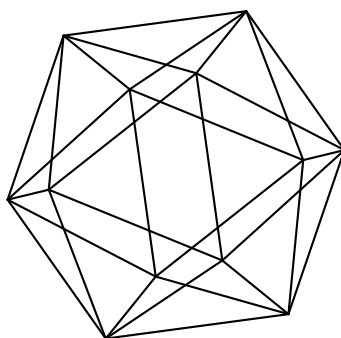


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LOGARITHMIC INTERPRETATION OF MULTIPLE ZETA VALUES IN POSITIVE CHARACTERISTIC

CHIEH-YU CHANG AND YOSHINORI MISHIBA

ABSTRACT. In this paper, we study multiple zeta values (MZV's) over rational function fields in positive characteristic. For each ∞ -adic MZV $\zeta_{\mathcal{A}}(\mathfrak{s})$ of weight n introduced by Thakur, we show that it is related to the n th coordinate of the logarithm of an explicitly constructed uniformizable t -module $G_{\mathfrak{s}}$ at a special point $\mathbf{v}_{\mathfrak{s}}$. Inspired by Furusho's definition of p -adic MZV's [F04], we define v -adic MZV's for every finite place v of the given rational function field. We further show that each v -adic MZV $\zeta_{\mathcal{A}}(\mathfrak{s})_v$ is related to the n th coordinate of the v -adic logarithm of the t -module $G_{\mathfrak{s}}$ at a special point constructed using $\mathbf{v}_{\mathfrak{s}}$. These two logarithmic interpretations completely generalize the work of Anderson-Thakur [AT90] to arbitrary depth MZV's. As an application, we show that v -adic MZV's satisfy the linear relations that their corresponding ∞ -adic MZV's satisfy.

1. INTRODUCTION

1.1. Logarithms and MZV's. Hilbert's seventh problem is to ask the transcendence of α^{β} for algebraic numbers α and β with $\alpha \notin \{0, 1\}$, $\beta \notin \mathbb{Q}$. It was solved by Gelfond-Schneider in the 1930's in terms of the $\overline{\mathbb{Q}}$ -linear independence of two logarithms at algebraic numbers which are linearly independent over \mathbb{Q} . The work of Gelfond-Schneider was fully generalized by Baker to arbitrary many logarithms at algebraic numbers in the 1960's. It is well known that Baker's celebrated theorem on linear forms in logarithms has many important applications in diophantine geometry, see [BW07, Wa08]. Baker's theorem was later on extensively generalized by Wüstholz [W89], whose theory is addressed as the analytic subgroup theorem stated as follows.

Theorem 1.1.1. *Let G be a connected commutative algebraic group defined over $\overline{\mathbb{Q}}$. Let \exp_G be the exponential map of G when regarding $G(\mathbb{C})$ as a Lie group. Let $\mathbf{u} \in \text{Lie } G(\mathbb{C})$ satisfy $\exp_G(\mathbf{u}) \in G(\overline{\mathbb{Q}})$, and put $T_{\mathbf{u}}$ to be the smallest linear subspace of $\text{Lie } G(\mathbb{C})$ that is defined over $\overline{\mathbb{Q}}$ and that contains the vector \mathbf{u} . Then $T_{\mathbf{u}} = \text{Lie } H$ for some algebraic subgroup H of G that is defined over $\overline{\mathbb{Q}}$.*

Let $A := \mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over the finite field \mathbb{F}_q of q elements with characteristic p . Let k be the fraction field of A equipped with the normalized absolute value $|\cdot|_{\infty}$ associated with the infinite place ∞ for which $|\theta|_{\infty} = q$. Let k_{∞} be the completion of k with respect to $|\cdot|_{\infty}$, and let \mathbb{C}_{∞} be the completion of a fixed algebraic closure of k_{∞} with respect to the canonical absolute value extending

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$|\cdot|_\infty$ on k_∞ . Let \bar{k} be the algebraic closure of k in \mathbb{C}_∞ . In the function field setting, Yu's sub- t -module theorem stated below is the analogue of Wüstholz's analytic subgroup theorem (see Sec. 6.3 for related definitions).

Theorem 1.1.2. ([Yu97, Thm. 0.1]) *Let G be a regular t -module defined over \bar{k} . Let Z be a vector in $\text{Lie } G(\mathbb{C}_\infty)$ such that $\exp_G(Z) \in G(\bar{k})$. Then the smallest linear subspace in $\text{Lie } G(\mathbb{C}_\infty)$ defined over \bar{k} , which is invariant under $\partial[t]$ and contains Z , is the tangent space at the origin of a sub- t -module of G over \bar{k} .*

Let \mathbb{N} be the set of positive integers. Recall that the classical multiple zeta values are defined for $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ with $s_1 > 1$,

$$\zeta(\mathfrak{s}) := \sum_{n_1 > n_2 > \dots > n_r \geq 1} \frac{1}{n_1^{s_1} \dots n_r^{s_r}} \in \mathbb{R}^\times.$$

The MZV $\zeta(\mathfrak{s})$ is a specialization at $(1, \dots, 1)$ of the r -variable multiple polylogarithm

$$\sum_{n_1 > n_2 > \dots > n_r \geq 1} \frac{z_1^{n_1} \dots z_r^{n_r}}{n_1^{s_1} \dots n_r^{s_r}},$$

which are generalizations of the classical logarithms. Classical MZV's and multiple polylogarithms have many interesting connections with various research topics, see [An04, Zh16].

There is a conjecture in the classical transcendence theory asserting that each multiple zeta value is transcendental over \mathbb{Q} . To date one only knows the transcendence of $\zeta(\mathfrak{s})$ when

$$\zeta(\mathfrak{s}) / (2\pi\sqrt{-1})^{\text{wt}(\mathfrak{s})} \in \mathbb{Q},$$

where $\text{wt}(\mathfrak{s}) := \sum_{i=1}^r s_i$. For example, $\zeta(2, \dots, 2)$ is transcendental over \mathbb{Q} .

1.2. The first main result. In what follows, we will review the ∞ -adic multiple zeta values initiated by Thakur [T04]. For a finite place v of k , we will define v -adic multiple zeta values in the next section and abbreviate them as “ v -adic MZV's”. To distinguish the difference between ∞ -adic and v -adic settings, throughout this paper we will use “MZV's” for Thakur's ∞ -adic multiple zeta values unless we particularly point out.

Fixing any r -tuple $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, Thakur [T04] defined the following positive characteristic MZV's:

$$(1.2.1) \quad \zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \dots a_r^{s_r}} \in k_\infty.$$

Here a_1, \dots, a_r run over all monic polynomials in A satisfying the strict inequalities: $|a_1|_\infty > |a_2|_\infty > \dots > |a_r|_\infty$. Note that since our absolute value $|\cdot|_\infty$ is non-archimedean, the series $\zeta_A(\mathfrak{s})$ converges ∞ -adically in k_∞ for all $\mathfrak{s} \in \mathbb{N}^r$. Furthermore, it is shown by Thakur [T09] that every $\zeta_A(\mathfrak{s})$ is non-vanishing. We call $\text{dep}(\mathfrak{s}) := r$ the depth and $\text{wt}(\mathfrak{s}) := \sum_{i=1}^r s_i$ the weight of the presentation of $\zeta_A(\mathfrak{s})$. Depth one MZV's were initiated by Carlitz [Ca35] and are called Carlitz zeta values.

In the seminal paper [AT90], Anderson and Thakur gave logarithmic interpretations for Carlitz zeta values and v -adic Goss zeta values, where v is a finite place of k . For each Carlitz zeta value $\zeta_A(s)$, we consider the s th tensor power of the Carlitz module denoted by $\mathbf{C}^{\otimes s}$ in (2.2.7). Anderson and Thakur explicitly constructed a special point $\mathbf{v}_s \in \mathbf{C}^{\otimes s}(k)$ and a vector $Z_s \in \text{Lie } \mathbf{C}^{\otimes s}(\mathbb{C}_\infty)$ so that up to an explicit multiple Γ_s in A , $\zeta_A(s)$ occurs as the s th coordinate of Z_s and $\exp_{\mathbf{C}^{\otimes s}}(Z_s) = \mathbf{v}_s$, where $\exp_{\mathbf{C}^{\otimes s}}$ is

the exponential map of $\mathbf{C}^{\otimes s}$ (see Sec. 2.2). Around that time, Yu [Yu91] developed a transcendence theory for the last coordinate of the logarithm of $\mathbf{C}^{\otimes s}$ at algebraic points. Combining the works of Anderson-Thakur and Yu, a significant consequence is to enable one to derive the transcendence of all Carlitz zeta values, which surpasses the classical situation. Later on, Yu extensively generalized the ∞ -adic transcendence theory in [Yu91] to the most general setting stated as Theorem 1.1.2. The transcendence of arbitrary MZV was obtained by the first author of the present paper [C14] using the current t-motivic transcendence theory, in particular the so-called ABP criterion, developed by Anderson, Brownawell and Papanikolas [ABP04].

The first main theorem in this paper is to generalize the result of Anderson-Thakur [AT90] to arbitrary MZV's (see Sec. 2.2 for the related definitions).

Theorem 1.2.2. *Given any r -tuple $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, we put $n := \text{wt}(\mathfrak{s})$. We explicitly construct a uniformizable t-module $G_{\mathfrak{s}}$ that is defined over k , a special point $\mathbf{v}_{\mathfrak{s}} \in G_{\mathfrak{s}}(k)$ and a vector $Z_{\mathfrak{s}} \in \text{Lie } G_{\mathfrak{s}}(\mathbf{C}_{\infty})$ so that*

- (a) $\Gamma_{s_1} \cdots \Gamma_{s_r} \zeta_A(\mathfrak{s})$ occurs as the n th coordinate of $Z_{\mathfrak{s}}$.
- (b) $\exp_{G_{\mathfrak{s}}}(Z_{\mathfrak{s}}) = \mathbf{v}_{\mathfrak{s}}$.

Here, Γ_{s_i} stands for the Carlitz factorials in A defined in (5.1.2). Note that the weaker version of the theorem above for Eulerian MZV's was obtained by the first author of the present paper in [C16] and it is the key point that enables one to compute the dimensions of double zeta values. For other instances about logarithmic interpretations of special zeta values (at small positive integers) coming from algebraic curves over \mathbb{F}_q , see [T92, A94, G17].

The following is a naive question as an analogue of Theorem 1.2.2.

Question 1.2.3. *Given a multiple zeta value $\zeta(\mathfrak{s})$, can one construct a connected commutative algebraic group $G_{\mathfrak{s}}$ over $\overline{\mathbb{Q}}$ and a vector $Z_{\mathfrak{s}} \in \text{Lie } G_{\mathfrak{s}}(\mathbb{C})$ for which*

- (a) up to a rational multiple, $\zeta(\mathfrak{s})$ occurs as certain coordinate of $Z_{\mathfrak{s}}$, and
- (b) $\exp_{G_{\mathfrak{s}}}(Z_{\mathfrak{s}}) \in G_{\mathfrak{s}}(\overline{\mathbb{Q}})$?

If the question above can be tackled, there may have hope toward proving the transcendence of multiple zeta values using Wüstholz's analytic subgroup theorem. A probing study of this question both for real-valued MZV's and p-adic MZV's may lead to a possible connection between real-valued MZV's and their corresponding p-adic MZV's, as realized in the function field setting by our paper.

As mentioned above, the result of Theorem 1.2.2 for depth one case was established by Anderson-Thakur [AT90]. However, we are not able to find a easier way to generalize their methods to higher depth case. In [AT90], there are two crucial points in the scheme of their proof:

- (1) Interpolation of power sums (see [AT90, (3.7.4)]).
- (2) Formulas for the right lower corner of coefficient matrices of the logarithm of $\mathbf{C}^{\otimes s}$ (see [AT90, Prop. 2.1.5]).

Property (1) enables one to connect $\zeta_A(\mathfrak{s})$ with a k-linear combination of the s th Carlitz polylogarithm at certain integral points, and (2) allows one to express the last coordinate of the logarithm of $\mathbf{C}^{\otimes s}$ at specific special point as an evaluation of the s th Carlitz polylogarithm. Note that the interpolation property (1) was used by the first author of the present paper to express each MZV $\zeta_A(\mathfrak{s})$ as a k-linear combination of the s th Carlitz multiple polylogarithm at integral points [C14].

Inspired by the period interpretation of MZV's in [AT09], for each MZV $\zeta_A(\mathfrak{s})$ the authors of [CPY14] constructed a t-module $E'_\mathfrak{s}$ defined over A and a special point $\mathbf{v}_\mathfrak{s} \in E'_\mathfrak{s}(A)$ to establish a criterion in terms of $(E'_\mathfrak{s}, \mathbf{v}_\mathfrak{s})$ for determining when $\zeta_A(\mathfrak{s})$ is a k -rational multiple of $\zeta_A(\text{wt}(\mathfrak{s}))$. It is natural to ask or predict whether $\zeta_A(\mathfrak{s})$ can be connected to the logarithm of $E'_\mathfrak{s}$. The difficulty along this direction is that in general the t-module $E'_\mathfrak{s}$ is complicated, and so far we do not know how to spell out a rule of writing it down explicitly except case by case (see [CPY14, Sec. 6.1.1]). Therefore, it is difficult to compute the coefficient matrices of the logarithm of $E'_\mathfrak{s}$ following Anderson-Thakur's methods, which involve recursive matrix calculations. For other instance involving calculations of the logarithm of a higher dimensional t-module, see [CM17, G17].

To circumvent the difficulty mentioned above, we introduce new techniques based on fiber coproducts of Anderson dual t-motives and we sketch the ideas how we prove Theorem 1.2.2 below. Fix an r -tuple $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ with $n := \text{wt}(\mathfrak{s})$.

- (I). Based on the formula [C14, Thm. 5.5.2] we further express $\Gamma_{s_1} \cdots \Gamma_{s_r} \zeta_A(\mathfrak{s})$ as an explicit A -linear combination of Carlitz multiple star polylogarithms at some integral points. See Theorem 5.2.5.
- (II). For each triple $(b_\ell, \mathfrak{s}_\ell, \mathbf{u}_\ell)$ occurring in the right hand side of the identity in Theorem 5.2.5, following [CM17] we explicitly construct a *uniformizable* t-module G_ℓ defined over k and a special point $\mathbf{v}_\ell \in G_\ell(k)$ and show that the logarithm \log_{G_ℓ} of G_ℓ converges at the special point \mathbf{v}_ℓ , and the n th coordinate of $\log_{G_\ell}(\mathbf{v}_\ell)$ gives $(-1)^{\text{dep}(\mathfrak{s}_\ell)-1} \text{Li}_{\mathfrak{s}_\ell}^*(\mathbf{u}_\ell)$. See Theorem 4.2.3.
- (III). We mention that G_ℓ comes from a rigid analytically trivial Anderson dual t-motive \mathcal{M}'_ℓ with $C^{\otimes n}$ as a sub-t-motive, where $C^{\otimes n}$ is the n th tensor power of the Carlitz t-motive (see Remark 4.1.10). We then define \mathcal{M} to be the fiber coproduct of those \mathcal{M}'_ℓ over $C^{\otimes n}$ and show that it is rigid analytically trivial in Proposition 2.5.3. Such \mathcal{M} corresponds to a uniformizable t-module $G_\mathfrak{s}$ and one has a natural morphism $\pi : \bigoplus_\ell G_\ell \rightarrow G_\mathfrak{s}$ defined over k (see Lemma 3.2.3). We then define $Z_\ell := \log_{G_\ell}(\mathbf{v}_\ell)$, $Z_\mathfrak{s} := \partial\pi((\partial[b_\ell(t)]Z_\ell)_\ell) \in \text{Lie } G_\mathfrak{s}(C_\infty)$, and $\mathbf{v}_\mathfrak{s} := \pi((\partial[b_\ell(t)]\mathbf{v}_\ell)_\ell) \in G_\mathfrak{s}(k)$, where $b_\ell \in A$ are given in Theorem 5.2.5 and $\partial[\cdot]$ is given in (2.2.1).
- (IV). In Lemma 3.3.2, we show that the n th coordinate of $Z_\mathfrak{s}$ is exactly the summation of the n th coordinate of $\partial[b_\ell(t)]Z_\ell$. Then by the formula in Theorem 5.2.5 the desired result follows.

1.3. The second main result and applications. We first briefly review Furusho's p -adic MZV's. Let p be a prime number. Given an r -tuple $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ with $s_1 \geq 2$, Furusho considered the one-variable p -adic multiple polylogarithm

$$\text{Li}_{(s_1, \dots, s_r)}(z) := \sum_{n_1 > n_2 > \dots > n_r \geq 1} \frac{z^{n_1}}{n_1^{s_1} \cdots n_r^{s_r}},$$

which is the same power series presentation as complex valued one-variable function, but we consider the p -adic convergence. This function converges on the open unit disk centered at 0 of \mathbb{C}_p , where \mathbb{C}_p is the p -adic completion of a fixed algebraic closure of \mathbb{Q}_p . We note that the open unit disk centered at 0 of \mathbb{C}_p and the one centered at 1 of \mathbb{C}_p are disjoint, and so it does not make sense when taking limit $z \rightarrow 1$ on \mathbb{C}_p . However, Furusho applied Coleman's p -adic iterated integration theory [Co82] to make an analytic

continuation of p -adic multiple polylogarithms and then defined the p -adic MZV $\zeta_p(\mathfrak{s})$ to be the limit value at 1 of the analytically continued p -adic multiple polylogarithm.

Now we turn back to the function field case. As we mention above, we have the formula in Theorem 5.2.5 for $\zeta_A(\mathfrak{s})$:

$$\zeta_A(\mathfrak{s}) = \frac{1}{\Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{\ell} b_{\ell} \cdot (-1)^{\text{dep}(\mathfrak{s}_{\ell})-1} \text{Li}_{s_{\ell}}^*(\mathbf{u}_{\ell}).$$

Fix a finite place v of k and let k_v be the completion of k at v . Let \mathbf{C}_v be the v -adic completion of a fixed algebraic closure of k_v . We mention that the CMSPL $\text{Li}_{s_{\ell}}^*$ converges v -adically on the open unit ball centered at the zero of $\mathbf{C}_v^{\text{dep}(\mathfrak{s}_{\ell})}$. In [CM17, Sec. 4], it is shown that $\text{Li}_{s_{\ell}}^*$ can be analytically continued to the closed unit ball centered at the zero of $\mathbf{C}_v^{\text{dep}(\mathfrak{s}_{\ell})}$, and so it can be evaluated v -adically at \mathbf{u}_{ℓ} and we denote by $\text{Li}_{s_{\ell}}^*(\mathbf{u}_{\ell})_v$ the v -adic value. We then define the v -adic MZV $\zeta_A(\mathfrak{s})_v$ in Definition 6.1.1 to be

$$\zeta_A(\mathfrak{s})_v := \frac{1}{\Gamma_{s_1} \cdots \Gamma_{s_r}} \sum_{\ell} b_{\ell} \cdot (-1)^{\text{dep}(\mathfrak{s}_{\ell})-1} \text{Li}_{s_{\ell}}^*(\mathbf{u}_{\ell})_v \in k_v.$$

The weight and depth of this presentation $\zeta_A(\mathfrak{s})_v$ are defined to be $\text{wt}(\mathfrak{s})$ and $\text{dep}(\mathfrak{s})$ respectively. Note that here we do not exclude the v -part and so for each $s \in \mathbb{N}$, $(1-v^{-s})\zeta_A(s)_v$ is identical to Goss' v -adic zeta value [Go79] at s (see [AT90, Theorem 3.8.3. (II)]). We further mention that Thakur [T04, Sec. 5.10] also defined v -adic MZV's by using Kummer congruences to interpolate the power sums at non-positive integers, and he remarked that his interpolated v -adic MZV's are not the same as ours defined above but they are expected to be related by certain linear relations.

The second main result in this paper stated as Theorem 6.2.4 is that for each v and $\mathfrak{s} \in \mathbb{N}^r$ with $n := \text{wt}(\mathfrak{s})$, the logarithm of the t -module $G_{\mathfrak{s}}$ converges v -adically at the point $[\mathbf{a}]_v$ in $G_{\mathfrak{s}}(k)$ for some nonzero $\mathbf{a} \in \mathbb{F}_q[t]$ and its n th coordinate gives $\alpha(\theta)\Gamma_{\mathfrak{s}}\zeta_A(\mathfrak{s})_v$. When $r = 1$, our result is the same as [AT90, Thm. 3.8.3 (II)]. As an application of these logarithmic interpretations, we apply Yu's sub- t -module theorem to create a route from the space of ∞ -adic MZV's to that of v -adic MZV's.

Theorem 1.3.1. *Let v be a finite place of k and fix an embedding $\bar{k} \hookrightarrow \mathbf{C}_v$. Let n be a positive integer and let $\overline{\mathcal{L}}_n$ be the \bar{k} -vector space spanned by all ∞ -adic MZV's of weight n , and $\overline{\mathcal{L}}_{n,v}$ be the \bar{k} -vector space spanned by all v -adic MZV's of weight n . Then we have a well-defined surjective \bar{k} -linear map*

$$\overline{\mathcal{L}}_n \twoheadrightarrow \overline{\mathcal{L}}_{n,v}$$

given by

$$\zeta_A(\mathfrak{s}) \mapsto \zeta_A(\mathfrak{s})_v$$

and its kernel contains the one-dimensional vector space $\bar{k} \cdot \zeta_A(n)$ when n is divisible by $q-1$.

Since by [Go79] we have $\zeta_A(n)_v = 0$ when n is divisible by $q-1$, an interesting consequence of the theorem above is that if $\zeta_A(\mathfrak{s})$ is "Eulerian", i.e., $\zeta_A(\mathfrak{s})$ is a k -multiple of the $\text{wt}(\mathfrak{s})$ th power of Carlitz period, then $\zeta_A(\mathfrak{s})_v = 0$. We mention that for fixed weight, the spirit of the result above is that v -adic MZV's satisfy the same \bar{k} -linear relations that their corresponding ∞ -adic MZV's satisfy. A stronger consequence is obtained in Corollary 6.4.3. Note that by [C14] the \bar{k} -linear relations among MZV's are generated by those k -linear relations among MZV's of the same weight.

The analogy of the theorem above in the classical setting between real-valued MZV's and p-adic MZV's is still unknown. However, we mention that both real-valued MZV's and p-adic MZV's satisfy the regularized double shuffle relations [IKZ06, FJ07], and conjecturally regularized double shuffle relations generate all the \mathbb{Q} -linear relations among the real-valued MZV's. Therefore, conjecturally p-adic MZV's satisfy the \mathbb{Q} -linear relations that their corresponding real-valued MZV's satisfy. Compared with this (classical) conjecture, our Theorem 1.3.1 provides stronger results as it is over algebraic coefficients.

1.4. Organization of this paper. We mention that one of our goals in writing this paper has been to introduce our techniques as general and robust as possible. Therefore we do not organize this paper in order matching the steps from (I) to (IV) above. We first review the related theory of Anderson t-modules and Anderson dual t-motives in Section 2, and then consider the fiber coproducts of Anderson dual t-motives in a setting as general as possible. The purpose of Section 3 is to establish the key result in Lemma 3.3.2 for handling tractable coordinates of logarithmic vectors with respect to the fiber coproduct in question. Then step (IV) above becomes a consequence of Lemma 3.3.2. Section 4 is devoted to verify Step (II) above. In Section 5 we set the stage for our MZV's: to any given MZV we associate a fiber coproduct family of Anderson dual t-motives satisfying the hypothesis of Lemma 3.3.2; furthermore an explicit integral point is picked up on the t-module associated to each of the t-motives in this coproduct family. This set up then enables us to prove Theorem 1.2.2.

Finally, we define v-adic MZV's in Section 6 and give a logarithmic interpretation for them (see Theorem 6.2.4). We then use these logarithmic interpretations for ∞ -adic and v-adic MZV's (Theorems 1.2.2 and 6.2.4) as well as Yu's sub-t-module theorem [Yu97] to prove Theorem 1.3.1. At the end we list three natural and interesting questions in Remark 6.4.4, which we will investigate in a future project.

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2. FIBER COPRODUCT OF ANDERSON DUAL t-MOTIVES

Throughout this paper, we will call *Anderson dual t-motives* for those called *dual t-motives* in [ABP04] and called *Anderson t-motives* in [P08].

2.1. Anderson dual t-motives. Let $\mathbb{C}_\infty((t))$ be the field of Laurent series in the variable t over \mathbb{C}_∞ . For an integer i , we define the i th fold twisting automorphism on $\mathbb{C}_\infty((t))$ given by $f \mapsto f^{(i)}$, where $f^{(i)} := \sum a_j^{q^i} t^j$ for $f = \sum a_j t^j \in \mathbb{C}_\infty((t))$. We extend the i th fold twisting to an operator on matrices with entries in $\mathbb{C}_\infty((t))$ by entry-wise action.

We define the twisted polynomial ring $\bar{k}[t, \sigma]$ generated by the two variables t and σ subject to the relations

$$\sigma f = f^{(-1)} \sigma \text{ for } f \in \bar{k}[t].$$

Definition 2.1.1. *An Anderson dual t-motive is a left $\bar{k}[t, \sigma]$ -module \mathcal{M} satisfying that*

- (1) \mathcal{M} is a free left $\bar{k}[t]$ -module of finite rank.
- (2) \mathcal{M} is a free left $\bar{k}[\sigma]$ -module of finite rank.

(3) $(t - \theta)^s \mathcal{M} \subset \sigma \mathcal{M}$ for all sufficiently large integers s .

For an Anderson dual t -motive \mathcal{M} of rank r over $\bar{k}[t]$ and of rank d over $\bar{k}[\sigma]$, we call the vector $\mathbf{x} = (x_1, \dots, x_r) \in \text{Mat}_{1 \times r}(\mathcal{M})$ (resp. $\mathbf{v} = (v_1, \dots, v_d) \in \text{Mat}_{1 \times d}(\mathcal{M})$) a $\bar{k}[t]$ -basis for \mathcal{M} (resp. a $\bar{k}[\sigma]$ -basis) if x_1, \dots, x_r (resp. v_1, \dots, v_d) form a $\bar{k}[t]$ -basis (resp. $\bar{k}[\sigma]$ -basis) of \mathcal{M} . Fixing a $\bar{k}[t]$ -basis \mathbf{x} for \mathcal{M} , then there exists a unique matrix $\Phi \in \text{Mat}_r(\bar{k}[t]) \cap \text{GL}_r(\bar{k}(t))$ satisfying that

$$\sigma \mathbf{x}^{\text{tr}} = \Phi \mathbf{x}^{\text{tr}},$$

where $\sigma \mathbf{x}^{\text{tr}}$ is defined via entry-wise action. We say that the matrix Φ represents multiplication by σ on \mathcal{M} with respect to \mathbf{x} (cf. [P08, Sec. 3.2.3]).

A typical example is the n th tensor power of the Carlitz t -motive denoted by $C^{\otimes n}$ for a positive integer n . The underlying module of $C^{\otimes n}$ is $\bar{k}[t]$, on which σ acts by

$$\sigma f := (t - \theta)^n f^{(-1)} \text{ for } f \in \bar{k}[t].$$

It is not hard to check that $C^{\otimes n}$ is an Anderson dual t -motive with a $\bar{k}[\sigma]$ -basis given by

$$\left((t - \theta)^{n-1}, \dots, (t - \theta), 1 \right).$$

As a left $\mathbb{F}_q[t]$ -module the quotient $C^{\otimes n}/(\sigma - 1)C^{\otimes n}$ gives the \bar{k} -valued points of the n th tensor power $C^{\otimes n}$ of the Carlitz module defined in (2.2.7). This fact was known by Anderson, and the reader can consult [T04] and [CPY14, Sec. 5.2].

2.2. Anderson t -modules. We quickly review the theory of t -modules developed by Anderson in [A86]. For any field extension L/k , we let $\tau : L \rightarrow L$ be the Frobenius q th power operator, and one naturally extends it to an operator on L^s by entry-wise action. Let $L[\tau]$ is the twisted polynomial ring generated by τ over L subject to the relation:

$$\tau \alpha = \alpha^q \tau \text{ for } \alpha \in L.$$

Given a d -dimensional additive algebraic group $G_{\alpha/L}^d$ over L , we denote by $\text{End}_{\mathbb{F}_q}(G_{\alpha/L}^d)$ the ring of endomorphisms of $G_{\alpha/L}^d$ that are \mathbb{F}_q -linear and defined over L , and we naturally identify $\text{End}_{\mathbb{F}_q}(G_{\alpha/L}^d)$ with the matrix ring $\text{Mat}_d(L[\tau])$.

A d -dimensional t -module defined over L is a pair $G = (G_{\alpha/L}^d, \rho)$, where $G_{\alpha/L}^d$ is the d -dimensional additive group G_{α}^d that is defined over L and ρ is an \mathbb{F}_q -linear ring homomorphism

$$\rho : \mathbb{F}_q[t] \rightarrow \text{End}_{\mathbb{F}_q}(G_{\alpha/L}^d)$$

so that $\partial \rho_t - \theta I_d$ is a nilpotent matrix, where $\partial \rho_t$ is defined to be the induced morphism of ρ_t at the identity on the Lie algebra $\text{Lie } G_{\alpha/L}^d$ of $G_{\alpha/L}^d$. For a nonzero polynomial $\alpha \in \mathbb{F}_q[t]$, we write $\rho_\alpha = \sum_{i=0}^m \alpha_i \tau^i$ with $\alpha_i \in \text{Mat}_d(L)$, where we understand that the symbols α_i and m depend on α . Then the differential of ρ_α at the identity is explicitly expressed as

$$(2.2.1) \quad \partial \rho_\alpha = \alpha_0.$$

Note that $G(F) = G_{\alpha}^d(F)$ has a left $\mathbb{F}_q[t]$ -module structure via the map ρ for any field extension F/L .

Given such a d -dimensional t -module G over L , Anderson [A86] showed the existence of a d -variable power series \exp_G with coefficients in L for which

$$(a) \exp_G(\mathbf{z}) \equiv \mathbf{z} \pmod{\deg q};$$

(b) for any $a \in \mathbb{F}_q[t]$, the following identity holds:

$$(2.2.2) \quad \rho_a \circ \exp_G = \exp_G \circ \partial \rho_a.$$

We mention that when we work over the field \mathbb{C}_∞ , $\exp_G : \text{Lie } G(\mathbb{C}_\infty) \rightarrow G(\mathbb{C}_\infty)$ is entire. Such as the classical terminology for Lie groups, we call \exp_G *the exponential map* of the t -module G . The formal inverse of the power series \exp_G is called the *logarithm* of G denoted by \log_G and it satisfies:

$$(2.2.3) \quad \exp_G \circ \log_G(\mathbf{z}) = \mathbf{z} = \log_G \circ \exp_G(\mathbf{z}) \text{ (as power series identities).}$$

$$(2.2.4) \quad \log_G \circ \rho_a = \partial \rho_a \circ \log_G \text{ for every } a \in \mathbb{F}_q[t].$$

Note that \log_G is the power series expansion around the origin of the multi-valued inverse map to \exp_G .

In fact, the exponential map \exp_G is functorial in G in the following sense. Let G and G' be two t -modules defined over L . By a morphism from G to G' over L , we mean a morphism as algebraic groups $\phi : G \rightarrow G'$ that is defined over L and that commutes with $\mathbb{F}_q[t]$ -actions. The functoriality property [A86, p. 473] means that we have the following functional equation:

$$(2.2.5) \quad \phi \circ \exp_G = \exp_{G'} \circ \partial \phi,$$

where $\partial \phi$ is the differential of the morphism ϕ at the identity. The functional equation for exponential maps and (2.2.3) imply the following functional equation for logarithms:

$$(2.2.6) \quad \log_{G'} \circ \phi = \partial \phi \circ \log_G.$$

An example of a t -module is the s th tensor power of the Calitz module denoted by $\mathbf{C}^{\otimes s} = (G_{a/k}^s, [-]_s)$ for any positive integer s . The underlying space of $\mathbf{C}^{\otimes s}$ is $G_{a/k}^s$ equipped with the $\mathbb{F}_q[t]$ -module structure given (and so uniquely determined) by

$$(2.2.7) \quad [t]_s = \begin{pmatrix} \theta & 1 & 0 & \cdots & 0 \\ & \theta & 1 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ \tau & & & & \theta \end{pmatrix} \in \text{Mat}_s(k[\tau]).$$

We call a t -module G over \bar{k} *uniformizable* if its exponential map $\exp_G : \text{Lie } G(\mathbb{C}_\infty) \rightarrow G(\mathbb{C}_\infty)$ is surjective. We mention that there are examples of t -modules which are not uniformizable, see [A86, Sec. 2.2]. Note that $\mathbf{C}^{\otimes s}$ is uniformizable for each $s \in \mathbb{N}$, see [Go96, Cor. 5.9.38].

2.3. From Anderson dual t -motives to t -modules. Here we review how one constructs a t -module from an Anderson dual t -motive following Anderson's approach (see [CPY14, Sec. 5.2], [BP16, Sec. 4.4] and [HJ16, Sec. 5.2]). Let \mathcal{M} be an Anderson dual t -motive with a $\bar{k}[t]$ -basis (x_1, \dots, x_r) , and a $\bar{k}[\sigma]$ -basis (v_1, \dots, v_d) . For any $y \in \mathcal{M}$, we express $y = \sum_{i=1}^d g_i v_i$ with $g_i \in \bar{k}[\sigma]$ and then define the map $\Delta : \mathcal{M} \rightarrow \text{Mat}_{d \times 1}(\bar{k})$ by

$$(2.3.1) \quad \Delta(g) := (\delta(g_1), \dots, \delta(g_d))^{\text{tr}} \in \text{Mat}_{d \times 1}(\bar{k}),$$

where for $g = \sum_j a_j \sigma^j (= \sum_j \sigma^j a_j^{q^j}) \in \bar{k}[\sigma]$, $\delta : \bar{k}[\sigma] \rightarrow \bar{k}$ is defined by

$$\delta(g) := \sum_j a_j^{q^j}.$$

It is clear that Δ is \mathbb{F}_q -linear and surjective. One further checks that $\text{Ker } \Delta = (\sigma - 1)\mathcal{M}$, and therefore we have the induced isomorphism

$$\Delta : \mathcal{M} / (\sigma - 1)\mathcal{M} \cong \text{Mat}_{d \times 1}(\bar{k}).$$

As $\mathbb{F}_q[t]$ is contained in the center of $\bar{k}[t, \sigma]$, $\mathcal{M} / (\sigma - 1)\mathcal{M}$ has a left $\mathbb{F}_q[t]$ -module structure, which allows us to equip with an $\mathbb{F}_q[t]$ -module structure on $\text{Mat}_{d \times 1}(\bar{k})$ from the isomorphism above. One thereby has a unique \mathbb{F}_q -linear ring homomorphism

$$\rho : \mathbb{F}_q[t] \rightarrow \text{Mat}_d(\bar{k}[\tau]),$$

whence defining a t -module $G = (\mathbb{G}_{a/\bar{k}}^d, \rho)$ associated to the Anderson dual t -motive \mathcal{M} since the group of \bar{k} -valued points is Zariski dense in $\mathbb{G}_{a/\bar{k}}^d$.

2.4. The fiber coproduct. In this section, we will construct a fiber coproduct of certain Anderson dual t -motives, which will play the key role when proving Theorem 1.2.2. Here, we deal with the situation as general as possible, and expect it to have wide applications for the related issues.

2.4.1. The set up. Let \mathcal{N} be an Anderson dual t -motive of rank r over $\bar{k}[t]$, and we fix a $\bar{k}[t]$ -basis $\mathbf{x} = (x_1, \dots, x_r) \in \text{Mat}_{1 \times r}(\mathcal{N})$ as well as a $\bar{k}[\sigma]$ -basis $\alpha = (\alpha_1, \dots, \alpha_n) \in \text{Mat}_{1 \times n}(\mathcal{N})$ for \mathcal{N} . Let $B := B_{\mathcal{N}} \in \text{Mat}_r(\bar{k}[t]) \cap \text{GL}_r(\bar{k}(t))$ be the matrix presenting multiplication by σ on \mathcal{N} with respect to \mathbf{x} , ie.,

$$\sigma \mathbf{x}^{\text{tr}} = B \mathbf{x}^{\text{tr}}.$$

Suppose that $\{\mathcal{M}'_\ell\}_{\ell=1}^T$ is a family of Anderson dual t -motives equipped with inclusions $\mathcal{N} \hookrightarrow \mathcal{M}'_\ell$ so that either

$$(2.4.1) \quad \mathcal{M}'_\ell \text{ is isomorphic to } \mathcal{N} \text{ as left } \bar{k}[t, \sigma]\text{-modules}$$

or

$$(2.4.2) \quad \mathcal{M}'_\ell \text{ fits into the short exact sequence of left } \bar{k}[t, \sigma]\text{-modules}$$

$$0 \rightarrow \mathcal{N} \rightarrow \mathcal{M}'_\ell \rightarrow \mathcal{M}''_\ell \rightarrow 0,$$

where \mathcal{M}''_ℓ is an Anderson dual t -motive of rank $m_\ell \geq 1$ over $\bar{k}[t]$. We let $\mathcal{T} = \{1, \dots, T\}$ and decompose it as the disjoint union

$$\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2,$$

where \mathcal{T}_1 consists of those indexes ℓ for which \mathcal{M}'_ℓ satisfies (2.4.1) and \mathcal{T}_2 consists of those indexes ℓ for which \mathcal{M}'_ℓ satisfies (2.4.2). We let $s := |\mathcal{T}_1|$, and for convenience we rearrange the indexes so that

$$\mathcal{T}_1 = \{1, \dots, s\} \text{ and } \mathcal{T}_2 = \{s+1, \dots, T\}.$$

It is allowed to be the case that $s = 0$, ie., $\mathcal{T}_1 = \emptyset$ and $\mathcal{T}_2 = \mathcal{T}$, or the case that $s = T$, ie., $\mathcal{T}_1 = \mathcal{T}$ and $\mathcal{T}_2 = \emptyset$. In the later case when $s = T$, it means that \mathcal{M}'_ℓ is isomorphic

to \mathcal{N} for all ℓ . In the former case when $s = 0$, every \mathcal{M}'_ℓ is an extension of \mathcal{M}''_ℓ by \mathcal{N} in (2.4.2).

By convenience we put $m_\ell = 0$ for $1 \leq \ell \leq s$. For each $1 \leq \ell \leq T$, we denote by $\mathbf{x}_\ell = (x_{\ell 1}, \dots, x_{\ell r}) \in \text{Mat}_{1 \times r}(\mathcal{M}'_\ell)$ the image of the $\bar{k}[t]$ -basis $\mathbf{x} = (x_1, \dots, x_r)$ for \mathcal{N} under the map $\mathcal{N} \hookrightarrow \mathcal{M}'_\ell$. Since \mathcal{M}''_ℓ is free of rank m_ℓ over $\bar{k}[t]$, there exist vectors $\mathbf{y}_\ell = (y_{\ell 1}, \dots, y_{\ell m_\ell}) \in \text{Mat}_{1 \times m_\ell}(\mathcal{M}'_\ell)$, where $\mathbf{y}_\ell = \emptyset$ for $1 \leq \ell \leq s$, so that $(\mathbf{x}_\ell, \mathbf{y}_\ell)$ is a $\bar{k}[t]$ -basis for \mathcal{M}'_ℓ . For the $\bar{k}[t]$ -basis $(\mathbf{x}_\ell, \mathbf{y}_\ell)$, the action of σ is given by the form

$$\sigma \begin{pmatrix} \mathbf{x}_\ell^{\text{tr}} \\ \mathbf{y}_\ell^{\text{tr}} \end{pmatrix} = \begin{pmatrix} B & 0 \\ D_\ell & \Phi_\ell'' \end{pmatrix} \begin{pmatrix} \mathbf{x}_\ell^{\text{tr}} \\ \mathbf{y}_\ell^{\text{tr}} \end{pmatrix}.$$

Here $\Phi_\ell'' \in \text{Mat}_{m_\ell}(\bar{k}[t]) \cap \text{GL}_{m_\ell}(\bar{k}(t))$ is the matrix representing multiplication by σ on \mathcal{M}''_ℓ with respect to the $\bar{k}[t]$ -basis as the image of \mathbf{y}_ℓ in \mathcal{M}''_ℓ .

For each $1 \leq \ell \leq T$, we denote by $\tilde{\alpha}_\ell := (\alpha_{\ell 1}, \dots, \alpha_{\ell n}) \in \text{Mat}_{1 \times n}(\mathcal{M}'_\ell)$ the image of the $\bar{k}[\sigma]$ -basis $\alpha = (\alpha_1, \dots, \alpha_n)$ for \mathcal{N} under the map $\mathcal{N} \hookrightarrow \mathcal{M}'_\ell$. We understand that for $1 \leq \ell \leq s$, $\tilde{\alpha}_\ell$ is a $\bar{k}[\sigma]$ -basis for \mathcal{M}'_ℓ since $\mathcal{N} \cong \mathcal{M}'_\ell$, and since \mathcal{M}''_ℓ is free of finite rank over $\bar{k}[\sigma]$ for $s+1 \leq \ell \leq T$, $\tilde{\alpha}_\ell$ can be extended to a $\bar{k}[\sigma]$ -basis $(\tilde{\alpha}_\ell, \beta_\ell)$ for \mathcal{M}'_ℓ for some $\beta_\ell \in \text{Mat}_{1 \times h_\ell}(\mathcal{M}'_\ell)$ with $h_\ell := \text{rank}_{\bar{k}[\sigma]} \mathcal{M}''_\ell$. Note that the image of β_ℓ under the quotient map $\mathcal{M}'_\ell \twoheadrightarrow \mathcal{M}''_\ell$ forms a $\bar{k}[\sigma]$ -basis for \mathcal{M}''_ℓ . By convenience, for $1 \leq \ell \leq s$ we put $h_\ell = 0$ and $\beta_\ell = \emptyset$.

We note that for the case applied to prove Theorem 1.2.2, the \mathcal{N} above would be the n th tensor power of the Carlitz t -motive and the \mathcal{M}'_ℓ would be the Anderson dual t -motive occurring from certain Carlitz multiple polylogarithms at certain integral points constructed from [C14, CPY14]. See Sec. 5.3.

2.4.2. The definition. We continue with the notation and set up as above. We define \mathcal{M} to be the fiber coproduct of all \mathcal{M}'_ℓ over \mathcal{N} denoted by

$$\mathcal{M} := \mathcal{M}'_1 \sqcup_{\mathcal{N}} \mathcal{M}'_2 \sqcup_{\mathcal{N}} \cdots \sqcup_{\mathcal{N}} \mathcal{M}'_T.$$

More precisely, as a left $\bar{k}[t]$ -module, \mathcal{M} is defined by the quotient:

$$(2.4.3) \quad \mathcal{M} := \left(\bigoplus_{\ell=1}^T \mathcal{M}'_\ell \right) / \left(\text{Span}_{\bar{k}[t]} \{x_{\ell i} - x_{\ell' i} \mid 1 \leq \ell, \ell' \leq T, 1 \leq i \leq r\} \right).$$

Without confusion, we denote by x_i the image of $x_{\ell i}$ in the quotient module \mathcal{M} for any ℓ , and $1 \leq i \leq r$. This is well defined from the description of \mathcal{M} above, and it makes sense to use the notation as one has the natural embedding $\mathcal{N} \hookrightarrow \mathcal{M}$. We still denote by $y_{\ell j}$ the image of $y_{\ell j}$ in the quotient module \mathcal{M} for $s+1 \leq \ell \leq T$, and $1 \leq j \leq m_\ell$, as it is well-defined due to (2.4.3). Under such notation, it is clear to see that \mathcal{M} is a free $\bar{k}[t]$ -module and

$$(2.4.4) \quad \mathbf{m} := (\mathbf{x}, \mathbf{y}_{s+1}, \dots, \mathbf{y}_T)$$

is a $\bar{k}[t]$ -basis for \mathcal{M} .

Proposition 2.4.5. *The left $\bar{k}[t]$ -module \mathcal{M} defined above is an Anderson dual t -motive.*

Proof. We first claim that the $\bar{k}[t]$ -submodule $\text{Span}_{\bar{k}[t]} \{x_{\ell i} - x_{\ell' i} \mid 1 \leq \ell, \ell' \leq T, 1 \leq i \leq r\}$ is stable under the σ -action, whence a left $\bar{k}[t, \sigma]$ -submodule of $\bigoplus_{\ell=1}^T \mathcal{M}'_\ell$. To show this, we note that $x_{\ell i} - x_{\ell' i}$ is the i th component of $(\mathbf{x}_\ell - \mathbf{x}_{\ell'})^{\text{tr}}$. By definition, $\sigma(\mathbf{x}_\ell - \mathbf{x}_{\ell'})^{\text{tr}}$ is the vector

$$B \cdot (\mathbf{x}_\ell - \mathbf{x}_{\ell'})^{\text{tr}}.$$

By expanding the above vector we see that

$$\sigma(x_{\ell i} - x_{\ell' i}) \in \text{Span}_{\bar{k}[t]} \{x_{\ell i} - x_{\ell' i} \mid 1 \leq \ell, \ell' \leq T, 1 \leq i \leq r\}.$$

To show that \mathcal{M} is free of finite rank over $\bar{k}[\sigma]$, we first note that the following matrix

$$(2.4.6) \quad \Phi := \begin{pmatrix} B & & & & \\ D_{s+1} & \Phi''_{s+1} & & & \\ \vdots & & \ddots & & \\ D_T & & & & \Phi''_T \end{pmatrix}$$

is the matrix representing the action of σ on \mathcal{M} with respect to the $\bar{k}[t]$ -basis \mathbf{m} given in (2.4.4). It follows that we have the following short exact sequence of Anderson dual t -motives:

$$0 \rightarrow \mathcal{N} \rightarrow \mathcal{M} \rightarrow \bigoplus_{\ell=s+1}^T \mathcal{M}_\ell'' \rightarrow 0.$$

By hypothesis, each \mathcal{M}_ℓ'' is an Anderson dual t -motive, so is $\bigoplus_{\ell=s+1}^T \mathcal{M}_\ell''$. Since \mathcal{N} and $\bigoplus_{\ell=s+1}^T \mathcal{M}_\ell''$ are Anderson dual t -motives, \mathcal{M} is a finitely generated $\bar{k}[\sigma]$ -module. By [ABP04, Prop. 4.3.4], we know that $\bar{k}[t]$ -torsion submodule of \mathcal{M} is as same as the $\bar{k}[\sigma]$ -torsion submodule of \mathcal{M} , and hence \mathcal{M} is free over $\bar{k}[\sigma]$ since \mathcal{M} is a free left $\bar{k}[t]$ -module.

Finally, one directly checks that $(t - \theta)^i \mathcal{M} \subset \sigma \mathcal{M}$ for sufficiently large integers i , whence \mathcal{M} is an Anderson dual t -motive. \square

2.5. Rigid analytic trivialization. Let $\mathbb{T} \subset \mathbb{C}_\infty((t))$ be the subring consisting of power series that are convergent on the closed unit disk centered at the zero of \mathbb{C}_∞ . More precisely, every element f in \mathbb{T} is of the form $f = \sum_{i=0}^{\infty} a_i t^i$ with the property that $|a_i|_\infty \rightarrow 0$ as $i \rightarrow \infty$. We follow [ABP04, P08] to introduce the following terminology (cf. [A86]).

Definition 2.5.1. *Let M be an Anderson dual t -motive of rank r over $\bar{k}[t]$. Let $\Phi \in \text{Mat}_r(\bar{k}[t]) \cap \text{GL}_r(\bar{k}(t))$ be the matrix representing multiplication by σ on certain $\bar{k}[t]$ -basis for M . We say that M is rigid analytically trivial if there exists a matrix $\Psi \in \text{GL}_r(\mathbb{T})$ so that*

$$\Psi^{(-1)} = \Phi \Psi.$$

Such a Ψ is called a rigid analytic trivialization of Φ .

Remark 2.5.2. If an Anderson dual t -motive is rigid analytically trivial, then its associated t -module is uniformizable. See [BP16, Sec. 4.5] and [HJ16, Thm. 5.2.8].

Proposition 2.5.3. *Let \mathcal{N} , $\{\mathcal{M}_\ell'\}_{\ell=1}^T$ be the Anderson dual t -motives given in Sec. 2.4.1 and suppose that all of them are rigid analytically trivial. Then so is the fiber coproduct \mathcal{M} of $\{\mathcal{M}_\ell'\}_{\ell=1}^T$ over \mathcal{N} .*

Proof. We continue with the above notation that B is the matrix representing multiplication by σ on \mathbf{x} for \mathcal{N} , and for each $s+1 \leq \ell \leq T$,

$$\begin{pmatrix} B & \\ D_\ell & \Phi_\ell'' \end{pmatrix}$$

is the matrix representing multiplication by σ on $(\mathbf{x}_\ell, \mathbf{y}_\ell)$ for \mathcal{M}_ℓ' .

Since \mathcal{N} and \mathcal{M}'_ℓ are rigid analytically trivial, there exist rigid analytic trivializations Q and

$$\begin{pmatrix} Q \\ R_\ell & \Psi''_\ell \end{pmatrix}$$

for which $Q^{(-1)} = BQ$ and

$$\begin{pmatrix} Q \\ R_\ell & \Psi''_\ell \end{pmatrix}^{(-1)} = \begin{pmatrix} B \\ D_\ell & \Phi''_\ell \end{pmatrix} \begin{pmatrix} Q \\ R_\ell & \Psi''_\ell \end{pmatrix}.$$

Since Φ given in (2.4.6) is the matrix representing multiplication by σ on \mathfrak{m} for \mathcal{M} , we put

$$\Psi := \begin{pmatrix} Q \\ R_{s+1} & \Psi''_{s+1} & & \\ \vdots & & \ddots & \\ R_T & & & \Psi''_T \end{pmatrix}$$

and find that Ψ is a rigid analytic trivialization of Φ . So the desired result follows. \square

3. THE KEY LEMMA

We continue with the setting and notation given in Sec. 2.4.1 and Sec. 2.4.2. As \mathcal{M} is a quotient of $\bigoplus_{\ell=1}^T \mathcal{M}'_\ell$, we have the natural projection map $\mu : \bigoplus_{\ell=1}^T \mathcal{M}'_\ell \rightarrow \mathcal{M}$. In fact, according to the definition of \mathcal{M} we can write down the map μ explicitly as

$$(3.0.1) \quad \mu \left(\left(\sum_{i=1}^r f_{\ell i}(t)x_{\ell i} + \sum_{j=1}^{m_\ell} f_{\ell j}(t)y_{\ell j} \right)_\ell \right) = \sum_{i=1}^r \left(\sum_{\ell=1}^T f_{\ell i}(t) \right) x_i + \sum_{\ell=s+1}^T \sum_{j=1}^{m_\ell} f_{\ell j}(t)y_{\ell j}.$$

According to the set up in Sec. 2.4.1 that α is identified with $\tilde{\alpha}_\ell$ in \mathcal{M}'_ℓ , it follows that

$$(3.0.2) \quad \text{Span}_{\bar{k}[t]} \{x_{\ell i} - x_{\ell' i} | 1 \leq \ell, \ell' \leq T, 1 \leq i \leq r\} = \text{Span}_{\bar{k}[\sigma]} \{\alpha_{\ell i} - \alpha_{\ell' i} | 1 \leq \ell, \ell' \leq T, 1 \leq i \leq n\},$$

hence it is well-defined so that we can denote by α_i the image of $\alpha_{\ell i}$ for any $1 \leq \ell \leq T$ and $1 \leq i \leq n$. Note that such a fact can be also seen from the definition of fiber coproduct.

We denote by α_i the image of $\alpha_{\ell i}$ in \mathcal{M} , by α the image of $\tilde{\alpha}_\ell \in \text{Mat}_{1 \times n}(\mathcal{M}'_\ell)$ in $\text{Mat}_{1 \times n}(\mathcal{M})$, and by $\beta_{\ell j}$ the image of $\beta_{\ell j} \in \mathcal{M}'_\ell$ in \mathcal{M} , which are well-defined by (3.0.2) and the condition of $\bar{k}[\sigma]$ -basis $(\tilde{\alpha}_\ell, \beta_\ell)$ for \mathcal{M}'_ℓ . From the setting in Sec. 2.4.1, we see that $(\alpha, \beta_{s+1}, \dots, \beta_T)$ is a $\bar{k}[\sigma]$ -basis for \mathcal{M} .

3.1. The setting. For each $1 \leq \ell \leq T$, we let G_ℓ be the t -module associated to the Anderson dual t -motive \mathcal{M}'_ℓ , i.e., we have the $\mathbb{F}_q[t]$ -module isomorphism

$$G_\ell(\bar{k}) \cong \mathcal{M}'_\ell / (\sigma - 1)\mathcal{M}'_\ell.$$

To simplify the notation, we denote by $[-]$ the $\mathbb{F}_q[t]$ -action on any t -module without confusions. We denote by H the t -module associated to the Anderson dual t -motive \mathcal{N} . By our hypothesis that $\mathcal{N} \cong \mathcal{M}'_\ell$ for $1 \leq \ell \leq s$ and the identification of $\bar{k}[\sigma]$ -bases α and $\tilde{\alpha}_\ell$, H is the t -module associated to \mathcal{M}'_ℓ for $1 \leq \ell \leq s$.

By Proposition 2.4.5 we know that \mathcal{M} is an Anderson dual t -motive. We let G be the t -module associated to \mathcal{M} , i.e., $G(\bar{k}) \cong \mathcal{M} / (\sigma - 1)\mathcal{M}$ as $\mathbb{F}_q[t]$ -modules. Recall

that $(\alpha, \beta_{s+1}, \dots, \beta_T)$ is a $\bar{k}[\sigma]$ -basis of \mathcal{M} and the rank of \mathcal{M}'_ℓ over $\bar{k}[\sigma]$ is $n + h_\ell$ for $s+1 \leq \ell \leq T$. So the dimension of G is

$$(3.1.1) \quad \dim G = n + h_{s+1} + \dots + h_T.$$

3.2. The main diagram.

Definition 3.2.1. Let n be the rank of \mathcal{N} over $\bar{k}[\sigma]$. For any integer $m \geq n$ and any vector $\mathbf{z} = (z_1, \dots, z_m)^{\text{tr}} \in \mathbb{C}_\infty^m$, we put

$$\hat{\mathbf{z}} := \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \quad \text{and} \quad \mathbf{z}_- := \begin{pmatrix} z_{n+1} \\ \vdots \\ z_m \end{pmatrix}$$

and so \mathbf{z} is expressed as

$$\mathbf{z} = \begin{pmatrix} \hat{\mathbf{z}} \\ \mathbf{z}_- \end{pmatrix}.$$

Definition 3.2.2. We define a morphism $\pi: \bigoplus_{\ell=1}^T G_\ell \rightarrow G$ of algebraic groups by

$$\pi((\mathbf{z}_1^{\text{tr}}, \dots, \mathbf{z}_T^{\text{tr}})^{\text{tr}}) = \left(\sum_{\ell=1}^T \hat{\mathbf{z}}_\ell^{\text{tr}}, \mathbf{z}_{1-}^{\text{tr}}, \dots, \mathbf{z}_{T-}^{\text{tr}} \right)^{\text{tr}}.$$

Recall that $\mu: \bigoplus_{\ell=1}^T \mathcal{M}'_\ell \rightarrow \mathcal{M}$ is the natural quotient map, which is a left $\bar{k}[\mathfrak{t}, \sigma]$ -module homomorphism by (3.0.2). Via μ we find from the following Lemma that π is indeed a morphism of \mathfrak{t} -modules.

Lemma 3.2.3. Let notation be given as above. Then the following diagram

$$\begin{array}{ccc} \bigoplus_{\ell=1}^T \mathcal{M}'_\ell & \xrightarrow{\Delta} & \bigoplus_{\ell=1}^T G_\ell(\bar{k}) \\ \mu \downarrow & & \downarrow \pi \\ \mathcal{M} & \xrightarrow{\Delta} & G(\bar{k}) \end{array}$$

commutes. In particular, π is a morphism of \mathfrak{t} -modules.

Proof. Recall that for each $1 \leq \ell \leq T$, $(\tilde{\alpha}_\ell, \beta_\ell)$ is a $\bar{k}[\sigma]$ -basis for \mathcal{M}'_ℓ . Since the maps Δ, π and μ are additive, it suffices to show the commutativity of the diagram on elements of the form

$$(3.2.4) \quad \omega = f_{\ell i}(\sigma) \alpha_{\ell i} \in \mathcal{M}'_\ell \hookrightarrow \bigoplus_{\ell=1}^T \mathcal{M}'_\ell \quad \text{for } 1 \leq \ell \leq T, 1 \leq i \leq n$$

and

$$(3.2.5) \quad \omega = g_{\ell i}(\sigma) \beta_{\ell i} \in \mathcal{M}'_\ell \hookrightarrow \bigoplus_{\ell=1}^T \mathcal{M}'_\ell \quad \text{for } s+1 \leq \ell \leq T, 1 \leq i \leq h_\ell.$$

Let $\omega = f_{\ell i}(\sigma) \alpha_{\ell i}$ be given in (3.2.4). We write

$$\Delta(\omega) = \left(\begin{array}{c} 0 \\ \vdots \\ \delta(f_{\ell i}(\sigma)) \\ \vdots \\ 0 \\ \vdots \\ 0 \end{array} \right) \left. \begin{array}{l} \left. \vphantom{\begin{pmatrix} 0 \\ \vdots \\ \delta(f_{\ell i}(\sigma)) \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \right\} i \\ \left. \vphantom{\begin{pmatrix} 0 \\ \vdots \\ \delta(f_{\ell i}(\sigma)) \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \right\} n \end{array} \right\} \in G_\ell \hookrightarrow \bigoplus_{\ell=1}^T G_\ell.$$

Then by the definition of π we have

$$\pi(\Delta(\omega)) = \left(\begin{array}{c} 0 \\ \vdots \\ \delta(f_{\ell i}(\sigma)) \\ \vdots \\ 0 \\ \vdots \\ 0 \end{array} \right) \left. \begin{array}{l} \left. \right\} i \\ \left. \right\} n \\ \left. \right\} h_{s+1} + \cdots + h_T \end{array} \right\} \in G.$$

On the other hand, we recall that the image of $\alpha_{\ell i}$ under the projection map $\mu : \bigoplus_{\ell=1}^T \mathcal{M}'_{\ell} \rightarrow \mathcal{M}$ is denoted by α_i . As μ is a left $\bar{k}[t, \sigma]$ -module homomorphism, we have

$$\mu(f_{\ell i}(\sigma)\alpha_{\ell i}) = f_{\ell i}(\sigma)\alpha_i.$$

Recall further that $(\alpha_1, \dots, \alpha_n, \beta_{s+1}, \dots, \beta_T)$ is a $\bar{k}[\sigma]$ -basis for \mathcal{M} . Hence by the definition of Δ , we see that

$$\Delta(\mu(\omega)) = \Delta(f_{\ell i}(\sigma)\alpha_i)$$

is equal to

$$\pi(\Delta(\omega)) = \left(\begin{array}{c} 0 \\ \vdots \\ \delta(f_{\ell i}(\sigma)) \\ \vdots \\ 0 \\ \vdots \\ 0 \end{array} \right) \left. \begin{array}{l} \left. \right\} i \\ \left. \right\} n \\ \left. \right\} h_{s+1} + \cdots + h_T \end{array} \right\} \in G.$$

Now we consider the case of $\omega = g_{\ell i}(\sigma)\beta_{\ell i}$ in (3.2.5). We denote by

$$\mathbf{z}_{\ell} := \Delta(\omega) = \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \vdots \\ \delta(g_{\ell i}(\sigma)) \\ \vdots \\ 0 \end{array} \right) \left. \begin{array}{l} \left. \right\} n \\ \left. \right\} i \\ \left. \right\} h_{\ell} \end{array} \right\} \in G_{\ell} \hookrightarrow \bigoplus_{\ell=1}^T G_{\ell}.$$

Since $s+1 \leq \ell \leq T$ and $1 \leq i \leq h_{\ell}$, by the definition of π we have

$$\pi(\Delta(\omega)) = \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \vdots \\ \mathbf{z}_{\ell-} \\ \vdots \\ 0 \end{array} \right) \left. \begin{array}{l} \left. \right\} n \\ \left. \right\} h_{s+1} + \cdots + h_{\ell-1} \\ \left. \right\} h_{\ell+1} + \cdots + h_T \end{array} \right\} \in G.$$

Recall that we identify $\mu(\beta_\ell)$ with β_ℓ . Since μ is a left $\bar{k}[t, \sigma]$ -module homomorphism, we have

$$\mu(\omega) = g_{\ell i}(\sigma)\mu(\beta_{\ell i}) = g_{\ell i}(\sigma)\beta_{\ell i}.$$

Since $(\alpha_1, \dots, \alpha_n, \beta_{s+1}, \dots, \beta_T)$ is a $\bar{k}[\sigma]$ -basis for \mathcal{M} , via this basis we see that $\Delta(\mu(\omega))$ is the same as $\pi(\Delta(\omega))$.

Since the map μ induces an $\mathbb{F}_q[t]$ -module homomorphism

$$\bigoplus_{\ell=1}^T (\mathcal{M}'_\ell / (\sigma - 1)\mathcal{M}'_\ell) \rightarrow \mathcal{M} / (\sigma - 1)\mathcal{M},$$

the diagram above shows that $\pi : \bigoplus_{\ell=1}^T G_\ell(\bar{k}) \rightarrow G(\bar{k})$ is a left $\mathbb{F}_q[t]$ -module homomorphism, and hence π is a morphism of t -modules since the group of \bar{k} -valued points is Zariski dense inside the algebraic group in question. \square

Corollary 3.2.6. *Let notation be given as above. Let ρ_ℓ be the map defining the $\mathbb{F}_q[t]$ -module structure of G_ℓ for $1 \leq \ell \leq T$, and ρ be the map defining the $\mathbb{F}_q[t]$ -module of G . If $\rho_{\ell t} \in \text{Mat}_{\dim G_\ell}(A[\tau])$ for every $1 \leq \ell \leq T$, then $\rho_t \in \text{Mat}_{\dim G}(A[\tau])$.*

Proof. It is clear to see that the map π is surjective. Since π is $\mathbb{F}_q[t]$ -linear, the result is derived from Definition 3.2.2. \square

3.3. The key lemma. In this section, we give a formula which is a crucial step in the proof of Theorem 1.2.2. However, we state and prove the formulation in the setting as general as possible. We follow Brownawell and Papanikolas to introduce the notion of tractable coordinates, to which Yu's sub- t -module theorem is most easily applied.

Definition 3.3.1. *Let L be a field extension over k and suppose that $L^d := \text{Mat}_{d \times 1}(L)$ has a left $\mathbb{F}_q[t]$ -module structure. The i th coordinate of L^d is called tractable if the i th coordinate of $\alpha \cdot \mathbf{z}$ is equal to $\alpha(\theta)z_i$ for any $\alpha \in \mathbb{F}_q[t]$ and any $\mathbf{z} = (z_1, \dots, z_d)^{\text{tr}} \in L^d$.*

Suppose that the affine variety $\mathbb{A}_{/L}^d$ has a left $\mathbb{F}_q[t]$ -module structure in the sense that for every field extension L'/L , $\mathbb{A}^d(L')$ has a left $\mathbb{F}_q[t]$ -module structure that is functorial in L' . We say that the i th coordinate of $\mathbb{A}_{/L}^d$ is tractable if for every field extension L'/L , the i th coordinate of $\mathbb{A}^d(L')$ is tractable.

Typical examples of tractable coordinates arise from the Lie algebras of tensor powers of the Carlitz module. For any positive integer s , we note that $\text{Lie } \mathbf{C}^{\otimes s}(L) \cong L^s$ has a left $\mathbb{F}_q[t]$ -module structure via $\partial[-]_s$ for a field extension L/k . From (2.2.7) we see that the s th coordinate of $\text{Lie } \mathbf{C}^{\otimes s}(L)$ is tractable.

The main result in this section is the following lemma.

Lemma 3.3.2. *Let \mathcal{N} , $\{\mathcal{M}'_\ell\}_{\ell=1}^T$ and \mathcal{M} be the Anderson dual t -motives with hypothesis given in Sec. 2.4. Let H be the n -dimensional t -module associated to \mathcal{N} , G_ℓ be the t -module associated to \mathcal{M}'_ℓ for $\ell = 1, \dots, T$, and G be the t -module associated to \mathcal{M} . Suppose that the n th coordinate of $\text{Lie } G_\ell(\mathbf{C}_\infty)$ is tractable for all $1 \leq \ell \leq T$. Let $Z_\ell \in \text{Lie } G_\ell(\mathbf{C}_\infty)$ be a vector with n th coordinate denoted by $\mathcal{L}_{\ell n}$. Let $\pi : \bigoplus_{\ell=1}^T G_\ell \rightarrow G$ be the morphism of t -modules given in Definition 3.2.2. For each $1 \leq \ell \leq T$, let $\mathbf{b}_\ell \in \mathbb{F}_q[\theta]$ be any polynomial and put $\mathbf{v}_\ell := \exp_{G_\ell}(Z_\ell) \in G_\ell(\mathbf{C}_\infty)$, $\mathbf{Z} := \partial\pi((\partial[\mathbf{b}_\ell(t)]Z_\ell)_\ell) \in \text{Lie } G(\mathbf{C}_\infty)$ and $\mathbf{v} := \pi(([\mathbf{b}_\ell(t)]\mathbf{v}_\ell)_\ell) \in G(\mathbf{C}_\infty)$. Then we have*

- (a) *The n th coordinate of \mathbf{Z} is equal to $\sum_{\ell=1}^T \mathbf{b}_\ell \mathcal{L}_{\ell n}$.*

(b) $\exp_G(Z)$ is equal to \mathbf{v} .

Proof. By the canonical identification $\mathrm{Lie}(\bigoplus_\ell G_\ell(\mathbb{C}_\infty)) = \bigoplus_\ell \mathrm{Lie} G_\ell(\mathbb{C}_\infty)$, we have the following commutative diagram according to (2.2.5):

$$(3.3.3) \quad \begin{array}{ccc} \bigoplus_{\ell=1}^T G_\ell(\mathbb{C}_\infty) & \xrightarrow{\pi} & G(\mathbb{C}_\infty) \\ \bigoplus_\ell \exp_{G_\ell} \uparrow & & \exp_G \uparrow \\ \bigoplus_{\ell=1}^T \mathrm{Lie} G_\ell(\mathbb{C}_\infty) & \xrightarrow{\partial\pi} & \mathrm{Lie} G(\mathbb{C}_\infty). \end{array}$$

The property (b) follows from the diagram above.

To prove (a), we note that the n th coordinate of $\partial[b_\ell(t)]Z_\ell$ is given by $b_\ell \mathcal{L}_{\ell n}$ since by hypothesis the n th coordinate of $\mathrm{Lie} G_\ell(\mathbb{C}_\infty)$ is tractable. By Definition 3.2.2 the morphism π has no τ -terms when expressing it as a matrix with entries in $\bar{k}[\tau]$. So the induced morphism $\partial\pi$ has the same form as π (see (2.2.1)), implying the desired property from the definition of Z . \square

Remark 3.3.4. If we take \mathcal{N} and all $\{\mathcal{M}'_\ell\}_{\ell=1}^T$ to be $\mathbb{C}^{\otimes n}$, then the fiber coproduct of $\{\mathcal{M}'_\ell\}_{\ell=1}^T$ over \mathcal{N} is $\mathbb{C}^{\otimes n}$ and hence its associated t-module G is $\mathbb{C}^{\otimes n}$. In this case, the morphism $\pi : \bigoplus_{\ell=1}^T \mathbb{C}^{\otimes n} \rightarrow \mathbb{C}^{\otimes n}$ is the sum of vectors. This special case would help the reader understand how one uses Lemma 3.3.2 to generalize [AT90, Thm. 3.8.3(I)] to higher depth MZV's in Sec. 5.3

4. THE CONVERGENCE OF $\log_{G_\ell}(\mathbf{v}_\ell)$

In this section, we consider the t-module and special point constructed in [CM17], and the primary goal is to show Theorem 4.2.3 asserting that the logarithm of the t-module in question converges ∞ -adically at the special point, and certain coordinates of the logarithm give Carlitz multiple star polylogarithms. To prove Theorem 1.2.2, the results presented in this section are applied in Section 5 to illustrate that all the conditions of Lemma 3.3.2 are satisfied for our setting.

4.1. The constructions of the t-module and special point. In what follows, we fix $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ and $\mathbf{u} = (u_1, \dots, u_r) \in (\bar{k}^\times)^r$. We will define a pair (G, \mathbf{v}) associated to \mathfrak{s} and \mathbf{u} , where G is a t-module defined over \bar{k} and $\mathbf{v} \in G(\bar{k})$.

Put $L_0 := 1$, and $L_i := (\theta - \theta^q) \cdots (\theta - \theta^{q^i})$ for $i \in \mathbb{N}$. We define the s th Carlitz multiple polylogarithm, abbreviated as CMPL, as follows (see [C14]):

$$(4.1.1) \quad \mathrm{Li}_{\mathfrak{s}}(z_1, \dots, z_r) := \sum_{i_1 > \dots > i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}}.$$

To avoid heavy notation on the subscript, we use the same notation $\mathrm{Li}_{\mathfrak{s}}$ in the function field setting. Since we no longer use the classical multiple polylogarithms in the later context, there will not be misunderstanding.

We also define the s th Carlitz multiple star polylogarithm, abbreviated as CMSPL, as follows (see [CM17]):

$$(4.1.2) \quad \mathrm{Li}_{\mathfrak{s}}^*(z_1, \dots, z_r) := \sum_{i_1 \geq \dots \geq i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{L_{i_1}^{s_1} \cdots L_{i_r}^{s_r}}.$$

Remark 4.1.3. For an r -tuple $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, we put

$$\mathbb{D}'_{\mathfrak{s}} := \left\{ (x_1, \dots, x_r) \in \mathbb{C}_{\infty}^r : |x_i|_{\infty} < q^{\frac{s_i q}{q-1}} \text{ for } i = 1, \dots, r \right\} \subset \mathbb{D}''_{\mathfrak{s}},$$

where

$$(4.1.4) \quad \mathbb{D}''_{\mathfrak{s}} := \left\{ (x_1, \dots, x_r) \in \mathbb{C}_{\infty}^r : |x_1|_{\infty} < q^{\frac{s_1 q}{q-1}} \text{ and } |x_i|_{\infty} \leq q^{\frac{s_i q}{q-1}} \text{ for } i = 2, \dots, r \right\}.$$

By [C14, Rem. 5.1.5] and the same estimate on general terms, $\text{Li}_{\mathfrak{s}}(\mathbf{x})$ and $\text{Li}_{\mathfrak{s}}^*(\mathbf{x})$ converge ∞ -adically for any $\mathbf{x} \in \mathbb{D}''_{\mathfrak{s}}$, and $\text{Li}_{\mathfrak{s}}(\mathbf{x})$ is non-vanishing for any $\mathbf{x} \in \mathbb{D}'_{\mathfrak{s}} \cap (\mathbb{C}_{\infty}^{\times})^r$. We mention that $\mathbb{D}''_{\mathfrak{s}}$ is used in Theorem 4.2.3.

For $1 \leq \ell \leq r$, we put $d_{\ell} := s_{\ell} + \dots + s_r$ and $d := d_1 + \dots + d_r$. Let B be a $d \times d$ -matrix of the form

$$\left(\begin{array}{c|ccc|c} B[11] & \cdots & & B[1r] \\ \hline \vdots & & & \vdots \\ \hline B[r1] & \cdots & & B[rr] \end{array} \right),$$

where $B[\ell m]$ is a $d_{\ell} \times d_m$ -matrix for each ℓ and m and we call $B[\ell m]$ the (ℓ, m) -th block sub-matrix of B .

For $1 \leq \ell \leq m \leq r$, we define the following matrices:

$$N_{\ell} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ & 0 & 1 & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix} \in \text{Mat}_{d_{\ell}}(\bar{\mathbb{k}}),$$

$$N := \begin{pmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_r \end{pmatrix} \in \text{Mat}_d(\bar{\mathbb{k}}),$$

$$E[\ell m] := \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix} \in \text{Mat}_{d_{\ell} \times d_m}(\bar{\mathbb{k}}) \quad (\text{if } \ell = m),$$

$$E[\ell m] := \begin{pmatrix} 0 & & \cdots & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & \ddots & \vdots \\ (-1)^{m-\ell} \prod_{e=\ell}^{m-1} u_e & 0 & \cdots & 0 \end{pmatrix} \in \text{Mat}_{d_{\ell} \times d_m}(\bar{\mathbb{k}}) \quad (\text{if } \ell < m),$$

$$E := \begin{pmatrix} E[11] & E[12] & \cdots & E[1r] \\ & E[22] & \ddots & \vdots \\ & & \ddots & E[r-1, r] \\ & & & E[rr] \end{pmatrix} \in \text{Mat}_d(\bar{\mathbb{k}}).$$

We further define

$$E_m := \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & E[mm] & 0 \\ \hline 0 & 0 & 0 \end{array} \right) \in \text{Mat}_d(\bar{k})$$

to be the $d \times d$ -matrix such that the (m, m) -th block sub-matrix is $E[mm]$ and the others are zero matrices.

We then define the t -module $G = G_{\mathfrak{s}, \mathbf{u}} := (G_{\mathfrak{s}}^d, \rho)$ by

$$(4.1.5) \quad \rho_t = \theta I_d + N + E\tau \in \text{Mat}_d(\bar{k}[\tau]),$$

and note that G depends only on u_1, \dots, u_{r-1} . Finally, we define the special point

$$(4.1.6) \quad \mathbf{v} := \mathbf{v}_{\mathfrak{s}, \mathbf{u}} := \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ (-1)^{r-1} u_1 \cdots u_r \\ 0 \\ \vdots \\ 0 \\ (-1)^{r-2} u_2 \cdots u_r \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ u_r \end{array} \right) \in G(\bar{k}).$$

Remark 4.1.7. If $\mathbf{u} \in A^r$, then $\rho_t \in \text{Mat}_d(A[\tau])$ and $\mathbf{v} \in G(A)$.

Remark 4.1.8. The t -module G above is the t -module associated to the Anderson dual t -motive \mathcal{M}' , where \mathcal{M}' is free of rank r over $\bar{k}[t]$ and the representing matrix by σ on certain $\bar{k}[t]$ -basis for \mathcal{M}' is given by

$$(4.1.9) \quad \Phi' := \left(\begin{array}{cccc} (t - \theta)^{s_1 + \cdots + s_r} & & & \\ \mathbf{u}_1^{(-1)} (t - \theta)^{s_1 + \cdots + s_r} & (t - \theta)^{s_2 + \cdots + s_r} & & \\ & \ddots & \ddots & \\ & & \mathbf{u}_{r-1}^{(-1)} (t - \theta)^{s_{r-1} + s_r} & (t - \theta)^{s_r} \end{array} \right) \in \text{Mat}_r(\bar{k}[t]),$$

where

$$\{(t - \theta)^{s_1 + \cdots + s_r}, (t - \theta)^{s_2 + \cdots + s_r}, \dots, (t - \theta)^{s_r}\}$$

are the diagonals and

$$\{\mathbf{u}_1^{(-1)} (t - \theta)^{s_1 + \cdots + s_r}, \dots, \mathbf{u}_{r-1}^{(-1)} (t - \theta)^{s_{r-1} + s_r}\}$$

are displayed below the diagonals. We note that \mathcal{M}' is an iterated extension of some tensor powers of the Carlitz t -motive.

Remark 4.1.10. The Anderson dual t-motive \mathcal{M}' contains $\mathbb{C}^{\otimes n}$ as a sub-Anderson dual t-motive. Moreover, \mathcal{M}' is rigid analytically trivial since a rigid analytic trivialization $\Psi' \in \mathrm{GL}_r(\mathbb{T})$ is given as the upper left square of Ψ given in [CPY14, (2.3.7)] by changing (Q_1, \dots, Q_r) to (u_1, \dots, u_r) .

4.2. The convergence. To study the ∞ -adic convergence issue about \log_G at \mathbf{v} , we adopt some techniques of [AT90, 2.4.3]. We denote by

$$\log_G = \sum_{i \geq 0} P_i \tau^i$$

the logarithm of the t-module G , where $P_0 = I_d$ and $P_i \in \mathrm{Mat}_d(\bar{k})$ for all i .

For a matrix $\gamma := (\gamma_{ij})$ with entries in \mathbb{C}_∞ , we put

$$|\gamma|_\infty := \max_{i,j} \{|\gamma_{ij}|_\infty\}.$$

Lemma 4.2.1. *Let $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ and $\mathbf{u} = (u_1, \dots, u_r) \in (\bar{k}^\times)^r$. If $|u_\ell|_\infty \leq q^{\frac{s_\ell q}{q-1}}$ for each $1 \leq \ell < r$, then we have*

$$|P_i N^{d_\ell - j} E_\ell|_\infty \leq q^{(d_\ell - j)q^i - (d_\ell q^i - d_1) \frac{q}{q-1}}$$

for each i, j, k with $i \geq 0$, $1 \leq \ell \leq r$, and $1 \leq j \leq d_\ell$.

Proof. Note that the $(d_1 + \dots + d_{\ell-1} + 1)$ th column of $P_i N^{d_\ell - j} E_\ell$ is the $(d_1 + \dots + d_{\ell-1} + j)$ th column of P_i , and the other columns are zero vectors. When $i = 0$, the inequality holds clearly. Let $i \geq 1$ and assume that the inequality holds for i . By [CM17, 3.2.4], we have

$$P_{i+1} N^{d_\ell - j} E_\ell = - \sum_{m=0}^{2d_1 - 2} \frac{1}{(\theta^{q^{i+1}} - \theta)^{m+1}} \sum_{n=0}^m (-1)^n \binom{m}{n} N^{m-n} P_i E^{(i)} N^{n+d_\ell - j} E_\ell.$$

Note that $E^{(i)} N^{n+d_\ell - j} E_\ell = 0$ for $n \neq j-1$, and $N^{m-n} = 0$ for $m-n \geq d_1$. Thus we have

$$\begin{aligned} P_{i+1} N^{d_\ell - j} E_\ell &= \sum_{m=j-1}^{d_1 + j - 2} \frac{(-1)^j}{(\theta^{q^{i+1}} - \theta)^{m+1}} \binom{m}{j-1} N^{m-j+1} P_i E^{(i)} N^{d_\ell - 1} E_\ell \\ &= \sum_{m=j-1}^{d_1 + j - 2} \frac{(-1)^j}{(\theta^{q^{i+1}} - \theta)^{m+1}} \binom{m}{j-1} N^{m-j+1} \sum_{n=1}^{\ell} (-1)^{\ell-n} P'_{i,\ell,n} \prod_{n \leq e \leq \ell-1} u_e^{q^i}, \end{aligned}$$

where $P'_{i,\ell,n}$ is the matrix such that the $(d_1 + \dots + d_{\ell-1} + 1)$ th column is the $(d_1 + \dots + d_{n-1} + d_n)$ th column of P_i , and the other columns are zero vectors.

By the induction hypothesis, we obtain

$$\begin{aligned} \left| P'_{i,\ell,n} \prod_{n \leq e \leq \ell-1} u_e^{q^i} \right|_\infty &\leq q^{(d_n - d_n)q^i - (d_n q^i - d_1) \frac{q}{q-1}} \cdot \prod_{n \leq e \leq \ell-1} q^{\frac{s_e q}{q-1} \cdot q^i} \\ &= q^{-(d_n q^i - d_1) \frac{q}{q-1}} \cdot q^{(d_n - d_\ell) \frac{q^{i+1}}{q-1}} \\ &= q^{-(d_\ell q^i - d_1) \frac{q}{q-1}}. \end{aligned}$$

Therefore we have

$$\begin{aligned}
|P_{i+1}N^{d_\ell-j}E_\ell|_\infty &\leq \max_{j-1 \leq m \leq d_1+j-2} \left\{ q^{-(m+1)q^{i+1}} \right\} \cdot q^{-(d_\ell q^i - d_1) \frac{q}{q-1}} \\
&= q^{-jq^{i+1}} \cdot q^{-(d_\ell q^i - d_1) \frac{q}{q-1}} \\
&= q^{(d_\ell-j)q^{i+1} - (d_\ell q^{i+1} - d_1) \frac{q}{q-1}}.
\end{aligned}$$

□

Proposition 4.2.2. *Assume $|u_\ell|_\infty \leq q^{\frac{s_\ell q}{q-1}}$ for each $1 \leq \ell < r$. Take a point $\mathbf{x} = (x_m) \in G(\mathbb{C}_\infty)$ such that*

$$|x_{d_1+\dots+d_{\ell-1}+j}|_\infty < q^{-(d_\ell-j)+\frac{d_\ell q}{q-1}}$$

for each j, ℓ with $1 \leq \ell \leq r$ and $1 \leq j \leq d_\ell$. Then $\log_G(\mathbf{x})$ converges in $\text{Lie } G(\mathbb{C}_\infty)$.

Proof. By Lemma 4.2.1, we have

$$\begin{aligned}
|P_i \mathbf{x}^{(i)}|_\infty &\leq \max_{j, \ell} \left\{ q^{(d_\ell-j)q^i - (d_\ell q^i - d_1) \frac{q}{q-1}} \cdot |x_{d_1+\dots+d_{\ell-1}+j}|_\infty^{q^i} \right\} \\
&= \max_{j, \ell} \left\{ q^{\frac{d_1 q}{q-1}} \cdot \left(|x_{d_1+\dots+d_{\ell-1}+j}|_\infty / q^{-(d_\ell-j)+\frac{d_\ell q}{q-1}} \right)^{q^i} \right\} \\
&\rightarrow 0 \quad (i \rightarrow \infty).
\end{aligned}$$

□

Theorem 4.2.3. *Given any $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, we put $\tilde{\mathfrak{s}} := (s_r, \dots, s_1)$ and let $\mathbb{D}_{\tilde{\mathfrak{s}}}''$ be defined in (4.1.4). Suppose that we have $\mathbf{u} = (u_1, \dots, u_r) \in (\bar{\mathbb{k}}^\times)^r$ for which $\tilde{\mathbf{u}} := (u_r, \dots, u_1) \in \mathbb{D}_{\tilde{\mathfrak{s}}}''$. Let G and \mathbf{v} be defined as above associated to \mathfrak{s} and \mathbf{u} . Then \log_G converges ∞ -adically at \mathbf{v} and we have the formula*

$$\log_G(\mathbf{v}) = \left(\begin{array}{c} * \\ \vdots \\ * \\ (-1)^{r-1} \text{Li}_{(s_r, \dots, s_1)}^*(\mathbf{u}_r, \dots, \mathbf{u}_1) \\ * \\ \vdots \\ * \\ (-1)^{r-2} \text{Li}_{(s_r, \dots, s_2)}^*(\mathbf{u}_r, \dots, \mathbf{u}_2) \\ \vdots \\ * \\ \vdots \\ * \\ \vdots \\ * \\ \text{Li}_{s_r}^*(\mathbf{u}_r) \end{array} \right) \begin{array}{l} \left. \vphantom{\begin{array}{c} * \\ \vdots \\ * \\ (-1)^{r-1} \text{Li}_{(s_r, \dots, s_1)}^*(\mathbf{u}_r, \dots, \mathbf{u}_1) \\ * \\ \vdots \\ * \\ (-1)^{r-2} \text{Li}_{(s_r, \dots, s_2)}^*(\mathbf{u}_r, \dots, \mathbf{u}_2) \\ \vdots \\ * \\ \vdots \\ * \\ \text{Li}_{s_r}^*(\mathbf{u}_r) \end{array}} \right\} d_1 \\ \left. \vphantom{\begin{array}{c} * \\ \vdots \\ * \\ (-1)^{r-1} \text{Li}_{(s_r, \dots, s_1)}^*(\mathbf{u}_r, \dots, \mathbf{u}_1) \\ * \\ \vdots \\ * \\ (-1)^{r-2} \text{Li}_{(s_r, \dots, s_2)}^*(\mathbf{u}_r, \dots, \mathbf{u}_2) \\ \vdots \\ * \\ \vdots \\ * \\ \text{Li}_{s_r}^*(\mathbf{u}_r) \end{array}} \right\} d_2 \\ \vdots \\ \left. \vphantom{\begin{array}{c} * \\ \vdots \\ * \\ (-1)^{r-1} \text{Li}_{(s_r, \dots, s_1)}^*(\mathbf{u}_r, \dots, \mathbf{u}_1) \\ * \\ \vdots \\ * \\ (-1)^{r-2} \text{Li}_{(s_r, \dots, s_2)}^*(\mathbf{u}_r, \dots, \mathbf{u}_2) \\ \vdots \\ * \\ \vdots \\ * \\ \text{Li}_{s_r}^*(\mathbf{u}_r) \end{array}} \right\} d_r \end{array} \in \text{Lie } G(\mathbb{C}_\infty).$$

In particular, the $(s_1 + \dots + s_r)$ th coordinate of $\log_G(\mathbf{v})$ is $(-1)^{\text{dep}(\mathfrak{s})-1} \text{Li}_{\tilde{\mathfrak{s}}}^*(\tilde{\mathbf{u}})$.

Proof. For each $1 \leq \ell \leq r$, the $(d_1 + \dots + d_{\ell-1} + d_\ell)$ th component of \mathbf{v} is $(-1)^{r-\ell} u_\ell u_{\ell+1} \cdots u_r$, and we have

$$|(-1)^{r-\ell} u_\ell u_{\ell+1} \cdots u_r|_\infty < q^{\frac{s_\ell q}{q-1}} \cdot q^{\frac{s_{\ell+1} q}{q-1}} \cdots q^{\frac{s_r q}{q-1}} = q^{\frac{d_\ell q}{q-1}} = q^{-(d_\ell - d_\ell) + \frac{d_\ell q}{q-1}}.$$

Thus $\log_G(\mathbf{v})$ converges ∞ -adically by Proposition 4.2.2.

Arguments of proving the second assertion are entirely the same as the calculations in the proof of [CM17, 3.3.3], where we just change the v -adic convergence to ∞ -adic convergence. \square

Remark 4.2.4. We do not know the precise forms of those coordinates not interpreted on the right hand side of the identity above.

5. PROOF OF THEOREM 1.2.2

5.1. Formulae for MZV's via CMPL's. When $r = 1$ and $s = 1 \in \mathbb{N}$, the series (4.1.1) is called Carlitz logarithm, which is the formal inverse of the exponential map of the Carlitz module \mathbf{C} . For $r = 1$ and any $s \in \mathbb{N}$, the series Li_s in (4.1.1) is called the s th Carlitz polylogarithm studied in [AT90]. Unlike the classical case there is a simple identity between $\zeta(s)$ and certain specialization of the classical multiple polylogarithms, $\zeta_{\mathcal{A}}(s)$ is in fact a k -linear combination of Li_s at some integral points, which will be reviewed in the following section. It turns out that such an identity for $\zeta_{\mathcal{A}}(s)$ is a crucial connection that enables us to give a logarithmic interpretation for $\zeta_{\mathcal{A}}(s)$ in Theorem 1.2.2.

Remark 5.1.1. For recent advances of transcendence theory for CMPL's, see [CY07, M17].

We now recall the Carlitz factorials. We set $D_0 := 1$, and $D_i := \prod_{j=0}^{i-1} (\theta^{q^i} - \theta^{q^j}) \in A$ for $i \in \mathbb{N}$. Given a non-negative integer n , we express n as $n = \sum_{i \geq 0} n_i q^i$ for $0 \leq n_i \leq q-1$. The Carlitz factorial is defined as

$$(5.1.2) \quad \Gamma_{n+1} := \prod_i D_i^{n_i} \in A.$$

To introduce the formula of $\zeta_{\mathcal{A}}(s)$ in terms of Li_s , we need to review the Anderson-Thakur polynomials [AT90, AT09]. Let t be an independent variable from θ . We put $F_0 := 1$ and define polynomials $F_i \in A[t]$ for $i \in \mathbb{N}$ by the product

$$F_i = \prod_{j=1}^i (t^{q^i} - \theta^{q^j}).$$

We then define the sequence of Anderson-Thakur polynomials $H_n \in A[t]$ (for non-negative integers n) by the generating function identity

$$\left(1 - \sum_{i=0}^{\infty} \frac{F_i}{D_i|_{\theta=t}} x^{q^i} \right)^{-1} = \sum_{n=0}^{\infty} \frac{H_n}{\Gamma_{n+1}|_{\theta=t}} x^n.$$

Define the sup-norm $\|f\| := \max_i \{|a_i|_{\infty}\}$ for polynomials $f = \sum_i a_i t^i \in \mathbb{C}_{\infty}[t]$, and note that the Anderson-Thakur polynomials have the following property

$$(5.1.3) \quad \|H_{n-1}(t)\| < |\theta|_{\infty}^{\frac{nq}{q-1}}$$

for every $n \in \mathbb{N}$.

Remark 5.1.4. The bound above comes from [AT90, (3.7.3)]. However, we shall mention about the difference of notation. Our $H_n(t)$ is exactly the same as $H_n(y, T)$ in [AT90] replacing y by θ and replacing T by t . One can compare with [AT09], where their T is referred to our t and their t is referred to our θ .

In what follows, we fix an r -tuple of positive integers $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$. For each $1 \leq i \leq r$, we expand the Anderson-Thakur polynomial $H_{s_i-1}(t) \in A[t]$ as

$$(5.15) \quad H_{s_i-1}(t) = \sum_{j=0}^{m_i} u_{ij} t^j,$$

where $u_{ij} \in A$ with $u_{im_i} \neq 0$ and by (5.1.3) it satisfies

$$(5.16) \quad |u_{ij}|_\infty < q^{\frac{s_i q}{q-1}} \text{ for } j = 0, \dots, m_i.$$

We define

$$J_{\mathfrak{s}} := \{0, 1, \dots, m_1\} \times \dots \times \{0, 1, \dots, m_r\}.$$

For each $\mathbf{j} = (j_1, \dots, j_r) \in J_{\mathfrak{s}}$, we set

$$\mathbf{u}_{\mathbf{j}} := (u_{1j_1}, \dots, u_{rj_r}) \in A^r,$$

and

$$a_{\mathbf{j}} := a_{\mathbf{j}}(t) := t^{j_1 + \dots + j_r}.$$

Note that by (5.1.6), we have $\mathbf{u}_{\mathbf{j}} \in \mathbb{D}'_{\mathfrak{s}}$ for every $\mathbf{j} \in J_{\mathfrak{s}}$.

Set $\Gamma_{\mathfrak{s}} := \Gamma_{s_1} \cdots \Gamma_{s_r} \in A$. The first author of the present paper established the following formula that extends the work of Anderson-Thakur [AT90] for $r = 1$.

Theorem 5.1.7. ([C14, Thm. 5.5.2]) *For each $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, let $J_{\mathfrak{s}}$, $a_{\mathbf{j}}$ and $\mathbf{u}_{\mathbf{j}}$ be defined as above. Then the following identity holds.*

$$\Gamma_{\mathfrak{s}} \zeta_A(\mathfrak{s}) = \sum_{\mathbf{j} \in J_{\mathfrak{s}}} a_{\mathbf{j}}(\theta) \text{Li}_{\mathfrak{s}}(\mathbf{u}_{\mathbf{j}}).$$

In the following section, we have to express the right hand side of the identity in Theorem 5.1.7 in terms of CMSPL's since such a formulation plays a crucial role in the proof of Theorem 1.2.2.

5.2. Formulae for MZV's via CMSPL's. To express CMPL in terms of CMSPL's, one just needs the inclusion-exclusion principle on the set

$$\{i_1 > \dots > i_r \geq 0\}.$$

Let us take a simple example for $r = 2$, which would simply allow one to understand what we do in the more general setting. Since we have

$$\{i_1 > i_2 \geq 0\} = \{i_1 \geq i_2 \geq 0\} \setminus \{i_1 = i_2 \geq 0\},$$

it follows that

$$\text{Li}_{(s_1, s_2)}(z_1, z_2) = \text{Li}_{(s_1, s_2)}^*(z_1, z_2) - \text{Li}_{s_1+s_2}^*(z_1 + z_2).$$

So one can obtain that $\text{Li}_{\mathfrak{s}}$ can be expressed as a linear combination of CMSPL's, which is presented in Proposition 5.2.3.

In what follows, the main target is to express $\Gamma_{\mathfrak{s}} \zeta_A(\mathfrak{s})$ explicitly as an A -linear combination of some CMSPL's at certain integral points. The details of the procedure below are to explain that s_ℓ , b_ℓ and \mathbf{u}_ℓ in Theorem 5.2.5 can be written down explicitly, and that will explain why the constructions of $G_{\mathfrak{s}}$ and $\mathbf{v}_{\mathfrak{s}}$ in Theorem 1.2.2 are explicit. So we suggest the reader to skip the following ahead to Theorem 5.2.5 unless one needs explicit examples of $G_{\mathfrak{s}}$ and $\mathbf{v}_{\mathfrak{s}}$.

An index of depth r is defined to be an r -tuple $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$.

Definition 5.2.1. Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be an index of depth $r > 1$. Let S be the set consisting of the two symbols ‘,’ (comma) and ‘+’ (addition) and S^\times be the set consisting of the two symbols ‘,’ and ‘ \times ’ (multiplication).

- (1) We define a map $\lambda := (\mathbf{w} \mapsto \mathbf{w}^\times) : S^{r-1} \rightarrow S^{\times r-1}$ by leaving ‘,’ be fixed and changing ‘+’ to ‘ \times ’. That’s, if $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_{r-1})$, then $\mathbf{w}_i^\times := ‘,$ ’ if $\mathbf{w}_i = ‘,$ ’; otherwise $\mathbf{w}_i^\times := ‘\times$ ’.
- (2) For any $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_{r-1}) \in S^{r-1}$, we define $\mathbf{w}(\mathfrak{s}) := (s_1 \mathbf{w}_1 s_2 \mathbf{w}_2 \cdots \mathbf{w}_{r-1} s_r)$. That’s, $\mathbf{w}(\mathfrak{s})$ is a tuple of positive integers obtained from (s_1, \dots, s_r) by inserting the symbol \mathbf{w}_i between s_i and s_{i+1} for $i = 1, \dots, r-1$.
- (3) For any $\mathbf{w} = (\mathbf{w}_1, \dots, \mathbf{w}_{r-1}) \in S^{r-1}$ and $\mathbf{u} = (u_1, \dots, u_r) \in \bar{k}^r$, we define $\mathbf{w}^\times(\mathbf{u}) := (u_1 \mathbf{w}_1^\times u_2 \mathbf{w}_2^\times \cdots \mathbf{w}_{r-1}^\times u_r)$. That’s, $\mathbf{w}^\times(\mathbf{u})$ is the tuple of algebraic elements over k obtained from (u_1, \dots, u_r) by inserting the symbol \mathbf{w}_i^\times between u_i and u_{i+1} for $i = 1, \dots, r-1$.

For example, let $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$ with $\mathbf{w}_1 = ‘,$ ’ and $\mathbf{w}_2 = ‘+’$. Then for $\mathfrak{s} = (s_1, s_2, s_3) \in \mathbb{N}^3$, we have

$$\mathbf{w}(\mathfrak{s}) = (s_1, s_2 + s_3).$$

Furthermore, for $\mathbf{u} = (u_1, u_2, u_3) \in \bar{k}^3$, we have

$$\mathbf{w}^\times(\mathbf{u}) = (u_1, u_2 u_3).$$

Finally, we define $\nu(\mathbf{w})$ to be the number of ‘+’ in \mathbf{w} .

Proposition 5.2.2. Fix an index $\mathfrak{s} \in \mathbb{N}^r$ with $r > 1$. Then for any $\mathbf{u} \in \mathbb{D}'_{\mathfrak{s}}$ (resp. $\mathbf{u} \in \mathbb{D}''_{\mathfrak{s}}$) and $\mathbf{w} \in S^{r-1}$, we have that $\mathbf{w}^\times(\mathbf{u}) \in \mathbb{D}'_{\mathbf{w}(\mathfrak{s})}$ (resp. $\mathbf{w}^\times(\mathbf{u}) \in \mathbb{D}''_{\mathbf{w}(\mathfrak{s})}$). In particular, $\text{Li}_{\mathbf{w}(\mathfrak{s})}^*(\mathbf{w}^\times(\mathbf{u}))$ converges by Remark 4.1.3.

Proof. The assertion follows immediately from the non-archimedean property of $|\cdot|_\infty$. \square

In order to make our formula of MZV’s convenient for use, for $r = 1$ we simply define $S^{r-1} = S^0 := \{\text{identity}\}$, and denote by $P(\mathfrak{s}) = s$, $P^\times(\mathbf{u}) = u$ and $\nu(P) := 0$ for $P \in S^0$.

Applying the inclusion-exclusion principle on the set $\{i_1 > \cdots > i_r \geq 0\}$, we have the following identity.

Proposition 5.2.3. Let r be a positive integer, $\mathfrak{s} \in \mathbb{N}^r$ be an index and z_1, \dots, z_r be r independent variables. Putting $\mathbf{z} = (z_1, \dots, z_r)$. Then the following identity holds:

$$\text{Li}_{\mathfrak{s}}(\mathbf{z}) = \sum_{P \in S^{r-1}} (-1)^{\nu(P)} \text{Li}_{P(\mathfrak{s})}^*(P^\times(\mathbf{z})).$$

Remark 5.2.4. The similar statement of Proposition 5.2.3 for classical MZV’s can be seen in [Ya13].

Recall that the special points \mathbf{u}_j in Theorem 5.1.7 belong to $\mathbb{D}'_{\mathfrak{s}}$ for every $\mathbf{j} \in J_{\mathfrak{s}}$, and so $\text{Li}_{P(\mathfrak{s})}^*(P^\times(\mathbf{u}_\ell))$ converges by Proposition 5.2.2. Combining Theorem 5.1.7 and the proposition above, we have the following expression for $\zeta_A(\mathfrak{s})$ in terms of CMSPL’s.

Theorem 5.2.5. For any depth r index $\mathfrak{s} \in \mathbb{N}^r$, there are explicit tuples $\mathfrak{s}_\ell \in \mathbb{N}^{\text{dep}(\mathfrak{s}_\ell)}$ with $\text{wt}(\mathfrak{s}_\ell) = \text{wt}(\mathfrak{s})$, $\text{dep}(\mathfrak{s}_\ell) \leq r$, explicit coefficients $b_\ell \in A$ and vectors $\mathbf{u}_\ell \in A^{\text{dep}(\mathfrak{s}_\ell)}$ so that

$$\Gamma_{\mathfrak{s}} \zeta_A(\mathfrak{s}) = \sum_{\ell} b_\ell \cdot (-1)^{\text{dep}(\mathfrak{s}_\ell)-1} \text{Li}_{\mathfrak{s}_\ell}^*(\mathbf{u}_\ell).$$

Remark 5.2.6. Precisely, we have

$$\begin{aligned}
\Gamma_{\mathfrak{s}} \zeta_{\Lambda}(\mathfrak{s}) &= \sum_{\mathbf{j} \in \mathcal{J}_{\mathfrak{s}}} \mathbf{a}_{\mathbf{j}}(\theta) \operatorname{Li}_{\mathfrak{s}}(\mathbf{u}_{\mathbf{j}}) \\
&= \sum_{\mathbf{j} \in \mathcal{J}_{\mathfrak{s}}} \mathbf{a}_{\mathbf{j}}(\theta) \sum_{P \in S^{r-1}} (-1)^{\nu(P)} \operatorname{Li}_{P(\mathfrak{s})}^{\star}(P^{\times}(\mathbf{u}_{\mathbf{j}})) \\
&= \sum_{\mathbf{j} \in \mathcal{J}_{\mathfrak{s}}} \sum_{P \in S^{r-1}} (-1)^{r-1} \mathbf{a}_{\mathbf{j}}(\theta) \cdot (-1)^{\operatorname{dep}(P(\mathfrak{s}))-1} \operatorname{Li}_{P(\mathfrak{s})}^{\star}(P^{\times}(\mathbf{u}_{\mathbf{j}})),
\end{aligned}$$

where we use the equality $\nu(P) + \operatorname{dep}(P(\mathfrak{s})) = r$ for each $P \in S^{r-1}$. Let T be the cardinality of the terms in the right hand side of the identity above. Then for convenience we renumber the indexes ℓ of $(\mathbf{b}_{\ell}, \mathfrak{s}_{\ell}, \mathbf{u}_{\ell})$ for which

$$(5.2.7) \quad \{(\mathbf{b}_{\ell}, \mathfrak{s}_{\ell}, \mathbf{u}_{\ell}) \mid 1 \leq \ell \leq T\} = \{((-1)^{r-1} \mathbf{a}_{\mathbf{j}}(\theta), P(\mathfrak{s}), P^{\times}(\mathbf{u}_{\mathbf{j}})) \mid \mathbf{j} \in \mathcal{J}_{\mathfrak{s}}, P \in S^{r-1}\},$$

and $\operatorname{dep}(\mathfrak{s}_{\ell}) = 1$ for $1 \leq \ell \leq s$, and $\operatorname{dep}(\mathfrak{s}_{\ell}) \geq 2$ for $s+1 \leq \ell \leq T$. Note further that when $r = 1$, i.e., $\mathfrak{s} = s \in \mathbb{N}$, we have $\operatorname{Li}_{\mathfrak{s}} = \operatorname{Li}_{\mathfrak{s}}^{\star}$ and so the formula above for $\Gamma_{\mathfrak{s}} \zeta_{\Lambda}(\mathfrak{s})$ is the same as Theorem 5.1.7, which was established previously by Anderson-Thakur [AT90].

Note that the terms $(-1)^{\operatorname{dep}(\mathfrak{s}_{\ell})-1} \operatorname{Li}_{\mathfrak{s}_{\ell}}^{\star}(\mathbf{u}_{\ell})$ in the identity above occur as certain coordinates of the logarithm of the t-module considered in Theorem 4.2.3.

5.3. Proof of Theorem 1.2.2. Let r be a positive integer and fix any index $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$. Let $n := \operatorname{wt}(\mathfrak{s})$. We identify the set of triples $(\mathbf{b}_{\ell}, \mathfrak{s}_{\ell}, \mathbf{u}_{\ell})$ occurring in Theorem 5.2.5 as the set

$$\mathcal{T} = \{1, \dots, T\},$$

where we understand that each element $\ell \in \mathcal{T}$ corresponds to a triple $(\mathbf{b}_{\ell}, \mathfrak{s}_{\ell}, \mathbf{u}_{\ell})$. We further rearrange the indexes to decompose the disjoint union

$$\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$$

so that \mathcal{T}_1 consists of those indexes ℓ for which $\operatorname{dep}(\mathfrak{s}_{\ell}) = \operatorname{dep}(\mathbf{u}_{\ell}) = 1$, and \mathcal{T}_2 consists of those indexes for which $\operatorname{dep}(\mathfrak{s}_{\ell}) = \operatorname{dep}(\mathbf{u}_{\ell}) > 1$.

Put $s := |\mathcal{T}_1|$ and note that due to cancellations of the right hand side of the identity in Theorem 5.2.5, we allow s to be either zero or T .

For each $\ell \in \mathcal{T}$ equipped with $(\mathbf{b}_{\ell}, \mathfrak{s}_{\ell}, \mathbf{u}_{\ell})$, we let G_{ℓ} be the t-module that is defined in (4.1.5), and $\mathbf{v}_{\ell} \in G_{\ell}(k)$ be the special point defined in (4.1.6) that are constructed using the pair $(\tilde{\mathfrak{s}}_{\ell}, \tilde{\mathbf{u}}_{\ell})$, where $\tilde{\cdot}$ is defined to reverse the order of components (see the definition in Theorem 4.2.3). Note that G_{ℓ} is the t-module associated to the Anderson dual t-motive \mathcal{M}_{ℓ}' that is associated to $(\tilde{\mathfrak{s}}_{\ell}, \tilde{\mathbf{u}}_{\ell})$ and is defined in Remark 4.1.8. So by Remark 4.1.10 \mathcal{M}_{ℓ}' is rigid analytically trivial for each $\ell \in \mathcal{T}$. Note that $\operatorname{wt}(\mathfrak{s}_{\ell}) = \operatorname{wt}(\mathfrak{s}) = n$ for every $\ell \in \mathcal{T}$. Therefore, by Theorem 4.2.3 the n th coordinate of $\log_{G_{\ell}}(\mathbf{v}_{\ell})$ is

$$(-1)^{\operatorname{dep}(\tilde{\mathfrak{s}}_{\ell})-1} \operatorname{Li}_{\tilde{\mathfrak{s}}_{\ell}}^{\star}(\tilde{\mathbf{u}}_{\ell}) = (-1)^{\operatorname{dep}(\mathfrak{s}_{\ell})-1} \operatorname{Li}_{\mathfrak{s}_{\ell}}^{\star}(\mathbf{u}_{\ell}).$$

Put $\mathcal{N} := \mathbf{C}^{\otimes n}$, the n th tensor power of the Carlitz t-motive, and note that $\mathbf{C}^{\otimes n}$ is its corresponding t-module (see [CPY14, Sec. 5.2]). By the definition of $\mathbf{C}^{\otimes n}$, we see that the n th coordinate of $\operatorname{Lie} \mathbf{C}^{\otimes n}(\mathbf{C}_{\infty})$ is tractable.

Note that for $\ell \in \mathcal{T}_1$, \mathcal{M}_{ℓ}' is isomorphic to \mathcal{N} and for $\ell \in \mathcal{T}_2$, \mathcal{M}_{ℓ}' fits into the short exact sequence of left $\bar{k}[t, \sigma]$ -modules

$$0 \rightarrow \mathcal{N} \rightarrow \mathcal{M}_{\ell}' \rightarrow \mathcal{M}_{\ell}'' \rightarrow 0,$$

where \mathcal{M}_ℓ'' is an Anderson dual t-motive. Let \mathcal{M} be the fiber coproduct of $\{\mathcal{M}_\ell'\}_{\ell=1}^T$ over \mathcal{N} and so by Proposition 2.5.3 \mathcal{M} is rigid analytically trivial. Let $G_\mathfrak{s}$ be the t-module associated to \mathcal{M} , ie., $G_\mathfrak{s}(\bar{k}) \cong \mathcal{M}/(\sigma-1)\mathcal{M}$ as $\mathbb{F}_q[t]$ -modules. Hence $G_\mathfrak{s}$ is uniformizable by Remark 2.5.2.

Recall that every \mathbf{u}_j belongs to $\mathbb{D}'_\mathfrak{s}$ for $\mathbf{j} \in J_\mathfrak{s}$, and hence we have that for every $\ell \in \mathcal{T}$, \mathbf{u}_ℓ belongs to $\mathbb{D}'_{\mathfrak{s}_\ell}$ and hence $\tilde{\mathbf{u}}_\ell \in \mathbb{D}'_{\mathfrak{s}_\ell}$. Since $(G_\ell, \mathbf{v}_\ell)$ are constructed using $(\tilde{\mathfrak{s}}_\ell, \tilde{\mathbf{u}}_\ell)$, which satisfy the conditions of Theorem 4.2.3, $\log_{G_\ell}(\mathbf{v}_\ell)$ converges ∞ -adically for every $\ell \in \mathcal{T}$.

Note that since all \mathbf{u}_ℓ are integral points (see (5.2.7)), by Remark 4.1.7 the t-modules $\{G_\ell\}_{\ell=1}^T$ are defined over k and hence $G_\mathfrak{s}$ is also defined over k by Corollary 3.2.6.

Now we let $\pi : \bigoplus_{\ell=1}^T G_\ell \rightarrow G_\mathfrak{s}$ be the morphism of t-modules over k given in Definition 3.2.2. Recall that to simplify notation, we use $[a]$ for the action of $a \in \mathbb{F}_q[t]$ on any t-module without confusion. For each $\ell \in \mathcal{T}$, we define

$$(5.3.1) \quad Z_\ell := \log_{G_\ell}(\mathbf{v}_\ell) \in \text{Lie } G_\ell(\mathbb{C}_\infty),$$

and further set

$$(5.3.2) \quad Z_\mathfrak{s} := \partial\pi((\partial[b_\ell(t)]Z_\ell)_\ell) \in \text{Lie } G_\mathfrak{s}(\mathbb{C}_\infty),$$

and

$$(5.3.3) \quad \mathbf{v}_\mathfrak{s} := \pi([\mathbf{b}_\ell(t)]\mathbf{v}_\ell)_\ell \in G_\mathfrak{s}(k),$$

where $b_\ell \in A$ are given in Theorem 5.2.5. We note that by the functional equation (2.2.3) we have

$$\exp_{G_\ell}(Z_\ell) = \mathbf{v}_\ell$$

and therefore by Lemma 3.3.2 we have

$$\exp_{G_\mathfrak{s}}(Z_\mathfrak{s}) = \mathbf{v}_\mathfrak{s}.$$

On the other hand, by Theorem 4.2.3 the n th coordinate of Z_ℓ is given by $(-1)^{\text{dep}(\mathfrak{s}_\ell)-1} \text{Li}_{\mathfrak{s}_\ell}^*(\mathbf{u}_\ell)$. By Lemma 3.3.2 and the formula in Theorem 5.2.5 we see that the n th coordinate of $Z_\mathfrak{s}$ is $\Gamma_\mathfrak{s}\zeta_A(\mathfrak{s})$.

5.4. Examples.

Example 5.4.1. Take q to be a power of any prime number p and let $\mathfrak{s} = (1, 1, 2)$. In this case, we have $\Gamma_1 = \Gamma_2 = 1$, $H_{1-1} = H_{2-1} = 1$, $J_{(1,1,2)} = \{(0, 0, 0)\}$, $\mathbf{u}_{(0,0,0)} = (1, 1, 1)$, $\alpha_{(0,0,0)} = 1$. Thus we have

$$\begin{aligned} \zeta_A(1, 1, 2) &= \text{Li}_{(1,1,2)}(1, 1, 1) = \text{Li}_{(1,1,2)}^*(1, 1, 1) - \text{Li}_{(2,2)}^*(1, 1) - \text{Li}_{(1,3)}^*(1, 1) + \text{Li}_4^*(1) \\ &= (-1)^{1-1} \text{Li}_4^*(1) + (-1)^{2-1} \text{Li}_{(1,3)}^*(1, 1) \\ &\quad + (-1)^{2-1} \text{Li}_{(2,2)}^*(1, 1) + (-1)^{3-1} \text{Li}_{(1,1,2)}^*(1, 1, 1), \end{aligned}$$

and $(b_1, \mathfrak{s}_1, \mathbf{u}_1) = (1, 4, 1)$, $(b_2, \mathfrak{s}_2, \mathbf{u}_2) = (1, (1, 3), (1, 1))$, $(b_3, \mathfrak{s}_3, \mathbf{u}_3) = (1, (2, 2), (1, 1))$, $(b_4, \mathfrak{s}_4, \mathbf{u}_4) = (1, (1, 1, 2), (1, 1, 1))$.

For $\ell = 1$, we have $G_1 = \mathbb{C}^{\otimes 4}$, and hence its t-action on G_α^4 is given by

$$\mathbf{C}_t^{\otimes 4} = \begin{pmatrix} \theta & 1 & & \\ & \theta & 1 & \\ & & \theta & 1 \\ \tau & & & \theta \end{pmatrix}.$$

We further have $\mathbf{v}_1 = (0, 0, 0, 1)^{\text{tr}} \in \mathbf{C}^{\otimes 4}(\mathbf{k})$, and $Z_1 = (*, *, *, \text{Li}_4^*(1))^{\text{tr}} \in \text{Lie } \mathbf{C}^{\otimes 4}(\mathbf{C}_\infty)$.
For $\ell = 2$, we have $G_2 = G_a^5$ with the \mathfrak{t} -action

$$[\mathfrak{t}] = \left(\begin{array}{ccc|c} \theta & 1 & & \\ & \theta & 1 & \\ & & \theta & 1 \\ \hline \tau & & \theta & -\tau \\ \hline & & & \theta + \tau \end{array} \right),$$

and

$$\mathbf{v}_2 = (0, 0, 0, -1, 1)^{\text{tr}} \in G_2(\mathbf{k}),$$

$$Z_2 = (*, *, *, -\text{Li}_{(1,3)}^*(1, 1), \text{Li}_1^*(1))^{\text{tr}} \in \text{Lie } G_2(\mathbf{C}_\infty).$$

For $\ell = 3$, we have $G_3 = G_a^6$ with the \mathfrak{t} -action

$$[\mathfrak{t}] = \left(\begin{array}{ccc|c} \theta & 1 & & \\ & \theta & 1 & \\ & & \theta & 1 \\ \hline \tau & & \theta & -\tau \\ \hline & & & \theta & 1 \\ & & & \tau & \theta \end{array} \right),$$

and points

$$\mathbf{v}_3 = (0, 0, 0, -1, 0, 1)^{\text{tr}} \in G_3(\mathbf{k}),$$

$$Z_3 = (*, *, *, -\text{Li}_{(2,2)}^*(1, 1), *, \text{Li}_2^*(1))^{\text{tr}} \in \text{Lie } G_3(\mathbf{C}_\infty).$$

For $\ell = 4$, we have $G_4 = G_a^7$ with the \mathfrak{t} -action

$$[\mathfrak{t}] = \left(\begin{array}{ccc|c|c} \theta & 1 & & & \\ & \theta & 1 & & \\ & & \theta & 1 & \\ \hline \tau & & \theta & -\tau & \tau \\ \hline & & & \theta & 1 \\ & & & \tau & \theta \\ \hline & & & & \theta + \tau \end{array} \right),$$

and

$$\mathbf{v}_4 = (0, 0, 0, 1, 0, -1, 1)^{\text{tr}} \in G_4(\mathbf{k}),$$

$$Z_4 = (*, *, *, \text{Li}_{(1,1,2)}^*(1, 1, 1), *, -\text{Li}_{(1,1)}^*(1, 1), *, \text{Li}_1^*(1))^{\text{tr}} \in \text{Lie } G_4(\mathbf{C}_\infty).$$

Therefore we have $G_{(1,1,2)} = G_a^{10}$ with the \mathfrak{t} -action

$$[\mathfrak{t}] = \left(\begin{array}{ccc|c|c|c} \theta & 1 & & & & \\ & \theta & 1 & & & \\ & & \theta & 1 & & \\ \hline \tau & & \theta & -\tau & -\tau & -\tau & \tau \\ \hline & & & \theta + \tau & & & \\ \hline & & & & \theta & 1 & \\ & & & & \tau & \theta & \\ \hline & & & & & \theta & 1 \\ & & & & & \tau & \theta & -\tau \\ & & & & & & & \theta + \tau \end{array} \right),$$

and

$$\mathbf{v}_{(1,1,2)} = \pi(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = (0, 0, 0, 0, 1, 0, 1, 0, -1, 1)^{\text{tr}} \in \mathbf{G}_{(1,1,2)}(\mathbf{k}),$$

$$Z_{(1,1,2)} = (*, *, *, \zeta_A(1, 1, 2), \text{Li}_1^*(1), *, \text{Li}_2^*(1), *, -\text{Li}_{(1,1)}^*(1, 1), \text{Li}_1^*(1))^{\text{tr}} \in \text{Lie } \mathbf{G}_{(1,1,2)}(\mathbf{C}_\infty).$$

Example 5.4.2. Take $q = 2$ and $\mathfrak{s} = (1, 3)$. In this case, we have $\Gamma_1 = 1$, $\Gamma_3 = \theta^2 + \theta$, $H_{1-1} = 1$, $H_{3-1} = t + \theta^2$, $J_{(1,3)} = \{(0, 0), (0, 1)\}$, $\mathbf{u}_{(0,0)} = (1, \theta^2)$, $\mathbf{u}_{(0,1)} = (1, 1)$, $\alpha_{(0,0)} = 1$, $\alpha_{(0,1)} = t$. Thus we have

$$\begin{aligned} (\theta^2 + \theta)\zeta_A(1, 3) &= \text{Li}_{(1,3)}(1, \theta^2) + \theta \text{Li}_{(1,3)}(1, 1) \\ &= \text{Li}_{(1,3)}^*(1, \theta^2) - \text{Li}_4^*(\theta^2) + \theta \text{Li}_{(1,3)}^*(1, 1) - \theta \text{Li}_4^*(1) \\ &= (-1)^{1-1} \text{Li}_4^*(\theta^2) + \theta \cdot (-1)^{1-1} \text{Li}_4^*(1) \\ &\quad + (-1)^{2-1} \text{Li}_{(1,3)}^*(1, \theta^2) + \theta \cdot (-1)^{2-1} \text{Li}_{(1,3)}^*(1, 1), \end{aligned}$$

and $(\mathbf{b}_1, \mathfrak{s}_1, \mathbf{u}_1) = (1, 4, \theta^2)$, $(\mathbf{b}_2, \mathfrak{s}_2, \mathbf{u}_2) = (\theta, 4, 1)$, $(\mathbf{b}_3, \mathfrak{s}_3, \mathbf{u}_3) = (1, (1, 3), (1, \theta^2))$, $(\mathbf{b}_4, \mathfrak{s}_4, \mathbf{u}_4) = (\theta, (1, 3), (1, 1))$.

For $\ell = 1$, we have $\mathbf{G}_1 = \mathbf{C}^{\otimes 4}$, and points

$$\mathbf{v}_1 = (0, 0, 0, \theta^2)^{\text{tr}} \in \mathbf{C}^{\otimes 4}(\mathbf{k}),$$

$$Z_1 = (*, *, *, \text{Li}_4^*(\theta^2))^{\text{tr}} \in \text{Lie } \mathbf{C}^{\otimes 4}(\mathbf{C}_\infty).$$

For $\ell = 2$, we have $\mathbf{G}_2 = \mathbf{C}^{\otimes 4}$, and points

$$\mathbf{v}_2 = (0, 0, 0, 1)^{\text{tr}} \in \mathbf{C}^{\otimes 4}(\mathbf{k}),$$

$$Z_2 = (*, *, *, \text{Li}_4^*(1))^{\text{tr}} \in \text{Lie } \mathbf{C}^{\otimes 4}(\mathbf{C}_\infty).$$

We also have

$$[t]\mathbf{v}_2 = (0, 0, 1, \theta)^{\text{tr}} \in \mathbf{C}^{\otimes 4}(\mathbf{k}).$$

For $\ell = 3$, we have $\mathbf{G}_3 = \mathbf{G}_a^5$ with the t -action

$$[t] = \left(\begin{array}{ccc|c} \theta & 1 & & \\ & \theta & 1 & \\ & & \theta & 1 \\ \hline \tau & & \theta & -\theta^2\tau \\ & & & \theta + \tau \end{array} \right),$$

and points

$$\mathbf{v}_3 = (0, 0, 0, -\theta^2, 1)^{\text{tr}} \in \mathbf{G}_3(\mathbf{k}),$$

$$Z_3 = (*, *, *, -\text{Li}_{(1,3)}^*(1, \theta^2), \text{Li}_1^*(1))^{\text{tr}} \in \text{Lie } \mathbf{G}_3(\mathbf{C}_\infty).$$

For $\ell = 4$, we have $\mathbf{G}_4 = \mathbf{G}_a^5$ with the t -action

$$[t] = \left(\begin{array}{ccc|c} \theta & 1 & & \\ & \theta & 1 & \\ & & \theta & 1 \\ \hline \tau & & \theta & -\tau \\ & & & \theta + \tau \end{array} \right),$$

and

$$\mathbf{v}_4 = (0, 0, 0, -1, 1)^{\text{tr}} \in \mathbf{G}_4(\mathbf{k}),$$

$$Z_4 = (*, *, *, -\text{Li}_{(1,3)}^*(1, 1), \text{Li}_1^*(1))^{\text{tr}} \in \text{Lie } \mathbf{G}_4(\mathbf{C}_\infty).$$

We also have

$$[t]\mathbf{v}_4 = (0, 0, 1, \theta + 1, \theta + 1)^{\text{tr}} \in G_4(k).$$

Therefore we have $G_{(1,3)} = G_a^6$ with the t -action

$$[t] = \left(\begin{array}{ccc|cc} \theta & 1 & & & \\ & \theta & 1 & & \\ & & \theta & 1 & \\ \tau & & & \theta & -\theta^2\tau & -\tau \\ \hline & & & & \theta + \tau & \\ \hline & & & & & \theta + \tau \end{array} \right),$$

and

$$\mathbf{v}_{(1,3)} = \pi(\mathbf{v}_1, [t]\mathbf{v}_2, \mathbf{v}_3, [t]\mathbf{v}_4) = (0, 0, 0, 1, 1, \theta + 1)^{\text{tr}} \in G_{(1,3)}(k),$$

$$Z_{(1,3)} = (*, *, *, (\theta^2 + \theta)\zeta_A(1, 3), \text{Li}_1^*(1), \theta \text{Li}_1^*(1))^{\text{tr}} \in \text{Lie } G_{(1,3)}(\mathbb{C}_\infty).$$

6. ν -ADIC MULTIPLE ZETA VALUES

Throughout this section, we fix a finite place ν of k corresponding to a monic irreducible polynomial of A that is still denoted by ν for convenience, and then fix an embedding $\bar{k} \hookrightarrow \mathbb{C}_\nu$. Let $|\cdot|_\nu$ be the normalized ν -adic absolute value on \mathbb{C}_ν . For a matrix $\gamma = (\gamma_{ij})$ with entries $\gamma_{ij} \in \mathbb{C}_\nu$, we define

$$|\gamma|_\nu := \max_{i,j} \{|\gamma_{ij}|_\nu\}.$$

In this section, we will define ν -adic multiple zeta values inspired by Furusho's definition of p -adic multiple zeta values in [F04]. The primary goal of this section is to give a logarithmic interpretation for ν -adic MZV's. Together with Theorem 1.2.2, we apply Yu's sub- t -module theorem [Yu97] to prove Theorem 1.3.1.

In what follows, for a t -module G defined over \bar{k} we denote by $\log_G(\mathbf{x})_\nu$ the ν -adic convergence value of \log_G at $\mathbf{x} \in G(\mathbb{C}_\nu)$ whenever $\log_G(\mathbf{x})_\nu$ converges, i.e., \log_G converges ν -adically at \mathbf{x} .

6.1. Definition of ν -adic MZV's.

6.1.1. *The set up.* Fix an index $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ with $n := \text{wt}(\mathfrak{s})$. As in Sec. 5.3, we identify the set of triples $(b_\ell, \mathfrak{s}_\ell, \mathbf{u}_\ell)$ occurring in Theorem 5.2.5 as the set

$$\mathcal{T} = \{1, \dots, T\},$$

where we understand that each element $\ell \in \mathcal{T}$ corresponds to a triple $(b_\ell, \mathfrak{s}_\ell, \mathbf{u}_\ell)$. Recall that each \mathbf{u}_ℓ is an integral point in $A^{\text{dep}(\mathfrak{s}_\ell)}$. We let G_ℓ be the t -module defined over k (Sec. 4.1) and \mathbf{v}_ℓ be the special point in $G_\ell(k)$ constructed using the pair $(\tilde{\mathfrak{s}}_\ell, \tilde{\mathbf{u}}_\ell)$. We then let $G_\mathfrak{s}$ be the t -module associated to the fiber coproduct \mathcal{M} of the Anderson dual t -motives $\{\mathcal{M}'_\ell\}_{\ell=1}^T$ over $\mathbb{C}^{\otimes n}$. Finally, we define $\mathbf{v}_\mathfrak{s} := \pi([\mathbf{b}_\ell(t)]\mathbf{v}_\ell)_\ell \in G_\mathfrak{s}(k)$.

Note that $G_\mathfrak{s}$ has dimension $d := d_\mathfrak{s} := n + h_{s+1} + \dots + h_T$ (see (3.1.1)), where $n + h_\ell$ is the dimension of G_ℓ for $s+1 \leq \ell \leq T$.

6.1.2. *v*-adic analytic continuation of Li_s^* . For each $\ell \in \mathcal{T}$, we consider the CMSPL $\text{Li}_{s_\ell}^*$ and its *v*-adic convergence. We note that $\text{Li}_{s_\ell}^*$ converges on the open unit ball centered at the zero of $\mathbb{C}_v^{\text{dep}(s_\ell)}$ and it is shown in [CM17, Sec. 4.1] that $\text{Li}_{s_\ell}^*$ can be analytically continued to the closed unit ball centered at the zero of $\mathbb{C}_v^{\text{dep}(s_\ell)}$. Since \mathbf{u}_ℓ is an integral point in $A^{\text{dep}(s_\ell)}$, we have $|\mathbf{u}_\ell|_v \leq 1$ and hence $\text{Li}_{s_\ell}^*$ is defined at \mathbf{u}_ℓ in the sense of *v*-adic convergence. We denote by $\text{Li}_{s_\ell}^*(\mathbf{u}_\ell)_v$ the *v*-adic convergence value of $\text{Li}_{s_\ell}^*$ at \mathbf{u}_ℓ , where we add the subscript *v* to emphasize the *v*-adic convergence. More precisely, $\text{Li}_{s_\ell}^*(\mathbf{u}_\ell)_v$ is the value $\frac{(-1)^{\text{dep}(s_\ell)-1}}{\alpha(\theta)}$ multiplied by the *n*th coordinate of $\log_{G_\ell}([\mathbf{a}]\mathbf{v}_\ell)_v$ for some nonzero polynomial $\alpha \in \mathbb{F}_q[t]$ with $|\alpha(\theta)|_v < 1$. Since the coefficients of \log_{G_ℓ} are matrices with entries in k (see [CM17, (3.2.4)]), we have $\text{Li}_{s_\ell}^*(\mathbf{u}_\ell)_v \in k_v$.

6.1.3. *The definition.* Now we are ready to define *v*-adic MZV's using $\text{Li}_{s_\ell}^*(\mathbf{u}_\ell)_v$.

Definition 6.1.1. For any index $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, let notation be given in Theorem 5.2.5. We define the *v*-adic MZV $\zeta_{\mathfrak{A}}(\mathfrak{s})_v$ to be the following value:

$$\zeta_{\mathfrak{A}}(\mathfrak{s})_v := \frac{1}{\Gamma_{\mathfrak{s}}} \sum_{\ell} \mathfrak{b}_\ell \cdot (-1)^{\text{dep}(s_\ell)-1} \text{Li}_{s_\ell}^*(\mathbf{u}_\ell)_v \in k_v.$$

We call $\text{wt}(\mathfrak{s}) := \sum_{i=1}^r s_i$ the weight and $\text{dep}(\mathfrak{s}) := r$ the depth of the presentation $\zeta_{\mathfrak{A}}(\mathfrak{s})_v$.

6.2. **Logarithmic interpretation of *v*-adic MZV's.** The primary goal in this subsection is to give a logarithmic interpretation for $\zeta_{\mathfrak{A}}(\mathfrak{s})_v$, where the depth one case was established in [AT90].

6.2.1. *The v*-adic convergence of \log_{G_s} .

Proposition 6.2.1. Fix any index $\mathfrak{s} \in \mathbb{N}^r$. For any $\mathbf{x} \in G_s(\mathbb{C}_v)$ with $|\mathbf{x}|_v < 1$, we have that \log_{G_s} converges *v*-adically at \mathbf{x} in $\text{Lie } G_s(\mathbb{C}_v)$.

Proof. We write $\log_{\mathbb{C}^{\otimes n}} = \sum_{i=0}^{\infty} R_i \tau^i$ ($R_i \in \text{Mat}_n(k)$) for the logarithm of $\mathbb{C}^{\otimes n}$. We denote the logarithms of G_s and G_ℓ by

$$\log_{G_s} = \sum_{i \geq 0} Q_i \tau^i \quad (Q_i \in \text{Mat}_d(k)) \quad \text{and} \quad \log_{G_\ell} = \sum_{i \geq 0} Q_{\ell i} \tau^i \quad (Q_{\ell i} \in \text{Mat}_{n+h_\ell}(k))$$

respectively. For each $s+1 \leq \ell \leq T$, we can write

$$Q_{\ell i} = \begin{pmatrix} R_i & R'_{\ell i} \\ & R''_{\ell i} \end{pmatrix} \quad (R'_{\ell i} \in \text{Mat}_{n \times h_\ell}(k), \quad R''_{\ell i} \in \text{Mat}_{h_\ell}(k)).$$

Then Q_i is expressed as

$$Q_i = \begin{pmatrix} R_i & R'_{s+1,i} & R'_{s+2,i} & \cdots & R'_{T,i} \\ & R''_{s+1,i} & R''_{s+2,i} & & \\ & & & \ddots & \\ & & & & R''_{T,i} \end{pmatrix}$$

for each i since it forces the functional equation

$$\partial \pi \circ \left(\bigoplus_{\ell=1}^T \log_{G_\ell} \right) = \log_{G_s} \circ \pi.$$

Let $\mathbf{x} = (\mathbf{x}_0^{\text{tr}}, \mathbf{x}_{s+1}^{\text{tr}}, \mathbf{x}_{s+2}^{\text{tr}}, \dots, \mathbf{x}_T^{\text{tr}})^{\text{tr}} \in G_s(\mathbb{C}_v)$ with $\mathbf{x}_0 \in \mathbb{C}_v^n$, $\mathbf{x}_\ell \in \mathbb{C}_v^{h_\ell}$ for $s+1 \leq \ell \leq T$, and $|\mathbf{x}|_v < 1$. Note that by [CM17, Sec. 3.3] we have

$$\left\{ \left| Q_{s+1,i} \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_{s+1} \end{pmatrix}^{(i)} \right|_v \right\} \rightarrow 0 \text{ and } \left\{ \left| Q_{\ell i} \begin{pmatrix} \mathbf{0} \\ \mathbf{x}_\ell \end{pmatrix}^{(i)} \right|_v \right\} \rightarrow 0 \text{ as } i \rightarrow \infty.$$

It follows that

$$|Q_i \mathbf{x}^{(i)}|_v \leq \max_{s+2 \leq \ell \leq T} \left\{ \left| Q_{s+1,i} \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_{s+1} \end{pmatrix}^{(i)} \right|_v, \left| Q_{\ell i} \begin{pmatrix} \mathbf{0} \\ \mathbf{x}_\ell \end{pmatrix}^{(i)} \right|_v \right\} \rightarrow 0 \text{ as } i \rightarrow \infty.$$

□

Proposition 6.2.2. *For any index $\mathfrak{s} \in \mathbb{N}^r$, we continue with the notation as above. Then there is a precise nonzero polynomial $\alpha \in \mathbb{F}_q[t]$ (depending on \mathfrak{s} and v) so that $|[\alpha] \mathbf{v}_\mathfrak{s}|_v < 1$, hence $\log_{G_\mathfrak{s}}([\alpha] \mathbf{v}_\mathfrak{s})_v$ converges.*

Proof. Write $\mathfrak{s}_\ell = (s_{\ell 1}, \dots, s_{\ell r_\ell})$ and set

$$\alpha_\ell := (v(t)^{s_{\ell 1} + s_{\ell 2} + \dots + s_{\ell r_\ell}} - 1)(v(t)^{s_{\ell 1} + s_{\ell 2} + \dots + s_{\ell, r_\ell - 1}} - 1) \dots (v(t)^{s_{\ell 1}} - 1) \in \mathbb{F}_q[t]$$

and

$$\alpha := \prod_{\ell=1}^T \alpha_\ell \in \mathbb{F}_q[t],$$

where $v(t) := v|_{\theta=t}$. Since $\mathbf{u}_\ell \in A^{\text{dep}(\mathfrak{s}_\ell)}$ for each ℓ , by Remark 4.1.7 we have that for each $\alpha \in \mathbb{F}_q[t]$, the coefficient matrices of τ^i of $[\alpha]$ are in $\text{Mat}_{\dim G_\ell}(A)$. It follows that $|[\alpha]([\mathbf{b}_\ell(t)] \mathbf{v}_\ell)|_v = |[\mathbf{b}_\ell(t)]([\alpha] \mathbf{v}_\ell)|_v \leq |[\alpha] \mathbf{v}_\ell|_v \leq |[\alpha_\ell] \mathbf{v}_\ell|_v < 1$, where the last inequality is from the proof of [CM17, Prop. 4.1.1]. So by [CM17, Sec. 3.3] again $\log_{G_\ell}([\alpha]([\mathbf{b}_\ell(t)] \mathbf{v}_\ell))_v$ converges in $\text{Lie } G_\ell(\mathbb{C}_v)$. Therefore we have

$$(6.2.3) \quad |[\alpha] \mathbf{v}_\mathfrak{s}|_v = |\pi([\alpha]([\mathbf{b}_\ell(t)] \mathbf{v}_\ell))_v| \leq \max_{\ell} |[\alpha]([\mathbf{b}_\ell(t)] \mathbf{v}_\ell)|_v < 1,$$

where the first inequality comes from Definition 3.2.2. It follows that

$$\log_{G_\mathfrak{s}}([\alpha] \mathbf{v}_\mathfrak{s})_v = \partial \pi([\alpha]([\mathbf{b}_\ell(t)] \mathbf{v}_\ell))_v$$

converges in $\text{Lie } G_\mathfrak{s}(\mathbb{C}_v)$. □

Theorem 6.2.4. *Fix a finite place v of k . Given an index $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, we put $n := \text{wt}(\mathfrak{s})$ and let $\{(\mathbf{b}_\ell, \mathfrak{s}_\ell, \mathbf{u}_\ell)\}_{\ell=1}^T$ be the set of triples in (5.2.7). Let G_ℓ be the t -module defined over k and $\mathbf{v}_\ell \in G_\ell(k)$ be the special point which are constructed using the pairs $(\tilde{\mathfrak{s}}_\ell, \tilde{\mathbf{u}}_\ell)$, and $G_\mathfrak{s}$ be the t -module over k and $\mathbf{v}_\mathfrak{s} \in G_\mathfrak{s}(k)$ be constructed as above. We take a nonzero $\alpha \in \mathbb{F}_q[t]$ for which $|[\alpha] \mathbf{v}_\mathfrak{s}|_v < 1$. Then the n th coordinate of $\log_{G_\mathfrak{s}}([\alpha] \mathbf{v}_\mathfrak{s})_v$ is given by $\alpha(\theta) \Gamma_\mathfrak{s} \zeta_A(\mathfrak{s})_v$.*

Remark 6.2.5. Since the n th coordinate of $\text{Lie } G_\mathfrak{s}(\mathbb{C}_v)$ is tractable, it is enough to show that the statement of Theorem 6.2.4 holds for some α . Indeed, assume that the statement holds for α , and let $\alpha' \in \mathbb{F}_q[t]$ be another nonzero polynomial with $|[\alpha'] \mathbf{v}_\mathfrak{s}|_v < 1$. Then we have

$$\begin{aligned} \alpha(\theta) \times n\text{th coordinate of } \log_{G_\mathfrak{s}}([\alpha'] \mathbf{v}_\mathfrak{s})_v &= n\text{th coordinate of } \log_{G_\mathfrak{s}}([\alpha][\alpha'] \mathbf{v}_\mathfrak{s})_v \\ &= \alpha'(\theta) \times n\text{th coordinate of } \log_{G_\mathfrak{s}}([\alpha] \mathbf{v}_\mathfrak{s})_v \\ &= \alpha(\theta) \alpha'(\theta) \Gamma_\mathfrak{s} \zeta_A(\mathfrak{s})_v. \end{aligned}$$

Proof of Theorem 6.2.4. Let $\mathcal{T} = \{1, \dots, T\}$ be given as before in Sec. 6.1.1. We first take a nonzero polynomial $\alpha \in \mathbb{F}_q[t]$ so that

- $|[\alpha(t)]\mathbf{v}_s|_v < 1$.
- $|[\alpha(t)]\mathbf{v}_\ell|_v < 1$ for all $\ell \in \mathcal{T}$.

Note that the second property can be obtained using the same arguments in (6.2.3). It follows by Proposition 6.2.1 that $\log_{G_s}([\alpha]\mathbf{v}_s)_v$ converges, and by [CM17, Thm. 3.3.3] that every $\log_{G_\ell}([\alpha]([\mathbf{b}_\ell(t)]\mathbf{v}_\ell))_v$ converges for every $\ell \in \mathcal{T}$. We have seen that

- $\alpha(\theta)\mathbf{b}_\ell \times (-1)^{\text{dep}(s_\ell)-1} \text{Li}_{s_\ell}^*(\mathbf{u}_\ell)_v$ is the n th coordinate of $\log_{G_\ell}([\alpha]([\mathbf{b}_\ell(t)]\mathbf{v}_\ell))_v$ (see [CM17, Def. 4.1.2]);
- The n th coordinate of $\text{Lie } G_\ell$ is tractable (see (4.1.5)).

Recall by (2.2.6) that we have the following functional equation:

$$(6.2.6) \quad \log_{G_s} \circ \pi = \partial\pi \circ \left(\bigoplus_{\ell=1}^T \log_{G_\ell} \right).$$

Recall by (5.3.3) $\mathbf{v}_s := \pi([\mathbf{b}_\ell(t)]\mathbf{v}_\ell)_\ell \in G_s(\mathbb{k})$. Now we consider the specialization at the point $([\alpha]([\mathbf{b}_\ell(t)]\mathbf{v}_\ell))_\ell \in \bigoplus_{\ell=1}^T G_\ell(\mathbb{C}_v)$ of both sides of (6.2.6) under the v -adic convergence. The LHS of (6.2.6) evaluated at $([\alpha]([\mathbf{b}_\ell(t)]\mathbf{v}_\ell))_\ell$ is the vector $\log_{G_s}([\alpha]\mathbf{v}_s)_v$, which is identical to $\partial\pi([\log_{G_\ell}([\alpha]([\mathbf{b}_\ell(t)]\mathbf{v}_\ell))]_\ell)$ from the RHS of (6.2.6) evaluated at $([\alpha]([\mathbf{b}_\ell(t)]\mathbf{v}_\ell))_\ell$. By Definition 3.2.2 we see that the n th coordinate of $\partial\pi([\log_{G_\ell}([\alpha]([\mathbf{b}_\ell(t)]\mathbf{v}_\ell))]_\ell)$ is given by

$$\sum_{\ell=1}^T \text{nth coordinate of } \log_{G_\ell}([\alpha]([\mathbf{b}_\ell(t)]\mathbf{v}_\ell)),$$

which is exactly the value (by [CM17, Def. 4.1.2] and Definition 6.1.1)

$$\sum_{\ell=1}^T \alpha(\theta) \times \mathbf{b}_\ell(\theta) \cdot (-1)^{\text{dep}(s_\ell)-1} \text{Li}_{s_\ell}^*(\mathbf{u}_\ell)_v = \alpha(\theta) \Gamma_s \zeta_A(s)_v.$$

6.3. Review of Yu's sub-t-module theorem. The following notion of *regular* t-modules is due to Yu [Yu97, p. 218].

Definition 6.3.1. *Let G be a t-module defined over $\bar{\mathbb{k}}$. We say that G is regular if there is a positive integer ν for which the α -torsion submodule of $G(\bar{\mathbb{k}})$ is free of rank ν over $\mathbb{F}_q[t]/(\alpha)$ for every nonzero polynomial $\alpha \in \mathbb{F}_q[t]$.*

Note that every n th tensor power of Carlitz module $\mathbf{C}^{\otimes n}$ and Drinfeld modules defined over $\bar{\mathbb{k}}$ are regular [Yu97, p. 217]. Other examples of regular t-modules arising from special Γ -values, see [S97, BP02].

Proposition 6.3.2. *Given an index $s \in \mathbb{N}^r$, we let G_s be the t-module constructed in Section 6.1.1. Then G_s is regular.*

Proof. Note that G_s is the t-module associated to the rigid analytically trivial Anderson dual t-motive \mathcal{M} . Let ν be the rank of \mathcal{M} over $\bar{\mathbb{k}}[t]$. Since G_s is unifromizable by Proposition 2.5.3 and Remark 2.5.2, we have the following $\mathbb{F}_q[t]$ -module isomorphism via \exp_{G_s} :

$$\text{Lie } G_s(\mathbb{C}_\infty) / \Lambda_s \cong G_s(\mathbb{C}_\infty),$$

where $\Lambda_{\mathfrak{s}} := \text{Ker exp}_{G_{\mathfrak{s}}} \subset \text{Lie } G_{\mathfrak{s}}(\mathbb{C}_{\infty}) \cong \mathbb{C}_{\infty}^{\dim G_{\mathfrak{s}}}$ is a discrete free $\mathbb{F}_q[t]$ -submodule of rank ν by [HJ16, Thm. 5.28] (cf. [A86, Thm. 4]). It follows that the \mathfrak{a} -torsion submodule of $G_{\mathfrak{s}}(\bar{k})$ is isomorphic to

$$\partial[\mathfrak{a}(t)]^{-1}\Lambda_{\mathfrak{s}}/\Lambda_{\mathfrak{s}} \cong (\mathbb{F}_q[t]/(\mathfrak{a}))^{\nu}$$

for any nonzero polynomial $\mathfrak{a} \in \mathbb{F}_q[t]$. \square

Let G be a t -module defined over \bar{k} . Any connected algebraic subgroup of G that is defined over \bar{k} and that is invariant under the $\mathbb{F}_q[t]$ -action is called a sub- t -module of G over \bar{k} . The spirit of Yu's sub- t -module theorem stated in Theorem 1.1.2 is that for a given logarithmic vector Z of an algebraic point on a regular t -module G defined over \bar{k} , the smallest $\partial[t]$ -invariant vector subspace over \bar{k} in $\text{Lie } G(\mathbb{C}_{\infty})$ containing that logarithmic vector must be $\text{Lie } H(\mathbb{C}_{\infty})$ for some t -submodule $H \subset G$ over \bar{k} .

6.4. The main result. We call a positive integer n “even” if n is divisible by $q-1$; otherwise we call n “odd”. As mentioned in the introduction, $(1-\nu^{-n})\zeta_{\mathcal{A}}(n)_{\nu}$ is identical to Goss' ν -adic zeta value at n , by [Go79] we know that $\zeta_{\mathcal{A}}(n)_{\nu} = 0$ for n “even”, and by [Yu91] $\zeta_{\mathcal{A}}(n)_{\nu}$ is transcendental over k for n “odd”. The main result of this section is as follows.

Theorem 6.4.1. *Let ν be a finite place of k and fix a positive integer n . Let $\overline{\mathcal{Z}}_n$ be the \bar{k} -vector space spanned by all ∞ -adic MZV's of weight n , and $\overline{\mathcal{Z}}_{n,\nu}$ be the \bar{k} -vector space spanned by all ν -adic MZV's of weight n . Then we have a well-defined surjective \bar{k} -linear map*

$$\overline{\mathcal{Z}}_n \twoheadrightarrow \overline{\mathcal{Z}}_{n,\nu}$$

given by

$$\zeta_{\mathcal{A}}(\mathfrak{s}) \mapsto \zeta_{\mathcal{A}}(\mathfrak{s})_{\nu},$$

and its kernel contains the one-dimensional vector space $\bar{k} \cdot \zeta_{\mathcal{A}}(n)$ when n is “even”.

In other words, the theorem above shows that the ν -adic MZV's of weight n satisfy the same \bar{k} -linear relations that their corresponding ∞ -adic MZV's of weight n satisfy.

Proof of Theorem 6.4.1. Suppose that we have a non-trivial \bar{k} -linear relations among some MZV's of weight n

$$c_1\zeta_{\mathcal{A}}(\mathfrak{s}_1) + \cdots + c_m\zeta_{\mathcal{A}}(\mathfrak{s}_m) = 0,$$

which we rewrite as

$$(6.4.2) \quad \epsilon_1\Gamma_{\mathfrak{s}_1}\zeta_{\mathcal{A}}(\mathfrak{s}_1) + \cdots + \epsilon_m\Gamma_{\mathfrak{s}_m}\zeta_{\mathcal{A}}(\mathfrak{s}_m) = 0,$$

where $\left\{ \epsilon_i := \frac{c_i}{\Gamma_{\mathfrak{s}_i}} \right\}_{i=1}^m$ are not all zero. For each index \mathfrak{s}_i , let $G_{\mathfrak{s}_i}$ be the t -module defined over k , $\mathbf{v}_{\mathfrak{s}_i} \in G_{\mathfrak{s}_i}(k)$ be the special point and $Z_{\mathfrak{s}_i} \in \text{Lie } G_{\mathfrak{s}_i}(\mathbb{C}_{\infty})$ be the vector given in Theorem 1.2.2. We identify $\text{Lie } G_{\mathfrak{s}_i}$ with $\mathbb{A}^{\dim G_{\mathfrak{s}_i}}_{/k}$, the affine variety of dimension $g_i := \dim G_{\mathfrak{s}_i}$ over k , and let $\mathbf{X}_i := (X_{i1}, \dots, X_{ig_i})^{\text{tr}}$ be the coordinates of $\text{Lie } G_{\mathfrak{s}_i}$. Let $G := G_{\mathfrak{s}_1} \oplus \cdots \oplus G_{\mathfrak{s}_m}$ be the t -module as direct sum of $\{G_{\mathfrak{s}_i}\}_{i=1}^m$ and so $\text{Lie } G$ is identified with $\mathbb{A}^{g_1+\cdots+g_m}_{/k}$ with coordinates

$$\mathbf{X} = (\mathbf{X}_1^{\text{tr}}, \dots, \mathbf{X}_m^{\text{tr}})^{\text{tr}}.$$

Note that since the n th coordinate of $\text{Lie } G_{\mathfrak{s}_i}$ is tractable for each i , so is the n th coordinate of $\text{Lie } G$ by Lemma 3.2.3.

Let V be the smallest linear subspace of $\text{Lie } G(\mathbb{C}_{\infty})$ defined over \bar{k} for which

- V contains the vector $Z := (Z_{s_1}^{\text{tr}}, \dots, Z_{s_m}^{\text{tr}})^{\text{tr}} \in \text{Lie } G(\mathbb{C}_\infty)$.
- V is invariant under the $\partial[t]$ -action.

We define the following hyperplane over \bar{k}

$$W := \{\epsilon_1 X_{1n} + \dots + \epsilon_m X_{mn} = 0\} \subset \text{Lie } G,$$

where we simply use $\text{Lie } G$ for the base change of $\text{Lie } G$ over \bar{k} when it is clear from the contents, and note that $Z \in W(\mathbb{C}_\infty)$. We further note that since the n th coordinate of $\text{Lie } G$ is tractable, W is invariant under $\partial[t]$. Since the n th coordinate of $\text{Lie } G_{s_i}$ is tractable for each i , we see that $V \subset W(\mathbb{C}_\infty)$. By Theorem 1.1.2 there exists a sub- t -module $H \subset G$ over \bar{k} for which

$$V = \text{Lie } H(\mathbb{C}_\infty).$$

Let $\mathcal{V} \subset \text{Lie } G$ be the linear sub-variety underlying V . That is, \mathcal{V} is the variety defined by the defining equations of $V \subset \text{Lie } G(\mathbb{C}_\infty)$ over \bar{k} . So we have $\mathcal{V}(\mathbb{C}_\infty) = V = \text{Lie } H(\mathbb{C}_\infty)$, and hence $\mathcal{V} = \text{Lie } H$.

For each s_i , by Proposition 6.2.2 we are able to pick a nonzero $a_i \in \mathbb{F}_q[t]$ for which $|[a_i] \mathbf{v}_{s_i}|_v < 1$. Put $\mathbf{a} := \prod_{i=1}^m a_i$. Note that by Corollary 3.2.6 the action $[t]$ on each G_{s_i} has coefficient matrices with entries in A . Therefore, we have that

$$|[a] \mathbf{v}_{s_i}|_v \leq |[a_i] \mathbf{v}_{s_i}|_v < 1,$$

and hence by Proposition 6.2.1 $\log_{G_{s_i}}([a] \mathbf{v}_{s_i})_v$ converges for each $1 \leq i \leq m$. Define

$$\mathbf{v} := \begin{pmatrix} \mathbf{v}_{s_1} \\ \vdots \\ \mathbf{v}_{s_m} \end{pmatrix} \in G(\mathbf{k}).$$

Since V is invariant under $\partial[t]$ -action, we have that $\partial[a]Z \in \mathcal{V}(\mathbb{C}_\infty) = \text{Lie } H(\mathbb{C}_\infty)$, whence

$$\exp_G(\partial[a]Z) = [a]\mathbf{v} \in G(\mathbf{k}) \cap H(\bar{\mathbf{k}}) \subset H(\bar{\mathbf{k}}).$$

Let $\phi : H \hookrightarrow G$ be the natural embedding morphism of t -modules. Fix an isomorphism $H \cong G_a^h$ over \bar{k} . By using this identification, we write

$$\phi = \sum_{i=0}^N A_i \tau^i : G_a^h \cong H \hookrightarrow G = G_a^{g_1 + \dots + g_m}, \quad A_i \in \text{Mat}_{(g_1 + \dots + g_m) \times h}(\bar{\mathbf{k}})$$

and

$$\log_H = \sum_{i \geq 0} B_i \tau^i, \quad B_i \in \text{Mat}_h(\bar{\mathbf{k}}).$$

We also write

$$\log_G = \sum_{i \geq 0} C_i \tau^i, \quad C_i \in \text{Mat}_{g_1 + \dots + g_m}(\bar{\mathbf{k}}).$$

By the functional equation (2.2.6), we have

$$A_0 \circ \left(\sum_{i \geq 0} B_i \tau^i \right) = \left(\sum_{i \geq 0} C_i \tau^i \right) \circ \left(\sum_{i=0}^N A_i \tau^i \right)$$

as formal power series. Since $\text{Lie } H \hookrightarrow \text{Lie } G$ is an injective \bar{k} -linear map ($\text{Lie}(\cdot)$ is a left exact functor), we know that $\text{rank } A_0 = h$. Therefore, the convergence domain of $\sum_{i \geq 0} B_i \tau^i$ is the same as that of $(\sum_{i \geq 0} C_i \tau^i) \circ (\sum_{i=0}^N A_i \tau^i)$. Since \log_G converges

v -adically at $[a]\mathbf{v}$ from the choice of a , and H is invariant under the $[a]$ -action, we can pull back $[a]\mathbf{v}$ via the embedding ϕ , and it is contained in the convergence domain of $(\sum_{i \geq 0} C_i \tau^i) \circ (\sum_{i=0}^N A_i \tau^i)$. This implies that

$$\bigoplus_{i=1}^m \log_{G_{s_i}}([a]\mathbf{v}_{s_i})_v = \log_G([a]\mathbf{v})_v = \log_H([a]\mathbf{v})_v \in \text{Lie } H(\mathbf{C}_v).$$

Note that since $\mathcal{V}(\mathbf{C}_\infty) = V \subset W(\mathbf{C}_\infty)$, we have $\text{Lie } H = \mathcal{V} \subset W$. It follows that the vector $\bigoplus_{i=1}^m \log_{G_{s_i}}([a]\mathbf{v}_{s_i})_v$ belongs to $W(\mathbf{C}_v)$, whence satisfying the \bar{k} -linear relations

$$\epsilon_1 X_{1n} + \cdots + \epsilon_m X_{mn} = 0.$$

By Theorem 6.2.4 the n th coordinate of $\log_{G_{s_i}}([a]\mathbf{v}_{s_i})$ is the value $a(\theta)\Gamma_{s_i}\zeta_A(s_i)_v$ and hence we obtain the desired identity

$$c_1 \zeta_A(s_1)_v + \cdots + c_m \zeta_A(s_m)_v = 0.$$

Note that by [Go79] we have $\zeta_A(n)_v = 0$ for $n \in \mathbb{N}$ “even” as our v -adic zeta value at n is Goss v -adic zeta value at n multiplied by $(1 - v^{-n})^{-1}$ (see [AT90, Theorem 3.8.3. (II)]). Therefore the second assertion follows immediately.

Corollary 6.4.3. *Let v be a finite place of k . Let $\overline{\mathcal{Z}} := \sum_{n=1}^{\infty} \overline{\mathcal{Z}}_n$ be the \bar{k} -vector space spanned by all MZV's, and let $\overline{\mathcal{Z}} := \sum_{n=1}^{\infty} \overline{\mathcal{Z}}_{n,v}$ be the \bar{k} -vector space spanned by all v -adic MZV's. Then we have the \bar{k} -linear map*

$$\overline{\mathcal{Z}} \rightarrow \overline{\mathcal{Z}}_v$$

given by

$$\zeta_A(\mathfrak{s}) \mapsto \zeta_A(\mathfrak{s})_v.$$

Proof. By [C14, Thm. 2.2.1], we have a natural isomorphism $\bigoplus_n \overline{\mathcal{Z}}_n \cong \overline{\mathcal{Z}}$ of \bar{k} -algebras. Thus the \bar{k} -linear maps $\overline{\mathcal{Z}}_n \rightarrow \overline{\mathcal{Z}}_{n,v}$ in Theorem 6.4.1 imply the \bar{k} -linear map

$$\overline{\mathcal{Z}} \cong \bigoplus_n \overline{\mathcal{Z}}_n \rightarrow \bigoplus_n \overline{\mathcal{Z}}_{n,v} \rightarrow \overline{\mathcal{Z}}_v.$$

□

Remark 6.4.4. Based on the results above, the following are some natural questions which need additional work.

- (1) Does $\overline{\mathcal{Z}}_v$ have an algebra structure?
- (2) For each positive integer n , what is the kernel of the above map $\overline{\mathcal{Z}}_n \rightarrow \overline{\mathcal{Z}}_{n,v}$?
- (3) Is the above map $\overline{\mathcal{Z}} \rightarrow \overline{\mathcal{Z}}_v$ an algebra homomorphism? If so, what is its kernel?

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