrow A^* P$$

(iii) There are two filtrations associated to Ω_X, $\mathcal{F}$ respectively,

$$0 = T_0 \subset T_1 \subset \ldots \subset T_{n-1} = \Omega_X$$

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{n-1} = \mathcal{F}$$

such that T_{i+1}/T_i and $\mathcal{F}_{i+1}/\mathcal{F}_i$ are invertible sheaves on P and $\mathcal{F}_{i+1}/\mathcal{F}_i \cong (T_{i+1}/T_i)^{\otimes p^m}$.

According to splitting principle, if we assume

$$C(\Omega_X) = \prod_{i=1}^{n-1} (1 + x_i)$$

formally, then

$$C(\mathcal{F}) = \prod_{i=1}^{n-1} (1 + p^m x_i) .$$

Developing $c(\mathcal{F})$ we get what we expect.

Theorem 3.3. Let X be hypersurface in $\mathbb{P}^n$ with $b_2(X) = p^m$ and a finite number of p^m–inflection then the hyperosculating space I as a 0–cycle is

$$[I] = C_{n-1}(\mathcal{F}(L) - R) \cap [X] =$$

$$\sum_{k=0}^{n-1} \sum_{i=0}^{k} \begin{bmatrix} n-1-k+i \\ i \end{bmatrix} p^{(k-i)m} C_{k-i}(\Omega_X) \cdot C_1(L)^i \cdot$$

$$(C_1(\Omega_X) + 3C_1(L))^{n-1-k} \cap [X] .$$

Proof. We saw that I is defined by the ideal $F^{\begin{bmatrix} p^m + n - 2 \\ n-2 \end{bmatrix} - 1}(\operatorname{coker} i)$, where $i : R \longrightarrow S^{p^m}\Omega_X(L)$. From Lemma 3.1 we have a diagram

$$\begin{array}{ccc} R & \xrightarrow{\quad i \quad} & S^{p^m}\Omega_X(L) \\ & {\scriptstyle j}\searrow \quad \nearrow & \\ & \mathcal{F}(L) & \end{array}$$

where $\mathcal{F}(L)$ is a locally free $(n-1)$–subsheaf and a local direct factor of $S^{p^m}\Omega_X(L)$. Therefore, we have

$$F^{\begin{bmatrix} p^m + n - 2 \\ n-2 \end{bmatrix} - 1}(\operatorname{coker} i) = F^{n-2}(\operatorname{coker} j)$$

and hence by Porteaus' formula,

$$[I] = C_{n-1}(\mathscr{F}(L) - R)$$

$$= \sum_{k=0}^{n-1} \sum_{i=0}^{k} \begin{bmatrix} n-1-k+i \\ i \end{bmatrix} p^{(k-i)m} C_{k-i}(\Omega_X) \cdot C_1(L)^i (C_1(\Omega_X) + 3C_1(L)) \cap [X] \ .$$

We can also express $\deg [I]$ by $\deg X$:

$$\deg [I] = \sum_{k=0}^{n-1} \sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^j \begin{bmatrix} n+1 \\ j \end{bmatrix} \begin{bmatrix} n-1-k+i \\ i \end{bmatrix} p^{(k-i)m}$$

$$(2 - n + \deg X)^{n-1-k} (\deg X)^{k-i-j+1} \ .$$

Finally we give a proposition about when a hypersurface as above has a finite number of p^m–inflections. The proof of the following proposition is like that of Theorem 3.3 in [16], so we only give a sketch here.

Proposition 3.4. Let X be a hypersurface with $b_2 = p^m$ and $\deg X = 1 + kp^m$. Then every X with $k = 1$ has only a finite number of p^m–inflections. Furthermore, the conclusion is true for generic X too.

Proof.

(a) From (iv) in the proof of Theorem 2.2, if C is an irreducible curve on X with $C \subset I$, we shall have the following assertion, i.e.

$$(***)_i \ , \ i = 0, \dots, n-1$$

are valid on C.

Therefore by an argument similar to [16], on C we have

$$(****)\qquad \sum_{j=0}^{n} X_j F_j^{p^m}{}_i = 0\,,\ i = 0, \dots ,n\,.$$

(b) If $\deg X = 1 + p^m$, then $F_i = \sum_{j=0}^{n} a_{ij} X_j$ and $\det(a_{ij})$ is invertible, and hence the solution of $(****)$ contains no curves. By (a) we get the conclusion.

(c) The space of all hypersurfaces X with $b_2 = p^m$ and $\deg X = 1 + kp^m$, $k \geq 1$, has dimension $(n+1)\left[\begin{matrix} k+n \\ n \end{matrix}\right] - 1$, and we can show, by counting dimensions, that for a generic one, $(****)$ will have a solution of dimension zero only. Then the conclusion follows from (a).

§ 4. Case $b_2 = 2$

The case is a bit subtler than § 3.

In the section we always assume that $p = 0$ or p does not divide $\deg X - 1$. From Corollary to Proposition 2.1 we see that the assumption implies $b_2 = 2$ and hence includes the most cases about $b_2 = 2$.

Let $0 \longrightarrow R \xrightarrow{\ i\ } S^2\Omega_X$ be the morphism determined by (A1), (A2). We saw many times that I is determined by $F^{n-1}(\operatorname{coker} i)$.

Locally, e.g., on $U_{0n} = \{X_0 \neq 0\} \cap \{G_n \neq 0\}$

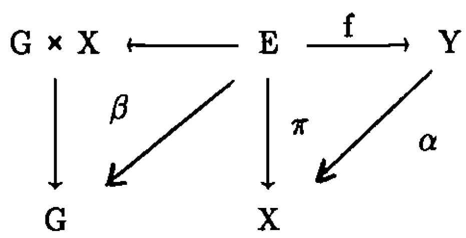
$$i(z) = \sum_{i+j=2}^{n-1} p_{ij} dx_i dx_j \, , \quad p_{ij} = p_{ji} \, , \quad 1 \leq i,j \leq n-1 \, ,$$

where $p_{ij} = \frac{X_0}{G_n^3} (G_n(G_n G_{ij} - G_i G_{jn} - G_j G_{in}) + G_i G_j G_{nn})$. Now from (A1) we define

$$G = Gr(n-2, V^{\vee}) \, ,$$
$$E = Gr(n-2, \mathscr{E}^{\vee}) \, ,$$
$$Y = Gr(n-2, \Omega_X(L)^{\vee}) \, ,$$

where $G(n-2, \mathscr{F})$ denotes the Grassmannian scheme of locally free $(n-2)$–quotient sheaves of $\mathscr{F}$. Thus $G_r(n-2, V_x^{\vee}) = G \times_K X$.

From (A1) we have a commutative diagram



where f is a rational mapping defined by

$$\mathscr{E} \longrightarrow \Omega_X(L) \longrightarrow 0 \, .$$

In fact, we have

Lemma 4.1. The set of definition for f is an open set U such that its complement in E is rationally equivalent to the special Schubert cycle $\sigma = \Delta_\lambda \cap [E]$, $\lambda = (2,0 \ldots 0)$ and hence has codimension 2 in E .

Proof. We have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longleftarrow & R^\vee & \overset{r_2}{\longleftarrow} & \mathscr{E}^\vee & \longleftarrow & \Omega_X(L)^\vee & \longleftarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle \gamma_1} & \swarrow & & & \\ & & & & \mathscr{H} & & & & \end{array}$$

Then for any locally free $(n-2)$–quotient of $\mathscr{E}^\vee$ it is a quotient of $\Omega_X(L)^\vee$ via the composition shown in the diagram if and only if $\ker\gamma_1 + \ker\gamma_2 = \mathscr{E}^\vee$. For the ranks to agree we see that this is a direct summation. Geometrically this means in each fiber of π , the $(n-2)$–space represented by $\mathscr{H}$ does not contain the 1–space represented by $R^\vee$. From [1] we obtain the lemma.

Now we turn to β , which is defined by the composition of $\mathscr{E} \longrightarrow V_x$ and the natural projection.

Lemma 4.2. Codim_E (the set of singularities of β) ≥ 2 .

Proof. Since every fiber of β is the intersection of a 2–plane in $\mathbb{P}^n$ with X , they have the same Hilbert polynomial. Then β is flat and hence that β is smooth at $x \in E$ if and only if x is a regular point of $\beta^{-1}\beta(x)$ [2].

Taking an arbitrary point $x \in X$, the intersection of the tangent hyperplane H at x with X is a subvariety with singularity x . Then every 2–plane passing x and being contained in H meets X always in a singular curve.

Firstly we count the dimension of the set of that 2–planes as x varying on X . In fact, every such a 2–plane at x may be a same kind of 2–planes at a different point y . In spite of that, the dimension always less than $(n-1)+3(n-3)-(n-3)$ and hence the codimension of the set in E is greater than 2.

On the other hand, when a hyperplane H is transverse to X at x , then the intersection variety X_0 is regular at x . But on H we have a $(n-2)$–plane H_1 which is tangent to X_0 at x . Hence every 2–plane contained in H and passing x must cut out a singular curve. When H varies with passing x , such a 2–plane H_2 may be the same kind of 2–plane of other H' .

So, $\dim\{H_2 \mid H$ varies with x in it$\} \leq (n-1)+n+3(n-4)-(n-2)$, hence its codimension in E is also greater than or equal to 2.

We could continue our "stratification" and then exhaust all singularities of β . But evidently there are only a finite number of steps and each step always gives the codimension of the set of singularities being greater than 2.

Now we come to the point.

Let

$$A : \pi^* R \longrightarrow \pi^* S^2\Omega_X(L) \longrightarrow S^2\Omega_E(\pi^* L) \longrightarrow S^2\Omega_{E/G}(\pi^* L)$$

be the composition of homomorphism as above.

It is known that $S^2\Omega_{E/G}(\pi^* L)$ is locally free at any smooth point of β. Then at such a point we have a neighborhood $\cong A^1 \times U' \subset \beta^{-1}(U')$ and a local coordinate $(t, w_1, \dots, w_{2(n-2)})$. In fact, the curve $\beta^{-1}\beta(x)$ is cut out by a 2–plane in $\mathbb{P}^n$, thus on X its coordinates can be written as

$$x_i = \lambda_i t + \dots , \quad i = 1, \dots , n-1 ,$$

where $\{\lambda_i\}, i = 1, \dots , n-1$ is taken as a part of the Plücker coordinates of $(n-1)$–subspace in $\check{\mathbb{P}}^n$ consisting of the tangent hyperplane at x and some other elements in the fiber of π at x.

We have

$$A(z) = (\Sigma\, p_{ij}\lambda_i\lambda_j)dt ,$$

thus

$$F^0(\mathrm{im}\ A) = (\Sigma\, p_{ij}\lambda_i\lambda_j)$$

and it determines a divisor J on W, which denotes the open set where β is smooth.

By Lemma 4.2, $\mathrm{codim}_E(E - W) \geq 2$, thus J extends uniquely to E. Let us write out the expression of the ideal of J.

Since $\text{codim}_E J = 1$, $A : R \longrightarrow S^2\Omega_{E/G}\big|_W$ has a correct dimension for its degeneracy locus. Then we have

$$\mathcal{O}_W(J) \sim \Omega_{E/G}(\pi^* L)^{\otimes 2} \otimes \pi^* R^{-1}|_W .$$

On the other hand, we have an exact sequence:

$$0 \longrightarrow \beta^* \Omega_G \longrightarrow \Omega_E \longrightarrow \Omega_{E/G} \longrightarrow 0$$

and the tautological sequence on G :

$$0 \longrightarrow P_G \longrightarrow \mathcal{O}_G^{n+1} \longrightarrow Q_G \longrightarrow 0 ,$$

where Q_G is the universal $(n-2)$–bundle , furthermore,

$$\Omega_G = \mathcal{H}om(Q_G, P_G) \cong P_G \otimes Q_G^* .$$

Now, for any locally free sheaf M with rank r , we denote $\Lambda^r M$ by K_M (but K_X , where X is a variety, still represent the canonical sheaf of X) and we will adopt the convention in the sequel.

Then

$$K_{E/G} = \beta^* K_G^{-1} \otimes K_E ,$$

but

$$\beta^* Q_G = Q_E ,$$

where Q_E is the universal $(n-2)$–bundle of E, thus

$$\beta^* K_{Q_G} = K_{Q_E} .$$

Moreover, from

$$0 \longrightarrow \pi^* \Omega_X \longrightarrow \Omega_E \longrightarrow \Omega_{E/X} \longrightarrow 0$$

and

$$\Omega_{E/X} = \mathcal{H}om(Q_E, P_E)$$

we have

$$\mathcal{O}(J) \cong \pi^* K_X{}^3 \otimes \pi^* L^{3(n-1)} \otimes K_{Q_E}^2 .$$

We shall pass to Y as we did in [16]. But at present, these $\{\lambda_i\}$ in the expression of J are Plücker coordinates, so we have to identify Y with its image under Plücker morphism, namely

$$Y = Gr(n-2, \Omega_X(L)^\vee) \xrightarrow{\ \sim\ } \mathbb{P}(\Lambda^{n-2}\Omega_X(L)^\vee) .$$

From now on, we always take Y as $\mathbb{P}(\Lambda^{n-2}\Omega_X(L)^\vee)$.

Recalling that U is the open set of definition for f, on $U \cap W$ we have

$$f^*(\alpha^* K_X^3 \otimes \alpha^* L^{3(n-1)} \otimes K_{Q_Y}^2) = \pi^* K_X^3 \otimes \pi^* L^{3(n-1)} \otimes K_{Q_E}^2 .$$

On the other hand, by using the Plücker coordinates $\lambda_1, \dots, \lambda_{n-1}$, we see that the scheme–theoretic image of J under f is determined by $\Sigma p_{ij}\lambda_i\lambda_j$, the same form which J is defined by an E. It means the scheme–theoretic inverse image of f(J) is J.

Furthermore, because $\mathrm{codim}_E(E - U) \geq 2$, we have

$$\mathrm{Pic}\, U \simeq \mathrm{Pic}\, E \simeq \mathbb{Z} \cdot \Delta_{1,0 \dots 0} \cap [E] \oplus \mathbb{Z} \cdot \pi^* \mathrm{Pic}\, X ,$$

where $\Delta_{\mu_1, \dots, \mu_k}$ is a typical symbol of some Chern class [1], here is just $C_1(Q_E)$.

Moreover, on Y we have

$$\mathrm{Pic}\, Y \simeq \mathbb{Z} \cdot C_1(\mathcal{O}_Y(1)) \cap [Y] \oplus \alpha^* \mathrm{Pic}\, X ,$$

but

$$f^* C_1(\mathcal{O}_Y(1)) = C_1(f^*(\Lambda^{n-2} Q_Y)) = C_1(\Lambda^{n-2} Q_E) = C_1(K_{Q_E}) = C_1(Q_E) ,$$

so

$$\varphi^* : \mathrm{Pic}\, Y \longrightarrow \mathrm{Pic}\, U$$

is injective.

Now from $f^{-1}(\mathrm{im}\, J) = J$ and

$$\mathcal{O}(J) = \pi^* K_X^3 \otimes \pi^* L^{3(n-1)} \otimes K_{Q_E}^2 \simeq f^*(\alpha^* K_X^3 \otimes \alpha^* L^{3(n-1)} \otimes K_{Q_Y}^2)$$

we obtain

$$\mathcal{O}(\mathrm{im}\, J) = \alpha^* K_X^3 \otimes \alpha^* L^{3(n-1)} \otimes K_{Q_Y}^2 .$$

Let $\mathrm{im}\, J = Z$. We consider the sheaf $\Omega_{Z/X}$. On Z , there is an exact sequence

$$\mathcal{O}_Z(-Z) \xrightarrow{d_{Y/X}} \Omega_{Y/X}|Z \longrightarrow \Omega_{Z/X} \longrightarrow 0 .$$

We see from this that $\Omega_{Z/X}$ locally is the quotient of $\mathcal{O}_X^{\oplus n-2}$ by the submodule generated by $d_{Y/X}(\Sigma\, p_{ij}\lambda_i/\lambda_k)$ for some $\lambda_k \neq 0$, or homogeneously it is isomorphic to $\bigoplus_{i=1}^{n-1} \mathcal{O}_X d\lambda_i / \{\Sigma\, p_{ij}\lambda_j d\lambda_i\}$. This means that locally $F^{n-3}(\Omega_{Z/X})$ is generated by

$$\left[\sum_{j=1}^{n-1} p_{ij}\lambda_j \right], \quad i = 1, \dots, n-1 .$$

It is worth noting that we only expressed p_{ij} on $U_{0n} = \{X_0 \neq 0\} \cap \{G_n \neq 0\}$ at the beginning of this section. But generally on arbitrary $U_{k\ell} = \{X_k \neq 0\} \cap \{G_\ell \neq 0\}$ we have

$$(*) \qquad p_{ij} = \frac{X_k}{G_\ell^3}(G_\ell(G_\ell G_{ij} - G_i G_{\ell j} - G_j G_{\ell i}) - G_i G_j G_{\ell\ell})$$

for $0 \leq i,j \leq n$ but $i,j \neq k,\ell$.

Let T be the scheme defined by $F^{n-3}(\Omega_{Z/X})$. We shall show that T is an equidimensional scheme of dimension $n-2$. For that we need some simple relations among p_{ij} .

We always work with U_{0n} . Firstly we define formally p_{0i} and p_{ni} for $i = 0, \dots ,n$ by using the same formula $(*)$.

We have

(a) $p_{ni} = 0$ for all i

and

(b) $\sum_{i=1}^{n-1} X_i p_{ij} = -X_0 p_{0j}$, $j = 0, \dots ,n$.

Lemma 4.3. Assume $p = 0$ or $p \nmid \deg X - 1$, then on every $U_{k\ell}$,

$$\det(p_{ij})_{1 \leq i,j \leq n-1} \neq 0 .$$

Proof. We shall prove inductively with $n \geq 2$.

For $n = 2$, $\det(p_{ij}) = p_{11}$ and from $b_2(X) = 2$, we have $p_{11} \neq 0$.

Now we assume Lemma is true for $n - 1$ and false for n .

Then we could assert that the rank of (p_{ij}) equals to $n - 2$. The reason is that, if we use $X_{n-1} = 0$ to cut X (recalling that we always assume that each coordinate hyperplane is transverse to X) , then on $X_{n-1} = 0$ we obtain a smooth hypersurface with $p = 0$ or $p \nmid (\deg X - 1)$. So by the inductive hypothesis that $\det(p'_{ij}) \neq 0$, which is just the restriction of the principal $(n-2)$–minors corresponding to $p_{n-1,n-1}$ of (p_{ij}) to $X_{n-1} = 0$, hence our assertion.

Now take a point $q \in U_{0n}$ such that $rk(p_{ij}(q)) = n - 2$ and assume that q has coordinates $(*,0, \dots ,0,*)$.

Let $q_{ij} = G_n(G_nG_{ij} - G_iG_{nj} - G_jG_{ni}) + G_iG_jG_{nn}$ for $i,j = 0, \dots ,n$.

We consider the equations

$$(**) \qquad \sum_{i=1}^{n-1} \lambda_i q_{ij} = 0 \,, \quad j = 1, \dots ,n-1$$

and find out about the solutions of them in $U_{0n} \times \mathbb{P}^n$. So we extend $(**)$ to $X \times \mathbb{P}^n$ firstly and then we see from the assumption of $\det(q_{ij}) = 0$ that $(**)$ determines a subscheme $\subset X \times \mathbb{P}^n$ with dimension at least $n + 1$. In a neighborhood u of q , since $rk(q_{ij}(q)) = n - 2$, every point has a fiber which is a 2–dimensional linear variety in the solution of (**) . Then by a theorem in [12] we see that over u , there is an irreducible component Γ of the solution of (**) in $X \times \mathbb{P}^n$ with dimension $n+1$. Γ meets the diagonal Δ of X in $X \times \mathbb{P}^n$ at $q = (*,0,\dots,0,*)$, i.e., $\Delta \cap \Gamma \neq \phi$ and hence intersect in a scheme with dimension at least 1. So we obtain a curve C passing q and which is a solution of (**) and lies on Δ . This means

$$\sum_{i=1}^{n-1} X_i q_{ij}(X_0, \dots, X_n) = 0 \; , \; j = 1, \dots, n-1$$

are valid on C .

By (6), this implies $q_{0j} = 0$ on C for $j = 1, \dots, n-1$ and hence for $j = 0,n$.

It is easy to check that under a non singular linear transformation

$$\text{(E)} \qquad \begin{cases} X_i = \displaystyle\sum_{i=0}^{n-1} a_{ij} T_j & i = 0, \dots, n-1 \\ X_n = T_n \end{cases} ,$$

we have

$$(q'_{ij}) = A^*(q_{ij})A \; ,$$

where q'_{ij} denotes the q_{ij} defined by (*) but under the new coordinates $T_0, \dots, T_n$, $A = (a_{ij})$ and A^* is the transposition of A .

Then we have

(1) $rk(q'_{ij}) = rk(q_{ij})$, $0 \le i,j \le n-1$; then by (6) we know $\det(q'_{ij}) = 0$, $1 \le i,j \le n-1$.

(2) the curve C is still a solution of

$$(**)' \qquad \sum_{i=1}^{n-1} T_i q'_{ij} = 0 \ , \ j = 1, \dots, n-1 \ .$$ Our reason is as follows.

Since $q_{0i}\big|_C = 0$, $A^*(q_{ij})_C A = \begin{bmatrix} 0, \dots, 0 \\ \vdots \quad B^*(q_{ij})B \\ 0 \end{bmatrix}_C$, where $B = (a_{ij})_{1 \le i,j \le n-1}$.

So,

$$(T_0, \dots, T_{n-1}) \cdot (q'_{ij})\big|_C = (X_0, \dots, X_{n-1})A^{*-1}A^*(q_{ij})A\big|_C$$

$$= (X_0, \dots, X_{n-1}) \begin{bmatrix} 0, \dots, 0 \\ \vdots \quad q_{ij} \\ 0 \end{bmatrix} A\big|_C$$

$$= (0, X_1, \dots, X_{n-1}) \begin{bmatrix} 0, \dots, 0 \\ \vdots \quad (q_{ij}) \cdot B \\ 0 \end{bmatrix}\Big|_C$$

$$= 0 \ .$$

Now we can use transformation (E) such that on C none of $G_0, \dots, G_{n-1}$ has a zero in common with G_n (noting that C must meet with each $G_i = 0$, since $G_i \not\equiv 0$ is an affine open set in X) . This can be done because $G_0, \dots, G_n$ have no common zero, so we may take them as the coordinates in $\check{\mathbb{P}}^n$.

So we may assume now (1) $q_{0i} = G_n(G_nG_{0i} - G_0G_{ni} - G_iG_{n0}) + G_0G_iG_{nn}$ vanishes on a curve C , which pass through U_{0n} ; (2) none of $G_0, \dots, G_{n-1}$ has a zero in common with G_n on C .

Now $q_{00} = G_n(G_nG_{00} - 2G_0G_{n0}) + G_0^2G_{nn} = 0$ on C , then $G_n = 0$ implies $G_0^2G_{nn} = 0$, but on C divisors $[G_n]$ and $[G_0]$ have no common component, so $[G_n] \le [G_{nn}]$. On the other hand, $\deg[G_n] > \deg[G_{nn}]$, thus $G_{nn}|_C = 0$. Now we obtain on C

$$G_nG_{00} - 2G_0G_{n0} = 0 ,$$

and by the same argument we have $G_{n0} = 0$, $G_{00} = 0$.

Taking arbitrary q_{0j} , on C we have

$$G_n(G_nG_{0j} - G_0G_{nj} - G_jG_{n0}) + G_0G_jG_{nn} = 0 ,$$

then $G_nG_{0j} - G_0G_{nj} = 0$.

By the same argument again we obtain

$$G_{nj} = 0 \quad \text{for} \quad j = 0, \dots, n .$$

Finally, $(d-1)G_n = \sum_{j=0}^{n} X_j G_{nj} = 0$, but since $p \nmid d-1$, $G_n(C) = 0$. This is impossible since C passes through U_{0n} by our construction.

As a digression we have

Corollary. For smooth hypersurfaces in $\mathbb{P}^n$ we have the following classification:

(i) If $p = 0$ or $p \nmid \deg X - 1$, then biduality is valid for X .

(ii) If p divides $\deg X - 1$ but G can be written as

$$\Sigma X_i F_i(X_0, \ldots, X_n)^{p^m}, \quad m \geq 1,$$

then the biduality is false for X .

(iii) For X not belonging to (i) (ii), the biduality is undefinite.

Proof. [13] showed that biduality is valid for X if and only if the dual mapping $\hat{\varphi} = (G_0, \ldots, G_n) : X \longrightarrow X^\vee$ is separably generated, that is equivalent to that $\Omega_{k(X)/k(X^\vee)}$ has rank $\mathrm{tr}\deg_{k(X^\vee)} k(X)$, where $k(X)$, $k(X^\vee)$ denote the rational fields of X and $X^\vee$ respectively. But naturally
$\dim X - \mathrm{rk}(p_{ij}) \geq \mathrm{rk}(\Omega_{k(X)/k(X^\vee)}) \geq \mathrm{tr} \cdot \deg_{k(X^\vee)} k(X)$ [14]. By Lemma
$\dim X - \mathrm{rk}(p_{ij}) = 0$, so the conclusion follows.

(ii) Theorem 3.3 in [14].

(iii) We have the following examples.

(a) In $\mathbb{P}^3$, the surface

$$X_0^2X_1^{p-1} + X_0X_1^p + X_1X_2^p + X_2X_3^p + X_3X_0^p = 0$$

is reflexive, i.e., biduality is valid for it.

(b) In $\mathbb{P}^3$, the surface

$$X_3^{p+1} - X_0^{p+1} - X_0^2X_1^{p+1} - X_1^{p+1} - X_2^{p+1} = 0$$

is non–reflexive.

These two examples both have $b_2 = 2$, but in (a) $\det(p_{ij}) \neq 0$ and in (b) $\det(p_{ij}) = 0$.

Lemma 4.4. $\{\det(p_{ij}) = 0\}$ and the scheme defined by all $(n-2)$–minors of (p_{ij}) have no common component.

Proof. We shall show inductively with $n \geq 3$.

For $n = 3$, suppose there is a curve C such that every p_{ij} vanishes on it. We knew already that under a transformation of coordinates the new p_{ij}'s still have C in their zero locus. So we may assume that any two of $G_0, \ldots, G_3$ have no common nullpoint on C. By the argument we used in the proof of Lemma 4.3, we deduce that $G_{ij} = 0$ on C for $0 \le i,j \le 3$ and hence $G_0 = \ldots = G_3 = 0$ on C. It is absurd.

Now we assume Lemma is true for $n-1$ but false for n. Therefore, there is a prime divisor D on X such that every $(n-2)$–minor vanishes on it. We choose a point q in D such that $rk(p_{ij}(q)) = n-2$; this is possible. The reason for this is as follows. We take a section, e.g., $\{X_{n-1} = 0\}$, the corresponding (p_{ij}) for it is just the principal minor of $p_{n-1,n-1}$ restricted to $\{X_{n-1} = 0\}$. Then by the inductive hypothesis, its $(n-3)$–minors have no common component with this principal minor. This means we can find such a point q in D.

Now we shall proceed along the same line as in the proof of Lemma 4.3 to get a solution of $q_{0i} = 0$ with dimension at least 1 and then $G_n = 0$ hence a contradiction.

<u>Proposition 4.5</u>. T is an equidimensional scheme of dimension $n-2$.

<u>Proof</u>. Let V_i be the subscheme of X which is defined by all $(i+1)$–minors of (p_{ij}), $1 \le i,j \le n-1$. By Lemma 4.3 $\dim V_{n-2} = n-2$ and by Lemma 4.4 $\dim V_{n-3} \le n-3$. In fact, we conclude that $\dim V_i \le i$ for $i = 0,1, \ldots, n-2$. Otherwise we assume $\dim X_i \ge i+1$ for some i , then all the i–th minors of (p_{ij}) would have a common component D with each $(i+1)$–th minors, where $\dim D \ge i+1$.

Now we cut X with $X_{n-1} = X_{n-2} = \ldots = X_{n-i-2} = 0$ and make them have a non empty intersection with D . Then we obtain a smooth hypersurface in $\mathbb{P}^{i+2}$ with degree $\deg X$, so by Lemma 4.4, its $\det(p_{ij})$ and the i–th minors have no common component. Noting that the new (p'_{ij}) is nothing other than the restriction of some principal $(i+1)$–minor of (p_{ij}), then we have a contradiction.

On the other hand, T is defined by the resolution

$$\mathcal{O}_Z(-Z) \longrightarrow \Omega_{Y/X}|_Z \longrightarrow \Omega_{Z/X} \longrightarrow 0\ ,$$

so $\mathrm{codim}_Z T \leq n-2$, i.e. $\dim T \geq n-2$.

From the local expression for T :

$$(**) \qquad \Sigma\, p_{ij}\lambda_j = 0\ ,\ i = 1, \dots, n-1\ ,$$

we see that over each $(V_i - V_{i-1})$ we have as solution for λ_i a linear space with dimension $n-2-i$, thus $\dim T|_{\alpha^{-1}(V_i - V_{i-1})} \leq n-2$ and hence $\dim T|_{\alpha^{-1}(V_i)} = n-2$. The proof is complete.

Theorem 4.6.

$$[I] \simeq \sum_{i=0}^{n-2} \sum_{k=0}^{i} 3^{n-1-i}(2-n)^{i-k} C_k(\Lambda^{n-2}\Omega_X^{\vee}) C_1(L)^{i-k}$$

$$(C_1(\Omega_X) + (n-1)C_1(L))^{n-1-i} \cap [X]\ .$$

Proof. Since T is $(n-2)$–equidimensional, from the resolution as shown above we have

$$[T] \sim C_{n-2}(\Omega_{Y/X} - \mathcal{O}(-Z)) \cap [Z] =$$

$$= \{C(\alpha^{*}\Lambda^{n-2}\Omega_X(L)^{\vee}(-1)) \cdot (1 + C_1(\mathcal{O}(Z)) + C_1^2(\mathcal{O}(Z)) + \dots)C_1(\mathcal{O}(Z))\}_{n-1} \cap [Y]$$

$$= \sum_{i=0}^{n-2} \sum_{k=0}^{i} \begin{bmatrix} n-1-k \\ i-k \end{bmatrix} C_k(\Lambda^{n-2}\alpha^{*}\Omega_X^{\vee})(-(n-2)C_1(\alpha^{*}L) - C_1(\mathcal{O}_Y(1))^{i-k}$$

$$(3C_1(\alpha^{*}K_X) + 3(n-1)C_1(\alpha^{*}L) + 2C_1(\mathcal{O}(1))^{n-1-i} \cap [Y] =$$

$$= \alpha^{*}D_0 + C_1(\mathcal{O}_Y(1)) \cap \alpha^{*}D_1 + \dots + C_1(\mathcal{O}_Y(1))^{n-2} \cap \alpha^{*}D_{n-2} ,$$

where D_i denotes a certain i–cycle on X .

Now it is enough to show that $D_0 = V_0$. (In fact, we can prove $D_i = V_i$ for all i) .

We know that locally T is expressed by the solution of (**) . Over U_{0n} , $Y \simeq U_{0n} \times \mathbb{P}^{n-2}$ and $C_1(\mathcal{O}(1))^{n-2}$ act on a cycle on Y is equivalent to use a generic 0–plane to meet T . At present case the intersection point is actually the solution of (**) for generic $(\lambda_1, \dots ,\lambda_{n-1})$, so it has to satisfy $p_{ij} = 0$, so $D_0 \subset V_0$. It is obvious that $D_0 \supset V_0$, therefore the assertion follows.

From the expression for [T] we see

$$[V_0] = [I] \simeq \sum_{i=0}^{n-2} \sum_{k=0}^{i} 3^{n-1-i}(2-n)^{i-k} C_k(\Lambda^{n-2}\Omega_X^{\vee}) C_1(L)^{i-k}$$

$$(C_1(\Omega_X) + (n-1)C_1(L))^{n-1-i} \cap [X] \; ,$$

and furthermore, if we wish,

$$\#[I] = \sum_{i=0}^{n-2} \sum_{k=0}^{i} \sum_{s+t+u=k} 3^{n-1-i} \begin{bmatrix} n-1-k \\ i-k \end{bmatrix} \begin{bmatrix} n+1 \\ s \end{bmatrix} (2-n)^{i-k}$$

$$(\deg X - n - 1)^t (2 \deg X - n - 1)^u (n - \deg X)^S \cdot (\deg X - 2)^{n-i-1} \deg X \; .$$

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