The new approach to the local embedding theorem of CR-structures, the local embedding theorem for $n \geq 4$

By

Takao Akahori

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26

5300 Bonn 3

Ryukyu University
Naha
OKINAWA
Japan

MPI/SFB 84/34

the local embedding theorem for $n \geq 4$

By

Takao Akahori

Max-Planck-Institüt für Mathematik

and

Ryukyu University

Introduction . The purpose of this paper is to establish the local embedding theorem of CR-structures for the real 7-dimensional case . Let $(M, {}^{\circ}T'')$ be an abstract strongly pseudo convex CR-structure and p_o be a reference point . We study the local embedding problem of $(M, {}^{\circ}T'')$ at p_o . As is well known , in the case $\dim_R M = 2n-1 \geq 9$,i.e., $n \geq 5$, this problem was solved affirmatively by Kuranishi ((1),(2),(3)) . On the other hand for the case $\dim_R M = 2n-1 = 3$,i.e.,$n=2$, there is the famous Nirenberg's counter example ((4)) . So the cases $n=3$ and $n=4$ were left open . In this paper , we settle the case $n=4$. To see our approach , we recall Kuranishi's approach .

This work is supported by Max-Planck Institute and SFB 40 in Bonn .

For a given strongly pseudo convex CR-structure $(M, {}^{\circ}T'')$, we take an approximate C^{∞}-embedding f^{o} satisfying ;

$$j^{(k)}(Df^{o})(p_{o}) = 0 \quad ,$$

where D is the induced $\bar{\partial}_{b}$-operator by $(M, {}^{\circ}T'')$, $j^{(k)}$ means k-th jets, and p_{o} is a reference point. We modify f^{o}. Namel we want to solve

$$Du = Df^{o} \text{ on a suitable neighborhood } U(p_{o}) \text{ of } p_{o}$$

satisfying

1) u is estimated by Df^{o} in a certain sense.

If the above is solved, then

$$f^{o}-u$$

satisfies

2) $D(f^{o}-u) = 0$,

3) by 1) , $f^{o}-u$ is a C^{∞}-embedding .

Hence for establishing CR-embedding theorem , it suffices to solve " D-Neumann problem " on a suitable neighborhood . For " D-Neumann problem " , Kuranishi's approach is divided into two parts .

Part 1 . For any approximate C^{∞}-embedding f , he shows that for the CR-structure $(M, {}^{f}T'')$, if $\dim_R M = 2n-1 \geq 7$, on $\wedge({}^{f}T'')^{*}$, D^{f}-Neumann problem can be solved , and if $\dim_R M = 2n-1 \geq 9$, on $\wedge^{2}({}^{f}T'')^{*}$, D^{f}-Neumann problem can be solved , where D^{f} means the induced operator by $(M, {}^{f}T'')$ and

$${}^{f}T'' = \left\{ X : X \in CTM , \ f_{*}X \in T''C^{n} \right\}$$, i.e., the induced CR-

structure by f .

Part 2 . To find f satisfying the above partial differential equation , he used Nash-Moser's process . Namely by induction , he constructed a sequence of neighborhoods $U_{r_{\mu}}$, and a C^{∞}-embeddings $f^{(\mu)}$ of $U_{r_{\mu}}$ into C^{n} as follows .

$$f^{(1)} = f^{(0)} - M_1 D^{(0)*} N^{(0)} D f^{(0)} \quad \text{on} \quad U_{r_0}$$

$$f^{(\mu+1)} = f^{(\mu)} - M_{\mu} D^{(\mu)*} N^{(\mu)} D f^{(\mu)} \quad \text{on} \quad U_{r_{\mu}} ,$$

where $N^{(\mu)} = N^{(1)}$, obtained in Part 1 and $D^{(\mu)*}$ means the adjoint operator of $D^{(\mu)}$. And if $\dim_R M = 2n-1 \geq 9$, this sequence converges and satisfies

$$Df = 0 \quad .$$

I must explain why he imposed the assumption $\dim_R M \geq 9$. Roughly speaking , to prove the convergence for $f^{(\mu)}$ by Nash-Moser' process is to show that $Df^{(\mu+1)}$ can be estimated by the quadratic of $Df^{(\mu)}$. And in his proof for Part 2 , he used

$$\begin{aligned} & D(f^{(\mu)} - D^{(\mu)*}N^{(\mu)}Df^{(\mu)}) \\ = \; & (D^{(\mu)} - D)D^{(\mu)*}N^{(\mu)}Df^{(\mu)} + D^{(\mu)*}D^{(\mu)}N^{(\mu)}Df^{(\mu)} \\ = \; & (D^{(\mu)} - D)D^{(\mu)*}N^{(\mu)}Df^{(\mu)} + D^{(\mu)*}N^{(\mu)}D^{(\mu)}Df^{(\mu)} \\ = \; & (D^{(\mu)} - D)D^{(\mu)*}N^{(\mu)}Df^{(\mu)} + D^{(\mu)*}N^{(\mu)}(D^{(\mu)} - D)Df^{(\mu)} \end{aligned}$$

Here $(D^{(\mu)}-D)Df^{(\mu)}$ behaves like the quadratic of $Df^{(\mu)}$.
In this equality , he used the assumption $\dim_R M = 2n-1 \geq 9$. It is needless to say that

$$D^{(\mu)*}D^{(\mu)}N^{(\mu)}Df^{(\mu)} = N^{(\mu)}D^{(\mu)*}D^{(\mu)}Df^{(\mu)}$$

doesn't make sense because of the boundary condition (we recall that at each step the domain U_{r_μ} varies) . So in his method , the assumption , $\dim_R M = 2n-1 \geq 9$ is necessary . Therefore in order to bypass this difficulty , it is very natural to try to reduce our problem to so called " D_b-Neumann problem " , where the boundary condition does not appear . I must explain this part precisely .

First , we take a C^∞-embedding ψ satisfying ;

$$j^{(\ell)}(D\psi)(p_o) = 0 \quad ,$$

and

$$o(\psi) \in \Gamma(M, {}^o\bar{T}'' \otimes ({}^oT'')^*) \quad ,$$

where $(M, {}^{o(\psi)}T'')$ is the induced CR-structure by ψ and

$${}^{o(\psi)}T'' = \left\{ X' \ ; \ X' = X + o(\psi)(X) \ , \ X \in {}^oT'' \right\}$$

(this result corresponds to the local triviality of deformations of contact structures) . Next we modify ψ . Namely , we want to find a C^∞ - embedding f^o satisfying ;

$$o(f^o) \text{ is of } \Gamma(M, {}^o\bar{T}'' \otimes ({}^oT'')^*) \quad ,$$

$$o(f^o)(p_o) = 0$$

and

$(1/b(f^{o}))^{2\ell} Df^{o}$ is bounded near at p_o , where ℓ is an integer satisfying

$$\ell \geq 30(3n+3)$$

and

$$b(f^{o}) = \sqrt{\sum_{i=1}^{n-1} |Y_i^{o(f^{o})} t \circ f^{o}|^2}$$

(this is proved in Sect.1.7) . Now we consider D - Neumann problem on a neighborhood of p_o . We must introduce a nice neighborhood for the differential operator D . For this , we take a holomorphic function h on C^n satisfying that $t \circ f^{o} = \mathrm{Re}\, h \circ f^{o}$ is an admissible distance function for $(M, {}^{o(f^{o})}T'')$ (for the notation , see (2.12) Definition in (3)) . And consider the domain defined by

$$\{ p \; ; \; p \in M \; , t \circ f^{o}(p) < r \} \; .$$

However this domain is still not enough for solving D - Neumann problem .

In following the Kohn's approach for $\bar{\partial}$ -operator , in our case there is one difficulty . Because

$$f\text{-}\dim_C (CTM/({}^{\circ}T_b'' + {}^{\circ}\bar{T}_b'')) = 2 ,$$

where ${}^{\circ}T_b'' = \{ X' ; X \in {}^{\circ}T'' , X(t\circ f^{\circ}) = 0 \}$.
And in treating with the bracket $[W_i, \bar{W}_j]$,
where W_i means the projection of Y_i " along $t\circ f^{\circ}$ "
namely ,

$$W_i = Y_i - (Y_i(t\circ f^{\circ})/b(f^{\circ}))\sum_{\ell=1}^{n-1} (\bar{Y}_{\ell}(t\circ f^{\circ})/b(f^{\circ}))Y_{\ell} ,$$

X°-term and $Y^{\circ}-\bar{Y}^{\circ}$ -term might appear , where

$$X^{\circ} = \sqrt{-1}\, b(f^{\circ})S + \bar{\gamma}_f \circ Y^{\circ} - \gamma_f \circ \bar{Y}^{\circ}$$

and

$$Y^{\circ} = \sum_{j=1}^{n-1} (\bar{Y}_j(t\circ f^{\circ})/b(f^{\circ}))Y_j .$$

And for X°-term , by the standard argument , we can control this term . But for $Y^{\circ}-\bar{Y}^{\circ}$ -term , we have no way to control this . Therefore we must modify f° .
For this , first , we consider C^{∞}-embedding f satisfying

(A) $(1/b(f^{\circ})) \,|\, j_f^{(1)}(f-f^{\circ})| < c_1(f^{\circ})$ on $U_r(f^{\circ})$, For this f , we see that D_b^f - Neumann problem can be solved on $({}^f T_b'')^*$ if $\dim_R M = 2n-1 \geq 7$, and D_b^f - operator is defined as follows . We set

$$W_i^f = Y_i^f - ((Y_i^f(t\cdot f))/b(f)) \sum_{\ell=1}^{n-1} (\bar{Y}_\ell^f(t\cdot f)/b(f)) Y_\ell^f ,$$

where $\{Y_i^f\}_{1 \leq i \leq n-1}$ is an orthonormal base of ${}^{o(f)}T''$ (this is determined canonically) . We note that by (A) , this makes sense on $M-C$. And defines ${}^f T_b''$ by the sub-vector bundle of ${}^f T''$, generated by W_i^f , $i=1,2,,,,n-1$.
Then we have a differential subcomplex , D_b^f - complex of D^f - complex .

Then , our problem , i.e. , to find a nice neighborhood for solving D - Neumann problem is reduced to

(B) $D_b f = 0$ along $t \circ f$, namely

$$(Y_i - ((Y_i t \circ f)/\sqrt{\sum_i |Y_i t \circ f|^2})\sum_{\ell=1}^{n-1}((\bar{Y}_\ell t \circ f)/\sqrt{\sum_i |Y_i t \circ f|^2})Y_\ell) f_\alpha$$

$= 0$, $\alpha = 1, 2, \ldots, n$,

where f satisfies (A) .

The condition (B) is equivalent to

" in $[W_i, \bar{W}_j]$, $Y^o - \bar{Y}^o$ -term doesn't appear " .

Therefore our problem is reduced to finding the solution of the non-linear differential equation . Hence it is very natural to rely on Nash-Moser's process . And this is carried out in Chapter 8 .

Chapter 1

1.1. Deformation theory of CR-structures

Let $(M, {}^{\circ}T'')$ be an abstract CR-manifold and p_o be a point of M. This means that ${}^{\circ}T''$ is a subbundle of complexfied tangent bundle CTM satisfying

1.1.1) ${}^{\circ}T'' \cap {}^{\circ}\bar{T}'' = 0$, $f\text{-}\dim_C((CTM)/({}^{\circ}T''+{}^{\circ}\bar{T}'')) = 1$,

1.1.2) $[\Gamma(M, {}^{\circ}T''), \Gamma(M, {}^{\circ}T'')] \subset \Gamma(M, {}^{\circ}T'')$,

where $\Gamma(M, {}^{\circ}T'')$ means the space of ${}^{\circ}T''$-valued global C^{∞}-sections . We take a C^{∞}-vector bundle decomposion of CTM (not unique but exists):

1.1.3) $CTM = {}^{\circ}T'' + {}^{\circ}\bar{T}'' + CS$, where S is a global vector field on M satisfying ; $S \notin {}^{\circ}T'' + {}^{\circ}\bar{T}''$ at each point of M .

By using 1.1.3) , we introduce the Levi-form $c_S(X,Y)$ by

$$c_S(X,Y) = -\sqrt{-1}\,[X,\bar{Y}]_S \quad \text{for} \quad X , Y \in {}^{\circ}T'' ,$$

where $[X,\bar{Y}]_S$ denotes the projection of $[X,\bar{Y}]$ to S-part according to 1.1.3) . If this Levi-form is positive definite , we call $(M, {}^{\circ}T'')$ strongly pseudo convex . From now on we assume that $(M, {}^{\circ}T'')$ is strongly pseudo convex .

Next we recall deformation theory of CR-structures (cf.(1),(2)) .

Definition 1.1.1. The pair (M,E) is called an almost CR-structure which is of finite distance from $(M, {}^{\circ}T'')$ if and only if E is a subbundle of

$\mathbb{C}TM$ satisfying ; $E \cap \bar{E} = 0$ and

1.1.4)
$$\begin{tikzcd} E \subset \mathbb{C}TM = {}^{\circ}T'' + {}^{\circ}\bar{T}'' + \mathbb{C}S \arrow[d] \arrow[dr, phantom] \\ {}^{\circ}T'' \end{tikzcd}$$

the induced map from E to ${}^{\circ}T''$ is an isomorphism map .

Then we have

Proposition 1.1.2. An almost CR-structure,which is of finite distance , corresponds to an element $o(\phi)$ of $\Gamma(M,T'\otimes({}^{\circ}T'')^*)$, where $T'={}^{\circ}\bar{T}''+S$, bijectively . The correspondence is that for $o(\phi)$ of $\Gamma(M,T'\otimes({}^{\circ}T'')^*)$,

$$^{\phi}T'' = \left\{X' \;;\; X' = X + o(\phi)(X) \;,\; X \in {}^{\circ}T''\right\} ,$$

(see Proposition 1.1 in (1)) .

And

Proposition 1.1.3. An almost CR-structure (M, T'') is integrable , i.e., CR-structure if and only if ϕ satisfies

$$P(\phi) = 0$$

(see Proposition 1.2 in (1)) .

As is well known , the local embedding theorem holds in the formal category . In terms of deformation theory we will write down this fact as follows . Let $\{Y_i\}_{1\leq i\leq n-1}$ be an orthonormal base of ${}^{\circ}T''$ on a neighborhood of p_o with respect to the Levi-form defined by 1.1.4)

Then for any integer k , there are C^{∞}-functions $z_1^k, \dots, z_n^k$ satisfying that ; if $p \leq k$, the p-th coefficient of Taylor expansion of $Y_i z_\alpha^k$ at p_o vanishes and

$$\{dz_\alpha^k\}_{\alpha=1,2,\dots,n}$$

are independent over C at p_o . So we consider the following CR-structure

$$\{Y' \ ; \ Y' \in \Gamma(M, CTM) \ , \ Y' z_\alpha^k = 0 \ , \alpha = 1,2,\dots,n \ \text{ on } M\}$$

In terms of deformation theory , this CR-structure corresponds to an element $o(\phi)$ of $\Gamma(M, T' \otimes ({}^\circ T'')^*)$, defined by ; for $Y \in \Gamma(M, {}^\circ T'')$

$$(Y + o(\phi)(Y)) z_\alpha^k = 0 \quad \text{on } M \quad , \alpha = 1,2,\dots,n .$$

We easily have that the p-th degree coefficient of Taylor expansion of $o(\phi)$ at p_o vanishes . Namely , we have

Theorem 1.1.3. Let $(M, {}^\circ T'')$ be a CR-structure . Then , for any integer k , there is a CR-structure $(M, {}^\circ T'')$ which is embeddable as a real hypersurface in a euclidean space , satisfying that the p-th degree coefficient of Taylor expansion of $o(\phi)$ at p_o vanishes , where $p \leq k$.

1.2. Reducing to an element of $\Gamma(M, {}^{\circ}\bar{T}'' \otimes ({}^{\circ}T'')^*)$

In 1.1 , we showed that for any strongly pseudo convex CR-structure $(M, {}^{\circ}T'')$ and for any integer k , there is a CR-structure $(M, {}^{\phi}T'')$ which can be embedded as a real hypersurface satisfying ; coefficients of p-th degree ($p \leq k$) of $o(\phi)$ vanish at p_o and

$$o(\phi) \in \Gamma(M, T' \otimes ({}^{\circ}T'')^*)$$

In this section , we see that this $o(\phi)$ can be reduced to an element of $\Gamma(M, {}^{\circ}\bar{T}'' \otimes ({}^{\circ}T'')^*)$. Namely we have

Theorem 1.2.1. For any strongly pseudo convex CR-structure $(M, {}^{\circ}T'')$ and for any integer k , there is a CR-structure $(M, {}^{\psi}T'')$, which can be embedded as a real hypersurface , satisfying ; coefficients of p-th degree ($p \leq k$) of $o(\psi)$ vanish at p_o and

$$o(\psi) \in \Gamma(M, {}^{\circ}\bar{T}'' \otimes ({}^{\circ}T'')^*) .$$

Proof . Let $(M, {}^{\phi}T'')$ be as in Theorem 1.1.3 for $k+1$ (we assume $k \geq 2$) . Then $(M, {}^{\phi}T'')$ defines a deformation of the contact structure (M, D) , $(M, {}^{\phi'}D)$, where $D = \{X' \ ; \ X' = \mathrm{Re}\, X , X \in {}^{\circ}T''\}$. On the other hand , by the existence theorem of the canonical form for contact structures , there is a diffeomorphism map of M , f , satisfying

$$f(p_o) = p_o$$

and

$$f_* {}^{\phi'}D = D \quad .$$

This is a well known result , but it is necessary to check f's value at p_o , so we briefly sketch the proof . Let θ be the 1-form which corresponds to (M,D) and $\theta(\phi')$ be the 1-form which corresponds to $(M, {}^{\phi'}D)$. The correspondence is that

$$D = \{ X ; \ X \in TM \ , \ \theta(X) = 0 \}$$

and

$${}^{\phi'}D = \{ X' ; \ X' \in TM \ , \ \theta(\phi')(X) = 0 \} \quad .$$

Then it is enough to see that there is a diffeomorphism map f of M satisfying

$$f(p_o) = p_o$$

and

$$f^* \theta(\phi') = \theta \quad .$$

We see this . We define the vector field ξ_t of D by

$$(d\theta(\phi') + t(d\theta - d\theta(\phi'))(\xi_t, X)$$

$$= -(\theta(\phi') - \theta)(X) \qquad \text{for } X \in D .$$

Since $d\theta(\phi') + t(d\theta - d\theta(\phi'))$ is non-generate(because of strongly pseudo convexity) , ξ_t uniquely exists . And we have ;
(1.2.1) if ϕ' vanishes at p_o up to order k+1 , ξ_t vanishes also at p_o up to order k+1 . Now we consider the 1-parameter group α_t integrated by ξ_t . We claim

$$(d/dt)(\alpha_t^*(\theta(\phi') + t(\theta - \theta(\phi'))) = 0$$

and

$$\alpha_1^*\theta = \theta \quad .$$

Because

$$(d/dt)(\alpha_t^*(\theta(\phi') + t(\theta - \theta(\phi')))$$

$$= \alpha_t^*((d/dt)(\theta(\phi') + t(\theta - \theta(\phi')))$$

$$+ \alpha_t^*(\mathcal{L}_{\xi_t}(\theta(\phi') + t(\theta - \theta(\phi'))) ,$$

where $\mathcal{L}_{\xi_t}$ means the Lie derivation .

Namely ,

$$(d/dt)\,(\alpha_t^*(\theta(\phi')) + t(\theta - \theta(\phi')))$$

$$= \alpha_t^*\,(\,(\theta - \theta(\phi')) + (\,d\theta(\phi') + t(\,d\theta - d\theta(\phi'))\circ\xi_t\,))$$

$$= 0\ .$$

And obviously ,

$$\alpha_1(x) = x \quad \text{and} \quad D_2\alpha(x) = \text{identity}\ .$$

So

$$\alpha_1^*\theta = \theta\ .$$

Furthermore as ξ_0 vanishes at p_0 for order $k+1$,

$$\alpha_0 - \text{identity}$$

vanishes at p_0 for order $k+1$. This means that there is a local diffeomorphism map f ($f=\alpha_0$) , satisfying

$$f(p) = p$$

and

$$f_*{}^{\phi'}D = D\ .$$

So we set $\psi = \phi \circ f$. Then obviously (M/T'') can be embedded as a real hypersurface and satisfying

$$o(\psi) \in \Gamma(M, {}^{o}\bar{T}'' \otimes ({}^{o}T'')^{*}) \quad .$$

We must check that coefficients of p-th degree $(p \leq k)$ of ψ vanish at p_o . By the definition of ${}^{\phi'}D$,

$${}^{\phi'}D = \left\{ Z' \; ; \; Z' = \mathrm{Re}\, X' \; , \; X' \in {}^{\phi}T'' \right\} ,$$

and

$${}^{\phi}T'' = \left\{ X' \; ; \; X' = X + \phi_1(X) + \phi_2(X) \; , \; X \in {}^{o}T'' \right\} .$$

So

$${}^{\phi'}D = \left\{ Z' \; ; \; Z' = \mathrm{Re}((X + \phi_1(X)) + \phi_2(X)) \; , \; X \in {}^{o}T'' \right\} .$$

We that this ϕ' can be expressed by ϕ_1 and ϕ_2 . First we consider the following map . For $X' = \mathrm{Re}\, X$, X ${}^{o}T''$, we set $K_1(\phi_1)^{-1}$ by

$$K_1(\phi_1)^{-1}(X') = \mathrm{Re}\,(\, X + \phi_1(X)\,) \quad .$$

This defines an automorphism map of D , which depends on ϕ_1 . We use the notation $k_1(\phi_1)$ for its inverse map . And we set a homomorphism map $\chi_2(\phi_2)$ from D to S by ; for $X' = \mathrm{Re}\, X$, $X \in {}^{\circ}T''$,

$$\chi_2(\phi_2)(X') = \mathrm{Re}\, \phi_2(X) .$$

Then our ϕ' is expressed by

$$\phi' = \chi_2(\phi_2)\chi_1(\phi_1) .$$

Since coefficients of p-th degree ($p \leqq k+1$) of ϕ_1 and ϕ_2, i.e., coefficients of p-th degree (p k+1) of ϕ' vanishes at p_o. Therefore we have (1.2.1) . With this , we are going to show that coefficients of p-th degree ($p \leqq k$) of ψ vanishes at p_o. We recall the definition of the induced CR-structure ,

$$f_*^{\phi}T'' = {}^{\psi}T'' ,$$

namely for $X \in \Gamma(M, {}^{\circ}T'')$, there is a Z in $\Gamma(M, {}^{\circ}T'')$ satisfying

$$f_*(X + \phi(X)) = Z + \psi(Z) .$$

So we have ; for $X \in \Gamma(M, {}^{\circ}T'')$

$$f_*(X + \phi(X)) = (f_*(X + \phi(X)))_{{}^{\circ}T''} + \psi((f_*(X + \phi(X)))_{{}^{\circ}T''}) ,$$

where $(f_*(X + \phi(X)))_{{}^{\circ}T''}$ means the projection of $f_*(X + \phi(X))$ to ${}^{\circ}T''$

according to the vector bundle decomposition of CTM in Definition 1.1.1 . At $p=p_0$, we check its value . Since coefficients of p-th degree $(p \leqq k+1)$ of $f-\psi$ vanish and coefficients of p-th degree $(p \leqq k)$ of $o(f)$ vanish , and so coefficients of p-th degree $(p \leqq k)$ of $o(\psi)$ vanish .

So we have our theorem . Q.E.D.

1.3. The orthonormal base of $(M, {}^{\psi}T'')$

We assume that $o(\psi)$ is of $\Gamma(M, {}^{o}\bar{T}'' \otimes ({}^{o}\bar{T}'')^*)$. In this section we construct a moving frame $\{Y_i^{o(\psi)}\}_{1\leq i\leq n-1}$ of $(M, {}^{\psi}T'')$ satisfying

$$-\sqrt{-1}\,[Y_i^{o(\psi)}, \bar{Y}_j^{o(\psi)}]_F = \delta_{i,j} ,$$

where $[Y_i^{o(\psi)}, \bar{Y}_j^{o(\psi)}]_F$ means the orthogonal projection of $[Y_i^{o(\psi)}, \bar{Y}_j^{o(\psi)}]$ to F-part according to the vector bundle decomposition (1.1.3) . Namely , we want to find a u of $\Gamma(M, {}^{o}T'' \otimes ({}^{o}T'')^*)$ satisfying

$$-\sqrt{-1}\left[Y_i+u(Y_i)+o(\psi)(Y_i+u(Y_i)) \;,\; \overline{Y_j+u(Y_j)+o(\psi)(Y_j+u(Y_j))}\right]_F = \delta_{i,j}$$

Let

$$u(Y_i) = \sum_\ell u_{\ell,i} Y_\ell$$

Then the above is

$$-\sqrt{-1}\left[Y_i+u(Y_i)+o(\psi)(Y_i+u(Y_i)) \;,\; \overline{Y_j+u(Y_j)+o(\psi)(Y_j+u(Y_j))}\right]_F$$

$$= -\sqrt{-1}\Big\{\left[Y_i+o(\psi)(Y_i) \;,\; \overline{Y_j+o(\psi)(Y_j)}\right]_F + \sum_\ell u_{\ell,i}\left[Y_\ell+o(\psi)(Y_\ell) \;,\; \overline{Y_j+o(\psi)(Y_j)}\right]_F$$

$$+ \sum_k \bar{u}_{k,j}\left[Y_i+o(\psi)(Y_i) \;,\; \overline{Y_k+o(\psi)(Y_k)}\right]_F$$

$$+ \sum_{\ell,k} u_{\ell,i}\bar{u}_{k,j}\left[Y_\ell+o(\psi)(Y_\ell) \;,\; \overline{Y_k+o(\psi)(Y_k)}\right]_F\Big\} = \delta_{i,j} .$$

So we let

$$c_{i,j}(o(\psi)) = \left[Y_i + o(\psi)(Y_i) \; , \; \overline{Y_j + o(\psi)(Y_j)} \right]_F \quad ,$$

then

$$c_{i,j}(o) = \delta_{i,j}$$

and

$$c_{i,j}(o) = c_{i,j}(o(\psi)) + \Sigma_\ell u_{\ell,i} c_{\ell j}(o(\psi))$$

$$+ \Sigma_k \bar{u}_{k,j} c_{ik}(o(\psi)) + \Sigma_{\ell,k} u_{\ell,i} \bar{u}_{k,j} c_{\ell k}(o(\psi)) \quad .$$

So if we assume

$$u_{ij} = \bar{u}_{ji} \quad ,$$

we can solved u_{ij} by the inverse function theorem and obviously u_{ij} depends on $o(\psi)$ real analytically .

1.4 . The induced CR-structure $(M, {}^{f}T'')$ and the admissible distance function

Let f be a C^{∞}-embedding of M into C^n, which is sufficiently close to ψ. This means that coefficients of p-thdegree($p \leq \ell$) of $f-\psi$ vanish at p_o. And let $(M, {}^{f}T'')$ be its induced CR-structure. Namely ${}^{f}T''$ is a subbundle of CTM, defined by

$$ {}^{f}T'' = \{ X' \ ; \ X' \in CTM \ , \ f_{*}X' \in T''C^n \} \ . $$

Since f is sufficiently close to ψ, $(M, {}^{f}T'')$ defines a CR-structure(if necessary, we must schrink M). For this CR-structure, we, also, define a C -vector bundle decomposition

$$ \text{(1.4.1)} \qquad CTM = {}^{f}T'' + {}^{f}\overline{T}'' + F \ . $$

According to this decopamosition, we introduce the Levi-form with respect to (1.4.1) as follows.

$$ c_F(f)(X,Y) = -\sqrt{-1}[X,\overline{Y}]_F \qquad \text{for} \quad X \ , \ Y \in {}^{f}T'' \ , $$

where $[X,\overline{Y}]_F$ denotes the projection of $[X,\overline{Y}]$ to F according to the vector bundle decomposition (1.4.1). Next we introduce, so called " Hessian ". For any holomorphic function h on C^n,

$$ \begin{aligned} [X,\overline{Y}]h\circ f &= [X,\overline{Y}]_{{}^{f}T''}h\circ f + [X,\overline{Y}]_{{}^{f}\overline{T}''}h\circ f + [X,\overline{Y}]_F\, h\circ f \\ &= [X,\overline{Y}]_{{}^{f}\overline{T}''}h\circ f + [X,\overline{Y}]_F h\circ f \qquad \text{for any } X \ , \ Y \in \Gamma(M, {}^{f}T'') \ . \end{aligned} $$

On the other hand

$$[X,\bar{Y}]\, h\circ f \;=\; X(\bar{Y}\, h\circ f) \qquad \text{for any } X\ ,\ Y \in \Gamma(M,{}^{f}T'') \ .$$

So we have

$$(X\bar{Y} - [X,\bar{Y}]_{f_{\bar{T}''}})h\circ f \;=\; [X,\bar{Y}]_{F} h\circ f$$

$$= c_F(X,Y)\,(\sqrt{-1}S(h\circ f)) \quad \text{for any } X\ ,\ Y \in \Gamma(M,{}^{f}T'') \ .$$

Hence we put

$$t_f = 2\mathrm{Re}\, h\circ f \ .$$

Then ,

$$(1.4.2) \quad (X\bar{Y} - [X,\bar{Y}]_{f_{\bar{T}''}})t_f = (X\bar{Y} - [X,\bar{Y}]_{f_{\bar{T}''}})h\circ f$$

$$= c_F(X,Y)\,(\sqrt{-1}S(h\circ f)) \quad \text{for any } X\ ,\ Y \in \Gamma(M,{}^{f}T'') \ .$$

Now we would like to find a nice holomorphic function h satisfying ;

(i) $t_f(p) \gtrsim 0$ for all p in M and $t_f(p)=0$ if and only if $p=p_o$

(ii) the gradient of t at p is zero if and only if $p=p_o$

(iii) $\sqrt{-1}\, S(h \circ f)(p_o) \neq 0$

(iv) if $X, Y \in \Gamma(M, {}^{f}T'')$, $XYt_f = 0$ at p_o .

For this purpose we recall Theorem 1.2.1 . Namely , there is a C^∞-embedding ψ of M into C^n

$$\psi(p_o) \in \psi(M) \subset C^n$$

satisfying

$$\psi \in \Gamma(M, {}^{o}\bar{T}'' \otimes ({}^{o}T'')^*)$$

and coefficients of p-th degree ($p \leq k$) of ψ vanish at p_o . By a biholomorphic transformation of C^n , we can assume

$$\psi(p_o) = 0$$

and

$$\psi(M) = \left\{ (z_n, z) \ ; \ z \in C^{n-1} \ , \ \mathrm{Im}\, z_n - k(z, \mathrm{Re}\, z_n) = 0 \right\} \ ,$$

where $k(z, \mathrm{Re}\, z_n)$ is a real valued C^∞-function and ,

$$k(z, \mathrm{Re}\, z_n) = \sum_{i,j} (\partial^2 k / \partial z_i \partial \bar{z}_j)(0) z_i \bar{z}_j + O(z_\ell, \bar{z}_\ell, \mathrm{Re}\, z_n) \ ,$$

where $(\partial^2 k / \partial z_i \partial \bar{z}_j)_{1 \leq i, j \leq n-1}$ is positive definite and $O(z_\ell, \bar{z}_\ell, \mathrm{Re}\, z_n)$

means the higher order term than 3 (here we regard z_ℓ , $\bar{z}_\ell$ as an order 1 and Re z_n as an order 2) . Now we set

$$h = (1/2i)z_n + z_n^2 ,$$

then we have (i) , (ii) , (iii) and (iv) (see Kuranishi in (3)) .

Finally , in this section we introduce the notation

$$C = \{ q ; q \in M , X' t_\psi(q) = 0 , X' \in \Gamma(M, {}^\circ T'') \}$$

and

$$b(\psi) = \sqrt{\sum_{\ell=1}^{n-1} | Y_\ell t_\psi |^2}$$

1.5 . The orthonormal base of $(M, {}^{f}T'')$

Let f be as in Sect.1.4 . In this section we will construct the orthonormal base $\{{}^{f}Y_i^{o}(\psi)\}_{1\leq i\leq n-1}$ of ${}^{f}T''$ satisfying

$$ {}^{\psi}Y_i^{o}(\psi) = Y_i^{o}(\psi) \ . $$

Since f is sufficiently close to ψ , $(M, {}^{f}T'')$ defines a deformation of CR-structure $(M, {}^{\psi}T'')$, namely there is an element of $\Gamma(M, (F + {}^{\psi}\overline{T}'') \otimes ({}^{\psi}T'')^{*})$ satisfying

$$ {}^{f}T'' = \left\{ X' \ ; \ X' = X + \omega(f,\psi)(X) \ , \ X \in {}^{\psi}T'' \right\} \ , $$

where $\omega(f,\psi)$ is defined by

$$ (X + \omega(f,\psi)(X))f_{\alpha} = 0 \quad \text{for} \ X \in {}^{\psi}T'' \ , \ \alpha = 1,2,\dots,n \ \text{and} \ f = (f_1,\dots,f_{\alpha},\dots f_n) \ . $$

We want to find out a u of $\Gamma(M, {}^{\psi}\overline{T}'' \otimes ({}^{\psi}T'')^{*})$ satisfying

$$ -\sqrt{-1}\Big[Y_i^{o}(\psi) + u(Y_i^{o}(\psi)) + \omega(f,\psi)(Y_i^{o}(\psi) + u(Y_i^{o}(\psi)) \ , $$
$$ \overline{Y_j^{o}(\psi) + u(Y_j^{o}(\psi)) + \omega(f,\psi)(Y_j^{o}(\psi) + u(Y_j^{o}(\psi)))}\Big]_F = \delta_{i,j} \ , $$

where $\Big[Y_i^{o}(\psi) + u(Y_i^{o}(\psi)) + \omega(f,\psi)(Y_i^{o}(\psi) + u(Y_i^{o}(\psi)) \ ,$

$$ \overline{Y_j^{o}(\psi) + u(Y_j^{o}(\psi)) + \omega(f,\psi)(Y_j^{o}(\psi) + u(Y_j^{o}(\psi)))}\Big]_F $$

means

the projection of

$$[Y_i^o(\psi) + u(Y_i^o(\psi)) + \omega(f,\psi)(Y_i^o(\psi) + u(Y_i^o(\psi))) ,$$

$$\overline{Y_j^o(\psi) + u(Y_j^o(\psi)) + \omega(f,\psi)(Y_j^o(\psi) + u(Y_j^o(\psi)))}]$$

to F-part according to the vector bundle decomposition (1.4.1) .

Let

$$u(Y_i^o(\psi)) = \sum_\ell u_{\ell,i} Y_\ell^o(\psi) .$$

Then

$$-\sqrt{-1}[Y_i^o(\psi) + u(Y_i^o(\psi)) + \omega(f,\psi)(Y_i^o(\psi) + u(Y_i^o(\psi))) ,$$

$$\overline{Y_j^o(\psi) + u(Y_j^o(\psi)) + \omega(f,\psi)(Y_j^o(\psi) + u(Y_j^o(\psi)))}]_F$$

$$= -\sqrt{-1}\Big\{[Y_i^o(\psi) + \omega(f,\psi)(Y_i^o(\psi)) , \overline{Y_j^o(\psi) + \omega(f,\psi)(Y_j^o(\psi))}]_F$$

$$+ \sum_\ell u_{\ell,i}[Y_\ell^o(\psi) + \omega(f,\psi)(Y_\ell^o(\psi)) , \overline{Y_j^o(\psi) + \omega(f,\psi)(Y_j^o(\psi))}]_F$$

$$+ \sum_k \bar{u}_{k,j}[Y_i^o(\psi) + \omega(f,\psi)(Y_i^o(\psi)) , \overline{Y_k^o(\psi) + \omega(f,\psi)(Y_k^o(\psi))}]_F$$

$$+ \sum_{\ell,k} u_{\ell,i}\bar{u}_{k,j}[Y_\ell^o(\psi) + \omega(f,\psi)(Y_\ell^o(\psi)) , \overline{Y_k^o(\psi) + \omega(f,\psi)(Y_k^o(\psi))}]_F$$

So let

$$c_{i,j}(f) = -\sqrt{-1}[Y_i^o(\psi) + \omega(f,\psi)(Y_i^o(\psi)) , \overline{Y_j^o(\psi) + \omega(f,\psi)(Y_j^o(\psi))}]_F ,$$

then $\qquad c_{i,j}(\psi) = \delta_{i,j} \qquad .$

And

$$c_{i,j}(\psi) = c_{i,j}(f) + \sum_{\ell} u_{\ell,i} c_{\ell,j}(f) + \sum_{k} \bar{u}_{k,j} c_{i,k}(f)$$

$$+ \sum_{\ell,k} u_{\ell,i} \bar{u}_{k,j} c_{\ell,k}(f)$$

So if we assume

$$u_{i,j} = \bar{u}_{j,i} \quad ,$$

we can solve $u_{i,j}$ in the term of $c_{i,j}(f)$, $\bar{c}_{i,j}(f)$. And obviously $u_{i,j}$ is real analytic with respect to $Y_i^{\psi}\omega(f,\psi)_{\alpha,j}$, $\alpha=1,2,\ldots,n$.
So we set

$$f_{Y_i^{\rho}(\psi)} = Y_i^{\rho}(\psi) + u(Y_i^{\rho}(\psi)) + \omega(f,\psi)(Y_i^{\rho}(\psi) + u(Y_i^{\rho}(\psi))) \quad .$$

Hence $f_{Y_i^{\rho}(\psi)}$ depends on $Y_{\alpha}^{\rho}(\psi) f_{\beta}$, $\bar{Y}_{\alpha}^{\rho}(\psi) f_{\beta}$, $(Y_i^{\rho}(\psi) f_r) Sf_{\alpha}$ real analytically . And we have

$$\psi_{Y_i^{\rho}(\psi)} = Y_i^{\rho}(\psi)$$

$$-\sqrt{-1}\left[f_{Y_i^{\rho}(\psi)} , \bar{f}_{Y_j^{\rho}(\psi)} \right]_P = \delta_{i,j} \quad .$$

From now on , we use abbreviations

$$\psi_{Y_i} = \psi_{Y_i^{\rho}(\psi)}$$

$$f_{Y_i} = f_{Y_i^{\rho}(\psi)} \quad .$$

1.6. Deformation theory with preserving the curve C

In this section we consider the deformation theory with preserving the curve C . For a CR-structure $(M, {}^{\psi}T'')$,

where

$$o(\psi) \in \Gamma(M, {}^{o}\bar{T}'' \otimes ({}^{o}T'')^{*}) ,$$

we consider a C^{∞} - embedding f satisfying

(1.6.1)

$$\operatorname*{Max}_{i, U_r(\psi)} (1/b(\psi)) j^{(1)}(f_i - \psi_i) < c_{\psi} \quad \text{on} \quad U_r(\psi)$$

here c_{ψ} is a sufficiently small constant . Then we have

$$K_1 b(\psi)^2 \leq \Sigma_i |Y_i^f t_f|^2 < K_2 b(\psi)^2 ,$$

where K_1 and K_2 are positive constants which don't depend on f . In fact ,

since

$$t_f = \mathrm{Re}\,((1/2i) f_n + f_n^2)$$

and

$$t_{\psi} = \mathrm{Re}\,((1/2i)\psi_n + \psi_n^2) ,$$

$$|(Y_i^f - Y_i^\psi)t_f| \leq c_\psi c' b(\psi) \quad \text{on} \quad U_r(\psi)$$

$$|Y_i^\psi(t_f - t_\psi)| \leq c_\psi c'' b(\psi) \quad \text{on} \quad U_r(\psi) \quad ,$$

where c' , c'' are constants depends only on ψ .
So if c_ψ is chosen sufficiently small ,

$$K_1 b(\psi)^2 \leq \sum_i |Y_i^f t_f|^2 \leq K_2 b(\psi)^2 \quad \text{on} \quad U_r(\psi) \quad .$$

Hence on $U_r(\psi) - C$, we can define a differential operator

(1.6.2) $$Y^f = \sum_{\ell=1}^{n-1} ((\bar{Y}_\ell^f t_f)/b(f)) Y_\ell^f$$

and

(1.6.3) $$X^f = \sqrt{-1}\, b(f) S + \bar{\gamma}_f Y^f - \gamma_f \bar{Y}^f \quad ,$$

for f satisfying (1.6.1) , where

(1.6.4) $$\gamma_f = \sqrt{-1}\, S(h \cdot f) \quad .$$

And similarly , we can define a differential operator

(1.6.5) $$W_i^f = Y_i^f - ((Y_i^f t_f)/b(f)) Y^f \quad .$$

And we can define D_b^f - operator . Namely , for u in $\Gamma(U_r(f)-C\ ,1)$, we set $D_b^f u$ in $\Gamma(U_r(f)-C,({}^f T_b''))$ by

$$D_b^f u(W_i^f) = W_i^f u \quad .$$

Then we have

$$0 \to \Gamma(U_r(f)-C,1) \xrightarrow{D_b^f} \Gamma(U_r(f)-C,({}^f T_b'')^*) \xrightarrow{D_b^f} \quad .$$

Furthermore we can define D_b - operator on $\Gamma(U_r(f)-C, \wedge^p({}^f T_b'')^*)$, $p=1,2,,,..$. Namely for u in $\Gamma(U_r(f)-C,1)$, we set $D_b u$ in $\Gamma(U_r(f)-C,({}^f T_b'')^*)$ by

$$D_b u(W_i^f) = W_i u \quad ,$$

where $W_i = Y_i - ((Y_i t_f)/\widetilde{b}(f))Y^o$ and $b(f) = \sqrt{\sum_{j=1}^{n-1} |Y_j t_f|^2}$,

$$Y^o = \sum_{j=1}^{n-1} (\bar{Y}_j t_f/\widetilde{b}(f))Y_j$$

(because of (1.6.1) , this definition makes sense). And like the case for scalar valued differential forms , we have $D_b^{(p)}$- operator from $\Gamma(U_r(f)-C,\wedge^p({}^f T_b'')^*)$ to $\Gamma(U_r(f)-C, \wedge^{p+1}({}^f T_b'')^*)$. From now on we use the following notation . Namely , the notation ;

$$D_b u = 0 \text{ along } t_f$$

means that

$$W_i u = (Y_i - ((Y_i t_f)/\ b(f))Y^o)u = 0$$

$$\text{on } U_r(f) - C \ , \quad i = 1, 2, \ldots, n-1 \ .$$

1.7. An approximate embedding

We want to find out a C^∞-embedding f^o satisfying ; $o(f^o)$ is of $\Gamma(M, {}^\circ\bar{T}'' \otimes ({}^\circ T'')^*)$,

$$o(f^o)(p_o) = 0$$

and

$$(1/b(f^o))^{2\ell} Df^o \quad \text{is bounded} \quad ,$$

where ℓ is an integer satisfying

$$\ell \geqq 30(2n+3) \quad .$$

Let ψ be a C^∞-embedding satisfying ;

$o(\psi)$ is of $\Gamma(M, {}^\circ\bar{T}'' \otimes ({}^\circ T'')^*)$,

and

$$j^{(2\ell+1)}(o(\psi))(p_o) = 0 \quad .$$

However it is not sure whether $(1/b(\psi))^{2\ell} D\psi$ is bounded . So we must modify ψ along C (C is defined by t_ψ) . We set

$$y_j = (1/2)(\bar{Y}_j t_\psi + Y_j t_\psi) \quad , \quad y_{j+n-1} = (1/2i)(\bar{Y}_j t_\psi - Y_j t_\psi) \quad .$$

Then these y_j , y_{j+n-1} and s are coordinates of a neighborhood of p_o , satisfying

$$C = \{ (s, y_1, \ldots, y_{n-1}, y_n, \ldots, y_{2n-2}) \ , \ y_1 = 0 \ , \ \ldots\ldots , \ y_{2n-2} = 0 \ \} \ .$$

We show

Lemma 1.7.1 . There are C^∞-functions $u_{\alpha,i}(s)$, $u_{\alpha+n-1,i}(s)$ $1 \leq \alpha \leq n-1$, satisfying

$$Y_j(Y_i t_\psi - \Sigma_\alpha u_{\alpha,i}(s) Y_\alpha t_\psi - \Sigma_\alpha u_{\alpha+n-1,i}(s) \bar{Y}_\alpha t_\psi) \equiv 0 \qquad \mathrm{mod} \ \ y_k$$

$$\bar{Y}_j(Y_i t_\psi - \Sigma_\alpha u_{\alpha,i}(s) Y_\alpha t_\psi - \Sigma_\alpha u_{\alpha+n-1,i}(s) \bar{Y}_\alpha t_\psi) \equiv \delta_\psi \delta_{ji} \qquad \mathrm{mod} \ \ y_k$$

where $\delta_\psi = \sqrt{-1}\ S(h\psi)$.

Proof . By the definition of y_k , it is enough to see

$$\Sigma_\alpha u_{\alpha,i}(s)(Y_j Y_\alpha t_\psi) + \Sigma_\alpha u_{\alpha+n-1,i}(s)(Y_j \bar{Y}_\alpha t_\psi) \equiv Y_j Y_i t_\psi \qquad \mathrm{mod} \ \ y_k$$

$$\Sigma_\alpha u_{\alpha,i}(s)(\bar{Y}_j Y_\alpha t_\psi) + \Sigma_\alpha u_{\alpha+n-1,i}(s)(\bar{Y}_j \bar{Y}_\alpha t_\psi) \equiv \bar{Y}_j Y_i t \ - \delta_\psi \delta_{ji} \qquad \mathrm{mod} \ \ y_k \ .$$

We see the matrix

$$\begin{pmatrix} Y_j Y_\alpha t_\psi , & Y_j \bar{Y}_\alpha t_\psi \\ \bar{Y}_j Y_\alpha t_\psi , & \bar{Y}_j \bar{Y}_\alpha t_\psi \end{pmatrix}$$

at p_o, namely

$$\begin{pmatrix} 0 & , & \gamma_\psi I \\ \bar{\gamma}_\psi I & , & 0 \end{pmatrix}.$$

Hence we can solve $u_{\alpha,i}(s)$. Q.E.D.

We note that this $u_{\alpha,i}(s)$ satisfies

$$u_{\alpha,i}(0) = 0$$

because of $Y_j Y_i t_\psi(0) = 0$ and $(\bar{Y}_j Y_i t_\psi - \bar{\gamma}_\psi \delta_{ji})(0) = 0$.

So we set

$$w_i = Y_i t_\psi - \sum_\alpha u_{\alpha,i}(s) Y_\alpha t \quad - \sum_\alpha u_{\alpha+n-1,i}(s) \bar{Y}_\alpha t \quad ,$$

and

$$w_{i+n-1} = \bar{w}_i \quad ,$$

then $Y_j w_i \equiv 0 \mod w_k$, $\quad Y_j w_{i+n-1} \equiv \gamma_\psi \delta_{ji} \mod w_k$.

By using w_j , we have

Lemma 1.7.2. There is a C^{∞}-embedding u satisfying ;

$$\bar{Y}_j u_\alpha(x,s) \equiv 0 \mod w_k$$

$$Y_j(\psi_\alpha(x,s) - u_\alpha(x,s)) \equiv 0 \mod (w_k)^2, \alpha = 1,2,\ldots,n ,$$

where $u = (u_1(x,s), \ldots u_n(x,s))$, $\psi(x,s) = (\psi_1(x,s), \ldots\ldots, \psi_n(x,s))$.

Proof . Let

$$u_\gamma(x,s) = \Sigma_\alpha u_{\alpha,\gamma} w_\alpha + \Sigma_{\alpha\beta} u_{\alpha\beta,\gamma} w_\alpha w_\beta + \text{(higher order term)} .$$

We determine $u_{\alpha,\gamma}$, $u_{\alpha\beta,\gamma}$, successively .

We set

$$u_{\alpha,\gamma} = 0 \quad \text{if} \quad 1 \leq \alpha \leq n-1 .$$

For $u_{\alpha+n-1,\gamma}$, we set

$$u_{\alpha+n-1,\gamma} = Y_j \psi_\gamma(x,s)(1/\gamma_\varphi) .$$

Then we have

$$Y_j(\Sigma_\alpha u_{\alpha+n-1,\gamma} \bar{w}_\alpha) \equiv Y_j \psi_\gamma(x,s) \mod w_k .$$

Next we will determine $u_{\alpha\beta,\gamma}$.

Let

$$u^{(1)}_{\gamma}(x,s) = \sum_{\alpha} u_{\alpha,\gamma} w_{\alpha} \quad .$$

We want to solve

$$Y_j v_{\gamma}(x,s) \equiv Y_j(\psi_{\gamma}(x,s) - u^{(1)}_{\gamma}(x,s))$$

$$\mod \quad (w_k)^2 \quad ,$$

where $v_{\gamma}(x,s) \equiv \sum_{\alpha,\beta} v_{\alpha,\beta,\gamma} w_{\alpha} w_{\beta} \quad \mod \quad (w_k)^2 \quad .$

Namely , since

$$Y_j v_{\gamma}(x,s) \equiv v_{\alpha,\beta,\gamma}((Y_j w_{\alpha}) w + w_{\alpha}(Y_j w_{\beta}))$$

$$\mod \quad (w_k)^2 \quad ,$$

the above equation becomes

$$v_{\alpha,\beta,\gamma}((Y_j w_{\alpha}) w_{\beta} + w_{\alpha}(Y_j w_{\beta})) \equiv Y_j(\psi_{\gamma}(x,s) - u^{(1)}_{\gamma}(x,s))$$

$$\mod \quad (w_k)^2 \quad .$$

So

$$2 \sum_{\beta} v_{j,\beta,\gamma} w_{\beta} \equiv Y_j(\psi_{\gamma}(x,s) - u^{(1)}_{\gamma}(x,s))$$

$$\mod \quad (w_k)^2$$

We define $c_{j,\alpha,\gamma}$ by

$$Y_j(\psi_\gamma(x,s) - u_\gamma^{(1)}(x,s)) \equiv \sum_\alpha c_{j,\alpha,\gamma} w_\alpha$$

$$\mod (w_k)^2 ,$$

and set

$$v_{j,\beta,\gamma} = (1/2)\, c_{j,\beta,\gamma} .$$

Furthermore if α or $\beta \geq n-1$, set

$$v_{\alpha,\beta,\gamma} = 0 .$$

However there is a problem ; if $1 \leq j, \beta \leq n-1$,

$$v_{j,\beta} = v_{\beta,j}$$

must hold . We check this point . By the definition of $c_{j,\alpha,\gamma}$,

$$Y_\beta Y_j(\psi_\gamma(x,s) - u_\gamma^{(1)}(x,s)) \equiv \sum_\alpha c_{j,\alpha,\gamma} Y_\beta w_\alpha$$

$$\mod w_k$$

and

$$Y_j Y_\beta(\psi_\gamma(x,s) - u_\gamma^{(1)}(x,s)) \equiv \sum_\alpha c_{\beta,\alpha,\gamma} Y_j w_\alpha$$

$$\mod w_k .$$

While by integrablity condition ; $[Y_\beta, Y_j] = \sum_\ell a_{\ell,(\beta,j)} Y_\ell$,

$$\sum_\alpha c_{j,\alpha,\gamma} Y_\beta w_\alpha - \sum_\alpha c_{\beta,\alpha,\gamma} Y_j w_\alpha \equiv \sum_\ell a_{\ell,(\beta,j)} Y_\ell (\psi_\gamma(x,s) - u_\gamma^{(1)}(x,s))$$

$$\bmod\ w_k$$

$$\equiv\ 0\ \text{(by the definition of } u_\gamma^{(1)}(x,s))$$

$$\bmod\ w_k$$

Namely

$$c_{\beta,\alpha,\gamma} = c_{\alpha,\beta,\gamma} .$$

Hence we have

$$v_{j,\beta,\gamma} = v_{\beta,j,\gamma} .$$

So our definition of $u^{(2)}_{\alpha_1,\alpha_2}$, makes sense . Successively we can determine

$$u^{(k)}_{\alpha_1,\dots,\alpha_k} .$$

Hence we have our lemma . Q.E.D.

Now we set

$$\psi' = \psi - u \quad .$$

Obviously ψ' defines a C^{∞}-embedding .

And

$$t_{\psi'} = \mathrm{Re}\ \left((1/2i)(\psi_n - u_n) + \lambda(\psi_n - u_n)^2\right)$$

$$= \mathrm{Re}\ \left((1/2i)\psi_n + \lambda\psi_n^2 - (1/2i)u_n - 2\lambda\psi_n u_n + \lambda u_n^2\right)$$

$$= t_{\psi} - \mathrm{Re}\ \left((1/2i)u_n^2 + \lambda u_n^2 + 2\lambda u_n(\psi_n - u_n)\right)$$

$$= t_{\psi} - \Big\{(1/2i)u_n - (1/2i)\bar{u}_n + \lambda u_n^2 + \bar{\lambda}\bar{u}_n^2 + 2\lambda u_n(\psi_n - u_n)$$

$$+ 2\bar{\lambda}\bar{u}_n(\bar{\psi}_n - \bar{u}_n) \quad .$$

So

$$Y_j t_{\psi'} = Y_j t_{\psi} - \Big\{(1/2i)Y_j u_n - (1/2i)Y_j\bar{u}_n + \lambda(Y_j u_n)2u_n$$

$$+ \bar{\lambda}(Y_j\bar{u}_n)2\bar{u}_n + 2\lambda(Y_j u_n)(\psi_n - u_n) + 2\lambda u_n Y_j(\psi_n - u_n)$$

$$+ 2\bar{\lambda}(Y_j\bar{u}_n)(\bar{\psi}_n - \bar{u}_n) + 2\bar{\lambda}\bar{u}_n Y_j(\bar{\psi}_n - \bar{u}_n) \quad .$$

We note

$$Y_j\psi_n(x,s) \equiv 0 \quad \mathrm{mod} \quad (w_k) \quad .$$

In fact ,

$Y_i = Z_i + o(\psi)(Z_i)$

and

$$o(\psi) \in \Gamma(M, {}^{\circ}\bar{T}'' \otimes ({}^{\circ}T'')^*) \quad .$$

By the definition of C ,

$$(Y_i + o(\psi)(Y_i))\,\mathrm{Re}((1/2i)\psi_n + \psi_n^2) = 0 \quad \text{on} \quad C \quad .$$

And trivially ,

$$(Y_i + o(\psi)(Y_i))((1/2i)\psi_n + \lambda\psi_n^2) = 0 \quad .$$

Hence

$$\overline{(Y_i + o(\psi)(Y_i))}((1/2i)\psi_n + \lambda\psi_n^2) = 0 \quad ,$$

$$(Y_i + o(\psi)(Y_i))((1/2i)\psi_n + \lambda\psi_n^2) = 0 \quad \text{on} \quad C \quad .$$

Since $o(\psi)$ is of $\Gamma(M, {}^{\circ}\bar{T}'' \otimes ({}^{\circ}T'')^*)$,

$$Y_i((1/2i)\psi_n + \lambda\psi_n^2) = 0 \quad \text{and} \quad \bar{Y}_i((1/2i)\psi_n + \lambda\psi_n^2) = 0$$

on C .

Hence

$$Y_j\psi_n = 0 \quad \text{on} \quad C \quad \text{(a sufficiently small neighborhood of } p_o)$$

and

$$\bar{Y}_j\psi_n = 0 \quad \text{on} \quad C \quad \text{(a sufficiently small neighborhood of } p_o)$$

On the other hand ,

$$Y_j\psi_n(x,s) - Y_j u_n \equiv 0 \mod (w_k)^2 \quad .$$

So

$$\begin{aligned} Y_j u_n &\equiv Y_j\psi_n(x,s) && \mod (w_k)^2 \\ &\equiv - o(\psi)_j\psi_n(x,s) && \mod (w_k)^2 \\ &\equiv - o(\psi)_{\alpha,j}(\bar{Y}_\alpha\psi_n(x,s)) && \mod (w_k)^2 \end{aligned}$$

$o(\psi')$ is defined by

$$(Y_i + o(\psi')(Y_i))(\psi - u) = 0 .$$

So by the above results ,

$$k_1 b(\psi) \leqq b(\psi') \leqq k_2 b(\psi) ,$$

where k_1 and k_2 are positive constants .
And

$$(1/b(\psi))^{2\ell} D(\psi - u)$$

is bounded . So

$$(1/b(\psi'))^2 D\psi'$$

is bounded . We set

$$f^o = \psi' \circ \exp \xi .$$

Then $(1/b(f^o))^{2\lambda} Df^o$ is of L^2 and

$$o(f^o) \in \Gamma(M, {}^{\circ}\bar{T}'' \otimes ({}^{\circ}T'')^*)$$

Chapter 2 . An apriori estimate for D_b^{ψ}

In this chapter , we will introduce D_b^{ψ}-complex , where we assume that $o(\psi)$ is an element of $\Gamma(M, {}^{\circ}\bar{T}'' \otimes ({}^{\circ}T'')^*)$ and $o(\psi)(p_o)=0$, and show an apriori estimate for this complex .

2.1 . D_b^{ψ}- complex with respect to t_{ψ}

We recall the definition of ${}^{\psi}T_b''$.

$$ {}^{\psi}T_b'' = \{ X \; ; \; X \in {}^{\psi}T'' \; , \; Xt_{\psi} = 0 \} $$

Obviously on(a sufficiently small neighborhood of p_o) $\cap$ M - C , ${}^{\psi}T_b''$ is a C^{∞}-vector bundle of rank n-2 and is generated by

$$ W_j^{\psi} = Y_j^{\psi} - (Y_j^{\psi} t_{\psi}/b(\psi))(\textstyle\sum_{\ell=1}^{n-1} (\bar{Y}_{\ell}^{\psi} t_{\psi}/b(\psi)) Y_{\ell}^{\psi}) $$

$$ j=1,2,\ldots\ldots,n-1 \; . $$

So on $U_r(\psi)-C$, where

$$ U_r(\psi) = \{ p \; ; \; p \in M \; , \; t_{\psi}(p) < r \} \; , $$

we can define D_b^{ψ}-operator with respect to t_{ψ} as follows (if necessary , we choose r sufficiently small) . For u in $\Gamma(U_r(\psi)-C,1)$, we set $D_b^{\psi}u \in \Gamma(U_r(\psi)-C, ({}^{\psi}T_b'')^*)$ by

$$ D_b^{\psi}u(W_j^{\psi}) = W_j^{\psi}u \quad . $$

Then because of

$$[\Gamma(U_r(\psi)-C, {}^{\psi}T_b''), \Gamma(U_r(\psi)-C, {}^{\psi}T_b'')] \subset \Gamma(U_r(\psi)-C, {}^{\psi}T_b'')$$

we have a differential complex

$$0 \longrightarrow \Gamma(U_r(\psi)-C, 1) \longrightarrow \Gamma(U_r(\psi)-C, ({}^{\psi}T_b'')^*) \xrightarrow{D_b^{\psi}} \Gamma(U_r(\psi)-C, \Lambda^2({}^{\psi}T_b'')^*)$$

$$\rightarrow \Gamma(U_r(\psi)-C, \Lambda^p({}^{\psi}T_b'')^*) \xrightarrow{D_b^{\psi}} \Gamma(U_r(\psi)-C, \Lambda^{p+1}({}^{\psi}T_b'')^*)$$

like the case for usual differential forms . We call this complex D_b^{ψ}- complex with respect to t_{ψ} .

2.2 . An a priori estimate for D_b^{ψ}-complex with respect to t_{ψ}

In 2.1 , we introduce D_b^{ψ}-complex with respect to t_{ψ} . In this section we show an a priori estimate for this complex . We ,first , set

$$Y^{\psi} = \sum_{\ell=1}^{n-1} (\bar{Y}_{\ell}^{\psi} t_{\psi}/b(\psi))Y_{\ell}^{\psi}$$

and

$$X^{\psi} = \sqrt{-1}\, b(\psi)S + \bar{\gamma}_{\psi} Y^{\psi} - \gamma_{\psi}\bar{Y}^{\psi} ,$$

where

$$b(\psi) = \sqrt{\sum_{i=1}^{n-1} |Y_i^{\psi} t_{\psi}|^2}$$

and

$$\gamma_{\psi} = \sqrt{-1}\, S(h\circ\psi) .$$

Then , there are C^{∞}-functions $a_{\ell,(i,j)}(\psi)$, $b_{\ell,(i,j)}(\psi)$ and $c_{\ell,(i,j)}$ on $U_r(\psi)-C$ satisfying

$$(2.2.1)\quad [W_i^{\psi}, W_j^{\psi}] = \gamma_{\psi}(Y_i^{\psi} t_{\psi}/b(\psi)^2)W_j^{\psi} - \gamma_{\psi}(Y_j^{\psi} t_{\psi}/b(\psi)^2)W_i^{\psi}$$

$$+ \sum_{\ell} a_{\ell,(i,j)}(\psi)W_{\ell}^{\psi} ,$$

where

$$\sum_{\ell} (Y_{\ell}^{\psi} t_{\psi}) a_{\ell,(i,j)}(\psi) = 0 ,$$

and

$$(2.2.2)\quad [W_i^\psi, \bar{W}_j^\psi] = -\sqrt{-1}\,(\delta_{i,j} - ((Y_i^\psi t_\psi)(\bar{Y}_j^\psi t_\psi)/b(\psi)^2)\,b(\psi)^{-1}X^\psi$$

$$+ \sum_\ell b_{\ell,(i,j)}(\psi) W_\ell^\psi + \sum_\ell c_{\ell,(i,j)}(\psi)\bar{W}_\ell^\psi ,$$

where

$$\sum_\ell (Y_\ell^\psi t_\psi) b_{\ell,(i,j)}(\psi) = 0 \quad \text{and} \quad \sum_\ell (\bar{Y}_\ell^\psi t_\psi) c_{\ell,(i,j)}(\psi) = 0 .$$

We want to compute $a_{\ell,(i,j)}(\psi)$, $b_{\ell,(i,j)}(\psi)$ and $c_{\ell,(i,j)}(\psi)$
By (2.2.1) ,

$$[[W_i^\psi, W_j^\psi], \bar{W}_k^\psi]_F = [\,\gamma_\psi(Y_i^\psi t_\psi/b(\psi)^2) W_j^\psi - \gamma_\psi(Y_j^\psi t_\psi/b(\psi)^2) W_i^\psi$$

$$+ \sum_\ell a_{\ell,(i,j)}(\psi) W_\ell^\psi ,\ \bar{W}_k^\psi]_F \quad .$$

The right hand side of this is ;

$$\gamma_\psi(Y_i^\psi t_\psi/b(\psi)^2) Q_{kj}(\psi) - \gamma_\psi(Y_j^\psi t_\psi/b(\psi)^2) Q_{ki}(\psi) + a_{k,(i,j)}(\psi) ,$$

where

$$Q_{\ell k}(\psi) = \overline{[W_\ell^\psi, \bar{W}_k^\psi]_F}$$

$$= \delta_{\ell k} - ((Y_k^\psi t_\psi)(\bar{Y}_\ell^\psi t_\psi)/b(\psi)^2) \quad .$$

The left hand side of the above is as follows

$$[[W_i^\psi, W_j^\psi], \bar{W}_k^\psi]_F = [[\Sigma_\ell Q_{\ell i}(\psi) Y_\ell^\psi, \Sigma_m Q_{mj}(\psi) Y_m^\psi], \Sigma_s \bar{Q}_{sk}(\psi) \bar{Y}_s^\psi]_F$$

We compute ${}^\psi T''$-term of

$$[\Sigma_\ell Q_{\ell i}(\psi) Y_\ell^\psi, \Sigma_m Q_{mj}(\psi) Y_m^\psi] \quad .$$

Namely ,

$$[\Sigma_\ell Q_{\ell i}(\psi) Y_\ell^\psi, \Sigma_m Q_{mj}(\psi) Y_m^\psi]$$

$$= \Sigma_{\ell,m} Q_{\ell i}(\psi)(Y_\ell^\psi Q_{mj}(\psi)) Y_m^\psi - \Sigma_{\ell,m} Q_{mj}(\psi)(Y_m^\psi Q_{\ell i}(\psi)) Y_\ell^\psi$$

$$+ \Sigma_{\ell,m} Q_{\ell i}(\psi) Q_{mj}(\psi) [Y_\ell^\psi, Y_m^\psi] \quad .$$

While

$$[Y_\ell^\psi, Y_m^\psi] = \Sigma_s \, r_{s,(\ell,m)}(o(\psi)) Y_s^\psi \quad ,$$

and $r_{s,(\ell,m)}$ is a C^∞-function which depends on $o(\psi)$, $Y_\alpha o(\psi)$, $\bar{Y}_\alpha o(\psi)$, real analytically . In fact $r_{s,(\ell,m)}$ is written by

$$r_{s,(\ell,m)}(o(\psi)) = [[Y_\ell^\psi, Y_m^\psi], \bar{Y}_s^\psi]_F \quad ,$$

and Y_ℓ^ψ depends on $o(\psi)$ real analytically .. Hence

$$[\Sigma_\ell Q_{\ell i}(\psi)Y_\ell^\psi, \Sigma_m Q_{mj}(\psi)Y_m^\psi]$$

$$= \Sigma_{\ell,m} Q_{\ell i}(\psi)(Y_\ell^\psi Q_{mj}(\psi))Y_m^\psi - \Sigma_{\ell,m} Q_{mj}(\psi)(Y_m^\psi Q_{\ell i}(\psi))Y_\ell^\psi$$

$$+ \Sigma_{\ell,m} Q_{\ell i}(\psi)Q_{mj}(\psi)(\Sigma_s r_{s,(\ell,m)}(o(\psi))Y_s^\psi)$$

So

$$[[W_i^\psi, W_j^\psi], \bar{W}_k^\psi]_F = \Sigma_{\ell,m} Q_{\ell i}(\psi)(Y_\ell^\psi Q_{mj}(\psi))\bar{Q}_{mk}(\psi)$$

$$- \Sigma_{\ell,m} Q_{mj}(\psi)(Y_m^\psi Q_{\ell i}(\psi))\bar{Q}_{\ell k}(\psi)$$

$$+ \Sigma_{\ell,m} Q_{\ell i}(\psi)Q_{mj}(\psi)(\Sigma_s r_{s,(\ell,m)}(o(\psi))Q_{sk}(\psi))$$

For the term , $\Sigma_{\ell,m} Q_{\ell i}(\psi)(Y_\ell^\psi Q_{mj}(\psi))\bar{Q}_{mk}(\psi)$, we have

$$Y_\ell^\psi Q_{mj}(\psi) = Y_\ell^\psi(\delta_{mj} - ((Y_j^\psi t_\psi)(\bar{Y}_m^\psi t_\psi)/b(\psi)^2))$$

$$= -(1/b(\psi)^2)\left\{(Y_\ell^\psi Y_j^\psi t_\psi)(\bar{Y}_m^\psi t_\psi) + (Y_j^\psi t_\psi)(Y_\ell^\psi \bar{Y}_m^\psi t_\psi)\right\}$$

$$+ (1/b(\psi)^4)(Y_j^\psi t_\psi)(\bar{Y}_m^\psi t_\psi)(Y_\ell^\psi(b(\psi)^2)) \quad .$$

Hence

$$\Sigma_{\ell,m} Q_{\ell i}(\psi)(Y_\ell^\psi Q_{mj}(\psi))\bar{Q}_{mk}(\psi)$$

$$= \sum_{\ell,m} Q_{\ell i}(\psi)\left\{-(1/b(\psi)^2)\left\{(Y_\ell^\psi Y_j^\psi t_\psi)(\bar{Y}_m^\psi t_\psi) + (Y_j^\psi t_\psi)(Y_\ell^\psi \bar{Y}_m^\psi t_\psi)\right\}\right\}\bar{Q}_{mk}(\psi)$$

$$+ \sum_{\ell,m} Q_{\ell i}(\psi)(1/b(\psi)^4)(Y_j^\psi t_\psi)(\bar{Y}_m^\psi t_\psi)(Y_\ell^\psi(b(\psi)^2))\bar{Q}_{mk}(\psi)$$

$$= \sum_{\ell,m} Q_{\ell i}(\psi)\left\{-(1/b(\psi)^2)\left\{(Y_\ell^\psi Y_j^\psi t_\psi)(\bar{Y}_m^\psi t_\psi) + (Y_j^\psi t_\psi)(Y_\ell^\psi \bar{Y}_m^\psi t_\psi)\right\}\right\}\bar{Q}_{mk}(\psi)$$

(by $\sum_m (\bar{Y}_m^\psi t_\psi)\bar{Q}_{mk}(\psi) = 0$)

$$= \sum_\ell Q_{\ell i}(\psi)\left\{-(1/b(\psi)^2)(Y_j^\psi t_\psi)\gamma_\psi\right\}\bar{Q}_{\ell k}(\psi)$$

$$+ \sum_{\ell,m} Q_{\ell i}(\psi)\left\{-(1/b(\psi)^2)(Y_\ell^\psi Y_j^\psi t_\psi)(\bar{Y}_m^\psi t_\psi)\right\}\bar{Q}_{mk}(\psi)$$

$$+ \sum_{\ell,m} Q_{\ell i}(\psi)\left\{-(1/b(\psi)^2)(Y_j^\psi t_\psi)(Y_\ell^\psi \bar{Y}_m^\psi t_\psi - \delta_{m,\ell}\gamma_\psi)\right\}\bar{Q}_{mk}(\psi)$$

$$= -\gamma_\psi(Y_j^\psi t_\psi/b(\psi)^2)Q_{ki}(\psi)$$

$$+ \sum_{\ell,m} Q_{\ell i}(\psi)\left\{-(1/b(\psi)^2)(Y_\ell^\psi Y_j^\psi t\ )(\bar{Y}_m^\psi t_\psi)\right\}\bar{Q}_{mk}(\psi)$$

$$+ \sum_{\ell,m} Q_{\ell i}(\psi)\left\{-(1/b(\psi)^2)(Y_j^\psi t_\psi)(Y_\ell^\psi \bar{Y}_m^\psi t_\psi - \delta_{m,\ell}\gamma_\psi)\right\}\bar{Q}_{mk}(\psi)$$

By the same way ,

$$\sum_{\ell,m} Q_{mj}(\psi)(Y_m^\psi Q_{\ell i}(\psi))\bar{Q}_{\ell k}(\psi)$$

$$= -\gamma_\psi(Y_i^\psi t_\psi/b(\psi)^2)Q_{kj}(\psi)$$

$$+ \sum_{\ell,m} Q_{mj}(\psi)\left\{-(1/b(\psi)^2)(Y_m^\psi Y_i^\psi t_\psi)(\bar{Y}_\ell^\psi t_\psi)\right\}\bar{Q}_{\ell k}(\psi)$$

$$+ \sum_{\ell,m} Q_{mi}(\psi)\left\{-(1/b(\psi)^2)(Y_i^\psi t_\psi)(Y_m^\psi \bar{Y}_\ell^\psi t_\psi - \delta_{m,\ell}\gamma_\psi)\right\}\bar{Q}_{\ell k}(\psi) .$$

Hence

$$a_{k,(i,j)}(\psi) = \sum_{\ell,m} Q_{\ell i}(\psi)\left(-(1/b(\psi)^2)(Y_\ell^\psi Y_j^\psi t_\psi)(\bar{Y}_m^\psi t_\psi)\right)\bar{Q}_{mk}(\psi)$$

$$+ \sum_{\ell,m} Q_{\ell i}(\psi)\left(-(1/b(\psi)^2)(Y_j^\psi t_\psi)(Y_\ell^\psi \bar{Y}_m^\psi t_\psi - \delta_{m\ell}\gamma_\psi)\right)\bar{Q}_{mk}(\psi)$$

$$- \sum_{\ell,m} Q_{mj}(\psi)\left(-(1/b(\psi)^2)(Y_m^\psi Y_i^\psi t_\psi)(\bar{Y}_\ell^\psi t_\psi)\right)\bar{Q}_{\ell k}(\psi)$$

$$- \sum_{\ell,m} Q_{mi}(\psi)\left(-(1/b(\psi)^2)(Y_i^\psi t_\psi)(Y_m^\psi \bar{Y}_\ell^\psi t_\psi - \delta_{m\ell}\gamma_\psi)\right)\bar{Q}_{\ell k}(\psi)$$

$$+ \sum_{\ell,m} Q_{\ell i}(\psi) Q_{mj}(\psi)\left(\sum_s r_{s,(\ell,m)}(o(\psi)) Q_{sk}(\psi)\right) .$$

Next we compute

$$b_{\ell,(i,j)}(\psi) \quad \text{and} \quad c_{\ell,(i,j)}(\psi) .$$

By (2.2.2) ,

$$[[W_i^\psi, \bar{W}_j^\psi], \bar{W}_k^\psi]_F$$

$$= \Big[(\delta_{ij} - ((Y_i^\psi t_\psi)(\bar{Y}_j^\psi t_\psi)/b(\psi)^2))b(\psi)^{-1}X^\psi + \sum_\ell b_{\ell,(i,j)}(\psi) W_\ell^\psi$$

$$+ \sum_\ell c_{\ell,(i,j)}(\psi)\bar{W}_\ell^\psi , \bar{W}_k^\psi \Big]_F$$

$$= \left[(\delta_{ij} - ((Y_i^\psi t_\psi)(\bar{Y}_j^\psi t_\psi)/b(\psi)^2))b(\psi)^{-1}X^\psi , \bar{W}_k^\psi\right]_F + b_{k,(i,j)}(\psi)$$

While , the X^ψ-term of $[W_i^\psi, \bar{W}_j^\psi]$ is

$$(\delta_{ij} - ((Y_i^\psi t_\psi)(\bar{Y}_j^\psi t_\psi)/b(\psi)^2))b(\psi)^{-1}X^\psi .$$

And the ${}^{\psi}T"$ - term of $[W_i^{\psi},\bar{W}_j^{\psi}]$ is

$$[Y_i^{\psi}-(Y_i^{\psi}t_{\psi}/b(\psi))\sum_{\ell=1}^{n-1}(\bar{Y}_\ell^{\psi}t_{\psi}/b(\psi))Y_\ell^{\psi}\ ,\bar{Y}_j^{\psi}-\ (\bar{Y}_j^{\psi}t_{\psi}/b(\psi))\sum_{\ell=1}^{n-1}(Y_\ell^{\psi}t_{\psi}/b(\psi))\bar{Y}_\ell^{\psi}]_{\psi_{T"}}$$

$$= [Y_i^{\psi},\bar{Y}_j^{\psi}]_{\psi_{T"}} - [Y_i^{\psi},(\bar{Y}_j^{\psi}t_{\psi}/b(\psi))\sum_{\ell=1}^{n-1}(Y_\ell^{\psi}t_{\psi}/b(\psi))\bar{Y}_\ell^{\psi}]_{\psi_{T"}}$$

$$+\sum_{\ell=1}^{n-1}\bar{W}_j^{\psi}((Y_i^{\psi}t_{\psi})(\bar{Y}_\ell^{\psi}t_{\psi})/b(\psi)^2)Y_\ell^{\psi}$$

$$-\sum_{\ell=1}^{n-1}((Y_i^{\psi}t_{\psi})(\bar{Y}_\ell^{\psi}t_{\psi})/b(\psi)^2)[Y_\ell^{\psi},\bar{Y}_j^{\psi}]_{\psi_{T"}}$$

$$+\sum_{\ell=1}^{n-1}\sum_{m=1}^{n-1}((Y_i^{\psi}t_{\psi})(\bar{Y}_j^{\psi}t_{\psi})/b(\psi)^2)((\bar{Y}_\ell^{\psi}t_{\psi})(Y_m^{\psi}t_{\psi})/b(\psi)^2)[Y_\ell^{\psi},\bar{Y}_m^{\psi}]_{\psi_{T"}}$$

So

$$[[W_i^{\psi},\bar{W}_j^{\psi}]\ ,\ \bar{W}_k^{\psi}]_F$$

$$=[\ (\delta_{ij} - ((Y_i^{\psi}t_{\psi})(\bar{Y}_j^{\psi}t_{\psi})/b(\psi)^2)b(\psi)^{-1}X^{\psi}$$

$$+\ [Y_i^{\psi},\bar{Y}_j^{\psi}]_{\psi_{T"}} - [Y_i^{\psi},(\bar{Y}_j^{\psi}t_{\psi}/b(\psi))\sum_\ell (Y_\ell^{\psi}t_{\psi}/b(\psi))\bar{Y}_\ell^{\psi}]_{\psi_{T"}}$$

$$+\ \sum_\ell (\bar{W}_j^{\psi}((Y_i^{\psi}t_{\psi})(\bar{Y}_\ell^{\psi}t_{\psi})/b(\psi)^2)Y_\ell^{\psi}$$

$$-\ \sum_\ell ((Y_i^{\psi}t_{\psi})(\bar{Y}_\ell^{\psi}t_{\psi})/b(\psi)^2)\ [Y_\ell^{\psi},\bar{Y}_j^{\psi}]_{\psi_{T"}}$$

$$+\sum_{\ell,m}\ ((Y_i^{\psi}t_{\psi})(\bar{Y}_j^{\psi}t_{\psi})/b(\psi)^2)((\bar{Y}_\ell^{\psi}t_{\psi})(Y_m^{\psi}t_{\psi})/b(\psi)^2)[Y_\ell^{\psi},\bar{Y}_m^{\psi}]_{\psi_{T"}}$$

$$+\ (\ {}^{\psi}\bar{T}" - \text{term}\ )\ ,\ \bar{W}_k^{\psi}]_F$$

$$= [(\delta_{ij} - ((Y_i^\psi t_\psi)(\bar{Y}_j^\psi t_\psi)/b(\psi)^2)b(\psi)^{-1}X^\psi , \bar{W}_k^\psi]_F$$

$$+ [\Sigma_\ell(\bar{W}_j((Y_i^\psi t_\psi)(\bar{Y}_\ell^\psi t_\psi)/b(\psi)^2))Y_\ell^\psi , \bar{W}_k^\psi]_F$$

$$+ [Y_i^\psi,\bar{Y}_j^\psi]_{T''} - [Y_i^\psi,(\bar{Y}_j^\psi t_\psi/b(\psi))\Sigma_\ell \ (Y_\ell^\psi t_\psi/b(\psi))\bar{Y}_\ell^\psi]_{\psi T''}$$

$$- \Sigma_\ell ((Y_i^\psi t_\psi)(\bar{Y}_\ell^\psi t_\psi)/b(\psi)^2) [Y_\ell^\psi,\bar{Y}_j^\psi]_{\psi T''}$$

$$+ \Sigma_{\ell,m} ((Y_i^\psi t_\psi)(\bar{Y}_j^\psi t_\psi)/b(\psi)^2)((\bar{Y}_\ell^\psi t_\psi)(Y_m^\psi t_\psi)/b(\psi)^2)[Y_\ell^\psi,\bar{Y}_m^\psi]_{\psi T''},\bar{W}_k^\psi]_F$$

So

$$b_{k,(i,j)}(\psi) = [\Sigma_\ell(\bar{W}_j^\psi((Y_i^\psi t_\psi)(\bar{Y}_\ell^\psi t_\psi)/b(\psi)^2))Y_\ell^\psi , \bar{W}_k^\psi]_F$$

$$+ [[Y_i^\psi,\bar{Y}_j^\psi]_{\psi T''} - [Y_i^\psi,(\bar{Y}_j^\psi t_\psi/b(\psi))\Sigma_\ell \ (Y_\ell^\psi t_\psi/b(\psi))\bar{Y}_\ell^\psi]_{\psi T''}$$

$$- \Sigma_\ell ((Y_i^\psi t_\psi)(\bar{Y}_\ell^\psi t_\psi)/b(\psi)^2) [Y_\ell^\psi,\bar{Y}_j^\psi]_{\psi T''}$$

$$+ \Sigma_{\ell,m} ((Y_i^\psi t_\psi)(\bar{Y}_j^\psi t_\psi)/b(\psi)^2)((\bar{Y}_\ell^\psi t_\psi)(Y_m^\psi t_\psi)/b(\psi)^2)[Y_\ell^\psi,\bar{Y}_m^\psi]_{\psi T''}$$

$$\bar{W}_k^\psi]_F$$

$$= \Sigma_\ell [(Y_i^\psi t_\psi/b(\psi)^2)(\bar{W}_j^\psi \bar{Y}_\ell^\psi t_\psi)Y_\ell^\psi, \bar{W}_k^\psi]_F$$

$$+ [[Y_i^\psi,\bar{Y}_j^\psi]_{\psi T''} - [Y_i^\psi,(\bar{Y}_j^\psi t_\psi/b(\psi))\Sigma_\ell \ (Y_\ell^\psi t_\psi/b(\psi))\bar{Y}_\ell^\psi]_{\psi T''}$$

$$- \Sigma_\ell ((Y_i^\psi t_\psi)(\bar{Y}_\ell^\psi t_\psi)/b(\psi)^2) [Y_\ell^\psi,\bar{Y}_j^\psi]_{\psi T''}$$

$$+ \Sigma_{\ell,m} ((Y_i^\psi t_\psi)(\bar{Y}_j^\psi t_\psi)/b(\psi)^2)((\bar{Y}_\ell^\psi t_\psi)(Y_m^\psi t_\psi)/b(\psi)^2)[Y_\ell^\psi,\bar{Y}_m^\psi]_{\psi T''},\bar{W}_k^\psi]$$

While

Lemma 2.2.1 .

2.2.1.1) $[s, Y_i^{\psi}] = r_i^{\psi} s + \sum_{\ell} q_{\ell,i}^{\psi} Y_{\ell}^{\psi} - \sum_{\ell} \bar{q}_{\ell,i}^{\psi} \bar{Y}_{\ell}^{\psi}$,

2.2.1.2) $[Y_i^{\psi}, \bar{Y}_j^{\psi}] = \sqrt{-1}\, \delta_{i,j} s + \sum_{\ell} q_{\ell,(i,j)}^{\psi} Y_{\ell}^{\psi} - \sum_{\ell} \bar{q}_{\ell,(i,j)}^{\psi} \bar{Y}_{\ell}^{\psi}$

where r_i^{ψ} , $q_{\ell,i}^{\psi}$, $q_{\ell,(i,j)}^{\psi}$ depend on $s \circ (\psi)$, $Y_i \circ (\psi)$, $\bar{Y}_i \circ (\psi)$ real analytically .

Proof . Since Y_i depends on $\circ(\psi)$ real analyticall , 2.2.1.1) is obvious . For 2.2.1.2) , by considering

$$[[Y_i^{\psi}, \bar{Y}_j^{\psi}] , \bar{Y}_{\ell}^{\psi}]_F = [\sqrt{-1}\, \delta_{i,j} s + \sum_{\ell} q_{\ell,(i,j)}^{\psi} Y_{\ell}^{\psi} - \sum_{\ell} \bar{q}_{\ell,(i,j)}^{\psi} \bar{Y}_{\ell}^{\psi} , \bar{Y}_{\ell}^{\psi}]_F$$
$$= \sqrt{-1}\, \delta_{i,j} [s, \bar{Y}_{\ell}^{\psi}]_F + q_{\ell,(i,j)}^{\psi}$$

with 2.2.1.1) , the proof is obvious . Q.E.D.

For $c_{k,(i,j)}(\psi)$, we have a silmilar formula .

So

$$a_{\ell,(i,j)}(\psi) \quad , \quad b_{\ell,(i,j)}(\psi) \quad , \quad c_{\ell,(i,j)}(\psi)$$

are polynomials of

$$Y_k^{\psi} t_{\psi}/b(\psi) \quad , \quad \bar{Y}_k^{\psi} t_{\psi}/b(\psi) \quad , \ k=1,2,\ldots\ldots,n-1$$

which have coefficients as a linear combination of

$$(Y_s^{\psi} Y_t^{\psi} t_{\psi}/b(\psi)) \ , \ (Y_s^{\psi} \bar{Y}_t^{\psi} t_{\psi} - \gamma_{\psi}\delta_{s,t}/b(\psi)) \quad , \quad r_{s,(\ell,m)}(o(\psi)) \ ,$$

$q_{\ell,(i,j)}(o(\psi))$, $q_{\ell,i}(o(\psi))$ and their bar , namely

$$\sum_{s,t} c^{(1)}_{s,t}(Y_s^{\psi} Y_t^{\psi} t_{\psi}/b(\psi)) + \sum_{s,t} c^{(2)}_{s,t}(\bar{Y}_s^{\psi} \bar{Y}_t^{\psi} t_{\psi}/b(\psi))$$

$$+ \sum_{s,t} c^{(3)}_{s,t}(Y_s^{\psi} \bar{Y}_t^{\psi} t_{\psi} - \gamma_{\psi}\delta_{s,t}/b(\psi)) + \sum_{s,t} c^{(4)}_{s,t}(Y_s^{\psi} \bar{Y}_t^{\psi} t_{\psi} - \gamma_{\psi}\delta_{s,t}/b(\psi))$$

$$+ \sum_{s,\ell,m} c^{(5)}_{s,\ell,m} r_{s,(\ell,m)}(o(\psi)) + \sum_{s,\ell,m} c^{(6)}_{s,\ell,m} \bar{r}_{s,(\ell,m)}(o(\psi))$$

$$+ \sum_{\ell,i,j} c^{(7)}_{\ell,i,j}\, q_{\ell,(i,j)}(o(\psi)) + \sum_{\ell,i,j} c^{(8)}_{\ell,i,j}\, \bar{q}_{\ell,(i,j)}(o(\psi))$$

$+ \sum_{\ell,i} c^{(9)}_{\ell,i}\, q_{\ell,i}(o(\psi)) + \sum_{\ell,i} c^{(10)}_{\ell,i}\, \bar{q}_{\ell,i}(o(\psi))$ (here we note that coefficeints $c^{(1)}_{s,t}$, ...,$c^{(10)}_{\ell,i}$ don't depend on ψ .

From now on we use the notation $\Theta_0(\psi)$ for the vector space generated by polynomials of $(Y_k^\psi t_\psi/b(\psi))$, $(\bar{Y}_k^\psi t_\psi/b(\psi))$, $k=1,2,\ldots,n-1$ with coefficients as a linear combination of

$$(Y_s^\psi Y_t^\psi t_\psi/b(\psi)) \;,\quad (Y_s^\psi \bar{Y}_t^\psi t_\psi - \gamma_\psi \delta_{s,t}/b(\psi)) \;,\quad r_{s,(\ell,m)}(o(\psi)) \;,$$

$$q_{\ell,(i,j)}(o(\psi)) \;,\quad q_{\ell,i}(o(\psi)) \quad \text{and their bar} \;,$$

namely

$$\sum_{s,t} c_{s,t}^{(1)} (Y_s^\psi Y_t^\psi t_\psi/b(\psi)) + \sum_{s,t} c_{s,t}^{(2)} (\bar{Y}_s^\psi \bar{Y}_t^\psi t_\psi/b(\psi))$$

$$+ \sum_{s,t} c_{s,t}^{(3)} (Y_s^\psi \bar{Y}_t^\psi t_\psi - \gamma_\psi \delta_{s,t}/b(\psi)) + \sum_{s,t} c_{s,t}^{(4)} \overline{(Y_s^\psi \bar{Y}_t^\psi t_\psi - \gamma_\psi \delta_{s,t}/b(\psi))}$$

$$+ \sum_{s,\ell,m} c_{s,\ell,m}^{(5)} r_{s,(\ell,m)}(o(\psi)) + \sum_{s,\ell,m} c_{s,\ell,m}^{(6)} \bar{r}_{s,(\ell,m)}(o(\psi))$$

$$+ \sum_{\ell,i,j} c_{\ell,i,j}^{(7)} q_{\ell,(i,j)}(o(\psi)) + \sum_{\ell,i,j} c_{\ell,i,j}^{(8)} \bar{q}_{\ell,(i,j)}(o(\psi))$$

$$+ \sum_{\ell,i} c_{\ell,i}^{(9)} q_{\ell,i}(o(\psi)) + \sum_{\ell,i} c_{\ell,i}^{(10)} \bar{q}_{\ell,i}(o(\psi))$$

and we assume that coefficients $c_{s,t}^{(1)},\ldots\ldots,c_{\ell,i}^{(10)}$ don't depend on ψ . So by this notation ,

$$a_{\ell,(i,j)}(\psi) \;,\quad b_{\ell,(i,j)}(\psi) \;,\quad c_{\ell,(i,j)}(\psi) \in \Theta_0(\psi) \;.$$

Next we put on the L^2-metric on $\Gamma_c(U_r(\psi)-C, \Lambda^p({}^{\psi}T_b^{"})^*)$, where $\Gamma_c(U_r(\psi)-C, \Lambda^p({}^{\psi}T_b^{"})^*)$ means the space consisting of $\Lambda^p({}^{\psi}T_b^{"})^*$-forms with compact support . Namely , for $u \in \Gamma(U_r(\psi)-C, \Lambda^p({}^{\psi}T_b^{"})^*)$,

$$\| u \|^2_{U_r(\psi)} = \Sigma_I \int_{U_r(\psi)-C} u_I \bar{u}_I \, dv \quad ,$$

where u_I is defined by

$$u_I = u(W^{\psi}_{i_1}, \ldots\ldots, W^{\psi}_{i_p}) \quad , \quad I = (i_1, \ldots\ldots, i_p) \quad ,$$

and dv means the volume element defined by the Levi metric . Then we have

Lemma 2.2.2 . With respect to this L^2-metric , we have

$$W_i^{\psi *} = - \bar{W}_i^{\psi} + (n-2)(\bar{Y}_i^{\psi} t_{\psi}/b(\psi)^2)\bar{\gamma}_{\psi} + a_i(\psi)$$

and

$$a_i(\psi) \in \textcircled{H}_0(\psi)$$

where $W_i^{\psi *}$ means the formal adjoint operator of W_i^{ψ} .

Proof . Let $Y_i{}^*$ be the formal adjoint operator of Y_i with respect to the above metric . Then

$$Y_i{}^* = - \bar{Y}_i + q_i \quad ,$$

where q_i is a C^∞-function , and

$$\bar{Y}_i{}^* = -Y_i + \bar{q}_i \quad .$$

Now for C^∞-functions u , v , which have a compact support in $U_r(\psi)-C$,

$$(W_i^\psi u,v) = ((Y_i^\psi - (Y_i^\psi t_\psi/b(\psi))\sum_{\ell=1}^{n-1}(\bar{Y}_\ell^\psi t_\psi/b(\psi))Y_\ell^\psi)u \ , \ v)$$

$$= (Y_i^\psi u,v) - \sum_{\ell=1}^{n-1}((1/b(\psi)^2)(Y_i^\psi t_\psi)(\bar{Y}_\ell^\psi t_\psi)Y_\ell^\psi u \ ,v)$$

$$= (u,(Y_i^\psi)^* v) - \sum_{\ell=1}^{n-1}(u,(Y_\ell^\psi)^*(((\bar{Y}_i^\psi t_\psi)(Y_\ell^\psi t_\psi)/b^2(\psi))v) \quad .$$

On the other hand ,

$$Y_i^\psi = Y_i + u(o(\psi))(Y_i) + o(\psi)(Y_i + u(o(\psi))(Y_i))$$

Hence we set

$$u(o(\psi))(Y_i) = \sum_\alpha u_{\alpha,i}Y_\alpha \quad , \quad o(\psi)(Y_j) = \sum_\beta o(\psi)_{\beta,j}\bar{Y}_\beta \quad ,$$

then

$$(Y_i^\psi)^* = (Y_i + \sum_\alpha u_{\alpha,i}Y_\alpha + \sum_\beta o(\psi)_{\beta,i}\bar{Y}_\beta + \sum_{\beta,\alpha} o(\psi)_{\beta,\alpha}u_{\alpha,i}\bar{Y}_\beta)^* \ .$$

So

$$(Y_i^\psi)^* = -\bar{Y}_i + q_i - \Sigma_\alpha \bar{u}_{\alpha,i}\bar{Y}_\alpha + \Sigma_\alpha q_\alpha \bar{u}_{\alpha,i} - \Sigma_\beta\, \bar{o}(\psi)_{\beta,i} Y_\beta$$

$$+\Sigma_\beta \bar{q}_\beta \bar{o}(\psi)_{\beta,i} - \Sigma_{\beta,\alpha}\, \bar{o}(\psi)_{\beta,\alpha}\bar{u}_{\alpha,i} Y_\beta + \Sigma_\beta \bar{q}_\beta \Sigma_\alpha \bar{o}(\psi)_{\beta,\alpha}\bar{u}_{\alpha,i}$$

$$-\Sigma_\alpha \bar{Y}_\alpha u_{\alpha,i} - \Sigma_\beta Y_\beta (o(\psi)_{\beta,i}) - \Sigma_{\beta,\alpha}\, Y_\beta (o(\psi)_{\beta,\alpha} u_{\alpha,i})$$

$$= -\bar{Y}_i^\psi + q_i + \Sigma_\alpha q_\alpha \bar{u}_{\alpha,i} + \Sigma_\beta \bar{q}_\beta \bar{o}(\psi)_{\beta,i} + \Sigma_{\beta,\alpha} \bar{q}_\beta \bar{o}(\psi)_{\beta,\alpha}\bar{u}_{\alpha,i}$$

$$-\Sigma_\alpha \bar{Y}_\alpha u_{\alpha,i} - \Sigma_\beta Y_\beta (o(\psi)_{\beta,i}) - \Sigma_{\beta,\alpha}\, Y_\beta (o(\psi)_{\beta,\alpha} u_{\alpha,i})$$

$$= -\bar{Y}_i^\psi + \Theta_{-1}^{i}(o(\psi))$$

(here $\Theta_{-1}^{i}(o(\psi)) = q_i + \Sigma_\alpha q_\alpha \bar{u}_{\alpha,i} + \Sigma_\beta \bar{q}_\beta \bar{o}(\psi)_{\beta,i}$

$$+ \Sigma_{\beta,\alpha}\, \bar{q}_\beta \bar{o}(\psi)_{\beta\alpha} \bar{u}_{\alpha,i} - \Sigma_\alpha \bar{Y}_\alpha u_{\alpha,i}$$

$$-\Sigma_\beta Y_\beta (o(\psi)_{\beta,i}) - \Sigma_{\beta,\alpha}\, Y_\beta (o(\psi)_{\beta,\alpha} u_{\alpha,i}) \quad) .$$

Hence ,

$$(W_i^\psi u, v) = (u, (Y_i^\psi)^* v) - \Sigma_{\ell=1}^{n-1}(u, (Y_\ell^\psi)^*(((\bar{Y}_i^\psi t_\psi)(Y_\ell^\psi t_\psi)/b(\psi)^2)v)$$

$$= (u, -\bar{Y}_i^\psi v + \Sigma_{\ell=1}^{n-1}\Big\{((\bar{Y}_i^\psi t_\psi)(Y_\ell^\psi t_\ell)/b(\psi)^2)(\bar{Y}_\ell^\psi v)$$

$$+ (((\bar{Y}_\ell^\psi((\bar{Y}_i^\psi t_\psi)(Y_\ell^\psi t_\psi)))b(\psi)^2 - (\bar{Y}_i^\psi t_\psi)(Y_\ell^\psi t_\psi)(\bar{Y}_\ell^\psi b(\psi)^2)/b(\psi)^4)v$$

$$+ \Theta_{-1}^{i}(o(\psi))v) \quad .$$

We compute

$$((\bar{Y}^\psi_\ell((\bar{Y}^\psi_i t_\psi)(Y^\psi_\ell t_\psi)))b(\psi)^2 - (\bar{Y}^\psi_i t_\psi)(Y^\psi_\ell t_\psi)(\bar{Y}^\psi_\ell b(\psi)^2)/b(\psi)^4) \quad .$$

By (1.4.2) , we have

$$(Y^\psi_j \bar{Y}^\psi_k - [Y^\psi_j, \bar{Y}^\psi_k]\psi_{\bar{T}"})t_\psi = c_F(Y^\psi_j, Y^\psi_k)\delta_\psi \quad ,$$

$$(2.2.3) \qquad Y^\psi_j \bar{Y}^\psi_k t_\psi = \delta_{j,k}\delta_\psi + b(\psi)\,\Theta^i_{0,(j,k)}(\psi)$$

and $\Theta^i_{0,(j,k)}(\psi)$ is of $\Theta_0(\psi)$. In fact , since ψ is of $°\bar{T}"\otimes(°T")^*$ -valued ,

$$[Y^\psi_j, \bar{Y}^\psi_k]\psi_{\bar{T}"}t_\psi$$

is of $b(\psi)\,\Theta_0(\psi)$. So it is obvious . With this in mind ,

$$\sum_{\ell=1}^{n-1} (((\bar{Y}^\psi_\ell((\bar{Y}^\psi_i t_\psi)(Y^\psi_\ell t_\psi)))b(\psi)^2 - (\bar{Y}^\psi_i t_\psi)(Y^\psi_\ell t_\psi)(\bar{Y}^\psi_\ell b(\psi)^2))/b(\psi)^4)$$

$$= \sum_{\ell=1}^{n-1} (1/b(\psi)^2)\left\{(\bar{Y}^\psi_\ell \bar{Y}^\psi_i t_\psi)(Y^\psi_\ell t_\psi) + (\bar{Y}^\psi_i t_\psi)(\bar{Y}^\psi_\ell Y^\psi_\ell t_\psi)\right\}$$

$$- \sum_{\ell=1}^{n-1} (1/b(\psi)^4)(\bar{Y}^\psi_i t_\psi)(Y^\psi_\ell t_\psi)\left\{\sum_{k=1}^{n-1} (\bar{Y}^\psi_\ell \bar{Y}^\psi_k t_\psi)(Y^\psi_k t_\psi) + (\bar{Y}^\psi_k t_\psi)(\bar{Y}^\psi_\ell Y^\psi_k t_\psi)\right\}$$

$$= (n-1)(1/b(\psi)^2)(\bar{Y}^\psi_i t_\psi)\bar{\delta}_\psi + \sum_{\ell=1}^{n-1}\left\{(Y^\psi_\ell t_\psi/b(\psi))(\bar{Y}^\psi_\ell \bar{Y}^\psi_i t_\psi/b(\psi))\right.$$

$$\left. + (\bar{Y}^\psi_i t_\psi/b(\psi))(\bar{Y}^\psi_\ell Y^\psi_\ell t_\psi - \delta_\psi/b(\psi))\right\} - \sum_{\ell=1}^{n-1} (1/b(\psi)^4)(\bar{Y}^\psi_i t_\psi)(Y^\psi_\ell t_\psi)(\bar{Y}^\psi_\ell t_\psi)\bar{\delta}_\psi$$

$$- \sum_{\ell=1}^{n-1} (\bar{Y}^\psi_i t_\psi/b(\psi))(Y^\psi_\ell t_\psi/b(\psi)) \sum_{k=1}^{n-1} (Y^\psi_k t_\psi/b(\psi))(\bar{Y}^\psi_\ell \bar{Y}^\psi_k t_\psi/b(\psi))$$

$$- \sum_{\ell=1}^{n-1} (\bar{Y}^\psi_i t_\psi/b(\psi))(Y^\psi_\ell t_\psi/b(\psi)) \sum_{k=1}^{n-1} (\bar{Y}^\psi_k t_\psi/b(\psi))((\bar{Y}^\psi_\ell Y^\psi_k t_\psi - \delta_{\ell k}\delta_\psi/b(\psi))$$

$= (n-2)(1/b(\psi)^2)(\bar{Y}_i^{\psi} t_\psi)\bar{\delta}_\psi + \Theta_0^i(\psi)$,

where $\Theta_0^i(\psi)$ is of $\Theta_0(\psi)$.

So we have our lemma . Q.E.D.

From this lemma , we have

Lemma 2.2.3 . For $u \in \Gamma_c(U_r(\psi)-C, (T_b^{"})^*)$,

$$D_b^{\psi *}u = -\sum_j \bar{W}_j^{\psi} u_j + \sum_j a_j(\psi)u_j ,$$

where $D_b^{\psi *}$ is the formal adjoint operator of D_b^{ψ} and

$$a_j(\psi) \in \Theta_0(\psi) .$$

Proof . For $u \in \Gamma_c(U_r(\psi)-C, 1)$ and for $v \in \Gamma_c(U_r(\psi)-C, ({}^{\psi}T_b^{"})^*)$, we have

$$(D_b^{\psi}v,u) = \sum_i \int_{U_r(\psi)-C} (W_i^{\psi}v)\bar{u}_i \, dv \quad , \text{ where } u_i = u(W_i^{\psi})$$

$$= \sum_i (W_i^{\psi}v, u_i)$$

$$= \sum_i (v, (W_i^{\psi})^* u_i)$$

$$= \sum_i (v, -\bar{W}_i^{\psi}u_i + (n-2)(\bar{Y}_i^{\psi}t_\psi/b(\psi)^2)\bar{\delta}_\psi u_i + a_i(\psi)u_i)$$

(by Lemma 2.2.1)

$$= (v, -\sum_{i=1}^{n-1} \bar{W}_i^{\psi}u_i + \sum_{i=1}^{n-1} a_i(\psi)u_i)$$

(because $u_i = u(W_i^{\psi})$, so $\sum_{i=1}^{n-1} (\bar{Y}_i^{\psi}t_\psi)u_i = \sum_{i=1}^{n-1} (\bar{Y}_i^{\psi}t_\psi)u(W_i^{\psi})$

$= 0$) .

Therefore we have our lemma . Q.E.D.

With these preparations , we will compute

$$\| D_b^{\psi} u \|^2_{U_r(\psi)} + \| D_b^{\psi *} u \|^2_{U_r(\psi)}$$

for $u \in \Gamma(U_r(\psi) - C, ({}^{\psi}T_b'')^*)$ satisfying

1) $D_b^{\psi} u$, $D_b^{\psi *} u$ are of L^2 ,

2) $W_i^{\psi} u$, $i=1,2,\dots,n-1$ are of L^2 and $(1/b(\psi))u$ is of L^2 .

Namely we have

Theorem 2.2.4. The following inequality holds .

$$\| D_b^{\psi} u \|^2_{U_r(\psi)} + \| D_b^{\psi *} u \|^2_{U_r(\psi)} + \varepsilon \| u \|'^2_{U_r(\psi)} + (K/\varepsilon) \| \Theta_0^{(1)}(\psi) \|^2_{U_r(\psi)}$$

$$\geq \sum_{i,j} (n-3/n-2) \| W_j^{\psi} u_i \|^2_{U_r(\psi)} + \sum_{i,j} (1/n-2) \| \bar{W}_j^{\psi} u_i \|^2_{U_r(\psi)}$$

$$+ (n-3) \sum_i \| (\gamma_{\psi}/b(\psi)) u_i \|^2_{U_r(\psi)} \quad , \text{ for all } \varepsilon > 0 \;,$$

for all $u \in \Gamma(U_r(\psi) - C, ({}^{\psi}T_b'')^*)$ satisfying

1) $D_b^{\psi} u$, $D_b^{\psi *} u$ are of L^2 ,

2) $W_i^{\psi} u$, $i=1,2,\dots,n-1$ are of L^2 and $(1/b(\psi))u$ is of L^2 ,

where K is a constant which doesn't depend on ψ , ε , u and $\Theta_0^{(1)}(\psi)$ is an element of $\Theta_0(\psi)$, and

$$\| u \|'^2_{U_r(\psi)} = \sum_{i,j} \| W_j^{\psi} u_i \|^2_{U_r(\psi)} + \sum_{i,j} \| \bar{W}_j^{\psi} u_i \|^2_{U_r(\psi)}$$

$$+ \| (\gamma_{\psi}/b(\psi)) u \|^2_{U_r(\psi)} \;.$$

Proof . For $u \in \Gamma(U_r(\psi)-C, ({}^{\psi}T_b^{*})^{*})$,

$$D_b^{\psi}u(W_i^{\psi},W_j^{\psi}) = W_i^{\psi}u_j - W_j^{\psi}u_i - u([W_i^{\psi},W_j^{\psi}])$$

$$= W_i^{\psi}u_j - W_j^{\psi}u_i - \gamma_{\psi}(Y_i^{\psi}t_{\psi}/b(\psi)^2)u_j + \gamma_{\psi}(Y_j^{\psi}t_{\psi}/b(\psi)^2)u_i$$

$$- \Sigma_{\ell}\, a^{\psi}_{\ell,(i,j)}u_{\ell} \qquad \text{(by (2.2.1))}$$

Here $a^{\psi}_{\ell,(i,j)}$ are of $\Theta_0(\psi)$. So we have

$$\Sigma_{i\leq j}\| D_b^{\psi}u(W_i^{\psi},W_j^{\psi})\|^2_{U_r(\psi)} + \| D_b^{\psi *}u\|^2_{U_r(\psi)}$$

$$= \Sigma_{i\leq j}\| W_i^{\psi}u_j - W_j^{\psi}u_i - \gamma_{\psi}(Y_i^{\psi}t_{\psi}/b(\psi)^2)u_j + \gamma_{\psi}(Y_j^{\psi}t_{\psi}/b(\psi)^2)u_i$$

$$- \Sigma_{\ell}\, a^{\psi}_{\ell,(i,j)}u_{\ell}\|^2_{U_r(\psi)}$$

$$+ \| - \Sigma_j \bar{W}_j^{\psi}u_j - \Sigma_{\ell}\, a_{\ell}u_{\ell}\|^2_{U_r(\psi)}$$

$$= \Sigma_{i\leq j}\| W_i^{\psi}u_j - W_j^{\psi}u_i - \gamma_{\psi}(Y_i^{\psi}t_{\psi}/b(\psi)^2)u_j + \gamma_{\psi}(Y_j^{\psi}t_{\psi}/b(\psi)^2)u_i\|^2_{U_r(\psi)}$$

$$+ 2\mathrm{Re}\,\Sigma_{i\leq j}(W_i^{\psi}u_j - W_j^{\psi}u_i - \gamma_{\psi}(Y_i^{\psi}t_{\psi}/b(\psi)^2)u_j + \gamma_{\psi}(Y_j^{\psi}t_{\psi}/b(\psi)^2)u_i, -\Sigma_{\ell}\, a_{\ell,(i,j)}u_{\ell})$$

$$+ \| -\Sigma_{\ell}\, a_{\ell,(i,j)}u_{\ell}\|^2_{U_r(\psi)}$$

$$+ \| - \Sigma_j \bar{W}_j^{\psi}u_j \|^2_{U_r(\psi)} + 2\mathrm{Re}(-\Sigma_j \bar{W}_j^{\psi}u_j, -\Sigma_{\ell}\, a_{\ell}u_{\ell})$$

$$+ \| -\Sigma_{\ell}\, a_{\ell}u_{\ell}\|^2_{U_r(\psi)}$$

$$= \sum_{i\leq j} \| W_i^\psi u_j - W_j^\psi u_i - \gamma_\psi (Y_i^\psi t_\psi / b(\psi)^2) u_j + \gamma_\psi (Y_j^\psi t_\psi / b(\psi)^2) u_i \|^2_{U_r(\psi)}$$

$$+ \| - \sum_j \bar{W}_j^\psi u_j \|^2_{U_r(\psi)}$$

$$+ \quad 2\mathrm{Re} \sum_{i\leq j} (W_i^\psi u_j - W_j^\psi u_i - \gamma_\psi (Y_i^\psi t_\psi / b(\psi)^2) u_j + \gamma_\psi (Y_j^\psi t_\psi / b(\psi)^2) u_i , -\sum_\ell a_{\ell,(i,j)} u_\ell)$$

$$+ \quad 2\mathrm{Re}(-\sum_j \bar{W}_j^\psi u_j \, , -\sum_\ell a_\ell u_\ell)$$

$$+ \| -\sum_\ell a_{\ell,(i,j)} u_\ell \|^2_{U_r(\psi)} \quad + \| -\sum_\ell a_\ell u_\ell \|^2_{U_r(\psi)}$$

For the term ;

$$2\mathrm{Re} \sum_{i\leq j} (W_i^\psi u_j - W_j^\psi u_i - \gamma_\psi (Y_i^\psi t_\psi / b(\psi)^2) u_i + \gamma_\psi (Y_j^\psi t_\psi / b(\psi)^2) u_i , -\sum_\ell a_{\ell,(i,j)} u_\ell)$$

$$+ \quad 2\mathrm{Re}(-\sum_j \bar{W}_j^\psi u_j \, , -\sum_\ell a_\ell u_\ell) \, ,$$

this term can be estimated by

$$\varepsilon \| u \|^2_{\dot{U}_r(\psi)} + \sum_{\alpha=1}^{\ell} (2/\varepsilon) \| \Theta_0^{\alpha,(1)}(\psi) u \|^2_{U_r(\psi)}$$

where $\Theta_0^{\alpha,(1)}(\psi)$ is an element of $\Theta_0(\psi)$. (Here we used the Schwarz inequality .) There fore for the proof , it is sufficient to show ;

$$\sum_{i\leq j} \| W_i^\psi u_j - W_j^\psi u_i - \gamma_\psi (Y_i^\psi t_\psi / b(\psi)^2) u_j + \gamma_\psi (Y_j^\psi t_\psi / b(\psi)^2) u_i \|^2_{U_r(\psi)}$$

$$+ \quad \| - \sum_j \bar{W}_j^\psi u_j \|^2_{U_r(\psi)}$$

$$+ \quad \varepsilon \| u \|^2_{\dot{U}_r(\psi)} \quad + \sum_{\alpha=1}^{\ell} (2/\varepsilon) \| \Theta_0^{\alpha,(1)}(\psi) u \|^2_{U_r(\psi)}$$

$$\geq \sum_{i,j} ((n-3)/(n-2)) \| W_j^{\psi} u_i \|^2_{U_r(\psi)} + \sum_{i,j} (1/(n-2)) \| \bar{W}_j^{\psi} u_i \|^2_{U_r(\psi)}$$

$$+ \quad (n-3) \sum_i \| (\delta\psi/b(\psi)) u_i \|^2_{U_r(\psi)} \quad , \text{ for all } \varepsilon > 0 \quad .$$

For $\sum_{i \leq j} \| W_i^{\psi} u_j - W_j^{\psi} u_i - (Y_i^{\psi} t_{\psi}/b(\psi)^2) u_j + (Y_j^{\psi} t_{\psi}/b(\psi)^2) u_i \|^2_{U_r(\psi)}$

$$\sum_{i \leq j} \| W_i^{\psi} u_j - W_j^{\psi} u_i - \gamma_{\psi}(Y_i^{\psi} t_{\psi}/b(\psi)^2) u_j + \gamma_{\psi}(Y_j^{\psi} t_{\psi}/b(\psi)^2) u_i \|^2_{U_r(\psi)}$$

$$= \sum_{i \leq j} \Big\{ \| W_i^{\psi} u_j - W_j^{\psi} u_i \|^2_{U_r(\psi)} + 2\mathrm{Re}\,(W_i^{\psi} u_j \, , - \gamma_{\psi}(Y_i^{\psi} t_{\psi}/b(\psi)^2) u_j)$$

$$+ \quad 2\mathrm{Re}\,(W_i^{\psi} u_j \, , \gamma_{\psi}(Y_j^{\psi} t_{\psi}/b(\psi)^2) u_i) + 2\mathrm{Re}\,(- W_j^{\psi} u_i, - \gamma_{\psi}(Y_i^{\psi} t_{\psi}/b(\psi)^2) u_j)$$

$$+ \quad 2\mathrm{Re}\,(- W_j^{\psi} u_i, \gamma_{\psi}(Y_j^{\psi} t_{\psi}/b(\psi)^2) u_i)$$

$$+ \quad \| \gamma_{\psi}(Y_i^{\psi} t_{\psi}/b(\psi)^2) u_j - \gamma_{\psi}(Y_j^{\psi} t_{\psi}/b(\psi)^2) u_i \|^2_{U_r(\psi)} \Big\}$$

$$= \sum_{i \leq j} \| W_i^{\psi} u_j - W_j^{\psi} u_i \|^2_{U_r(\psi)} + \sum_{i \leq j} 2\mathrm{Re}\,(W_i^{\psi} u_j, - \gamma_{\psi}(Y_i^{\psi} t_{\psi}/b(\psi)^2) u_j)$$

$$+ \sum_{i \leq j} 2\mathrm{Re}\,(W_i^{\psi} u_j, \gamma_{\psi}(Y_j^{\psi} t_{\psi}/b(\psi)^2) u_i)$$

$$+ \sum_{i \leq j} 2\mathrm{Re}\,(- W_j^{\psi} u_i, - \gamma_{\psi}(Y_i^{\psi} t_{\psi}/b(\psi)^2) u_j)$$

$$+ \sum_{i \leq j} 2\mathrm{Re}\,(- W_j^{\psi} u_i, \gamma_{\psi} (Y_j^{\psi} t_{\psi}/b(\psi)^2) u_i)$$

$$+ \sum_{i \leq j} \| \gamma_{\psi}(Y_i^{\psi} t_{\psi}/b(\psi)^2) u_j - \gamma_{\psi} (Y_j^{\psi} t_{\psi}/b(\psi)^2) u_i \|^2_{U_r(\psi)}$$

$$+ \left\{ \sum_i 2\mathrm{Re}\,(W_i^{\psi} u_i ,\ -\gamma_{\psi}(Y_i^{\psi} t_{\psi}/b(\psi)^2) u_i) \right.$$

$$\left. + \quad \sum_{i,j} 2\mathrm{Re}(W_i^{\psi} u_j ,\ -\gamma_{\psi}(Y_i^{\psi} t_{\psi}/b(\psi)^2) u_j) \right\}$$

$$+ \left\{ \sum_i 2\mathrm{Re}\,(W_i^{\psi} u_i ,\ \gamma_{\psi}(Y_i^{\psi} t_{\psi}/b(\psi)^2) u_i) \right.$$

$$\left. + \quad \sum_{i,j} 2\mathrm{Re}(W_i^{\psi} u_j ,\ \gamma_{\psi}(Y_j^{\psi} t_{\psi}/b(\psi)^2) u_i) \right\}$$

$$+ \sum_{i\leq j} \| \gamma_{\psi}(Y_i^{\psi} t_{\psi}/b(\psi)^2) u_j - \gamma_{\psi}(Y_j^{\psi} t_{\psi}/b(\psi)^2) u_i \|^2_{U_r(\psi)}$$

Since

$$\sum_i (\bar{Y}_i^{\psi} t_{\psi}) W_i^{\psi} = 0 ,$$

$$\sum_{i,j} 2\mathrm{Re}\,(W_i^{\psi} u_j ,\ -\gamma_{\psi}(Y_i^{\psi} t_{\psi}/b(\psi)^2) u_j) = 0 .$$

So the above becomes

$$\sum_{i\leq j} \| W_i^{\psi} u_j - W_j^{\psi} u_i \|^2_{U_r(\psi)} + \sum_{i,j} 2\mathrm{Re}(W_i^{\psi} u_j ,\ \gamma_{\psi}(Y_j^{\psi} t_{\psi}/b(\psi)^2) u_i)$$

$$+ \sum_{i\leq j} \| \gamma_{\psi}(Y_i^{\psi} t_{\psi}/b(\psi)^2) u_j - \gamma_{\psi}(Y_j^{\psi} t_{\psi}/b(\psi)^2) u_i \|^2_{U_r(\psi)}$$

For $\sum_{i,j} 2\mathrm{Re}\,(W_i^{\psi} u_j , \gamma_{\psi}(Y_j^{\psi} t_{\psi}/b(\psi)^2) u_i)$,

$$\sum_{i,j} 2\mathrm{Re}\,(W_i^{\psi} u_j ,\ \gamma_{\psi}(Y_j^{\psi} t_{\psi}/b(\psi)^2) u_i)$$

$$= \sum_{i,j} 2\mathrm{Re}\ ((\bar{Y}_j^{\psi} t_{\psi}) W_i^{\psi} u_j ,\ (\gamma_{\psi}/b(\psi)^2) u_i)$$

$$\sum_{i,j} 2\mathrm{Re}\,((\bar{Y}_j^{\psi} t_\psi) W_i^{\psi} u_j,\ (\gamma_\psi / b(\psi)^2) u_i)$$

$$= -\sum_{i,j} 2\mathrm{Re}\,((W_i^{\psi} \bar{Y}_j^{\psi} t_\psi) u_j, (\gamma_\psi / b(\psi)^2) u_i) \ .$$

On the other hand ,

$$W_i^{\psi}\bar{Y}_j^{\psi} t_\psi = (Y_i^{\psi} - (Y_i^{\psi} t_\psi / b(\psi)) \sum_{\ell=1}^{n-1} (\bar{Y}_\ell^{\psi} t_\psi / b(\psi)) Y_\ell^{\psi}) \bar{Y}_j^{\psi} t_\psi$$

$$= (\delta_{i,j} - (Y_i^{\psi} t_\psi / b(\psi)) (\bar{Y}_j^{\psi} t_\psi / b(\psi))) \gamma_\psi + b(\psi) \Theta_0^{(1)}(\psi)$$

where $\Theta_0^{(1)}(\psi)$ is an element of $\Theta_0(\psi)$ (by (2.2.3)) .
So

$$-\sum_{i,j} 2\mathrm{Re}\,((W_i^{\psi} \bar{Y}_j^{\psi} t_\psi) u_j, (\gamma_\psi / b(\psi)^2) u_i)$$

$$= -\sum_{i,j} 2\mathrm{Re}\,((\delta_{i,j} - (Y_i^{\psi} t_\psi / b(\psi)) (\bar{Y}_j^{\psi} t_\psi / b(\psi))) \gamma_\psi u_j + b(\psi) \Theta_0^{(1)}(\psi) u\ ,$$

$$(\gamma_\psi / b(\psi)^2) u_i)$$

$$= -\sum_{i,j} 2\mathrm{Re}\,(\delta_{i,j} \gamma_\psi u_j,\ (\gamma_\psi / b(\psi)) u_i)$$

$$-\sum_{i,j} 2\mathrm{Re}\,(\Theta_0^{(1)}(\psi) u, (\gamma_\psi / b(\psi)) u_i) \quad (\text{by } \sum_j (\bar{Y}_j^{\psi} t_\psi) u_j = 0\ )$$

$$= -2 \sum_i \| (\gamma_\psi / b(\psi)) u_i \|^2_{U_r(\psi)} - \sum_{i,j} 2\mathrm{Re}\,(\Theta_0^{(1)}(\psi) u\ , (\gamma_\psi / b(\psi)) u_i)$$

$$\Sigma_{i\leq j}\| \gamma_\psi(Y_i^\psi t_\psi/b(\psi)^2)u_j - \gamma_\psi(Y_j^\psi t_\psi/b(\psi)^2)u_i \|^2_{U_r(\psi)}$$

$$= \Sigma_{i,j}\| \gamma_\psi(Y_i^\psi t_\psi/b(\psi)^2)u_j \|^2_{U_r(\psi)} - \Sigma_{i,j}\mathrm{Re}(\gamma_\psi(Y_i^\psi t_\psi/b(\psi)^2)u_j, \gamma_\psi(Y_j^\psi t_\psi/b(\psi)^2)$$

$$= \Sigma_i \| (\gamma_\psi/b(\psi))u_i \|^2_{U_r(\psi)} \quad (\text{by } \Sigma_j \ (\bar{Y}_j^\psi t_\psi)u_j = 0 \ \text{ and}$$

$$b(\psi)^2 = \Sigma_i \ (Y_i^\psi t_\psi)(\bar{Y}_i^\psi t_\psi) \ ) .$$

Hence

$$\Sigma_{i\leq j}\| W_i^\psi u_j - W_j^\psi u_i - \gamma_\psi(Y_i^\psi t_\psi/b(\psi)^2)u_j + \gamma_\psi(Y_j^\psi t_\psi/b(\psi)^2)u_i \|^2_{U_r(\psi)}$$

$$= \Sigma_{i\leq j}\| W_i^\psi u_j - W_j^\psi u_i \|^2_{U_r(\psi)} - \Sigma_i \| (\gamma_\psi/b(\psi))u_i \|^2_{U_r(\psi)}$$

$$- \Sigma_{i,j} \ 2\mathrm{Re}\,(\Theta_0^{(1)}(\psi)u, \ (\gamma_\psi/b(\psi))u_i)$$

So we must prove

$$\Sigma_{i\leq j}\| W_i^\psi u_j - W_j^\psi u_i \|^2_{U_r(\psi)} - \Sigma_i \| (\gamma_\psi/b(\psi))u_i \|^2_{U_r(\psi)}$$

$$+ \| - \Sigma_j \bar{W}_j^\psi u_j \|^2_{U_r(\psi)} + \varepsilon \| u \|'^2_{U_r(\psi)} + (K/\varepsilon)\Sigma_{\alpha=1}^{\ell} \| \Theta_0^{\alpha,(1)}(\psi)u \|^2_{U_r(\psi)}$$

$$\geq \Sigma_{i,j} \ ((n-3)/(n-2)) \| W_j^\psi u_i \|^2_{U_r(\psi)} + \Sigma_{i,j}(1/(n-2)) \| \bar{W}_j^\psi u_i \|^2_{U_r(\psi)}$$

$$+ (n-3)\Sigma_i \| (\gamma_\psi/b(\psi))u_i \|^2_{U_r(\psi)} \quad \text{for all } u \in \Gamma(U_r(\psi) - C, ({}^\psi T_b'')^*)$$

1) $D_b^{\psi}u$ and $D_b^{\psi *}u$ are of L^2 ,

2) $W_i^{\psi}u$, $i=1,2,\ldots,n-1$ are of L^2 and $(1/b(\psi))u$ is of L^2 (so by Lemma 2.2.2 , $\bar{W}_i^{\psi}u$ is also of L^2) ,

where $\Theta_0^{\alpha,(1)}(\psi)$ are of $\Theta_0(\psi)$. We see this . For the term ;

$$\sum_{i\leqq j}\| W_i^{\psi}u_j - W_j^{\psi}u_i\|^2_{U_r(\psi)} + \| - \sum_j \bar{W}_j^{\psi}u_j\|^2_{U_r(\psi)}$$

we have

Proposition 2.2.5.

$$\sum_{i\leqq j}\| W_i^{\psi}u_j - W_j^{\psi}u_i\|^2_{U_r(\psi)} + \| - \sum_j \bar{W}_j^{\psi}u_j\|^2_{U_r(\psi)}$$

$$+ \varepsilon \|u\|'^2_{U_r(\psi)} + (K/\varepsilon)\sum_{\alpha=1}^{\rho}\|\Theta_0^{\alpha,(2)}(\psi)u\|^2_{U_r(\psi)}$$

$$\geqq (n-2)\sum_i \|(\partial\psi/b(\psi))u_i\|^2_{U_r(\psi)}$$

$$+ \sum_{i,j}((n-3)/(n-2))\|W_j^{\psi}u_i\|^2_{U_r(\psi)} + \sum_{i,j}(1/(n-2))\|\bar{W}_j^{\psi}u_i\|^2_{U_r(\psi)}$$

for all $u \in \Gamma(U_r(\psi)-C,({}^{\psi}T_b'')^*)$ satisfying

1) $D_b u$ and $D_b^{\psi *}u$ are of L^2 ,

2) $W_i^{\psi}u$, $i=1,2,\ldots,n-1$ are of L^2 and $(1/b(\psi))u$ is of L^2 ,

where $\Theta_0^{\alpha,(2)}(\psi)$ are of $\Theta_0(\psi)$.

Proof .

$$\sum_{i\leqq j}\| W_i^{\psi}u_j - W_j^{\psi}u_i\|^2_{U_r(\psi)}$$

$$= \sum_{i,j}\| W_i^{\psi}u_j\|^2_{U_r(\psi)} - \mathrm{Re}\sum_{i,j} W_i^{\psi}u_j , W_j^{\psi}u_i$$

$$+\,\Theta_0^{(1)}(\psi)W_i^\psi u_j,\ u_i)\ ,$$

where $\Theta_0^{(1)}(\psi)$ is an element of $\Theta_0(\psi)$.

While

$$\|-\textstyle\sum_j \bar{W}_j^\psi u_j\|^2_{U_r(\psi)} = \mathrm{Re}\sum_{i,j}(\bar{W}_i^\psi u_i, \bar{W}_j^\psi u_j)$$

$$= \mathrm{Re}\sum_{i,j}((-W_j^\psi + (n-2)(Y_j^\psi t_\psi/b(\psi)^2)\gamma_\psi$$

$$+\,\Theta_0^{(1)}(\psi))\bar{W}_i^\psi u_i,\ u_j)$$

So

$$\sum_{i\leq j}\|W_i^\psi u_j - W_j^\psi u_i\|^2_{U_r(\psi)} + \|-\textstyle\sum_j \bar{W}_j^\psi u_j\|^2_{U_r(\psi)}$$

$$= \sum_{i,j}\|W_i^\psi u_j\|^2_{U_r(\psi)} + \mathrm{Re}\sum_{i,j}([\bar{W}_j^\psi, W_i^\psi]u_j, u_i)$$

$$+\ (n-2)\mathrm{Re}\sum_{i,j}((\bar{\gamma}_\psi/b(\psi)^2)(W_i^\psi\bar{Y}_j^\psi t_\psi)u_j, u_i)$$

$$= \sum_{i,j}\|W_i^\psi u_j\|^2_{U_r(\psi)} + \mathrm{Re}\sum_{i,j}([\bar{W}_j^\psi, W_i^\psi]u_j, u_i)$$

$$+\ (n-2)\sum_i\|(\gamma_\psi/b(\psi))u_i\|^2_{U_r(\psi)} + \sum_{i,j}(\Theta_0^{(1)}(\psi)u, (1/b(\psi))u_j)$$

$$= \Sigma_{i,j} \| W_i u_j \|_{U_r(\psi)} - \ldots \Sigma_i (\ldots _i, u_i)$$

$$+ (n-2) \Sigma_i \| (\gamma_\psi / b(\psi)) u_i \|^2_{U_r(\psi)} + \Sigma_{i,j} (\Theta_0^{(1)}(\psi) u, (1/b(\psi)) u_j)$$

(by (2.2.3) and $\Sigma_i (\bar{Y}_i^\psi t_\psi) u_i = 0$)

On the other hand , we have

Lemma 2.2.6 .

$$\Sigma_i \| W_i^\psi u_j \|^2_{U_r(\psi)} = \Sigma_i \| \bar{W}_i^\psi u_j \|^2_{U_r(\psi)}$$

$$+ \Sigma_i ([W_i^\psi, \bar{W}_j^\psi] u_j, u_j) + \Sigma_i (W_i^\psi u_j, \Theta_0^{(2)}(\psi) u_j),$$

where $\Theta_0^{(2)}(\psi)$ is an element of $\Theta_0(\psi)$.

Proof .

$$\Sigma_i \| W_i^\psi u_j \|^2_{U_r(\psi)} = \Sigma_i (W_i^\psi u_j , W_i^\psi u_j)$$

$$= \Sigma_i ((- \bar{W}_i^\psi + (n-2)(\bar{Y}_i^\psi t_\psi / b(\psi)^2) \bar{\gamma}_\psi + \Theta_0^{(3)\,i}(\psi)) W_i^\psi u_j$$

$$u_j)$$

(here $\Theta_0^{(3)\,j}(\psi)$ are elements of $\Theta_0(\psi)$)

$$= - \Sigma_i (\bar{W}_i^\psi W_i^\psi u_j , u_j) + \Sigma_i (W_i^\psi u_j , \bar{\Theta}_0^{(3)\,i} u_j)$$

$$= \sum_i([W_i^\psi,\bar{W}_i^\psi]u_j,u_j) + \sum_i \|\bar{W}_i^\psi u_j\|^2_{U_r(\psi)}$$

$$+ \sum_i (W_i u_j, \Theta_0^{(4)}(\psi) u_j) + \sum_i (\bar{W}_i u_j, \Theta_0^{(4)'}(\psi) u_j)$$

Here $\Theta_0^{(4)}(\psi)$, $\Theta_0^{(4)'}(\psi)$ are elements of $\Theta_0(\psi)$. So we have our lemma . Q.E.D.

We note

$$\sum_i [W_i^\psi,\bar{W}_i^\psi] = (n-2)X^\psi + \sum_\ell \Theta^{(5)}_{0\,\ell}(\psi) W_\ell^\psi + \sum_\ell \Theta^{(6)}_{0\,\ell}(\psi)\bar{W}_\ell^\psi \quad .$$

Therefore we have

$$\sum_{i\leq j} \| W_i^\psi u_j - W_j^\psi u_i \|^2_{U_r(\psi)} + \| - \sum_j \bar{W}_j^\psi u_j \|^2_{U_r(\psi)}$$

$$+ \varepsilon \| u \|'^2_{U_r(\psi)} + (K/\varepsilon) \| \Theta_0^{(6)}(\psi) u \|^2_{U_r(\psi)}$$

$$\geq (n-3) \sum_i \| (\gamma\psi/b(\psi)) u_i \|^2_{U_r(\psi)}$$

$$+ \sum_{i,j} ((n-3)/(n-2)) \| W_j^\psi u_i \|^2_{U_r(\psi)} + \sum_{i,j} (1/(n-2)) \|\bar{W}_j^\psi u_i \|^2_{U_r(\psi)}$$

Hence we have our theorem . Q.E.D.

Chapter 3 . Some estimates for $\square_b$

In Chapter 2 , we showed the existence of L^2- solution for D^{ψ}_b-operator . Here we proved some estimates for this solution in terms of $\| \ \|_{(\dot{\ell})}$-norm . For u in $\Gamma(U_r(\psi)-C,({}^{\psi}T''_b)^*)$,

$$\| u \|_{(\dot{\ell}),U_r(\psi)} = \sum_{k\leq \ell} \| L_{i_1}L_{i_2}\cdots L_{i_k}u \|_{U_r(\psi)}$$

where $L_i = W^{\psi}_j$, $\bar{W}^{\psi}_j$, Y^{ψ}, $\bar{Y}^{\psi}$, X^{ψ} and the 0-th order operator $1/b(\psi)$ and $\| \ \|_{U_r(\psi)}$ means the L^2-norm on $U_r(\psi)-C$. In this chapter , we want to prove

Main theorem . There are elements of $\Theta_0(\psi)$, $\Theta^{(1),k}_{0\ \ell}(\psi)$, $\Theta^{(2),k}_{0\ \ell}(\psi)$, $\Theta^{(3),k}_{0\ \ell}(\psi)$ satisfying ; there are constants C_ℓ , K_ℓ , K'_ℓ satisfying ; for any ε , $\delta > 0$,

$$\begin{aligned}
&(K_\ell/\delta)\| \square^{\psi}_b u \|_{(\dot{\ell}),U_r(\psi)} + \delta \| u \|_{(\ddot{\ell}),U_r(\psi)} \\
&+ \varepsilon \| u \|_{(\ddot{\ell}),U_r(\psi)} + (K'_\ell/\varepsilon)\Big\{ \| \Theta^{(1),k}_0(\psi) W^{\psi}_k u \|_{(\dot{\ell}),U_r(\psi)} \\
&+ \| \Theta^{(2),k}_0(\psi)\bar{W}^{\psi}_k u \|_{(\dot{\ell}),U_r(\psi)} + \| \Theta^{(3)}_0(\psi)(1/b(\psi))u \|_{(\dot{\ell}),U_r(\psi)}\Big\} \\
&\geq C_\ell \| u \|_{(\ddot{\ell}),U_r(\psi)}
\end{aligned}$$

for $u \in \Gamma(U_r(\psi)-C,({}^{\psi}T''_b)^*)$ satisfying ;

$$L_\alpha L_\beta L_{i_1}L_{i_2}\cdots\cdots L_{i_k}u \text{ is of } L^2$$

where $0\leq k\leq \ell+2$, $L_i = W^{\psi}_j$, $\bar{W}^{\psi}_j$, Y^{ψ}, $\bar{Y}^{\psi}$, X^{ψ} and the 0-th order operator $1/b(\psi)$ and

$$\| u \|''_{(\ell), U_r(\psi)} = \| (\delta_\psi / b(\psi)^2) u \|'_{(\ell), U_r(\psi)}$$

$$+ \sum_k \| (1/b(\psi)) W_k^\psi u \|'_{(\ell), U_r(\psi)}$$

$$+ \sum_k \| (1/b(\psi)) \bar{W}_k^\psi u \|'_{(\ell), U_r(\psi)}$$

$$+ \sum_{i,j} \Big\{ \| W_i^\psi W_j^\psi u \|'_{(\ell), U_r(\psi)} + \| W_i^\psi \bar{W}_j^\psi u \|'_{(\ell), U_r(\psi)}$$

$$+ \| \bar{W}_i^\psi W_j^\psi u \|'_{(\ell), U_r(\psi)} + \| \bar{W}_i^\psi \bar{W}_j^\psi u \|'_{(\ell), U_r(\psi)} \Big\}$$

where K_ℓ, C_ℓ do not depend on ε, ψ.

3.1 . Commutator relations ,I

Proposition 3.1.1.

$$(3.1.1)\quad [W_j^{\psi}, X^{\psi}] = a_j^{(1)}(\psi)X^{\psi} + |\gamma_{\psi}|^2 b(\psi)^{-1} W_j^{\psi}$$

$$+ \sum_{\ell} b_{\ell}^{(1)}(\psi) W_{\ell}^{\psi} + \sum_{\ell} c_{\ell}^{(1)}(\psi)\bar{W}_{\ell}^{\psi}$$

$$\sum_{\ell} (Y_{\ell}^{\psi} t_{\psi}) b_{\ell}^{(1)}(\psi) = 0 , \quad \sum_{\ell} (\bar{Y}_{\ell}^{\psi} t_{\psi}) c_{\ell}^{(1)}(\psi) = 0 ,$$

$$(3.1.2)\quad [W_i^{\psi}, W_j^{\psi}] = b(\psi)^{-2}\gamma_{\psi}(Y_i^{\psi} t_{\psi}) W_j^{\psi} - b(\psi)^{-2}\gamma_{\psi}(Y_j^{\psi} t_{\psi}) W_i^{\psi}$$

$$+ \sum_{\ell} a_{\ell,(i,j)}^{(2)}(\psi) W_{\ell}^{\psi}$$

$$\sum_{\ell} (Y_{\ell}^{\psi} t_{\psi}) a_{\ell,(i,j)}^{(2)}(\psi) = 0 ,$$

$$(3.1.3)\quad [W_i^{\psi}, \bar{W}_j^{\psi}] = -\sqrt{-1}\, b(\psi)^{-1}\gamma_{\psi}(\delta_{ij} - ((Y_i^{\psi} t_{\psi})(\bar{Y}_j^{\psi} t_{\psi})/b(\psi)^2)) X^{\psi}$$

$$+ \sum_{\ell} a_{\ell,(i,j)}^{(3)}(\psi) W_{\ell}^{\psi} + \sum_{\ell} b_{\ell,(i,j)}^{(3)}(\psi)\bar{W}_{\ell}^{\psi}$$

$$\sum_{\ell} (Y_{\ell}^{\psi} t_{\ell}) a_{\ell,(i,j)}^{(3)}(\psi) = 0 , \quad \sum_{\ell} (\bar{Y}_{\ell}^{\psi} t_{\psi}) b_{\ell,(i,j)}^{(3)}(\psi) = 0 ,$$

$$(3.1.4)\quad [W_j^{\psi}, Y^{\psi}] = b(\psi)^{-1}\gamma_{\psi} W_j^{\psi} + \sum_{\ell} a_{\ell,(i,j)}^{(4)}(\psi) W_{\ell}^{\psi}$$

$$+ a_j^{(4)}(\psi) Y^{\psi} ,$$

$$\sum_{\ell} (Y_{\ell}^{\psi} t_{\psi}) a_{\ell,(i,j)}^{(4)}(\psi) = 0 ,$$

(3.1.5) $[W_j^\psi, \bar{Y}^\psi] = \sum_\ell a_{\ell,j}^{(5)}(\psi) W_\ell^\psi + \sum_\ell b_{\ell,j}^{(5)}(\psi) \bar{W}_\ell^\psi + a_j^{(5)}(\psi) \bar{Y}^\psi$,

$\sum_\ell (Y_\ell^\psi t_\psi) a_{\ell,j}^{(5)}(\psi) = 0$, $\sum_\ell (\bar{Y}_\ell^\psi t_\psi) b_{\ell,j}^{(5)}(\psi) = 0$, where

$a_j^{(1)}(\psi)$, $b_j^{(1)}(\psi)$, $c_j^{(1)}(\psi)$, $a_{\ell,(i,j)}^{(2)}(\psi)$, $a_{\ell,(i,j)}^{(3)}(\psi)$, $b_{\ell,(i,j)}^{(3)}(\psi)$,

$a_{\ell,(i,j)}^{(4)}(\psi)$, $a_j^{(4)}(\psi)$, $a_{\ell,j}^{(5)}(\psi)$, $b_{\ell,j}^{(5)}(\psi)$, $a_j^{(5)}(\psi)$,

$(W_j^\psi b(\psi)/b(\psi)^2)$, $(\bar{W}_j^\psi b(\psi)/b(\psi)^2)$ are of $\Theta_0(\psi)$.

The proof . (3.1.2) and (3.1.3) were already proved in Chapter 2 (see (2.2.1) and (2.2.2)) . We show (3.1.1) , (3.1.4) , (3.1.5) .

The proof of (3.1.1)

Since X^ψ , Y^ψ , $\bar{Y}^\psi$, W_j^ψ , $\bar{W}_j^\psi$ $j=1,2,..,n-1$ generate CTM on M-C , there are C^∞-functions

$$a_j^{(1)}(\psi) , b_j^{(1)}(\psi) , c_j^{(1)}(\psi) , d_j(\psi) , e_j(\psi)$$

satisfying

$$[W_j^\psi, X^\psi] = a_j^{(1)}(\psi) X^\psi + \sum_\ell b_\ell^{(1)}(\psi) W_\ell^\psi + \sum_\ell c_\ell^{(1)}(\psi) \bar{W}_\ell^\psi + d_j(\psi) Y^\psi + e_j(\psi) \bar{Y}^\psi .$$

We first see

$$d_j(\psi) = 0 \text{ and } e_j(\psi) = 0 .$$

Because

$$[W_j^{\psi}, X^{\psi}]h\circ\psi = (a_j^{(1)}(\psi)X^{\psi} + \sum_j b_j^{(1)}(\psi)W_j^{\psi} + \sum_j c_j^{(1)}(\psi)\bar{W}_j^{\psi} + d_j(\psi)Y^{\psi} + e_j(\psi)\bar{Y}^{\psi})h\circ\psi$$

The left hand side is zero and the right hand side is $e_j(\psi)b(\psi)$. Hence

$$e_j(\psi) = 0 \ .$$

Similarly , from

$$[W_j^{\psi}, X^{\psi}]\bar{h}\circ\psi = (a_j^{(1)}(\psi)X^{\psi} + \sum_j b_j^{(1)}(\psi)W_j^{\psi} + \sum_j c_j^{(1)}(\psi)\bar{W}_j^{\psi} + d_j(\psi)Y^{\psi})\bar{h}\circ\psi$$

we have

$$d_j(\psi) = 0 \ .$$

So there are C^{∞}-functions $a_j^{(1)}(\psi)$, $b_j^{(1)}(\psi)$, $c_j^{(1)}(\psi)$ satisfying

$$[W_j^{\psi}, X^{\psi}] = a_j^{(1)}(\psi)X^{\psi} + |\gamma_{\psi}|^2 b(\psi)^{-1}W_j^{\psi} + \sum_{\ell} b_{\ell}^{(1)}(\psi)W_{\ell}^{\psi} + \sum_{\ell} c_{\ell}^{(1)}(\psi)\bar{W}_{\ell}^{\psi} \ ,$$

$$\Sigma_\ell (Y_\ell^\psi t_\psi) b_\ell^{(1)}(\psi) = 0$$

and

$$\Sigma_\ell (\bar{Y}_\ell^\psi t_\psi) c_\ell^{(1)}(\psi) = 0 \quad .$$

We recall

$$X^\psi = \sqrt{-1}\, b(\psi) S + \bar{\gamma}_\psi Y^\psi - \gamma_\psi \bar{Y}^\psi \quad .$$

So , by comparing S-term , we have

$$a_j^{(1)}(\psi) = W_j^\psi b(\psi)/b(\psi) \quad .$$

Hence

$$a_j^{(1)}(\psi) \in \Theta_0(\psi) \quad .$$

Next we compute $b_\ell^{(1)}(\psi)$.

$$\begin{aligned}[[W_j^\psi, X^\psi], \bar{W}_\kappa^\psi]_F &= [a_j^{(1)}(\psi) X^\psi + \Sigma_\ell b_\ell^{(1)}(\psi) W_\kappa^\psi \\ &\quad + \Sigma_\ell c_\ell^{(1)}(\psi) \bar{W}_\ell^\psi + d_j(\psi) Y^\psi + e_j(\psi) \bar{Y}^\psi, \bar{W}_\kappa^\psi]_F \\ &= [a_j^{(1)}(\psi) X^\psi, \bar{W}_\kappa^\psi]_F + b_\kappa^{(1)}(\psi)\end{aligned}$$

Hence in order to compute $[[W_j^\psi, X^\psi], \bar{W}_\kappa^\psi]_F$, we see

"T"-term and F-term of $[W_j^\psi, X^\psi]$

$$[W_j^\psi, \sqrt{-1}\, b(\psi)S + \bar{\gamma}_\psi Y^\psi - \gamma_\psi \bar{Y}^\psi]$$

$$= [W_j^\psi, \sqrt{-1}\, b(\psi)S] + (W_j^\psi \bar{\gamma}_\psi)Y^\psi - (W_j\gamma_\psi)\bar{Y}^\psi + \bar{\gamma}_\psi[W_j^\psi, Y^\psi] - \gamma_\psi[W_j^\psi, \bar{Y}^\psi]$$

$$= \sqrt{-1}\,(W_j^\psi b(\psi))S + \sqrt{-1}\, b(\psi)[W_j^\psi, S] + (W_j\bar{\gamma}_\psi)Y^\psi + \bar{\gamma}_\psi[W_j^\psi, Y^\psi]$$

$$- (W_j^\psi\gamma_\psi)\bar{Y}^\psi - \gamma_\psi[W_j^\psi, \bar{Y}^\psi]$$

While

$$[W_j^\psi, S] = [\Sigma_\kappa Q_{\kappa j}(\psi)Y_\kappa^\psi, S]$$

$$= \Sigma_\kappa (SQ_{\kappa j}(\psi))Y_\kappa^\psi + \Sigma_\kappa Q_{\kappa j}(\psi)[Y_\kappa^\psi, S]$$

And

$$SQ_{\kappa j}(\psi) = S(\delta_{\kappa j} - ((Y_j^\psi t_\psi)(\bar{Y}_\kappa^\psi t_\psi)/b(\psi)^2))$$

$$= -(((SY_j^\psi t_\psi)(\bar{Y}_\kappa^\psi t_\psi) + (Y_j^\psi t_\psi)(S\bar{Y}_\kappa^\psi t_\psi))/b(\psi)^2)$$

$$+ ((Y_j^\psi t_\psi)(\bar{Y}_\kappa^\psi t_\psi)/b(\psi)^2)(Sb(\psi)^2)$$

Furthermore

$$[Y^\psi, W_j^\psi]_F = 0 \quad \text{and} \quad [Y^\psi, \bar{W}_j^\psi]_F = 0$$

Hence if (3.1.4) and (3.1.5) are proved ,

$$[[W_j^{\psi}, X^{\psi}], \bar{W}_{\kappa}^{\psi}]_F = |\gamma_{\psi}|^2 b(\psi) + \Theta_{0,j\kappa}(\psi) \quad ,$$

where $\Theta_{0,j\kappa}(\psi)$ is an element of $\Theta_0(\psi)$.

Therefore $b_{\ell}^{(1)}(\psi)$ is an element of $\Theta_0(\psi)$. For the case $c_{\ell}^{(1)}(\psi)$, the proof is similar . So we omit this .

The proof of (3.1.4)

Since W^{ψ}_{ℓ} and Y^{ψ} generate ${}^{\psi}T''$, the existence of $a^{(4)}_{\ell,(i,j)}(\psi)$ $a^{(4)}_{j}(\psi)$ is obvious . So we must see that $a^{(4)}_{\ell,(i,j)}(\psi)$, $a^{(4)}_{j}(\psi)$ are of $\text{Ⓗ}_0(\psi)$. By (3.1.4) ,

$$[[W^{\psi}_{j}, Y^{\psi}], \bar{W}^{\psi}_{k}]_F = [b(\psi)^{-1}\gamma_{\psi}W^{\psi}_{j} + \sum_{\ell} a^{(4)}_{\ell,(i,j)}(\psi) + a^{(4)}_{j}(\psi)Y^{\psi}, \bar{W}^{\psi}_{k}]_F$$

$$= b(\psi)^{-1}\gamma_{\psi}[W^{\psi}_{j}, \bar{W}^{\psi}_{k}]_F + a^{(4)}_{k,(i,j)}(\psi) .$$

While

$$[W^{\psi}_{j}, Y^{\psi}_{k}] = [Y^{\psi}_{j} - \sum_{\ell}((Y^{\psi}_{j}t_{\psi})(\bar{Y}^{\psi}_{\ell}t_{\psi})/b(\psi)^2)Y^{\psi}_{\ell}, Y^{\psi}_{k}]$$

$$= [Y^{\psi}_{j}, Y^{\psi}_{k}] - \sum_{\ell}((Y^{\psi}_{j}t_{\psi})(\bar{Y}^{\psi}_{\ell}t_{\psi})/b(\psi)^2)[Y^{\psi}_{\ell}, Y^{\psi}_{k}]$$

$$+ \sum_{\ell} Y^{\psi}_{k}(((Y^{\psi}_{j}t_{\psi})(\bar{Y}^{\psi}_{\ell}t_{\psi})/b(\psi)^2)Y^{\psi}_{\ell}$$

$$= \sum_{\ell} Y^{\psi}_{k}(((Y^{\psi}_{j}t_{\psi})(\bar{Y}^{\psi}_{\ell}t_{\psi})/b(\psi)^2)Y^{\psi}_{\ell} + \sum_{s} r_{s,(j,k)}(o(\psi))Y^{\psi}_{s}$$

$$- \sum_{s,\ell}(Y^{\psi}_{j}t_{\psi}/b(\psi))(\bar{Y}^{\psi}_{\ell}t_{\psi}/b(\psi))r_{s,(\ell,k)}(o(\psi))Y^{\psi}_{s} .$$

And

$$[W_j^\psi, Y^\psi] = [W_j^\psi, \sum_s (\bar{Y}_s^\psi t_\psi / b(\psi)) Y_s^\psi]$$

$$= \sum_s (W_j^\psi (\bar{Y}_s^\psi t_\psi / b(\psi)) Y_s^\psi + (\bar{Y}_s^\psi t_\psi / b(\psi)) [W_j^\psi, Y_s^\psi])$$

Hence

$$[[W_j^\psi, Y^\psi], \bar{W}_k^\psi]_F$$

$$= [\sum_s W_j^\psi (\bar{Y}_s^\psi t_\psi / b(\psi)) Y_s^\psi, \bar{W}_k^\psi]_F + [\sum_s (\bar{Y}_s^\psi t_\psi / b(\psi)) [W_j^\psi, Y_s^\psi], \bar{W}_k^\psi]_F .$$

First we compute

$$[\sum_s W_j^\psi (\bar{Y}_s^\psi t_\psi / b(\psi)) Y_s^\psi, \bar{W}_k^\psi]_F .$$

$$[\sum_s W_j^\psi (\bar{Y}_s^\psi t_\psi / b(\psi)) Y_s^\psi, \bar{W}_k^\psi]_F = \sum_s W_j^\psi (\bar{Y}_s^\psi t_\psi / b(\psi)) Q_{k,s}(\psi)$$

$$(\quad Q_{k,s}(\psi) = \delta_{ks} - ((Y_s^\psi t_\psi)(\bar{Y}_k^\psi t_\psi) / b(\psi)^2) \quad)$$

$$= \sum_s (W_j^\psi \bar{Y}_s^\psi t_\psi / b(\psi)) Q_{k,s}(\psi)$$

$$(\text{ because of } \sum_s (\bar{Y}_s^\psi t_\psi) Q_{k,s}(\psi) = 0 \quad)$$

$$= \sum_{s,\ell} (Q_{\ell,j}(\psi) Y_\ell^\psi \bar{Y}_s^\psi t_\psi / b(\psi)) Q_{k,s}(\psi)$$

$$= (\gamma_\psi / b(\psi)) Q_{k,j}(\psi) +$$

$$\sum_{s,\ell} (Q_{\ell,j}(\psi)(Y_\ell^\psi \bar{Y}_s^\psi t_\psi - \delta_{\ell s}\gamma_\psi) / b(\psi)) Q_{k,s}(\psi)$$

Second we compute $\left[\sum_s (\bar{Y}_s^{\psi} t_\psi / b(\psi))[W_j^{\psi}, Y_s^{\psi}],\ \bar{W}_k^{\psi}\right]_F$.

$$\left[\sum_s (\bar{Y}_s^{\psi} t_\psi / b(\psi))[W_j^{\psi}, Y_s^{\psi}]\ ,\ \bar{W}_k^{\psi}\right]_F$$

$$= \sum_s (\bar{Y}_s^{\psi} t_\psi / b(\psi))\Big(\sum_\ell (Y_s^{\psi}((Y_j^{\psi} t_\psi)(\bar{Y}_\ell^{\psi} t_\psi)/b(\psi)^2) Y_\ell^{\psi}$$

$$+ \sum_{\ell,t} Q_{\ell j}(\psi) r_{t,(\ell,s)}(o(\psi)) Y_t^{\psi}\ ,\quad \bar{W}_k^{\psi}\Big]_F$$

$$= (\gamma_\psi (\bar{Y}_\ell^{\psi} t_\psi / b(\psi))(Y_j^{\psi} t_\psi / b(\psi)^2)[Y_\ell^{\psi}, \bar{W}_k^{\psi}]_F$$

$$+ \quad \Big(\sum_s (\bar{Y}_s^{\psi} t_\psi / b(\psi)) \sum_\ell ((Y_j^{\psi} t_\psi)(Y_s^{\psi}\bar{Y}_\ell^{\psi} t_\psi - \delta_{s,\ell}\gamma_\psi)/b(\psi)^2)[Y_\ell^{\psi}, \bar{W}_k^{\psi}]$$

$$+ \sum_{\ell,t} Q_{\ell j}(\psi) r_{t,(\ell,s)}(o(\psi))[Y_t^{\psi}\ , \bar{W}_k^{\psi}]$$

$$= \sum_s (\bar{Y}_s^{\psi} t_\psi / b(\psi)) \sum_\ell \quad ((Y_j^{\psi} t_\psi)(Y_s^{\psi}\bar{Y}_\ell^{\psi} t_\psi \ - \ \delta_{s,\ell}\gamma_\psi)/b(\psi)^2) Q_{k,\ell}(\psi)$$

$$+ \sum_{\ell,t} Q_{\ell j}(\psi) r_{t,(\ell,s)}(o(\psi)) Q_{k,t}(\psi)$$

Hence we have

$$a^{(4)}_{k,(i,j)}$$

is of $\Theta_0(\psi)$.

For $a_j^{(4)}(\psi)$, we have

$$[[W_j^{\psi}, Y^{\psi}],\ \bar{Y}^{\psi}]_F \ = [b(\psi)^{-1}\gamma_\psi W_j^{\psi} + \sum_\ell \ a^{(4)}_{\ell,(i,j)}(\psi) W_\ell^{\psi} + a_j^{(4)}(\psi)$$

$$= \ a_j^{(4)}(\psi)$$

$$[W_j^\psi, Y^\psi] = \sum_s \left\{ (W_j^\psi(\bar{Y}_s^\psi t_\psi / b(\psi))Y_s^\psi + (\bar{Y}_s^\psi t_\psi / b(\psi))[W_j^\psi, Y_s^\psi] \right\}$$

$$= \sum_s \left\{ (W_j^\psi(\bar{Y}_s^\psi t_\psi / b(\psi))Y_s^\psi + (\bar{Y}_s^\psi t_\psi / b(\psi)) \sum_\ell (Y_s^\psi(((Y_j^\psi t_\psi)(\bar{Y}_\ell^\psi t_\psi)/b(\psi)^2)Y_\ell^\psi \right.$$

$$\left. + \sum_k r_{k,(j,s)}(o(\psi))Y_k^\psi - \sum_{k,\ell}(Y_j^\psi t_\psi / b(\psi))(\bar{Y}_\ell^\psi t_\psi / b(\psi)) r_{k,(\ell,s)} \bar{Y}_k^\psi \right\}$$

Hence

$$(1/\sqrt{-1})[[W_j^\psi, Y^\psi], \bar{Y}^\psi]_F$$

$$= \sum_s \left\{ W_j^\psi(\bar{Y}_s^\psi t_\psi / b(\psi))(Y_s^\psi t_\psi / b(\psi)) \right.$$

$$+ (\bar{Y}_s^\psi t_\psi / b(\psi)) \sum_\ell Y_s^\psi((Y_j^\psi t_\psi)(\bar{Y}_\ell^\psi t_\psi)/b(\psi)^2)(Y_\ell^\psi t_\psi / b(\psi))$$

$$+ \sum_k r_{k,(j,s)}(o(\psi))(Y_k^\psi t_\psi / b(\psi))$$

$$- \sum_{k,\ell}(Y_j^\psi t_\psi / b(\psi))(\bar{Y}_\ell^\psi t_\psi / b(\psi)) r_{k,(\ell,s)}(o(\psi))(Y_k^\psi t_\psi / b(\psi))$$

Hence $a_j^{(4)}(\psi)$ is of $\Theta_0(\psi)$.

The proof of (3.1.5) .

There are C^∞-functions $a_{\ell,j}^{(5)}(\psi)$, $b_{\ell,j}^{(5)}(\psi)$, $a_j^{(5)}(\psi)$, $a_j(\psi)$, $c_j(\psi)$ satisfying

$$[W_j^\psi, \bar{Y}^\psi] = \sum_\ell a_{\ell,j}^{(5)}(\psi) W_\ell^\psi + \sum_\ell b_{\ell,j}^{(5)}(\psi) \bar{W}_\ell^\psi + a_j^{(5)}(\psi) \bar{Y}^\psi$$

$$+ a_j(\psi) Y^\psi + c_j(\psi) X^\psi.$$

Easily we have

$$a_j(\psi) = 0 \quad \text{and} \quad c_j(\psi) = 0 \quad .$$

In fact by comparing X^{ψ} term ,

$$[W_j^{\psi}, \bar{Y}^{\psi}]_{X^{\psi}} = c_j(\psi)$$

$$0 = c_j(\psi) \; .$$

And for $a_j^{(5)}(\psi)$, we have

$$[W_j^{\psi}, \bar{Y}^{\psi}]h\circ\psi = (\textstyle\sum_{\ell} a_{\ell,j}^{(5)}(\psi)W_{\ell}^{\psi} + \sum_{\ell} b_{\ell,j}^{(5)}(\psi)\bar{W}_{\ell}^{\psi} + a_j^{(5)}(\psi)\bar{Y}^{\psi})h\circ\psi$$

The right hand side of this is

$$a_j^{(5)}(\psi)b(\psi) \quad .$$

And since

$$[W_j^{\psi}, \bar{Y}^{\psi}]\bar{h}\circ\psi = (\textstyle\sum_{\ell} a_{\ell,j}^{(5)}(\psi)W_{\ell}^{\psi} + \sum_{\ell} b_{\ell,j}^{(5)}(\psi)\bar{W}_{\ell}^{\psi} + a_j^{(5)}(\psi)\bar{Y}^{\psi} + a_j(\psi)Y^{\psi})\bar{h}\circ\psi$$

$$0 = a_j(\psi)b(\psi) \quad .$$

Hence $a_j(\psi) = 0$.

Next we see that $a_j^{(5)}(\psi)$ is of $\Theta_0(\psi)$.

$$[W_j^{\psi}, \bar{Y}^{\psi}]h\circ\psi = (\textstyle\sum_{\ell} a_{\ell,j}^{(5)}(\psi)W_{\ell}^{\psi} + \sum_{\ell} b_{\ell,j}^{(5)}(\psi)\bar{W}_{\ell}^{\psi} + a_j^{(5)}(\psi)\bar{Y}^{\psi})h\circ\psi$$

So

$$W_j^{\psi} b(\psi) = a_j^{(5)}(\psi) b(\psi) \quad .$$

Hence

$$a_j^{(5)}(\psi) = (1/b(\psi)) W_j^{\psi} b(\psi) \quad .$$

Therefore $a_j^{(5)}(\psi)$ is of $\Theta_0(\psi)$. The proof for $a_{\ell,j}^{(5)}(\psi)$, $b_{\ell,j}^{(5)}(\psi)$ is similar as (3.1.4) . So we omit this .

Q.E.D.

3.2. Commutator relations , II

From Proposition 3.1.1 , the following relation follows .

Proposition 3.2.1. For $u \in \Gamma(U_r(\psi)-C, ({}^{\psi}T''_b)^*)$

3.2.1) $$[D^{\psi}_b, X^{\psi}]\, u = |\gamma_{\psi}|^2 b(\psi)^{-1} D^{\psi}_b u + \Theta_{o,1}(\psi) X^{\psi} u + \sum_j \Theta^j_{o,2}(\psi) W^{\psi}_j u + \sum_j \Theta^j_{o,3}(\psi) \bar{W}^{\psi}_j u + \Theta_{1,1}(\psi) u$$

3.2.2) $$[D^{\psi *}_b, X^{\psi}] u = |\gamma_{\psi}|^2 b(\psi)^{-1} D^{\psi *}_b u + \Theta_{o,4}(\psi) X^{\psi} u + \sum_j \Theta^j_{o,5}(\psi) W^{\psi}_j u + \sum_j \Theta^j_{o,6}(\psi) \bar{W}^{\psi}_j u + \Theta_{1,2}(\psi) u$$

3.2.3) $$[D^{\psi}_b, W^{\psi}_k] u(W^{\psi}_i, W^{\psi}_j) = - b(\psi)^{-2}\gamma_{\psi}(Y^{\psi}_k t_{\psi}) D^{\psi}_b u(W^{\psi}_i, W^{\psi}_j) + b(\psi)^{-2}\gamma_{\psi}(Y^{\psi}_i t_{\psi}) W^{\psi}_k u_j - b(\psi)^{-2}\gamma_{\psi}(Y^{\psi}_j t_{\psi}) W^{\psi}_k u_i + \sum_{\ell} \Theta^{k(i,j)}_{o,7,\ell}(\psi) W^{\psi}_{\ell} u + \sum_{\ell} \Theta^{k(i,j)}_{o,8,\ell}(\psi) \bar{W}^{\psi}_{\ell} u + \Theta^{k(i,j)}_{1,3}(\psi) u$$, where $u_{\ell} = u(W^{\psi}_{\ell})$

3.2.4) $$[D^{\psi}_b, \bar{W}^{\psi}_k] u(W^{\psi}_i, W^{\psi}_j) = b(\psi)^{-1}\gamma_{\psi}(\delta_{ik} - ((Y^{\psi}_i t_{\psi})(\bar{Y}^{\psi}_k t_{\psi})/b(\psi)^2)) X^{\psi} u_j - b(\psi)^{-1}\gamma_{\psi}(\delta_{jk} - ((Y^{\psi}_j t_{\psi})(\bar{Y}^{\psi}_k t_{\psi})/b(\psi)^2)) X^{\psi} u_i + \sum_{\ell} \Theta^{k(i,j)}_{o,9,\ell}(\psi) W^{\psi}_{\ell} u + \sum_{\ell} \Theta^{k(i,j)}_{o,10,\ell}(\psi) \bar{W}^{\psi}_{\ell} u + \Theta^{k(i,j)}_{1,4}(\psi) u$$

3.2.3)' $[D_b^{\psi *}, W_k^{\psi}]u = -\sum_\ell b(\psi)^{-1}\bar{\gamma}_\psi(\delta_{\ell k} - ((Y_\ell^{\psi} t_\psi)(\bar{Y}_k^{\psi} t_\psi)/b(\psi)^2))X^{\psi} u_\ell$

$$+ \sum_j \Theta_{o,11,j}^{(k)}(\psi) W_j^{\psi} u + \sum_j \Theta_{o,12,j}^{(k)}(\psi) \bar{W}_j^{\psi} u$$

$$+ \Theta_{1,5}^{(k)}(\psi) u$$

3.2.4)' $[D_b^{\psi *}, \bar{W}_k^{\psi}]u = - b(\psi)^{-2}\bar{\gamma}_\psi(\bar{Y}_k t_\psi) D_b^{\psi *} u$

$$+ \sum_j \Theta_{o,13,j}^{(k)}(\psi) W_j^{\psi} u + \sum_j \Theta_{o,14,j}^{(k)}(\psi) \bar{W}_j^{\psi} u$$

$$+ \Theta_{1,6}^{(k)}(\psi) u$$

3.2.5) $[D_b^{\psi}, Y^{\psi}] = b(\psi)^{-1}\gamma_\psi D_b^{\psi} + \sum_j \Theta_{o,15,j}(\psi) W_j^{\psi}$

$$+ \Theta_{o,16}(\psi) Y^{\psi} + \Theta_{1,7}(\psi)$$

3 2.6) $[D_b^{\psi}, \bar{Y}^{\psi}] = \sum_j \Theta_{o,17,j}(\psi) W_j^{\psi} + \sum_j \Theta_{o,18,j}(\psi) \bar{W}_j^{\psi}$

$$+ \Theta_{o,19}(\psi) Y^{\psi} + \Theta_{1,8}(\psi)$$

3.2.5)' $[D_b^{\psi *}, Y^{\psi}] = b(\psi)^{-1}\gamma_\psi D_b^{\psi *} + \sum_j \Theta_{o,20,j}(\psi) W_j^{\psi}$

$$+ \sum_j \Theta_{o,21,j}(\psi) \bar{W}_j^{\psi} + \Theta_{o,22}(\psi) \bar{Y}^{\psi} + \Theta_{1,9}(\psi)$$

3.2.6)' $[D_b^{\psi *}, \bar{Y}^{\psi}] = \sum_j \Theta_{o,23,j}(\psi) W_j^{\psi} + \sum_j \Theta_{o,24,j}(\psi) \bar{W}_j^{\psi}$

$$+ \Theta_{o,25}(\psi)\bar{Y}^{\psi} + \Theta_{1,10}(\psi)$$

and

3.2.7) $[D_b^{\psi}, (1/b(\psi))]$ is of $\Theta_1(\psi)$

3.2.7)' $[D_b^{\psi *}, (1/b(\psi))]$ is of $\Theta_1(\psi)$.

Here $\Theta_{o,1}(\psi) \sim \Theta_{o,24,j}(\psi)$ are of $\Theta_o(\psi)$ and $\Theta_{1,1}(\psi) \sim \Theta_{1,10}(\psi)$ are of $\Theta_1(\psi)$.
The proof is the direct computation . So we omit this .

3.3. ... $\| \; \|_{(0),U_r(\psi)}$ ... b

In this section we prove our main theorem for the case $\ell = 0$. Namely we have

Theorem 3.3.1 . There are elements of $\Theta_0(\psi)$, $\Theta_0^{(1),k}(\psi)$, $\Theta_0^{(2),k}(\psi)$, $\Theta_0^{(3),k}(\psi)$ satisfying ; there are constants C_0 , K_0 , K_0' satisfying ; for any ε , $\delta > 0$,

$$(K_0/\delta)\|\Box^{\psi}_b u\|_{(\dot{0}),U_r(\psi)} + \delta\|u\|_{(\ddot{0}),U_r(\psi)}$$

$$+\ \varepsilon\|u\|_{(\ddot{0}),U_r(\psi)} + (K_0'/\varepsilon)\Big\{\|\Theta_0^{(1),k}(\psi)W^{\psi}_k u\|_{(\dot{0}),U_r(\psi)}$$

$$+\ \|\Theta_0^{(2),k}(\psi)\bar{W}^{\psi}_k u\|_{(\dot{0}),U_r(\psi)} + \|\Theta_0^{(3)}(\psi)(1/b(\psi))u\|_{(\dot{0}),U_r(\psi)}\Big\}$$

$$\geqq\ C_0\|u\|_{(\ddot{0}),U_r(\psi)}, \quad \text{for } u \in \Gamma(U_r(\psi)-C,({}^{\psi}T''_b)^*)$$

satisfying $L_\alpha L_\beta L_{i_1} L_{i_2} u$ is of L^2 , where

$$\|u\|_{(\ddot{0}),U_r(\psi)} = \|(\mathfrak{X}_\psi/b(\psi)^2)u\|_{(0),U_r(\psi)}$$

$$+\ \Sigma_k\|(1/b(\psi))W^{\psi}_k u\|_{(0),U_r(\psi)}$$

$$+\ \Sigma_k\|(1/b(\psi))\bar{W}^{\psi}_k u\|_{(0),U_r(\psi)}$$

$$+\ \Sigma_{i,j}\Big\{\|W^{\psi}_i W^{\psi}_j u\|_{(0),U_r(\psi)} + \|W^{\psi}_i \bar{W}^{\psi}_j u\|_{(0),U_r(\psi)}$$

$$+\ \|\bar{W}^{\psi}_i W^{\psi}_j u\|_{(0),U_r(\psi)} + \|\bar{W}^{\psi}_i \bar{W}^{\psi}_j u\|_{(0),U_r(\psi)}\Big\}$$

and K_0, C_0 do not depend on ε , ψ and $L_i = W^{\psi}_j$, $\bar{W}^{\psi}_j$, Y^{ψ} $\bar{Y}^{\psi}$, X^{ψ} and the 0-th order operator $1/b(\psi)$.

Proof . We recall Theorem 2.2.4 . We put $(1/b(\psi))v$ the place of u in Theorem 2.2.4 , where we assume

$$L_i L_j v \text{ is of } L^2$$

(so this substitution makes sense) . Then we have

$$\| D_b^{\psi}((1/b(\psi))v) \|^2_{U_r(\psi)} + \| D_b^{*\psi}((1/b(\psi))v) \|^2_{U_r(\psi)}$$

$$+ \varepsilon \| (1/b(\psi))v \|^{'2}_{(0),U_r(\psi)} + (K_0/\varepsilon) \| \Theta_0^{(1)}(\psi)(1/b(\psi))v \|^2_{U_r(\psi)}$$

$$\geqq (n-3) \| (\delta\psi/b(\psi))(1/b(\psi))v \|^2_{U_r(\psi)} +$$

$$((n-3)/(n-2))\sum_{\ell} \| W_{\ell}^{\psi}((1/b(\psi))v) \|^2_{U_r(\psi)}$$

$$+ (1/(n-2))\sum_{\ell} \| \bar{W}_{\ell}^{\psi}((1/b(\psi))v) \|^2_{U_r(\psi)} \quad \text{for any } \varepsilon > 0$$

and for $u \in \Gamma(U_r(\psi)-C, ({}^{\psi}T_b'')^*)$.

We note that $[D_b, (1/b(\psi))]$, $[D_b^{*\psi}, (1/b(\psi))]$, $[W_{\ell}^{\psi}, (1/b(\psi))]$ and $[\bar{W}_{\ell}^{\psi}, (1/b(\psi))]$ are of $\Theta_1(\psi)$.

Hence from the above inequality , there is an element $\Theta_1^{(1)}(\psi)$ of $\Theta_1(\psi)$,

$$\| (1/b(\psi)) D_b^{\psi} v \|^2_{U_r(\psi)} + \| (1/b(\psi)) D_b^{\psi *} v \|^2_{U_r(\psi)}$$

$$+ \ \mathcal{E} \| (1/b(\psi)) v \|^2_{(\dot{0}), U_r(\psi)} + (K_0^{(1)}/\mathcal{E}) \| \Theta_1^{(1)}(\psi) v \|^2_{U_r(\psi)}$$

$$\geq (n-3) \| (\gamma\psi/b(\psi)^2) v \|^2_{U_r(\psi)} +$$

$$((n-3)/(n-2)) \sum_\ell \| (1/b(\psi)) W_\ell^{\psi} v \|^2_{U_r(\psi)}$$

$$+ \ (1/(n-2)) \sum_\ell \| (1/b(\psi)) \bar{W}_\ell^{\psi} v \|^2_{U_r(\psi)}$$

Furthermore

$$\begin{aligned}
\| (1/b(\psi)) D_b^{\psi} v \|^2_{U_r(\psi)} &= ((1/b(\psi)^2) D_b^{\psi} v \ , \ D_b^{\psi} v) \\
&= (D_b^{\psi}((1/b(\psi)^2) v) \ , \ D_b^{\psi} v) \\
&+ \ ([(1/b(\psi)^2), D_b^{\psi}] v \ , \ D_b^{\psi} v) \\
&= ((1/b(\psi)^2) v \ , \ D_b^{\psi *} D_b^{\psi} v) \\
&+ \ ([(1/b(\psi)^2), D_b^{\psi}] v, \ D_b^{\psi} v)
\end{aligned}$$

The term $([(1/b(\psi)^2), D_b^{\psi}] v \ , \ D_b^{\psi} v)$ can be estimated .

Because

$$| (W_i^{\psi}((1/b(\psi)^2) v) \ , \ \bar{W}_j^{\psi} v) |$$

$$= \left| \left(-\left((W_i^\psi b(\psi))2b(\psi)/b(\psi)^4\right)v \,,\, W_j^\psi v \right) \right|$$

$$= \left| \left(((2W_i^\psi b(\psi))/b(\psi)^2)v \,,\, (1/b(\psi))W_i^\psi v \right) \right|$$

$$\leqq (1/2\varepsilon) \,\| ((2W_i^\psi b(\psi))/b(\psi)^2)v \|^2_{U_r(\psi)} + \varepsilon \| (1/b(\psi))W_j^\psi v \|^2_{U_r(\psi)}$$

We note that

$$(2W_i^\psi b(\psi))/b(\psi)^2 \text{ is of } \Theta_1(\psi) \quad .$$

Hence there is an element $\Theta_1^{(2)}(\psi)$ of $\Theta_1(\psi)$ satisfying

$$\| (1/b(\psi))D_b^\psi v \|^2_{U_r(\psi)} \leqq \left((1/b(\psi)^2)v \,,\, D_b^{\psi *}D_b^\psi v \right)$$

$$+ (1/2\varepsilon) \| \Theta_1^{(2)}(\psi)v \|^2_{U_r(\psi)} + \varepsilon \| (1/b(\psi))W_j^\psi v \|^2_{U_r(\psi)}$$

Similarly there is an element $\Theta_1^{(3)}(\psi)$ of $\Theta_1(\psi)$

$$\| (1/b(\psi))D_b^{\psi *}v \|^2_{U_r(\psi)} \leqq \left((1/b(\psi)^2)v \,,\, D_b^\psi D_b^{\psi *}v \right)$$

$$+ (1/2\varepsilon) \| \Theta_1^{(3)}(\psi)v \|^2_{U_r(\psi)} + \varepsilon \| (1/b(\psi))\bar{W}_j^\psi v \|^2_{U_r(\psi)}$$

Hence with (3.2.1) ,

$$((1/b(\psi)^2)v \;,\; (D_b^{\psi *} D_b^{\psi} + D_b^{\psi} D_b^{\psi *})v)$$

$$+ \quad (1/2\varepsilon) \| \Theta_1^{(2)}(\psi) v \|^2_{U_r(\psi)} \quad + \quad (1/2\varepsilon) \| \Theta_1^{(3)}(\psi) v \|^2_{U_r(\psi)}$$

$$+ \quad \varepsilon \, (\textstyle\sum_j \| (1/b(\psi)) W_j^{\psi} v \|^2_{U_r(\psi)} \quad + \quad \sum_j \| (1/b(\psi)) \bar{W}_j^{\psi} v \|^2_{U_r(\psi)})$$

$$+ \quad (K_0/\varepsilon) \| \Theta_0^{(1)}(\psi) (1/b(\psi)) v \|^2_{U_r(\psi)}$$

$$\geqq c_0 \Big\{ \| (\gamma_\psi / b(\psi)^2) v \|^2_{U_r(\psi)} \quad + \textstyle\sum_\ell \| (1/b(\psi)) W_\ell^{\psi} v \|^2_{U_r(\psi)}$$

$$+ \textstyle\sum_\ell \| (1/b(\psi)) \bar{W}_\ell^{\psi} v \|^2_{U_r(\psi)} \Big\}$$

By the Schwarz inequality , for any $\delta > 0$,

$$(1/2\delta) \| \Box_b^{\psi} v \|^2_{(0), U_r(\psi)} \quad + \delta \| v \|''^2_{(0), U_r(\psi)}$$

$$+ \; \varepsilon \| v \|''^2_{(0), U_r(\psi)} \quad + \quad (K_0'/\varepsilon) \Big\{ \| \Theta_0^{(1),k}(\psi) W_k^{\psi} v \|'^2_{(0), U_r(\psi)}$$

$$+ \; \| \Theta_0^{(2),k}(\psi) \bar{W}_k^{\psi} v \|'^2_{(0), U_r(\psi)} \quad + \quad \| \Theta_0^{(3)}(\psi) (1/b(\psi)) v \|'^2_{(0), U_r(\psi)} \Big\}$$

$$\geqq c_0 \Big\{ \| (\gamma_\psi / b(\psi)^2) v \|^2_{U_r(\psi)}$$

$$+ \textstyle\sum_\ell \Big\{ \| (1/b(\psi)) W_\ell^{\psi} v \|^2_{U_r(\psi)} \quad + \| (1/b(\psi)) \bar{W}_\ell^{\psi} v \|^2_{U_r(\psi)} \Big\} \Big\}$$

for $v \in \Gamma(U_r(\psi) - C, ({}^{\psi}T_b'')^*)$ satisfying

$L_\alpha L_\beta v$ is of L^2 .

Next we put $W_j^\psi v$ in the place of u in Theorem 2.2.3 . Here we assume

$$L_\alpha L_\beta v \quad \text{is of} \quad L^2 \quad .$$

So this substitution makes sense . Then we have

$(3.3.3)_j$

$$\| D_b^\psi(W_j^\psi v) \|^2_{U_r(\psi)} + \| D_b^{\psi *}(W_j^\psi v) \|^2_{U_r(\psi)} + \varepsilon \| W_j^\psi v \|^2_{(0),U_r(\psi)}$$

$$+ \quad (K/\varepsilon) \| \Theta_0^{(1)}(\psi) W_j^\psi v \|^2_{U_r(\psi)}$$

$$\geqq (n-3) \| (\gamma\psi/b(\psi)) W_j^\psi v \|^2_{U_r(\psi)} + \sum_\ell ((n-3)/(n-2)) \| W_\ell^\psi W_j^\psi v \|^2_{U_r(\psi)}$$

$$+ \quad (1/(n-2)) \sum_\ell \| \bar{W}_\ell^\psi W_j^\psi v \|^2_{U_r(\psi)} \quad .$$

We compute

$$\| D_b^\psi(W_j^\psi v) \|^2_{U_r(\psi)}$$

and

$$\| D_b^{\psi *}(W_j^\psi v) \|^2_{U_r(\psi)} \quad .$$

Namely we show that these terms can be estimated by

$(K/\varepsilon)\| \Box^{\psi}_{h} v \|^{2}_{U_r(\psi)} + \varepsilon \| v \|^{*2}_{U_r(\psi)}$ for any $\varepsilon > 0$,

where K is apositive constant which does not depend on ψ , v

We show this . By integral by parts ,

$$(D^{\psi}_{b} W^{\psi}_{j} v \ , \ D^{\psi}_{b} W^{\psi}_{j} v)$$

$$= (D^{\psi}_{b} W^{\psi}_{j} v, W^{\psi}_{j} D^{\psi}_{b} v) \ + \ (D^{\psi}_{b} W^{\psi}_{j} v, [D^{\psi}_{b}, W^{\psi}_{j}] v \ )$$

The term can be estimated as follows .

$$|(D^{\psi}_{b} W^{\psi}_{j} v \ , [D^{\psi}_{b}, W^{\psi}_{j}] v \ )|$$

$$\leq \varepsilon \| D^{\psi}_{b} W^{\psi}_{j} v \|^{2}_{U_r(\psi)} + (K/2\varepsilon) \| [D^{\psi}_{b}, W^{\psi}_{j}] v \|^{2}_{U_r(\psi)}$$

$$\leq \varepsilon \| D^{\psi}_{b} W^{\psi}_{j} v \|^{2}_{U_r(\psi)} + \Sigma_{\ell} (K'/2\varepsilon) \| (1/b(\psi)) W^{\psi}_{\ell} v \|^{2}_{U_r(\psi)}$$

(by the commutator relation (3.2.4)) .

And we have already obtained , for $\| (1/b(\psi)) W^{\psi}_{\ell} v \|^{2}_{U_r(\psi)}$, for any $\delta > 0$,

$$(1/2\delta)\| \Box_b^{\psi} v \|^2_{(0),U_r(\psi)} + \delta \| v \|''^2_{(0),U_r(\psi)}$$

$$+ \varepsilon \| v \|''^2_{(0),U_r(\psi)} + (K_0'/\varepsilon)\Big\{ \| \Theta_0^{(1),k}(\psi) W_k^{\psi} v \|'^2_{(0),U_r(\psi)}$$

$$+ \| \Theta_0^{(2),k}(\psi) \bar{W}_k^{\psi} v \|'^2_{(0),U_r(\psi)} + \| \Theta_0^{(3)}(\psi)(1/b(\psi)) v \|'^2_{(0),U_r(\psi)} \Big\}$$

$$\geq C_0 \Big\{ \| (\delta_{\psi}/b(\psi)^2) v \|^2_{U_r(\psi)}$$

$$+ \sum_{\ell} \Big\{ \| (1/b(\psi)) W_{\ell}^{\psi} v \|^2_{U_r(\psi)} + \| (1/b(\psi)) \bar{W}_{\ell}^{\psi} v \|^2_{U_r(\psi)} \Big\}\Big\}$$

Furthermore

$$(3.3.4) \qquad (D_b^{\psi} W_j^{\psi} v , W_j^{\psi} D_b^{\psi} v)$$

$$= (W_j^{\psi} v, D_b^{\psi *} W_j^{\psi} D_b^{\psi} v)$$

$$= (W_j^{\psi} v, W_j^{\psi} D_b^{\psi *} D_b^{\psi} v) + (W_j^{\psi} v, D_b^{\psi *}, W_j^{\psi} D_b^{\psi} v)$$

$$= - (\bar{W}_j^{\psi} W_j^{\psi} v, D_b^{\psi *} D_b^{\psi} v) + (\Theta_0(\psi) W_j^{\psi} v, D_b^{\psi *} D_b^{\psi} v)$$

$$+ (W_j^{\psi} v, [D_b^{\psi *}, W_j^{\psi}] D_b^{\psi} v) \quad .$$

We note that for $\Lambda^2({}^{\psi}T_b'')^*$-form, α,

$$([D_b^{\psi *}, W_j^{\psi}]\alpha)_m = - \sum_k [\bar{W}_k^{\psi}, W_j^{\psi}]\alpha_{k,m} + \Theta_1(\psi)\alpha$$

$$= -\sqrt{-1}\, b(\psi)^{-1} \gamma_\psi (\delta_{jk} - ((Y_j^\psi t_\psi)(\bar{Y}_k^\psi t_\psi)/b(\psi)^2))\alpha_{k,m}$$

$$+ \sum_{\ell,k} a^{(3)}_{\ell,(i,j)}(\psi) W_\ell^\psi \alpha_{k,m}$$

$$+ \sum_{\ell,k} b^{(3)}_{\ell,(i,j)}(\psi) \bar{W}_\ell^\psi \alpha_{k,m}$$

$$+ \quad \Theta_1(\psi)\alpha$$

Hence

$$(W_j^\psi v, [D_b^{\psi *}, W_j^\psi] D_b^\psi v)$$

$$= (W_j^\psi v, -\sum_k \sqrt{-1}\, b(\psi)^{-1}\gamma_\psi(\delta_{jk} - ((Y_j^\psi t_\psi)(\bar{Y}_k^\psi t_\psi)/b(\psi)^2)) X^\psi\ ((D_b^\psi v)(W_k^\psi)))$$

$$+ (W_j^\psi v, \sum_{\ell,k} a^{(3)}_{\ell,(i,j)}(\psi) W_\ell^\psi ((D_b^\psi v)(W_k^\psi)))$$

$$+ (W_j^\psi v, \sum_{\ell,k} b^{(3)}_{\ell,(i,j)}(\psi) \bar{W}_\ell^\psi ((D_b^\psi v)(W_k^\psi)))$$

$$+ (W_j^\psi v, \quad \Theta_1(\psi)(D_b^\psi v))$$

Therefore the problem for our estimate is only

$$(W_j^\psi v, -1 \sum_k \sqrt{-1}\, b(\psi)^{-1}\gamma_\psi(\delta_{jk} - ((Y_j^\psi t_\psi)(\bar{Y}_k^\psi t_\psi)/b(\psi)^2)) X^\psi\ ((D_b^\psi v)(W_k^\psi))) \quad .$$

However by considering the sum on j from 1 to $n-1$,

$$\sum_j (W_j v, -1 \sum_k \sqrt{-1}\, b(\psi)^{-1}\gamma_\psi(\delta_{jk} - ((Y_j^\psi t_\psi)(\bar{Y}_k^\psi t_\psi)/b(\psi)^2)) X^\psi\ ((D_b^\psi v)(W_k^\psi)))$$

$$= \sum_k \ (W_k^\psi v, -\sqrt{-1}\, b(\psi)^{-1}\gamma_\psi X^\psi((D_b^\psi v)(W_k^\psi, W_m^\psi)))$$

Fur[illegible] (3.1.[illegible] , th[illegible] [illegible]es

$$\sum_k \left(W_k^\psi v, \sqrt{-1}\left((X^\psi(b(\psi)\gamma_\psi))/b(\psi)^2\right)(D_b^\psi v)(W_k^\psi, W_m^\psi)\right)$$

$$+\sum_k \left(X^\psi W_k^\psi v, \sqrt{-1}\, b(\psi)^{-1}\gamma_\psi (D_b^\psi v)(W_k^\psi, W_m^\psi)\right) \quad .$$

The first term is no problem .

We manipulate the second term . Namely the second term becomes

$$\sum_k \left(W_k^\psi X^\psi v, \sqrt{-1}\, b(\psi)^{-1}\gamma_\psi (D_b^\psi v)(W_k^\psi, W_m^\psi)\right)$$

$$+\sum_k \left([X^\psi, W_k^\psi] v, \sqrt{-1}\, b(\psi)^{-1}\gamma_\psi (D_b^\psi v)(W_k^\psi, W_m^\psi)\right) \quad \text{(in integral by parts)} .$$

By (3.1.1) , the second term of this is no problem .

And the first term becomes

$$-\sum_k \left(X^\psi v, \sqrt{-1}\,(\bar{W}_k^\psi(b(\psi)^{-1}\gamma_\psi))(D_b^\psi v)(W_k^\psi, W_m^\psi)\right)$$

$$-\sum_k \left(X^\psi v, \sqrt{-1}\, b(\psi)^{-1}\gamma_\psi \bar{W}_k^\psi((D_b^\psi v)(W_k^\psi, W_m^\psi))\right) \quad \text{(in integral by parts)} .$$

The first term of this is no problem , because of

$$\bar{W}_k^\psi(b(\psi)^{-1}\gamma_\psi) \in \textcircled{H}_1(\psi) \quad .$$

And for the second term of this ,

$$| - \sum_k ((1/b(\psi))X^{\psi}v, \sqrt{-1}\,\bar{W}_k^{\psi}((D_b^{\psi}v)(W_k^{\psi}, W_m^{\psi}))$$

$$+ ((1/b(\psi))X^{\psi}v, \Theta_0(\psi)(D_b^{\psi}v)(W_k^{\psi}, W_m^{\psi}))$$

$$= | - ((\bar{\gamma}_{\psi}/b(\psi))X^{\psi}v, D_b^{\psi *}D_b^{\psi}v)$$

$$+ \sum_k ((\bar{\gamma}_{\psi}/b(\psi))X^{\psi}v, \Theta_0(\psi)(D_b^{\psi}v)(W_k^{\psi}, W_m^{\psi})) |$$

$$\leqq \varepsilon \| (1/b(\psi))\bar{\gamma}_{\psi}X^{\psi}v \|^2_{U_r(\psi)} + (K/\varepsilon) \| D_b^{\psi *}D_b^{\psi} v \|^2_{U_r(\psi)}$$

$$+ (K/\varepsilon) \| \Theta_0(\psi) D_b^{\psi} v \|^2_{U_r(\psi)}$$

$$\leqq \varepsilon \sum_{i,j} (\| W_i^{\psi} W_j^{\psi} u \|^2_{U_r(\psi)} + \| W_i^{\psi} \bar{W}_j^{\psi} u \|^2_{U_r(\psi)} + \| \bar{W}_i^{\psi} W_j^{\psi} u \|^2_{U_r(\psi)}$$

$$+ \| \bar{W}_i^{\psi} \bar{W}_j^{\psi} u \|^2_{U_r(\psi)})$$

$$+ (K/\varepsilon) \| D_b^{\psi *}D_b^{\psi} v \|^2_{U_r(\psi)} + (K/\varepsilon) \| \Theta_0(\psi) D_b^{\psi} v \|^2_{U_r(\psi)}$$

So it does not bother us . The case $D_b^{\psi *}W_j^{\psi}v$ is the same . By putting $\bar{W}_j^{\psi}v$ in the place of u in Theorem 2.2.3 and following the above method , we have our theorem .

Q.E.D.

3.4. The $\| \|_{(\dot{\ell}),U_r(\psi)}$-estimate for $\Box'_b$

For $\| \|_{(\dot{\ell}),U_r(\psi)}$-estimate , we have the following theorem .

Theorem 3.4.1 . There are elements of $\Theta_0(\psi)$, $\Theta_{0\ \ell}^{(1),k}(\psi)$, $\Theta_{0\ \ell}^{(2),k}(\psi)$, $\Theta_{0\ \ell}^{(3),k}(\psi)$ satisfying ; there are constants C_ℓ , K_ℓ , K'_ℓ satisfying ; for any $\varepsilon, \delta > 0$,

$$(K_\ell/\delta)\| \Box^\psi_b u \|_{(\dot{\ell}),U_r(\psi)} + \delta\| u \|_{(\ddot{\ell}),U_r(\psi)} + \varepsilon\| u \|_{(\ddot{\ell}),U_r(\psi)}$$

$$+ (K'_\ell/\varepsilon)\Big\{\| \Theta_{0\ \ell}^{(1),k}(\psi)W^\psi_k u \|_{(\dot{\ell}),U_r(\psi)} + \| \Theta_{0\ \ell}^{(2),k}(\psi)\bar{W}^\psi_k u \|_{(\dot{\ell}),U_r(\psi)}$$

$$+ \| \Theta_{0\ \ell}^{(3),k}(\psi)(1/b(\psi))u \|_{(\dot{\ell}),U_r(\psi)}\Big\} \geqq$$

$$C_\ell \| u \|_{(\ddot{\ell}),U_r(\psi)} \quad \text{for all } u \in \Gamma(U_r(\psi)-C, ({}^\psi T''_b)^*)$$

satisfying ;

$$L_\alpha L_\beta L_{i_1} L_{i_2} \cdots L_{i_k} u \text{ is of } L^2 ,$$

where $0 \leq k \leq \ell+2$, $L_i = W^\psi_j, \bar{W}^\psi_j, Y^\psi, \bar{Y}^\psi, X^\psi$ and $1/b(\psi)$.

We show this theorem by induction . The case $\ell = 0$ is proved in Section 3.3 . We assume the case μ . Namely we assume (Theorem)$_\mu$ for $\varepsilon, \delta > 0$,

$$(K_\mu/\delta)\| \Box^\psi_b u \|_{(\dot{\mu}),U_r(\psi)} + \delta\| u \|_{(\ddot{\mu}),U_r(\psi)} + \varepsilon\| u \|_{(\ddot{\mu}),U_r(\psi)}$$

$$(K'_\mu/\varepsilon)\Big\{\| \Theta_{0\ \mu}^{(1),k}(\psi)W^\psi_k u \|_{(\dot{\mu}),U_r(\psi)} + \| \Theta_{0\ \mu}^{(2),k}(\psi)\bar{W}^\psi_k u \|_{(\dot{\mu}),U_r(\psi)}$$

$$+ \| \Theta_{0\ \mu}^{(3),k}(\psi)(1/b(\psi))u \|_{(\dot{\mu}),U_r(\psi)}\Big\} \geqq$$

$$C_\mu \| u \|_{(\ddot{\mu}),U_r(\psi)} \quad \text{for all } u \in \Gamma(U_r(\psi)-C, ({}^\psi T''_b)^*)$$

$$L_\alpha L_\beta L_{i_1} L_{i_2} \cdots\cdots L_{i_k} u \text{ is of } L^2 ,$$

where $0 \leq k \leq \ell + 2$, $L_i = W_j, \bar{W}_j, Y^\psi, \bar{Y}^\psi, X^\psi$ and $1/b(\psi)$.

Now we see the case $\mu+1$. For u satisfying

$$L_\alpha L_\beta L_{i_1} L_{i_2} \cdots\cdots L_{i_k} u \text{ is of } L^2 , \quad 0 \leq k \leq \mu+3 ,$$

we put $(1/b(\psi))u$ in the place of u in (Theorem)$_\mu$.
Namely we have

$$
\begin{aligned}
(3.4.1)\quad & (K_\mu/\delta) \| \Box^\psi_b((1/b(\psi))u) \|_{(\dot{\mu}), U_r(\psi)} \\
& + \delta \| (1/b(\psi))u \|_{(\ddot{\mu}), U_r(\psi)} + \varepsilon \| u \|_{(\ddot{\mu}), U_r(\psi)} \\
& + (K'_\mu/\varepsilon) \Big\{ \| \Theta_0^{(1),k}(\psi) W_k^\psi((1/b(\psi))u) \|_{(\dot{\mu}), U_r(\psi)} \\
& + \qquad \| \Theta_0^{(2),k}(\psi) \bar{W}_k^\psi((1/b(\psi))u) \|_{(\dot{\mu}), U_r(\psi)} \\
& + \qquad \| \Theta_0^{(3),k}(\psi) (1/b(\psi)^2) u \|_{(\dot{\mu}), U_r(\psi)} \Big\} \\
& \geq C_\mu \| (1/b(\psi))u \|_{(\ddot{\mu}), U_r(\psi)} \quad \text{for all } u \in \Gamma(U_r(\psi) - C, (^\psi T''_b)^*)
\end{aligned}
$$

satisfying ;

$$L_\alpha L_\beta L_{i_1} L_{i_2} \cdots\cdots L_{i_k} u \text{ is of } L^2 ,$$

where $0 \leq k \leq \mu + 3$, $L_i = W_j^\psi, \bar{W}_j^\psi, Y^\psi, \bar{Y}^\psi, X^\psi$ and $1/b(\psi)$.

Furthermore by

$$W_i^\psi W_j^\psi((1/b(\psi))v) = (1/b(\psi))W_i^\psi W_j^\psi v + \sum_k \Theta_1(\psi)W_k^\psi v + \sum_k \Theta_1(\psi)\bar{W}_k^\psi v + ((\text{const.})/b(\psi)^3)v ,$$

$$W_i^\psi \bar{W}_j^\psi((1/b(\psi))v) = (1/b(\psi))W_i^\psi \bar{W}_j^\psi v + \sum_k \Theta_1(\psi)W_k^\psi v + \sum_k \Theta_1(\psi)\bar{W}_k^\psi v + ((\text{const.})/b(\psi)^3)v ,$$

$$\bar{W}_i^\psi W_j^\psi((1/b(\psi))v) = (1/b(\psi))\bar{W}_i^\psi W_j^\psi v + \sum_k \Theta_1(\psi)W_k^\psi v + \sum_k \Theta_1(\psi)\bar{W}_k^\psi v + ((\text{const.})/b(\psi)^3)v ,$$

$$\bar{W}_i^\psi \bar{W}_j^\psi((1/b(\psi))v) = (1/b(\psi))\bar{W}_i^\psi \bar{W}_j^\psi v + \sum_k \Theta_1(\psi)W_k^\psi v + \sum_k \Theta_1(\psi)\bar{W}_k^\psi v + ((\text{const.})/b(\psi)^3)v .$$

We have already estimated

$$\| (\gamma_\psi/b(\psi)^3)v \|_{(\mu),U_r(\psi)} , \quad \| (1/b(\psi)^2)W_k^\psi v \|_{(\mu),U_r(\psi)}$$

$$\| (1/b(\psi)^2)\bar{W}_k^\psi v \|_{(\mu),U_r(\psi)} .$$

Hence we have

$$\square_b^{\psi}((1/b(\psi))u) = (1/b(\psi))\square_b^{\psi}u + \Sigma_k \Theta_1(\psi)W_k^{\psi}u$$

$$+ \Sigma_k \Theta_1(\psi)\bar{W}_k^{\psi}u + \Theta_2(\psi)u ,$$

$$W_k^{\psi}((1/b(\psi))u) = (1/b(\psi))W_k^{\psi}u + \Sigma_j \Theta_0(\psi)W_j^{\psi}u$$

$$+ \Sigma_j \Theta_0(\psi)\bar{W}_j^{\psi}u ,$$

$$\bar{W}_k^{\psi}((1/b(\psi))u) = (1/b(\psi))\bar{W}_k^{\psi}u + \Sigma_j \Theta_0(\psi)W_j^{\psi}u$$

$$+ \Sigma_j \Theta_0(\psi)\bar{W}_j^{\psi}u .$$

By these with (Theorem) ,

(3.4.2) $$(\tilde{K}_{\mu}/\delta)\|\square_b^{\psi}u\|_{(\mu'+1),U_r(\psi)} + \delta\|u\|_{(\mu''+1),U_r(\psi)}$$

$$+ (\tilde{K}'_{\mu}/\varepsilon)\Big\{\|\Theta_0^{(1),k}(\psi)W_k^{\psi}u\|_{(\mu'+1),U_r(\psi)} + \|\Theta_0^{(2),k}(\psi)\bar{W}_k^{\psi}u\|_{(\mu'+1),U_r(\psi)}$$

$$+ \|\Theta_0^{(3),k}(\psi)(1/b(\psi))u\|_{(\mu'+1),U_r(\psi)}\Big\} + \varepsilon\|u\|_{(\mu''+1),U_r(\psi)}$$

$$\geqq c_{\mu}\|(1/b(\psi))u\|_{(\mu''),U_r(\psi)} \quad \text{for all } u \in \Gamma(U_r(\psi)-C,({}^{\psi}T_b'')^*)$$

satisfying ;

$$L_{\alpha}L_{\beta}L_{i_1}L_{i_2}\cdots\cdots L_{i_k}u \ \text{ is of } L^2 ,$$

where $0\leqq k\leqq\mu+3$, $L_i = W_j^{\psi},\bar{W}_j^{\psi},Y^{\psi},\bar{Y}^{\psi},X^{\psi}$ and $1/b(\psi)$.

(I)

$$\|(1/b(\psi)^2)X^\psi u\|_{(\mu),U_r(\psi)} + \sum_k\Big\{\|(1/b(\psi))W_k^\psi X^\psi u\|_{(\mu),U_r(\psi)}$$

$$+ \|(1/b(\psi))\bar{W}_k^\psi X^\psi u\|_{(\mu),U_r(\psi)}\Big\}$$

$$+ \sum_{i,j}\Big\{\|W_i^\psi W_j^\psi X^\psi u\|_{(\mu),U_r(\psi)} + \|W_i^\psi \bar{W}_j^\psi X^\psi u\|_{(\mu),U_r(\psi)}$$

$$+ \|\bar{W}_i^\psi W_j^\psi X^\psi u\|_{(\mu),U_r(\psi)} + \|\bar{W}_i^\psi \bar{W}_j^\psi X^\psi u\|_{(\mu),U_r(\psi)}\Big\}$$

(II)

$$\|(1/b(\psi)^2)Y^\psi u\|_{(\mu),U_r(\psi)} + \sum_k\Big\{\|(1/b(\psi))W_k^\psi Y^\psi u\|_{(\mu),U_r(\psi)}$$

$$+ \|(1/b(\psi))\bar{W}_k^\psi Y^\psi u\|_{(\mu),U_r(\psi)}\Big\}$$

$$+ \sum_{i,j}\Big\{\|W_i^\psi W_j^\psi Y^\psi u\|_{(\mu),U_r(\psi)} + \|W_i^\psi \bar{W}_j^\psi Y^\psi u\|_{(\mu),U_r(\psi)}$$

$$+ \|\bar{W}_i^\psi W_j^\psi Y^\psi u\|_{(\mu),U_r(\psi)} + \|\bar{W}_i^\psi \bar{W}_j^\psi Y^\psi u\|_{(\mu),U_r(\psi)}\Big\}$$

(III)

$$\|(1/b(\psi)^2)\bar{Y}^\psi u\|_{(\mu),U_r(\psi)} + \sum_k\Big\{\|(1/b(\psi))W_k^\psi \bar{Y}^\psi u\|_{(\mu),U_r(\psi)}$$

$$+ \|(1/b(\psi))\bar{W}_k^\psi \bar{Y}^\psi u\|_{(\mu),U_r(\psi)}\Big\}$$

$$+ \sum_{i,j}\Big\{\|W_i^\psi W_j^\psi \bar{Y}^\psi u\|_{(\mu),U_r(\psi)} + \|W_i^\psi \bar{W}_j^\psi \bar{Y}^\psi u\|_{(\mu),U_r(\psi)}$$

$$+ \|\bar{W}_i^\psi W_j^\psi \bar{Y}^\psi u\|_{(\mu),U_r(\psi)} + \|\bar{W}_i^\psi \bar{W}_j^\psi \bar{Y}^\psi u\|_{(\mu),U_r(\psi)}\Big\}$$

$$\| (1/b(\psi)^2) W_m u \|_{(\acute{\mu}), U_r(\psi)} + \sum_k \{ \| (1/b(\psi)) W_k^\psi W_m^\psi u \|_{(\acute{\mu}), U_r(\psi)}$$

$$+ \| (1/b(\psi)) \bar{W}_k^\psi W_m^\psi u \|_{(\acute{\mu}), U_r(\psi)} \}$$

$$+ \sum_{i,j} \{ \| W_i^\psi W_j^\psi W_m^\psi u \|_{(\acute{\mu}), U_r(\psi)} + \| W_i^\psi \bar{W}_j^\psi W_m^\psi u \|_{(\acute{\mu}), U_r(\psi)}$$

$$+ \| \bar{W}_i^\psi W_j^\psi W_m^\psi u \|_{(\acute{\mu}), U_r(\psi)} + \| \bar{W}_i^\psi \bar{W}_j^\psi W_m^\psi u \|_{(\acute{\mu}), U_r(\psi)} \}$$

and

$(V)_m$

$$\| (1/b(\psi)^2) \bar{W}_m^\psi u \|_{(\acute{\mu}), U_r(\psi)} + \sum_k \{ \| (1/b(\psi)) W_k^\psi \bar{W}_m^\psi u \|_{(\acute{\mu}), U_r(\psi)}$$

$$+ \| (1/b(\psi)) \bar{W}_k^\psi \bar{W}_m^\psi u \|_{(\acute{\mu}), U_r(\psi)} \}$$

$$+ \sum_{i,j} \{ \| W_i^\psi W_j^\psi \bar{W}_m^\psi u \|_{(\acute{\mu}), U_r(\psi)} + \| W_i^\psi \bar{W}_j^\psi \bar{W}_m^\psi u \|_{(\acute{\mu}), U_r(\psi)}$$

$$+ \| \bar{W}_i^\psi W_j^\psi \bar{W}_m^\psi u \|_{(\acute{\mu}), U_r(\psi)} + \| \bar{W}_i^\psi W_j^\psi \bar{W}_m^\psi u \|_{(\acute{\mu}), U_r(\psi)} \}$$

by

$$(\tilde{\tilde{K}}_\mu / \delta) \| \Box_b^\psi u \|_{(\acute{\mu}+1), U_r(\psi)} + \delta \| u \|_{(\ddot{\mu}+1), U_r(\psi)}$$

$$+ \varepsilon \| u \|_{(\ddot{\mu}+1), U_r(\psi)} + (\tilde{\tilde{K}}_\mu / \varepsilon) \{ \| \Theta_0^{(1),k}(\psi) W_k^\psi u \|_{(\acute{\mu}+1), U_r(\psi)}$$

$$+ \quad \| \Theta_0^{(2),k}(\psi)\bar{W}_k^{\psi} u \|_{(\dot{\mu}+1), U_r(\psi)}$$

$$+ \quad \| \Theta_0^{(3),k}(\psi)(1/b(\psi))u \|_{(\dot{\mu}+1), U_r(\psi)}$$

for any δ , $\varepsilon > 0$, where $\tilde{\tilde{K}}_\mu$, $\tilde{\tilde{K}}'_\mu$ are constants independent of ε , δ , ψ and u .

In order to estimate (I) , we put $X^\psi u$ in the place of u in $(\text{Theorem})_\mu$. Then

$$(K_\mu/\delta)\| \Box_b^\psi X^\psi u \|_{(\acute{\mu}), U_r(\psi)} + \delta \| X^\psi u \|_{(\ddot{\mu}), U_r(\psi)}$$

$$+ \quad \varepsilon \| X^\psi u \|_{(\ddot{\mu}), U_r(\psi)}$$

$$+ \quad (K'_\mu/\varepsilon)\Big\{ \| \Theta_0^{(1),k}(\psi) W_k^\psi X^\psi u \|_{(\dot{\mu}), U_r(\psi)}$$

$$+ \quad \| \Theta_0^{(2),k}(\psi)\bar{W}_k^\psi X^\psi u \|_{(\dot{\mu}), U_r(\psi)}$$

$$+ \quad \| \Theta_0^{(3),k}(\psi)(1/b(\psi))X^\psi u \|_{(\dot{\mu}), U_r(\psi)} \Big\}$$

$$\geq c_\mu \| X^\psi u \|_{(\ddot{\mu}), U_r(\psi)} \quad .$$

The problem is the difference ,i.e., $[\Box_b^\psi, X^\psi] = \Box_b^\psi X^\psi - X^\psi \Box_b^\psi$

$$
\begin{aligned}
[\Box_b^{\psi}, X^{\psi}]u &= 2|\gamma_\psi|^2 b(\psi)^{-1}\Box_b^{\psi}u + \Theta_{0,j}^{(4.1)}(\psi)X^{\psi}W_j^{\psi}u \\
&+ \Theta_{0,j}^{(4.2)}(\psi)X^{\psi}\overline{W}_j^{\psi}u + \Sigma_{i,j}\Big\{ \Theta_0^{(4.3)}(i,j)(\psi)W_i^{\psi}W_j^{\psi}u \\
&+ \Theta_0^{(4.4)}(i,j)(\psi)W_i^{\psi}\overline{W}_j^{\psi}u + \Theta_0^{(4.5)}(i,j)(\psi)\overline{W}_i^{\psi}W_j^{\psi}u \\
&+ \Theta_0^{(4.6)}(i,j)(\psi)\overline{W}_i^{\psi}\overline{W}_j^{\psi}u\Big\} + \Sigma_i\Big\{\Theta_{1,i}^{(4.7)}(\psi)W_i^{\psi}u \\
&+ \Theta_{1,i}^{(4.8)}(\psi)\overline{W}_i^{\psi}u\Big\} + \Theta_1^{(4.9)}(\psi)X^{\psi}u + \Theta_2^{(4.10)}(\psi)u ,
\end{aligned}
$$

where $\Theta_{0,j}^{(4.1)}(\psi)$, $\Theta_{0,j}^{(4.2)}(\psi)$, $\Theta_0^{(4.3)}(i,j)(\psi)$, $\Theta_0^{(4.4)}(i,j)(\psi)$, $\Theta_0^{(4.5)}(i,j)(\psi)$, $\Theta_0^{(4.6)}(i,j)(\psi)$ are of $\Theta_0(\psi)$ and $\Theta_{1,i}^{(4.7)}(\psi)$, $\Theta_{1,i}^{(4.8)}(\psi)$, $\Theta_1^{(4.9)}(\psi)$ are of $\Theta_1(\psi)$ and $\Theta_2^{(4.10)}(\psi)$ is of $\Theta_2(\psi)$.

By the direct computation , the proof follows from 3.2.1) and 3.2.2) . So we omit this . By this lemma , we have (I) . In order to show (II) , we put $Y^{\psi}u$ in the place of u in $(\text{Theorem})_{\mu}$. Then we have

$$(K_\mu/\delta)\|\Box_b^\psi Y^\psi u\|_{(\dot\mu),U_r(\psi)} + \delta\|Y^\psi u\|_{(\ddot\mu),U_r(\psi)}$$

$$+ \varepsilon\|Y^\psi u\|_{(\ddot\mu),U_r(\psi)}$$

$$+ (K'_\mu/\varepsilon)\Big\{\sum_k \|\Theta_0^{(1),k}(\psi)W_k^\psi Y^\psi u\|_{(\dot\mu),U_r(\psi)}$$

$$+ \sum_k \|\Theta_0^{(2),k}(\psi)\bar W_k^\psi Y^\psi u\|_{(\dot\mu),U_r(\psi)}$$

$$+ \sum_k \|\Theta_0^{(3),k}(\psi)(1/b(\psi))Y^\psi u\|_{(\dot\mu),U_r(\psi)}\Big\}$$

$$\geq C_\mu \|Y^\psi u\|_{(\ddot\mu),U_r(\psi)}$$

The problem is the commutator

$$[\Box_b^\psi, Y^\psi] = \Box_b^\psi Y^\psi - Y^\psi \Box_b^\psi$$

For $[\Box_b^\psi, Y^\psi]$, we obtain

Lemma 3.4.2.

$$[\Box_b^\psi, Y^\psi]u = 2\chi_\psi b(\psi)^{-1}\Box_b^\psi u + \sum_j \Big\{\Theta_{0,j}^{(4.11)}(\psi)Y^\psi W_j^\psi u$$

$$+ \Theta_{0,j}^{(4.12)}(\psi)Y^\psi \bar W_j^\psi u\Big\} + \sum_{i,j}\Big\{\Theta_0^{(4.13)(i,j)}(\psi)W_i^\psi W_j^\psi u$$

$$+ \Theta_0^{(4.14)(i,j)}(\psi)W_i^\psi \bar W_j^\psi u + \Theta_0^{(4.15)(i,j)}(\psi)\bar W_i^\psi W_j^\psi u$$

$$+ \Theta_0^{(4.16)(i,j)}(\psi)\bar W_i^\psi \bar W_j^\psi u\Big\} + \sum_i \Big\{\Theta_{1,i}^{(4.17)}(\psi)W_i^\psi u$$

$$+ \Theta_{1,i}^{(4.18)}(\psi)\bar W_i^\psi u\Big\} + \Theta_1^{(4.19)}(\psi)X^\psi u + \Theta_2^{(4.20)}(\psi)u$$

So we omit this . The proof for (III) is the same as (II) . Hence we omit this . In order to show (IV) , we put $W_\ell^\psi u$ in the place of u in $(\text{Theorem})_\mu$. Then ,

$$(K_\mu/\delta)\| \Box_b^\psi W_\ell^\psi u \|_{(\dot\mu),U_r(\psi)} + \delta \| W_\ell^\psi u \|_{(\ddot\mu),U_r(\psi)}$$

$$+ \varepsilon \| W_\ell^\psi u \|_{(\ddot\mu),U_r(\psi)}$$

$$+ (K'_\mu/\varepsilon)\Big\{ \sum_k \| \Theta_0^{(1),k}(\psi) W_k^\psi W_\ell^\psi u \|_{(\dot\mu),U_r(\psi)}$$

$$+ \sum_k \| \Theta_0^{(1),k}(\psi) \bar W_k^\psi W_\ell^\psi u \|_{(\dot\mu),U_r(\psi)}$$

$$+ \sum_k \| \Theta_0^{(1),k}(\psi)\,(1/b(\psi)) W_\ell^\psi u \|_{(\dot\mu),U_r(\psi)} \Big\}$$

$$\geqq C_\mu \| W_\ell^\psi u \|_{(\ddot\mu),U_r(\psi)} .$$

The problem is the commutator

$$[\Box_b^\psi, W_\ell^\psi] = \Box_b^\psi W_\ell^\psi - W_\ell^\psi \Box_b^\psi .$$

$$\Sigma_\ell \ \| \Box_b^\psi W_\ell^\psi u \|^{\cdot 2}_{(\mu),U_r(\psi)}$$

$$= \Sigma_\ell (\Box_b^\psi W_\ell^\psi u \ , \ \Box_b^\psi W_\ell u \ )_{(\mu)}$$

$$= \Sigma_\ell ([\Box_b^\psi , W_\ell^\psi] u \ , \ \Box_b^\psi W_\ell^\psi u \ )_{(\mu)} + \Sigma_\ell (W_\ell^\psi \Box_b^\psi u \ , \Box_b^\psi W_\ell^\psi u \ )_{(\mu)}$$

where $(\ , \ )_{(\mu)}$ means the inner product induced by $\| \ \|_{(\dot\mu),U_r(\psi)}$ -norm .

On the other hand

$$\begin{aligned}
[\Box_b^\psi , W_\ell^\psi] u(W_i^\psi) &= (\Box_b^\psi W_\ell^\psi - W_\ell^\psi \Box_b) u(W_i^\psi) \\
&= (D_b^\psi D_b^\psi{}^* W_\ell^\psi - D_b^\psi W^\psi D_b^\psi{}^* + D_b^\psi W_\ell^\psi D_b^\psi{}^* - W_\ell^\psi D_b^\psi D_b^\psi{}^* \\
&\quad + D_b^\psi{}^* D_b^\psi W_\ell^\psi - D_b^\psi{}^* W_\ell^\psi D_b^\psi + D_b^\psi{}^* W_\ell^\psi D_b^\psi - W_\ell^\psi D_b^\psi{}^* D_b^\psi) u(W_i^\psi) \\
&= D_b^\psi [D_b^\psi{}^*, W_\ell^\psi] u(W_i^\psi) + [D_b^\psi, W_\ell^\psi] D_b^\psi{}^* u(W_i^\psi) \\
&\quad + D_b^\psi{}^* [D_b^\psi, W_\ell^\psi] u(W_i^\psi) + [D_b^\psi{}^*, W_\ell^\psi] D_b^\psi u(W_i^\psi) \ .
\end{aligned}$$

For the term

$$D_b^\psi [D_b^\psi{}^*, W_\ell^\psi] u(W_i^\psi) \ ,$$

by (3.2.3)' ,

$$D_b^\psi[D_b^{\psi *}, W_\ell^\psi]u(W_i^\psi)$$

$$= - \sum_j b(\psi)^{-1}\overline{\delta}_\psi(\delta_{j\ell} - ((Y_j^\psi t_\psi)(\overline{Y}_\ell^\psi t_\psi)/b(\psi)^2))D_b^\psi X^\psi u_j(W_i^\psi)$$

$$- \sum_j W_i^\psi(b(\psi)^{-1}\overline{\delta}_\psi(\delta_{j\ell} - ((Y_j^\psi t_\psi)(\overline{Y}_\ell^\psi t_\psi)/b(\psi)^2)X^\psi u_j$$

$$+ \sum_j W_i^\psi(\Theta_0(\psi)W_j^\psi u) + \sum_j W_i^\psi(\Theta_0(\psi)\overline{W}_j^\psi u)$$

$$+ W_i(\Theta_1(\psi)u) .$$

And we have already estimated terms , $b(\psi)^{-1}W_i X^\psi u_j$, $b(\psi)^{-2}X^\psi u_j$. So by considering

$$|(D_b^\psi[D_b^{\psi *}, W_\ell^\psi]u , \square_b^\psi W_\ell^\psi u)_{(\mu)}| \leq (2/\varepsilon)\| D_b^\psi[D_b^{\psi *}, W_\ell^\psi]u \|_{(\mu), U_r(\psi)}^{\prime 2}$$

$$+ \varepsilon\| \square_b^\psi W_\ell^\psi u \|_{(\mu), U_r(\psi)}^{\prime 2} ,$$

this term doesn't bother us (if we choose ε sufficiently small) . The term

$$[D_b^{\psi *}, W_\ell^\psi]D_b^\psi u$$

is similar . So we omit this . The problem is to controll

$$\sum_\ell [D_b^\psi, W_\ell^\psi]D_b^{\psi *}u \quad \text{and} \quad \sum_\ell D_b^{\psi *}[D_b^\psi, W_\ell^\psi]u .$$

$$\sum_\ell ([D_b^\psi, W_\ell^\psi] D_b^{\psi *}u \ , \ \Box_b^\psi W_\ell^\psi u)_{(\mu)}$$

$$= \sum_\ell (\ \sum_i \ (\ [W_i^\psi, W_\ell^\psi] D_b^{\psi *}u \ , \ (\Box_b^\psi W_\ell^\psi u \ )(W_i^\psi))_{(\mu)})$$

$$= \sum_\ell (\ \sum_i \ (\ (b(\psi)^{-2}\gamma_\psi(Y_i^\psi t_\psi) W_\ell^\psi - b(\psi)^{-2}\gamma_\psi(Y_\ell^\psi t_\psi) W_i^\psi$$

$$+ \sum_k a_{k,(i,\ell)}^{(2)}(\psi) W_k^\psi \ ) D_b^{\psi *}u \ , \ (\Box_b^\psi W_\ell^\psi u \ )(W_i^\psi))_{(\mu)})$$

$$= \sum_{\ell,i} \ (\ b(\psi)^{-2}\gamma_\psi(Y_i^\psi t_\psi) W_\ell^\psi D_b^{\psi *}u \ , \ (\Box_b^\psi W_\ell^\psi u)(W_i^\psi) \ )_{(\mu)}$$

$$- \sum_{\ell,i} \ (\ b(\psi)^{-2}\gamma_\psi(Y_\ell^\psi t_\psi) W_i^\psi D_b^{\psi *}u \ , \ (\Box_b^\psi W_\ell^\psi u)(W_i^\psi) \ )_{(\mu)}$$

$$+ \sum_{\ell,i} \ (\sum_k a_{k,(i,\ell)}^{(2)}(\psi) W_k^\psi D_b^{\psi *}u \ , \ (\Box_b^\psi W_\ell^\psi u)(W_i^\psi) \ )_{(\mu)}$$

The second term of this becomes

$$- \ \sum_\ell (\ b(\psi)^{-2} \ (Y_\ell^\psi t_\psi) D_b^\psi D_b^{\psi *}u \ , \ (\Box_b^\psi W_\ell^\psi u) \ )_{(\mu)} \ .$$

So this is estimated by

$$(K/\varepsilon) \| \ b(\psi)^{-1} D_b^\psi D_b^{\psi *}u \ \|_{(\mu'), U_r(\psi)} \ + \varepsilon \| \ \Box_b^\psi W_\ell^\psi u \ \|_{(\mu'), U_r(\psi)} \ ,$$

where K is a large constant which doesn't depend on ψ , u and ε . Hence it doesn't bother us .

$$(\sum_k a^{(2)}_{k,(i,\ell)}(\psi) W^{\psi}_k D^{\psi *}_b u \ , \ (\Box^{\psi}_b W^{\psi}_\ell u\ )(W^{\psi}_i))_{(\mu)}$$

is also . We see the first term . However , by

$$\sum_i (\bar{Y}^{\psi}_i t_\psi) W^{\psi}_i = 0 \quad ,$$

we can neglect

$$\sum_{\ell,i} (b(\psi)^{-2}\gamma_\psi (Y^{\psi}_i t_\psi) W^{\psi}_\ell D^{\psi *}_b u \ , \ (\Box^{\psi}_b W^{\psi}_\ell u\ )(W^{\psi}_i)\ )_{(\mu)} \quad .$$

Hence

$$\sum_\ell ([\Box^{\psi}_b, W^{\psi}_\ell] u \ , \Box^{\psi}_b W^{\psi}_\ell u\ )_{(\mu)}$$

is estimated by

$$(K/\delta)\|\Box^{\psi}_b u\|'_{(\mu+1),U_r(\psi)} + \delta\| u\|''_{(\mu+1),U_r(\psi)}$$

$$+ \ \varepsilon \| u\|''_{(\mu+1),U_r(\psi)}$$

$$+ \ (K'/\varepsilon)\Big\{\|\Theta^{(1),k}_{0,\mu+1}(\psi) W^{\psi}_k u\|'_{(\mu+1),U_r(\psi)}$$

$$+ \ \|\Theta^{(2),k}_{0,\mu+1}(\psi)\bar{W}^{\psi}_k u\|'_{(\mu+1),U_r(\psi)}$$

$$+ \ \|\Theta^{(3),k}_{0,\mu+1}(\psi)(1/b(\psi))u\|'_{(\mu+1),U_r(\psi)}\Big\}$$

of ψ , u , ε , δ .

For the term

$$\Sigma_{\ell} \left(W_{\ell}^{\psi} \Box_b^{\psi} u \, , \, \Box_b^{\psi} W_{\ell}^{\psi} u \right)_{(\mu)} \, ,$$

by considering

$$\left| \Sigma_{\ell} \left(W_{\ell}^{\psi} \Box_b^{\psi} u \, , \, \Box_b^{\psi} W_{\ell}^{\psi} u \right)_{(\mu)} \right|$$

$$\leqq \Sigma_{\ell} \left(\varepsilon \| \Box_b^{\psi} W u \|^{'2}_{(\mu), U_r(\psi)} + (K/\varepsilon) \| \Box_b^{\psi} u \|^{'2}_{(\mu+1), U_r(\psi)} \right) \, ,$$

we can neglect this . So we have (IV) . The proof for (V) is the same as for (IV) . So we omit this . Hence we have $(\text{Theorem})_{\mu}$. So it completes the proof . Q.E.D.

In this chapter , we will recall D_b^f-complex and s' an apriori estimate for this complex .

4.1. D_b^f-complex with respect to t_f

In Section 1.6 , we showed that if a C^∞-embedding f sa

$$\mathrm{Max}_i \left\{ (1/b(\psi))\left| Y_i^\psi(f_n-\psi_n)\right| , (1/b(\psi))\left| \bar{Y}_i^\psi(f_n-\psi_n)\right| \right\} \leqq C_\psi \text{ on } U_r(\psi) ,$$

$$f(p_0) = \psi(p_0) = 0 ,$$

and

$$\mathrm{Max}_{i,j} \left\{ \left| Y_j^\psi(f_i-\psi_i)\right| , \left| \bar{Y}_j^\psi(f_i-\psi_i)\right| , \left| S(f_i-\psi_i)\right| \right\} \leqq 1 \text{ on } U_r(\psi)-C ,$$

we can introduce D_b^f-complex . For this complex , we show an apriori estimate like the case D_b^ψ-complex . For this we recall D_b^f-complex .

For u in $\Gamma(U_r(f)-C,1)$, we set $D_b^f u$ in $\Gamma(U_r(f)-C,({}^fT_b'')^*)$ by

$$D_b^f u(W_i^f) = W_i^f u ,$$

(for the definition of W_i^f , see (1.6.5)) .

Then like the case for usual diferetial forms , we have

$$0 \to \Gamma(U_r(f)-C,1) \xrightarrow{D_b^f} \Gamma(U_r(f)-C,({}^fT_b'')^*) \xrightarrow{D_b^f} \text{"}$$

$$D_b^f v(W_i^f, W_j^f) = W_i^f v(W_j^f) - W_j^f v(W_i^f) - v([W_i^f, W_j^f]) \quad ,$$

and for $v \in \Gamma(U_r(f)-C, 1)$,

$$D_b^f v(W_i^f) = W_i^f v \quad .$$

Next we show that there are C^∞-functions $a_{\ell,(i,j)}(f), b_{\ell,(i,j)}(f)$ on $U_r(f)-C$ satisfying

$$[W_i^f, \bar{W}_j^f] = -\sqrt{-1}(\delta_{ij} - ((Y_i^f t_f)(\bar{Y}_j^f t_f)/b(f)^2))b(f)^{-1}X^f$$

$$+ \sum_\ell a_{\ell,(i,j)}(f) W_\ell^f + \sum_\ell b_{\ell,(i,j)}(f) \bar{W}_\ell^f \ .$$

Proposition 4.1. There are C^∞-functions $a_{\ell,(i,j)}(f)$, $b_{\ell,(i,j)}(f)$ on $U_r(f)-C$ satisfying ;

$$[W_i^f, \bar{W}_j^f] = -\sqrt{-1}(\delta_{ij} - ((Y_i^f t_f)(\bar{Y}_j^f t_f)/b(f)^2)b(f)^{-1}X^f$$

$$+ \sum_\ell a_{\ell,(i,j)}(f) W_\ell^f + \sum_\ell b_{\ell,(i,j)}(f) \bar{W}_\ell^f \quad ,$$

$$\sum_\ell (Y_\ell^f t_f) a_{\ell,(i,j)}(f) = 0 \quad , \sum_\ell (\bar{Y}_\ell^f t_f) b_{\ell,(i,j)}(f) = 0 \quad ,$$

where W_i^f is defined in (1.6.5) and $a_{\ell,(i,j)}(f)$, $b_{\ell,(i,j)}(f)$ depend on $j^{(1)}(f)$ real analytically , and especially

$$a_{\ell,(i,j)}(\psi) = a_{\ell,(i,j)} \quad \text{and} \quad a_{\ell,(i,j)}(\psi) = b_{\ell,(i,j)} \quad ,$$

where $a_{\ell,(i,j)}$ and $b_{\ell,(i,j)}$ are introduced in Chapter 2.2 .

$$\{ X' \; ; \; X' \in CTM \, , \; X'(h \circ f) = 0 \} \; .$$

Then , obviously ,

$$\dim_C CTM(f) = 2n-3$$

and

$$W_i^f \; , \; i=1,2,\ldots,n \quad \text{and} \quad X^f \quad \text{are of} \quad CTM(f) \; .$$

Furthermore $\bar{W}_i^f$, $i=1,2,\ldots,n$ are of $CTM(f)$. In fact ,

$$\bar{W}_i^f(t \circ f) = \bar{W}_i^f((1/2)(h \circ f + \overline{h \circ} f)$$

$$= (1/2) \bar{W}_i^f(h \circ f) \; (\text{because of} \; W_i^f f_\alpha = 0 \; , \; f=(f_1,\ldots,f_\alpha,\ldots,f_r$$

Hence by the definition of W_i^f ,

$$\bar{W}_i^f(t \circ f) = 0 \; .$$

So

$$\bar{W}_i^f(h \circ f) = 0 \; .$$

Hence $CTM(f)$ is generated by

$$\{ W_i^f \, , \, \bar{W}_i^f \, , \, X^f \} \; ,$$

because the dimension of the space generated by W_i , $\bar{W}_i$, X^- is 2n-3 . Hence

$$[W_i^f, \bar{W}_j^f]$$

is of CTM(f) .

Hence there are C^∞-functions $a_{\ell,(i,j)}(f)$, $b_{\ell,(i,j)}(f)$, c_{ij} on $U_r(f)-C$ satisfying

$$[W_i^f, \bar{W}_j^f] = c_{ij}(\sqrt{-1}\, b(f)S + \bar{\gamma}_f Y^f - \gamma_f \bar{Y}^f)$$

$$+ \sum_\ell a_{\ell,(i,j)}(f) W_\ell^f + \sum_\ell b_{\ell,(i,j)}(f) \bar{W}_\ell^f .$$

By comparing S-term with respect to the C^∞-vector bundle decomposition

$$CTM = {}^fT'' + {}^f\bar{T}'' + S ,$$

we have

$$c_{ij} = -\sqrt{-1}(\delta_{ij} - ((Y_i^f t_f)(\bar{Y}_j^f t_f)/b(f)^2)) .$$

We see $a_{\ell,(i,j)}(f)$, $b_{\ell,(i,j)}(f)$. However , the proof in (2.2.2) is valid to our case , to determine $a_{\ell,(i,j)}(f)$, $b_{\ell,(i,j)}(f)$. So we have our proposition . Q.E.D.

In this section, we see the a prioiri estimate for D_b^f-complex.

In order to this, we show

Proposition 4.2.1.

4.2.1) $$[W_i^f, W_j^f] = b(f)^{-1}\gamma_f(Y_i^f t_f/b(f))W_j^f - b(f)^{-1}\gamma_f(Y_j^f t_f/b(f))W_i^f + \sum_j \Theta_{0,j}^{(4)}(f)W_j^f$$

4.2.2) $$[W_i^f, \bar{W}_j^f] = b(f)^{-1}(\delta_{ij} - ((Y_i^f t_f)(\bar{Y}_j^f t_f)/b(f)^2)X^f + \sum_j \Theta_{0,j}^{(5)}(f)W_j^f + \sum_j \Theta_{0,j}^{(6)}(f)\bar{W}_j^f$$

4.2.3) $$[W_i^f, Y^f - \bar{Y}^f] = b(f)^{-1}\gamma_f W_i^f + \Theta_0(f)(Y^f - \bar{Y}^f) + \sum_j \Theta_{0,j}^{(7)}(f)W_j^f + \sum_j \Theta_{0,j}^{(8)}(f)W_j^f$$

where

$$W_i^f = Y_i^f - (Y_i^f t_f/b(f))\sum_{\ell=1}^{n-1}(\bar{Y}_\ell^f t_f/b(f))Y_\ell^f,$$

$$\gamma_f = \sqrt{-1}\, S(h\cdot f), \quad b(f) = \sqrt{\sum_{\ell=1}^{n-1} |Y^f t_f|^2},$$

$$X^f = \sqrt{-1}\, b(f)S + \bar{\gamma}_f Y^f - \gamma_f \bar{Y}^f.$$

The proof is the same as for D_b^ψ-complex. So we omit this.

And we have

Lemma 4.2.2.

$$W_i^{f*} = -\bar{W}_i^f + (\bar{Y}_i^f t_f/b(f)^2)\gamma_f + \Theta_0(f)$$

The proof is the same for D_b-complex . So we omit this .

With these , by the same method in the proof of Theorem 2.2.4 , we have

Theorem 4.2.3. Under the assumption for f in the beginning of Chapter 4 , the following inequality holds .

$$\|D_b^f u\|^2_{U_r(f)} + \|D_b^{f*}u\|^2_{U_r(f)} + \varepsilon\|u\|'^2_{U_r(f)} + (K/\varepsilon)\|\Theta_0^{(1)}(f)u\|^2_{U_r(f)}$$

$$\geq \sum_{i,j} ((n-3)/(n-2))\|W_j^f u_i\|^2_{U_r(f)} + \sum_{i,j} (1/(n-2))\|\bar{W}_j^f u_i\|^2_{U_r(f)}$$

$$+ (n-3)\sum_i \|(\gamma_f/b(f))u_i\|^2_{U_r(f)} \quad , \text{ for all } \varepsilon > 0 \quad ,$$

for all $u \in \Gamma(U_r(f)-C,({}^fT_b'')^*)$ satisfying

1) $D_b^f u$, $D_b^{f*}u$ are of L^2 ,

2) W_i^f , $i=1,2,\ldots,n-1$ are of L^2 and $(1/b(f))u$ is of L^2 ,

where K is a constant which doesn't depend on ψ, ε, u and $\Theta_0^{(1)}(f)$ is an element of $\Theta_0(f)$, and

$$\|u\|'^2_{U_r(f)} = \sum_{i,j} \|W_j^f u_i\|^2_{U_r(f)} + \sum_{i,j}\|\bar{W}_j^f u_i\|^2_{U_r(f)}$$

$$+ \|(\gamma_f/b(f))u\|^2_{U_r(f)} \quad .$$

Chapter 5 . Some estimates for $\Box_b$

In Chapter 4 , we showed the existence of L^2-solution for $\Box_b^f$-operator . Here we prove some estimates for this solution in terms of $\| \ \|'_{(s),U_r(f)}$ - norm . For u in $\Gamma(U_r(f)-C,({}^f T_b'')^*)$,

$$\| u \|'_{(s),U_r(f)} = \sum_{k \leq s} \| L_{i_1} L_{i_2} \cdots L_{i_k} u \|_{U_r(f)} \quad ,$$

where $L_i = W_j^f$, $\bar{W}_j^f$, Y^f , $\bar{Y}^f$, $1/b(f)$ and X^f and $\| \ \|$ means the L^2-norm on $U_r(f)-C$. Then , obviously our norm is equivalent to

$$\| \tilde{u} \|'_{(s),U_r(f)} = \sum_{k \leq s} \| P_{i_1} P_{i_2} \cdots P_{i_k} u \|_{U_r(f)}$$

where $P_i = Y_j^f$, $\bar{Y}_j^f$, $b(f)S$ and $1/b(f)$.

Our main theorem of this section is

Main theorem . For any integer ℓ , and for any embeddding satisfying

$$b(\psi)^{-1} j_{\psi}^{(1)}(f-\psi) < c_{\psi}(o) \quad \text{on} \quad U_{r_o}(\psi)-C$$

and

$$U_r(\zeta) \subset U_{r_o}(\psi) \quad ,$$

where $j_{\psi}^{(1)}(f-\psi)$ means

$$\mathrm{Max} \, | P_i(f-\psi) | \quad ,$$

where $P_i = Y_j^{\psi}$, $\bar{Y}_j^{\psi}$, $b(\psi)S$, $(1/b(\psi))$.

$$(K_\ell/\delta)\|\square_b^f u\|_{(\dot{\ell}),U_r(f)} + \delta\|u\|_{(\ddot{\ell}),U_r(f)}$$

$$+ (K'_\ell/\varepsilon)\Big\{\|\Theta_0^{(1),k}(f)W_k^f u\|_{(\dot{\ell}),U_r(f)} + \|\Theta_0^{(2),k}(f)\bar{W}_k^f u\|_{(\dot{\ell}),U_r(f)}$$

$$+ \|\Theta_0^{(3),k}(f)(1/b(f))u\|_{(\dot{\ell}),U_r(f)}\Big\} \geq$$

$c_\ell \|u\|_{(\ddot{\ell}),U_r(f)}$ for all $u \in \Gamma(U_r(f)-C,({}^fT_b'')^*)$

satisfying ;

$$L_\alpha L_\beta L_{i_1} L_{i_2} \cdots\cdots L_{i_k} u \text{ is of } L^2 , \quad 0 \leq k \leq 2\ell+2$$

For the proof , we show some commutator relations .

5.1. Commutator relations,I .

Proposition 5.1.1.

5.1.1) $$[W_j^f, X^f] = \Theta_0^{(1)}(f)X^f + |\gamma_f|^2 b(f)^{-1} W_j^f + \sum_j \Theta_{0,j}^{(2)}(f)W_j^f + \sum_j \Theta_{0,j}^{(3)}(f)\bar{W}_j^f$$

5.1.2) $$[W_i^f, W_j^f] = b(f)^{-2}\gamma_f(Y_i^f t_f)W_j^f - b(f)^{-2}\gamma_f(Y_j^f t_f)W_i^f + \sum_j \Theta_{0,j}^{(4)}(f)W_j^f$$

5.1.3) $$[W_i^f, \bar{W}_j^f] = b(f)^{-1}\gamma_f(\delta_{ij} - ((Y_i^f t_f)(\bar{Y}_j^f t_f)/b(f)^2))X^f + \sum_j \Theta_{0,j}^{(5)}(f)W_j^f + \sum_j \Theta_{0,j}^{(6)}(f)\bar{W}_j^f$$

5.1.4) $[W_j^f, Y^f] = b(f)^{-1}\gamma_f W_j^f + \sum_j \Theta_{0,j}^{(7)}(f) W_j^f + (W_j^f b(f)/b(f)) Y^f$

5.1.5) $[W_j^f, \bar{Y}^f] = \sum_j \Theta_{0,j}^{(8)}(f) W_j^f + \sum_j \Theta_{0,j}^{(9)}(f) \bar{W}_j^f + (W_j^f b(f)/b(f)) \bar{Y}$

5.1.6) $W_j^f b(f)/b(f)^2$ and $\bar{W}_j^f b(f)/b(f)^2$ are

of $\Theta_1(f)$.

And

5.1.7) $[D_b^f, (1/b(f))]$, $[D_b^{f*}, (1/b(f))]$ are of $\Theta_1(f)$.

From now on we use the notation

$$j_\psi^\ell(f-\psi) \quad .$$

This means

$$\sup_{U_r(\psi)} \max_{k \leq \ell} | P_{i_1} P_{i_2} \ldots . P_{i_k} (f-\psi) | \quad ,$$

where

$$P_i = Y_j \ , \ \bar{Y}_j \ , \ b(\psi)S \ , \ (1/b(\psi)) \quad .$$

Similarly , with Proposition 5.1.1 we have

Proposition 5.2.1. For $u \in \Gamma(U_r(f)-C,({}^fT_b'')^*)$,

5.2.1) $$[D_b^f, X^f]u = |\gamma_f|^2 b(f)^{-1} D_b^f u + \Theta_0^{(1)}(f) X^f u$$
$$+ \sum_j \Theta_{0,j}^{(2)}(f) W_j^f u + \sum_j \Theta_{0,j}^{(3)}(f) \bar{W}_j^f u + \Theta_1^{(4)}(f) u$$

5.2.2) $$[D_b^{f*}, X^f]u = |\gamma_f|^2 b(f)^{-1} D_b^{f*} u + \Theta_0^{(5)}(f) X^f u$$
$$+ \sum_j \Theta_{0,j}^{(6)}(f) W_j^f u + \sum_j \Theta_{0,j}^{(7)}(f) \bar{W}_j^f u + \Theta_1^{(8)}(f) u$$

5.2.3) $$[D_b^f, W_k^f]u(W_i^f, W_j^f) = - b(f)^{-2} \gamma_f (Y_k^f t_f) D_b^f u(W_i^f, W_j^f)$$
$$+ b(f)^{-2} \gamma_f (Y_i^f t_f) W_k^f u_j$$
$$- b(f)^{-2} \gamma_f (Y_j^f t_f) W_k^f u_i$$
$$+ \sum_j \Theta_{0,j}^{(9)}(f) W_j^f u + \sum_j \Theta_{0,j}^{(10)}(f) \bar{W}_j^f u$$
$$+ \Theta_1^{(11)}(f) u \quad , \text{ where } \quad u_j = u(W_j^f) \quad ,$$

5.2.4) $$[D_b^f, \bar{W}_k^f]u(W_i^f, W_j^f) = b(f)^{-1} \gamma_f (\delta_{ik} - ((Y_i^f t_f)(\bar{Y}_k^f t_f)/b(f)^2)) X^f u_j$$
$$- b(f) \quad \gamma_f (\delta_{jk} - ((Y_j^f t_f)(\bar{Y}_k^f t_f)/b(f)^2)) X^f u_i$$
$$+ \sum_j \Theta_{0,j}^{(12)}(f) W_j^f u + \sum_j \Theta_{0,j}^{(13)}(f) \bar{W}_j^f u$$
$$+ \Theta_1^{(14)}(f) u \quad ,$$

5.2.3)' $[D_b^{f}*, W_k]u = -\sum_\ell (f) \ldots \gamma_f \ldots_{\ell k} - ((Y_\ell t_f)(Y_k t_f)) \ldots_\ell$

$$+ \sum_j \Theta_{0,j}^{(15)}(f) W_j^f u + \sum_j \Theta_{0,j}^{(16)}(f) \bar{W}_j^f u$$

$$+ \Theta_1^{(17)}(f) u$$

5.2.4)' $$[D_b^f*, \bar{W}_k^f]u = -b(f)^{-2} \bar{\gamma}_f(\bar{Y}_i^f t_f) D_b^f * u + b(f)^{-2} \bar{\gamma}_f(\bar{Y}_i^f t_f) W_k^f u_j$$

$$- b(f)^{-2} \bar{\gamma}_f(Y_j^f t_f) \bar{W}_k^f u_i + \sum_j \Theta_{0,j}^{(18)}(f) W_j^f u$$

$$+ \sum_j \Theta_{0,j}^{(19)}(f) \bar{W}_j^f u + \Theta_1^{(20)}(f) u$$

5.2.5) $$[D_b^f, Y^f]u = b(f)^{-1} \gamma_f D_b^f u + \sum_j \Theta_{0,j}^{(21)}(f) W_j^f u +$$

$$+ \Theta_0^{(22)}(f) Y^f u + \Theta_1^{(23)}(f) u$$

5.2.6) $$[D_b^f, \bar{Y}^f]u = \sum_j \Theta_{0,j}^{(24)}(f) W_j^f u + \sum_j \Theta_{0,j}^{(25)}(f) \bar{W}_j^f u$$

$$+ \Theta_0^{(26)}(f) Y^f u + \Theta_1^{(27)}(f) u$$

5.2.5)' $$[D_b^f*, Y^f]u = b(f)^{-1} \gamma_f D_b^f * u + \sum_j \Theta_{0,j}^{(28)}(f) W_j^f u$$

$$+ \Theta_0^{(29)}(f) Y^f u + \Theta_1^{(30)}(f) u$$

5.2.6)' $$[D_b^f*, \bar{Y}^f]u = \sum_j \Theta_{0,j}^{(31)}(f) W_j^f u + \sum_j \Theta_{0,j}^{(32)}(f) \bar{W}_j^f u$$

$$+ \Theta_0^{(33)}(f) Y^f u + \Theta_1^{(34)}(f) u$$

5.1.7) $\left[D_b^f, (1/b(f))\right]$, $\left[D_b^{f*}, (1/b(f))\right]$ are of $\Theta_1(f)$.

In this section we prove the $\Box_b^f$-estimate for the case.

Namely we have

Theorem 5.3.1. For any embedding f satisfying

$$b(\psi)^{-1}|j_{\psi}^{(1)}(f-\psi)| \leqq C_1(\psi) \quad \text{on} \quad U_{r_o}(\psi)-C ,$$

and

$$U_r(f) \subset U_{r_o}(\psi) \quad ,$$

the following inequality holds. There are elements of $\Theta_0(f)$, $\Theta_0^{(1),k}(f)$, $\Theta_0^{(2),k}(f)$, $\Theta_0^{(3),k}(f)$ satisfying ; there are constants C_0, K_0, K_0' satisfying ; for any $\varepsilon, \delta > 0$,

$$(K_0/\delta)\|\Box_b^f u\|_{(\dot{0}),U_r(f)} + \delta \| u \|_{(\ddot{0}),U_r(f)}$$

$$+ \varepsilon \| u \|_{(\ddot{0}),U_r(f)} + (K_0'/\varepsilon)\Big\{\|\Theta_0^{(1),k}(f)W_k^f u\|_{(\dot{0}),U_r(f)}$$

$$+ \|\Theta_0^{(2),k}(f)\bar{W}_k^f u\|_{(\dot{0}),U_r(f)} + \|\Theta_0^{(3)}(f)(1/b(f))u\|_{(\dot{0}),U_r(f)}\Big\}$$

$$\geqq C_0\| u \|_{(\ddot{0}),U_r(f)} \quad , \quad \text{for} \quad u \in \Gamma(U_r(f)-C,({}^f T_b'')^*)$$

satisfying $L_\alpha L_\beta L_{i_1} L_{i_2} u$ is of L^2, where

$$
\begin{aligned}
\| u \|^{*}_{(0),U_r(f)} = {} & \| (\gamma_f/b(f)^2)u \|_{(0),U_r(f)} \\
& + \sum_k \| (1/b(f)) W_k^f u \|_{(0),U_r(f)} \\
& + \sum_k \| (1/b(f)) \bar{W}_k^f u \|_{(0),U_r(f)} \\
& + \sum_{i,j} \Big\{ \| W_i^f W_j^f u \|_{(0),U_r(f)} + \| W_i^f \bar{W}_j^f u \|_{(0),U_r(f)} \\
& + \| \bar{W}_i^f W_j^f u \|_{(0),U_r(f)} + \| \bar{W}_i^f \bar{W}_j^f u \|_{(0),U_r(f)} \Big\}
\end{aligned}
$$

and K_0 , C_0 do not depend on ε , f and $L_i = W_j^f$, $\bar{W}_j^f$, Y^f , $\bar{Y}^f$, X^f and 0-th order operator $1/b(f)$.

In this section we see the $\square^f_b$-estimate for the case ℓ

Namely we have

Theorem 5.4.1. For any embedding f satisfying

$$b(\psi)^{-1}|j^{(1)}_\psi(f-\psi)| \leqq c_1(\psi) \quad \text{on} \quad U_{r_o}(\psi)-C$$

and

$$U_r(f) \subset U_{r_o}(\psi) \quad ,$$

the following inequality holds . There are elements of $\Theta_0(f)$, $\Theta_0^{(1),k}(f)$, $\Theta_0^{(2),k}(f)$, $\Theta_0^{(3),k}(f)$ satisfying ; there are constants C_0 , K_0 , K'_0 satisfying ; for any $\varepsilon , \delta > 0$,

$$\begin{aligned}
&(K_\ell/\delta)\| \square^f_b u \|_{(\dot{\ell}),U_r(f)} + \delta \| u \|_{(\ddot{\ell}),U_r(f)} \\
&+ \varepsilon \| u \|_{(\ddot{\ell}),U_r(f)} + (K'_\ell/\varepsilon)\Big\{ \| \Theta_0^{(1),k}(f) W^f_k u \|_{(\dot{\ell}),U_r(f)} \\
&+ \| \Theta_0^{(2),k}(f)\bar{W}^f_k u \|_{(\dot{\ell}),U_r(f)} + \| \Theta_0^{(3)}(f)(1/b(f))u \|_{(\dot{\ell}),U_r(f)} \Big\} \\
&\geqq C_0 \| u \|_{(\ddot{\ell}),U_r(f)} \quad , \quad \text{for} \quad u \in \Gamma(U_r(f)-C,({}^f T''_b)^*)
\end{aligned}$$

satisfying $L_\alpha L_\beta L_{i_1} L_{i_2} \cdots L_{i_\ell +1} u$ is of L^2 , where

$$\| \text{[illegible]} \|_{(\ell),U_r(f)} \; \text{[illegible]} \; \| \text{[illegible]} u \|_{(\ell),U_r(f)}$$

$$+ \sum_k \|(1/b(f)) W_k^f u\|'_{(\ell),U_r(f)}$$

$$+ \sum_k \|(1/b(f)) \bar{W}_k^f u\|'_{(\ell),U_r(f)}$$

$$+ \sum_{i,j} \Big\{ \| W_i^f W_j^f u \|'_{(\ell),U_r(f)} + \| W_i^f \bar{W}_j^f u \|'_{(\ell),U_r(f)}$$

$$+ \| \bar{W}_i^f W_j^f u \|'_{(\ell),U_r(f)} + \| \bar{W}_i^f \bar{W}_j^f u \|'_{(\ell),U_r(f)} \Big\}$$

and K_0 , C_0 do not depend on ε, δ, f and $L_i = W_j^f$, $\bar{W}_j^f$, Y^f , $\bar{Y}^f$, X^f and 0-th order operator $1/b(f)$.

By the similar argument as in Proposition 4.2.4 ,we have

Theorem 5.4.2 . $U_r(f) \subset U_{r_0}(\varphi)$ and if r_0 is sufficient. we have

$$\| \square_b^f u \|'_{(\ell),U_r(f)} + \sup_{U_r(f)} j_{\varphi}^{(1)}(f-\varphi) \| u \|'_{(\ell),U_r(f)}$$

$$+ \sup_{U_r(f)} j_{\varphi}^{(2)}(f-\varphi) \| u \|'_{(\ell-1),U_r(f)}$$

$$+ \cdots\cdots$$

$$\geq \; c_\ell \| u \|''_{(\ell),U_r(f)}$$

where c_ℓ is independent of f .

$$H'_{(s)} = \left\{ u \; ; \; u \in \Gamma(U_r(\psi)-C,1) \; , \; \| u \|'_{(s),U_r(\psi)} < +\infty \right\}$$

and

$$K_{(s)} = \left\{ u \; ; \; u \in H'_{(s)} \; , \; L_{i_1} \cdots L_{i_k} u = 0 \text{ on } bU_r(\psi)-C \right.$$

$$\left. 0 \leqq k \leqq s-1 \; , \text{ where } L_j = Y_i, \bar{Y}_i, \text{ or } S \right\} .$$

On $K_{(s)}$, we consider a differential operator $\Box_b^{\psi}(\delta)$, $\delta > 0$, defined by

$$\Box_b^{\psi}(\delta) = \sum_{i=1}^{n-1} (W_i^{\psi *} W_i^{\psi} + \bar{W}_i^{\psi *} \bar{W}_i^{\psi})$$

$$+ \; ((n-2)/b(\psi)^2) \, |\gamma_{\psi}|^2 + \delta (Y^{\psi *} Y^{\psi} + \bar{Y}^{\psi *} \bar{Y}^{\psi} + X^{\psi *} X^{\psi}) .$$

Then we have

Proposition 6.1. There is a constant $\delta_s > 0$, satisfying ; for $0 \leqq t \leqq s/2$ and for $0 < \delta < \delta_s$,

$$c_1 \| u \|'_{(2t),U_r(\psi)} \leqq \sum_{k, k \leqq t} \| (\Box_b^{\psi}(\delta))^k u \|_{U_r(\psi)} \leqq c_2 \| u \|'_{(2t),U_r(\psi)}$$

for u in $K_{(s)}$.

Proof . We show this proposition by induction . First , we see the case s=1 . For u in $K_{(2)}$,

$$(n-2)\|(\gamma\psi/b(\psi))u\|^2_{U_r(\psi)} + \sum_{i=1}^{n-1}\{\|W_i^\psi u\|^2_{U_r(\psi)} + \|\bar{W}_i^\psi u\|^2_{U_r(\psi)}\}$$

$$+ \delta\|Y^\psi u\|^2_{U_r(\psi)} + \delta\|\bar{Y}^\psi u\|^2_{U_r(\psi)} + \delta\|X^\psi u\|^2_{U_r(\psi)}$$

$$= (\square_b^\psi(\delta)u, u)$$

So we have

$$(6.1)\quad (n-2)\|(\gamma\psi/b(\psi))u\|^2_{U_r(\psi)} + \sum_{i=1}^{n-1}\{\|W_i^\psi u\|^2_{U_r(\psi)} + \|\bar{W}_i^\psi u\|^2_{U_r(\psi)}\}$$

$$+ \delta\|Y^\psi u\|^2_{U_r(\psi)} + \delta\|\bar{Y}^\psi u\|^2_{U_r(\psi)} + \delta\|X^\psi u\|^2_{U_r(\psi)}$$

$$\leqq \|\square_b^\psi(\delta)u\|^2_{U_r(\psi)} + \|u\|^2_{U_r(\psi)} \quad , \text{ for } u \text{ in } K_{(2)} .$$

Hence if r is chosen sufficiently small ,

$$(6.2)\quad \|(\gamma\psi/b(\psi))u\|^2_{U_r(\psi)} + \sum_{i=1}^{n-1}\{\|W_i^\psi u\|^2_{U_r(\psi)} + \|\bar{W}_i^\psi u\|^2_{U_r(\psi)}\}$$

$$+ \delta\|Y^\psi u\|^2_{U_r(\psi)} + \delta\|\bar{Y}^\psi u\|^2_{U_r(\psi)} + \delta\|X^\psi u\|^2_{U_r(\psi)}$$

$$\leqq c\|\square_b^\psi(\delta)u\|^2_{U_r(\psi)} \quad , \text{ for } u \text{ in } K_{(2)} .$$

Then ,

$$(n-2)\,\|(\gamma\psi/b(\psi)^2)v\|^2_{U_r(\psi)} + \sum_{i=1}^{n-1}\Big\{\|W_i^\psi((1/b(\psi))v)\|^2_{U_r(\psi)}$$

$$+ \|\bar{W}_i^\psi((1/b(\psi))v)\|^2_{U_r(\psi)}\Big\} + \delta\|Y^\psi((1/b(\psi))v)\|^2_{U_r(\psi)}$$

$$+ \delta\|\bar{Y}^\psi((1/b(\psi))v)\|^2_{U_r(\psi)} + \delta\|X^\psi((1/b(\psi))v)\|^2_{U_r(\psi)}$$

$$= (\Box_b^\psi(\delta)((1/b(\psi))v)\ ,\ v\ ).$$

For $W_i^\psi((1/b(\psi))v)$,

$$\|W_i^\psi((1/b(\psi))v)\|_{U_r(\psi)} \geq \|(1/b(\psi))W_i^\psi v\|_{U_r(\psi)}$$

$$- \|(W_i^\psi b(\psi)/b(\psi)^2)v\|_{U_r(\psi)}\ .$$

For $\bar{W}_i^\psi((1/b(\psi))v)$,

$$\|\bar{W}_i^\psi((1/b(\psi))v)\|_{U_r(\psi)} \geq \|(1/b(\psi))\bar{W}_i^\psi v\|_{U_r(\psi)}$$

$$- \|(\bar{W}_i^\psi b(\psi)/b(\psi)^2)v\|_{U_r(\psi)}\ .$$

For $Y^\psi((1/b(\psi))v)$,

$$\|Y^\psi((1/b(\psi))v)\|_{U_r(\psi)} \geq \|(1/b(\psi))Y^\psi v\|_{U_r(\psi)}$$

$$- \|(Y^\psi b(\psi)/b(\psi)^2)v\|_{U_r(\psi)}\ .$$

$$\| \bar{Y}^{\psi}((1/b(\psi))v) \|_{U_r(\psi)} \geqq \| (1/b(\psi))\bar{Y}^{\psi} v \|_{U_r(\psi)}$$

$$- \| (\bar{Y}^{\psi} b(\psi)/b(\psi)^2)v \|_{U_r(\psi)} \quad .$$

For $X^{\psi}((1/b(\psi))v)$,

$$\| X^{\psi}((1/b(\psi))v) \|_{U_r(\psi)} \geqq \| (1/b(\psi))X^{\psi} v \|_{U_r(\psi)}$$

$$- \| (X^{\psi} b(\psi)/b(\psi)^2)v \|_{U_r(\psi)} \quad .$$

And for $\Box_b^{\psi}(\delta)((1/b(\psi))v)$,

$$\| \Box_b^{\psi}(\delta)((1/b(\psi))v) \|_{U_r(\psi)} \geqq \| (1/b(\psi))\Box_b^{\psi}(\delta)v \|_{U_r(\psi)}$$

$$- \| (\Box_b^{\psi}(\delta)(1/b(\psi)))v \|_{U_r(\psi)} \quad .$$

While ,

$W_i^{\psi} b(\psi)$, $\bar{W}_i^{\psi} b(\psi)$ are of $b(\psi)\Theta_0(\psi)$

and

$Y^{\psi} b(\psi) = (\gamma_{\psi}/2) + b(\psi)\Theta_0(\psi)$, $\bar{Y}^{\psi} b(\psi) = (\bar{\gamma}_{\psi}/2) + b(\psi)\Theta_0(\psi)$

$X^{\psi} b(\psi)$ is of $b(\psi)\Theta_0(\psi)$.

$$\Box_b^{\psi}(\delta)(1/b(\psi)) = \Big\{\sum_{i=1}^{n-1}(W_i^{\psi *}W_i^{\psi} + \bar{W}_i^{\psi *}\bar{W}_i^{\psi}) + ((n-2)/b(\psi)^2)|\delta_\psi|^2$$

$$+ \delta\,(Y^{\psi *}Y^{\psi} + \bar{Y}^{\psi *}\bar{Y}^{\psi} + X^{\psi *}X^{\psi})\Big\}(1/b(\psi))$$

$$= \Theta_1(\psi) \quad .$$

Hence

$$(n-2)\,\|(\delta_\psi/b(\psi)^2)u\|^2_{U_r(\psi)} + \sum_{i=1}^{n-1}\Big\{\|(1/b(\psi))W_i^{\psi}u\|^2_{U_r(\psi)}$$

$$+ \|(1/b(\psi))\bar{W}_i^{\psi}u\|^2_{U_r(\psi)}$$

$$- \|\Theta_1(\psi)u\|^2_{U_r(\psi)}\Big\}$$

$$+ \delta\Big\{\|(1/b(\psi))Y^{\psi}u\|^2_{U_r(\psi)} - \|(\delta_\psi/2b(\psi)^2)u\|^2_{U_r(\psi)}$$

$$- \|\Theta_1(\psi)u\|^2_{U_r(\psi)} + \|(1/b(\psi))\bar{Y}^{\psi}u\|^2_{U_r(\psi)}$$

$$- \|(\bar{\delta}_\psi/2b(\psi)^2)u\|^2_{U_r(\psi)} - \|\Theta_1(\psi)u\|^2_{U_r(\psi)}\Big\}$$

$$+ \delta\Big\{\|(1/b(\psi))X^{\psi}u\|^2_{U_r(\psi)} - \|\Theta_1(\psi)u\|^2_{U_r(\psi)}\Big\}$$

$$\leqq \|\Box_b^{\psi}(\delta)u\|^2_{U_r(\psi)} + \|\Theta_1(\psi)u\|^2_{U_r(\psi)} \quad \text{for } u \in K_{(2)}$$

If we choose $\delta > 0$ sufficiently small and choose r sufficiently small , we obtain

$$+\|(1/b(\psi))\bar{W}_i^{\psi}u\|^2_{U_r(\psi)}\}$$

$$+\ \|(1/b(\psi))Y^{\psi}u\|^2_{U_r(\psi)} + \|(1/b(\psi))\bar{Y}^{\psi}u\|^2_{U_r(\psi)}$$

$$+\ \|(1/b(\psi))X^{\psi}u\|^2_{U_r(\psi)} \leqq \quad c_1\|\Box_b^{\psi}(\delta)u\|^2_{U_r(\psi)}$$

for $u \in K_{(2)}$

Let $L = W_i^{\psi}$, $\bar{W}_i^{\psi}$, Y^{ψ}, $\bar{Y}^{\psi}$, X^{ψ}. And we put Lv in the place of u in (6.1). Namely,

$$(6.4)\quad (n-2)\|(\chi_{\psi}/b(\psi))Lv\|^2_{U_r(\psi)} + \sum_{i=1}^{n-1}\{\|W_i^{\psi}Lv\|^2_{U_r(\psi)}$$

$$+\|\bar{W}_i^{\psi}Lv\|^2_{U_r(\psi)}\}$$

$$+\delta\|Y^{\psi}Lv\|^2_{U_r(\psi)} + \delta\|\bar{Y}^{\psi}Lv\|^2_{U_r(\psi)} + \delta\|X^{\psi}Lv\|^2_{U_r(\psi)}$$

$$= (\Box_b^{\psi}(\delta)Lv, Lv), \quad v \in K_{(2)}.$$

Like the case $L = 1/b(\psi)$, we are going to estimate

$\|(1/b(\psi))Lv\|^2_{U_r(\psi)}$, $\|LW_i^{\psi}v\|^2_{U_r(\psi)}$, $\|L\bar{W}_i^{\psi}v\|^2_{U_r(\psi)}$,

$\|LY^{\psi}v\|^2_{U_r(\psi)}$, $\|L\bar{Y}^{\psi}v\|^2_{U_r(\psi)}$ and $\|LX^{\psi}v\|^2_{U_r(\psi)}$ by

$$\|\Box_b^{\psi}(\delta)v\|^2_{U_r(\psi)}.$$

$$L = W_j^\psi .$$

Then , we have

$$(6.5) \quad (n-2) \| (\gamma_\psi / b(\psi)) W_j^\psi v \|^2_{U_r(\psi)} + \sum_{i=1}^{n-1} \Big\{ \| W_j^\psi W_i^\psi v + [W_i^\psi, W_j^\psi] v \|^2_{U_r(\psi)} + \| W_j^\psi \bar{W}_i^\psi v + [\bar{W}_i^\psi, W_j^\psi] v \|^2_{U_r(\psi)} \Big\}$$

$$+ \delta \| W_j^\psi Y^\psi v + [Y^\psi, W_j^\psi] v \|^2_{U_r(\psi)}$$

$$+ \delta \| W_j^\psi \bar{Y}^\psi v + [\bar{Y}^\psi, W_j^\psi] v \|^2_{U_r(\psi)}$$

$$+ \delta \| W_j^\psi X^\psi v + [X^\psi, W_j^\psi] v \|^2_{U_r(\psi)}$$

$$= (W_j^\psi \Box_b^\psi(\delta) v , W_j^\psi v) + ([\Box_b^\psi(\delta), W_j^\psi] v , W_j^\psi v)$$

The right hand side of (6.5) becomes

$$(\Box_b^\psi(\delta) v , W_j^{\psi *} W_j^\psi v) + ([\Box_b^\psi(\delta), W_j^\psi] v , W_j^\psi v) .$$

and this is estimated by

$$\varepsilon \| W_j^{\psi *} W_j^\psi v \|^2_{U_r(\psi)} + (2/\varepsilon) \| \Box_b^\psi(\delta) v \|^2_{U_r(\psi)} .$$

$$[W_i^\psi, W_j^\psi] = (Y_j^\psi t_\psi/b(\psi))W_i^\psi - (Y_i^\psi t_\psi/b(\psi))W_j^\psi + \Sigma_\ell \Theta_0(\psi)W_\ell^\psi$$

$$[\bar{W}_i^\psi, W_j^\psi] = \Omega_{ji}(\psi)X^\psi + \Sigma_\ell \Theta_0(\psi)W_\ell^\psi + \Sigma_\ell \Theta_0(\psi)\bar{W}_\ell^\psi$$

$$[Y^\psi, W_j^\psi] = -(W_j^\psi b(\psi)/b(\psi))Y^\psi - (\gamma_\psi/b(\psi))W_j^\psi + \Sigma_\ell \Theta_0(\psi)W_\ell^\psi$$

$$[\bar{Y}^\psi, W_j^\psi] = -(W_j^\psi b(\psi)/b(\psi))\bar{Y}^\psi + \Sigma_\ell \Theta_0(\psi)W_\ell^\psi + \Sigma_\ell \Theta_0(\psi)\bar{W}_\ell^\psi$$

$$[X^\psi, W_j^\psi] = |\gamma_\psi|^2 b(\psi)^{-1}W_j^\psi + \Sigma_\ell \Theta_0(\psi)W_\ell^\psi + \Sigma_\ell \Theta_0(\psi)\bar{W}_\ell^\psi + \Theta_0(\psi)X^\psi$$

And

$$\begin{aligned}
[\Box_b^\psi(\delta), W_j^\psi] = -\Sigma_{i=1}^{n-1}\{ & (Y_j^\psi t_\psi/b(\psi))(\bar{W}_i^\psi W_i^\psi + W_i^\psi \bar{W}_i^\psi) \\
& + \Omega_{ji}(\psi)X^\psi W_i^\psi + \Omega_{ji}(\psi)W_i^\psi X^\psi \} \\
& + 2\delta(W_j^\psi b(\psi)/b(\psi))\bar{Y}^\psi Y^\psi + 2\delta(W_j^\psi b(\psi)/b(\psi))Y^\psi \bar{Y}^\psi \\
& + \delta(\gamma_\psi/b(\psi))\bar{Y}^\psi W_j^\psi + \delta(\gamma_\psi/b(\psi))W_j^\psi \bar{Y}^\psi \\
& + \delta |\gamma_\psi|^2 b(\psi)^{-1} W_j^\psi X^\psi \\
& + \delta |\gamma_\psi|^2 b(\psi)^{-1} X^\psi W_j^\psi \\
& + \text{a 1-st order differential operator} \\
& + \text{a 0-th order differential operator}
\end{aligned}$$

So we have

(6.6)

$$
\begin{aligned}
(n-2)\|(\gamma_\psi/b(\psi))W_j^\psi v\|^2_{U_r(\psi)} &+ (1/2)\sum_{i=1}^{n-1}\|W_j^\psi W_i^\psi v\|^2_{U_r(\psi)} \\
&- 3\sum_{i=1}^{n-1}\|[W_i^\psi, W_j^\psi]v\|^2_{U_r(\psi)} \\
&+ (1/2)\sum_{j=1}^{n-1}\|W_j^\psi \bar{W}_i^\psi v\|^2_{U_r(\psi)} \\
&- 3\sum_{i=1}^{n-1}\|[\bar{W}_i^\psi, W_j^\psi]v\|^2_{U_r(\psi)} \\
&+ (\delta/2)\quad \|W_j^\psi Y^\psi v\|^2_{U_r(\psi)} \\
&- 3\delta\quad \|[Y^\psi, W_j^\psi]v\|^2_{U_r(\psi)} \\
&+ (\delta/2)\quad \|W_j^\psi \bar{Y}^\psi v\|^2_{U_r(\psi)} \\
&- 3\delta\quad \|[\bar{Y}^\psi, W_j^\psi]v\|^2_{U_r(\psi)} \\
&+ (\delta/2)\quad \|W_j^\psi X^\psi v\|^2_{U_r(\psi)} \\
&- 3\delta\quad \|[X^\psi, W_j^\psi]v\|^2_{U_r(\psi)}
\end{aligned}
$$

$$
\begin{aligned}
\leq (n- \ )\|(\gamma_\psi/b(\psi))W_j^\psi v\|^2_{U_r(\psi)} &+ \sum_{i=1}^{n-1}\Big\{\|W_j^\psi W_i^\psi v + [W_i^\psi, W_j^\psi]v\|^2_{U_r(\psi)} \\
&+ \|W_j^\psi \bar{W}_i^\psi v + [\bar{W}_i^\psi, W_j^\psi]v\|^2_{U_r(\psi)}\Big\}
\end{aligned}
$$

$$
+ \delta\|W_j^\psi Y^\psi v + [Y^\psi, W_j^\psi]v\|^2_{U_r(\psi)} + \delta\|W_j^\psi \bar{Y}^\psi v + [\bar{Y}^\psi, W_j^\psi]v\|^2_{U_r(\psi)}
$$

$$
+ \delta\|W_j^\psi X^\psi v + [X^\psi, W_j^\psi]v\|^2_{U_r(\psi)}
$$

By commutator relations with (6.2) and (6.5), we have

$$(n-2)\,\|(\gamma_\psi/b(\psi))W_j^\psi v\|^2_{U_r(\psi)} + (1/2)\sum_{i=1}^{n-1}\|W_j^\psi W_i^\psi v\|^2_{U_r(\psi)}$$

$$+ (1/2)\sum_{i=1}^{n-1}\|W_j^\psi \bar{W}_i^\psi v\|^2_{U_r(\psi)}$$

$$+ (\delta/2)\|W_j^\psi Y^\psi v\|^2_{U_r(\psi)} + (\delta/2)\|W_j^\psi \bar{Y}^\psi v\|^2_{U_r(\psi)}$$

$$+ (\delta/2)\|W_j^\psi X^\psi v\|^2_{U_r(\psi)}$$

$$\leq (c + (2/\varepsilon))\|\Box_b^\psi(\delta)v\|^2_{U_r(\psi)}$$

$$+ \varepsilon\|W_j^\psi W_j^\psi v\|^2_{U_r(\psi)}$$

The cases $L = \bar{W}_j^\psi$, Y^ψ, $\bar{Y}^\psi$, X^ψ are similar. Namely we have

$$(6.7)\quad \|(1/b(\psi))Lv\|^2_{U_r(\psi)} + \sum_{i=1}^{n-1}\left\{\|LW_i^\psi v\|^2_{U_r(\psi)} + \|L\bar{W}_i^\psi v\|^2_{U_r(\psi)}\right\}$$

$$+ \|LY^\psi v\|^2_{U_r(\psi)} + \|L\bar{Y}^\psi v\|^2_{U_r(\psi)} + \|LX^\psi v\|^2_{U_r(\psi)}$$

$$\leq (2\ \ \ )\|\Box_b^\psi(\delta)v\|^2_{U_r(\psi)} + \varepsilon\|L^* Lv\|^2_{U_r(\psi)},$$

where $L = W_j^\psi$, $\bar{W}_j^\psi$, Y^ψ, $\bar{Y}^\psi$, X^ψ.

$$c_1 \| u \|'_{(2), U_r(\psi)} \leqq \| \Box^{\psi}_b(\delta) u \|_{U_r(\psi)} \quad .$$

On the other hand , the estimate

$$\| \Box^{\psi}_b u \|_{U_r(\psi)} \leqq \quad c_2 \| u \|'_{(2), U_r(\psi)}$$

is trivial . So we finish the proof for the case $s = 2$. We assume the estimate up to $2m$. We now show the case $2(m+1)$ Namely we want to prove ; if we choose δ_{m+1} sufficiently small ,

$$c'_{m+1} \| u \|'_{(2m+2), U_r(\psi)} \leqq \sum_{k, k \leqq m+1} \| (\Box^{\psi}_b(\delta))^k u \|_{U_r(\psi)}$$

for $u \in K_{(2m+2)} \quad , \quad 0 < \delta < \delta_{m+1}$

By the assumption for the case m , for $0 < \delta < \delta_m$

$$c'_m \| v \|'_{(2m), U_r(\psi)} \leqq \sum_{k, k \leqq m} \| (\Box^{\psi}_b(\delta))^k u \|_{U_r(\psi)} \quad .$$

We set

$$v = \Box^{\psi}_b(\delta) u \quad .$$

Then

$$c'_m \| \Box^{\psi}_b(\delta) u \|'_{(2m), U_r(\psi)} \leqq \sum_{k, k \leqq m+1} \| (\Box^{\psi}_b(\delta))^k u \|_{U_r(\psi)}$$

So it is sufficient to show

$$\| v \|'_{(2m+2),U_r(\psi)} \leq C \| \Box_b^{\psi}(\delta) v \|'_{(2m),U_r(\psi)}$$

for $v \in K_{(2m+2)}$, $0 < \delta < \delta_{m+1}$, if δ_{m+1} is chosen sufficiently small . However the proof is just a direct computation and the computation is similar as in Chapter 3 . The difference is that our $\Box_b^{\psi}(\delta)$ includes $\delta\, Y^{\psi*}Y^{\psi}$, $\delta\, \bar{Y}^{\psi*}\bar{Y}^{\psi}$, $\delta\, X^{\psi*}X^{\psi}$. And in the computation of the bracket , $[Y^{\psi}, \bar{Y}^{\psi}]$, $\delta\,(\gamma_{\psi}/b(\psi))Y^{\psi}$ appears , but we note that if δ_m is sufficiently small , the appearance of $(\gamma_{\psi}/b(\psi))Y^{\psi}$ doesn't bother us . Hence we have our proposition . Q.E.D.

We set $s = 2\ell$ and consider the eigenvalue expansion with respect to $\Box_b^{\psi}(\delta)$. Let

$$V_\lambda = \{u \; ; \; u \in K_{(2\ell)} \, , \; \Box_b^{\psi}(\delta)u = \lambda u\} \; .$$

Then by the standard argument with Proposition 6.1 , we have

1) $\dim_C V_\lambda < +\infty$

2) $\tilde{K}_{(2\ell)} = \sum_{k=1}^{+\infty} V_{\lambda_k} \quad , \quad 0 < \lambda_1 < \lambda_2 < \;\ldots\ldots < \lambda_k < \;\ldots$,

where $\tilde{K}_{(2\ell)}$ means the competion of $K_{(2\ell)}$ with respect to $\| \; \|'_{(2\ell), U_r(\psi)}$-norm . By using this decomposition , we set

$$M_{\varepsilon_\nu}^{2\ell} u = \sum_{\lambda_k < (1/\varepsilon_\nu)} (u, v_{\lambda_k}^{\alpha}) v_{\lambda_k}^{\alpha} \quad ,$$

where $v_{\lambda_k}^{\alpha}$, $\alpha = 1, 2, \ldots, \lambda_k$ is an orthonormal base of V_k with respect to L^2-nrom ($\alpha_k = \dim_C V_{\lambda_k}$) . Then by Proposition 6.1 ,

1) $\| M_{\varepsilon_\nu}^{2\ell} u \|'_{(i+j), U_r(\psi)} \leqq (1/\varepsilon_\nu^{\,i}) \| u \|'_{(j), U_r(\psi)}$

2) $\| u - M_{\varepsilon_\nu}^{2\ell} u \|'_{(j), U_r(\psi)} \leqq \varepsilon_\nu^{\,i} \| u \|'_{(i+j), U_r(\psi)}$

for $u \in K_{(2\ell)}$, $0 \leqq i$, j , $i+j \leqq 2\ell$ and i , j are even .

embeddings

With devices , prepared in Chapters 4 ~ 6 , we construct a sequence of local embeddings f^ν of $U_{r_\nu}(f^\nu)$ into C^n satisfying

$$|\, b(f^o)^{-1} j_{f^o}^{(\ell+2)}(f^\nu - f^o)| < C\,(f^o) \quad \text{on} \quad U_{r_\nu}(f^\nu) - C$$

where f^o is introduced in Section 1.7 ,

and show that this f^ν converges to an f satisfying

$$D_b f = 0 \quad \text{along} \quad t\circ f \quad .$$

From now on , we assume $\ell \geq 30(2n+3)$, where $\dim_R M = 2n-1$, and fix this ℓ . And for simplicity , we write $(M, {}^\nu T'')$ for the induced CR-structure by f^ν . And D_b^ν , $D_b^{\nu *}$ and N_b^ν means $D_b^{f^\nu}$, $D_b^{f^\nu *}$ and $N_b^{f^\nu}$ respectively . Now we are going to show our construction by induction .

Let ε_ν be a sequence defined by $\varepsilon_0 = r_0^p$, $\varepsilon_{\nu+1} = \varepsilon_\nu^{(3/2)}$ where $p \geq 2\ell$ and define r_ν by

$$r_{\nu+1} = r_\nu - 2\delta_\nu \quad , \quad \delta_\nu = \varepsilon_\nu^{(1/\ell)} \quad .$$

And set

$$U_{r_\nu}(f^\nu) = \{x \;\; ; \;\; x \in M \; , \;\; t\circ f^\nu(x) < r_\nu\}$$

and

$$U_{r_\nu - \delta_\nu}(f^\nu) = \{x \;\; ; \;\; x \in M \; , \;\; t\circ f^\nu(x) < r_\nu - \delta_\nu\} \quad .$$

of $U_{r_\nu}(f^\nu)$ into C^n satisfying

$$0)_\nu \quad U_{r_\nu}(f^\nu) \subset U_{r_{\nu-1}-\delta_{\nu-1}}(f^{\nu-1}) \subset U_{r_{\nu-1}}(f^{\nu-1})$$

$$\cdots\cdots \subset U_{r_o}(f^o) \quad \text{and}$$

$$\sum_{i,j}(\partial^2 t\circ f/\partial x_i \partial x_j)\xi_i \xi_j \geq (\lambda/2)\sum_i \xi_i^2 \qquad \text{on } U_{r_\nu}(f^\nu) ,$$

where $(x_1,\ldots,x_{2n-1})$ is a local coordinate of $U_{r_o}(f^o)$ and $(\xi_1,\ldots,\xi_{2n-1})$ is an element of R^{2n-1} (this implies that $U_{r_\nu}(f^\nu)$ is convex , so we can use the Sobolev lemma) , and λ is introduced in Sect. 1.3 .

$$1)_\nu \quad D_b f^\nu , \quad f^\nu - f^o \in H'_{(2\ell),U_{r_\nu}(f^\nu)} ,$$

$$2)_\nu \quad \| D_b f \|'_{(\ell+j),U_{r_\nu}(f^\nu)} , \ \| f^\nu - f^o \|'_{(\ell+j),U_{r_\nu}(f^\nu)} < \varepsilon_\nu^{-j-3n}$$

$$3)_\nu \quad \sup_{p \in U_{r_\nu}(f^\nu)} | b(f^o)^{-1} j_{f^o}^{(\ell+1)}(f^\nu - f^o)| < c_\ell(f^o)$$

and
$$U_{r_\nu}(f^\nu) \subset U_{r_o}(f^o)$$

$$4)_\nu \quad p_\nu \leqq c_\ell^{\#} (\varepsilon_\nu^{-s} p_{\nu-1}^2 + \varepsilon_\nu^\ell \varepsilon_{\nu-1}^{-(\ell+t)}) , \quad \text{where}$$

s and t are integers satisfying

$$0 < s \leq (1/30)\ell , \qquad 0 < t \leq (1/4)\ell .$$

Here
$$p_\nu = \| D_b f^\nu \|'_{(\ell),U_{r_\nu}(f^\nu)} .$$

1) $\chi(t) \geq 0$, $t \in R$

2) $\chi(s) = 1$, $s \geq 1$, and $\chi(s) = 0$, $s \leq 0$.

By using this , we define a C^∞ function

$$\chi_\nu = \chi((r_\nu - t \circ f_\nu)/\delta_\nu)$$

on M . Then ,

on $U_{r_\nu - \delta_\nu}(f^\nu)$, $\chi_\nu = 1$

outsider of $U_{r_\nu}(f^\nu)$, $\chi_\nu = 0$.

With this function , we construct $f^{\nu+1}$ as follows .

$$f^{\nu+1} = f^\nu - M_{\nu+1}\chi_\nu D_b^{\nu *} N_b^\nu D_b f^\nu$$

where $D_b f^\nu$ means $D_b f^\nu$ along t_{f^ν} (this notion is introduced in Sect.1.6) .

We see

(A) On $U_{r_o}(f^o)$, $f^{\nu+1}$ makes sense as a C^∞ map and defines the C^∞ embedding of $U_{r_\nu - \delta_\nu}(f^\nu)$, and $f^{\nu+1}$ satisfies $0)_{\nu+1}$.

(B) the above $f^{\nu+1}$ satisfies

$$1)_{\nu+1} \quad D_b f^{\nu+1} \ , \ f^{\nu+1} - f^o \in H'_{(2\ell), U_{r_\nu - \delta_\nu}(f^\nu)}$$

$2)_{\nu+1}$ $\quad \ldots \|_{(\ell+j),U_{r_\nu-\delta_\nu}(f^\nu)}$,

$$\|f^{\nu+1}-f^{o}\|_{(\ell+j),U_{r_\nu-\delta_\nu}(f^\nu)} < \varepsilon_{\nu+1}^{-j-3n-4}\ , \quad j=1,2,\ldots$$

$3)_{\nu+1}$ $$\sup_{p\in U_{r_o}(f^o)} \left| b(f^o)^{-1} j^{(\ell+2)}(f^{\nu+1}-f^o)\right| < C_\ell(f^o)$$

$4)_{\nu+1}$ $$U_{r_{\nu+1}}(f^{\nu+1}) \subset U_{r_\nu-\delta_\nu}(f^\nu)$$

$$p_{\nu+1} \leqq C_\ell^{\#}\left(\varepsilon_{\nu+1}^{-s}p_\nu^2 + \varepsilon_{\nu+1}^{\ell}\varepsilon_\nu^{-(\ell+t)}\right)$$

7.1 . The proof of (A)

We see this by induction on ν . Namely we claim that if f^{ν} is defined on $U_{r_o}(f^o)$, then $f^{\nu+1}$ is also . However by the definition of $M_{\nu+1}$ and

$$D_b f^o \in H'_{(2\ell), U_{r_o}(f^o)} ,$$

we have

$$D_b f^{\nu} \in H'_{(2\ell), U_{r_o}(f^o)} .$$

Hence , on $U_{r_o}(f^o)$,

$$f^{\nu+1} = f^{\nu} - M_{\nu+1} \chi_{\nu} D_b^{\nu *} N_b^{\nu} D_b f^{\nu}$$

makes sense . In Sect.7.5 , we will prove that this map defines the C^{∞}-embedding of $U_{r_{\nu+1}}(f^{\nu+1})$.

By the construction of $f^{\nu+1}$,

$$f^{\nu+1} = f^{o} \quad \text{near} \quad C$$

and

$$f^{\nu+1} \text{ is of } C^{\infty} \text{ on } U_{r_o}(f^{o})-C \quad .$$

Therefore by $D_b f^{o} \in H'_{(2\ell),U_{r_\nu}(f^{\nu})}$,

$$D_b f^{\nu+1} \quad , \quad f^{\nu+1} - f^{o} \in H'_{(2\ell),U_{r_\nu}(f^{\nu})} \quad .$$

Hence

$$D_b f^{\nu+1} \quad , \quad f^{\nu+1} - f^{o} \in H'_{(2\ell),U_{r_{\nu+1}}(f^{\nu+1})} \quad .$$

$\ldots, \ell-4n$

(I) The estimate for $\| D_b f^{\nu+1} \|_{(\ell+j), U_{r_\nu} - \delta_\nu (f^\nu)}$

For $\| D_b f^{\nu+1} \|_{(\ell+j), U_{r_\nu} - \delta_\nu (f^\nu)}$, we have

Proposition 7.3.1.

$$\| D_b f^{\nu+1} \|_{(\ell+j), U_{r_\nu} - \delta_\nu (f^\nu)} < \varepsilon_{\nu+1}^{-j-4-2n} \quad , j=1,2,\ldots\ldots,\ell-4n$$

Proof . We show this by induction on ν . By choosing r_o sufficiently small , the case $\nu = 1$ is obvious . We assume

$$\| D_b f^\nu \|_{(\ell+j), U_{r_{\nu-1}} - \delta_{\nu-1} (f^{\nu-1})} < \varepsilon_\nu^{-j-4-2n} \quad , j=1,2,\ldots\ldots,\ell-4n$$

With this , we show the $\nu+1$ case . By the definition of $f^{\nu+1}$,

$$f^{\nu+1} = f^\nu - M_{\nu+1} \chi_\nu D_b^{\nu*} N_b^\nu D_b f^\nu \quad .$$

Hence

$$\begin{aligned} \| D_b f^{\nu+1} \|_{(\ell+j), U_{r_\nu} - \delta_\nu (f^\nu)} &\leq \| D_b f^\nu \|_{(\ell+j), U_{r_\nu} - \delta_\nu (f^\nu)} \\ &\quad + \| M_{\nu+1} \chi_\nu D_b^{\nu*} N_b^\nu D_b^\nu f^\nu \|_{(\ell+j), U_{r_\nu} - \delta_\nu (f^\nu)} \\ &\leq \| D_b f^\nu \|_{(\ell+j), U_{r_{\nu-1}} - \delta_{\nu-1} (f^{\nu-1})} \\ &\quad + \| M_{\nu+1} \chi_\nu D_b^{\nu*} N_b^\nu D_b f^\nu \|_{(\ell+j), U_{r_o} (f^o)} \end{aligned}$$

$$\leqq \quad \varepsilon_{\nu}^{-j-4-2n}$$

$$+ \quad c_{\ell}\, \varepsilon_{\nu+1}^{-j-3-2n} \, \| \varkappa_{\nu} D_{b}^{\nu *} N_{b}^{\nu} D_{b} f^{\nu} \|_{(\ell'-2n), U_{r_o}(f^o)}$$

$$\leqq \quad \varepsilon_{\nu}^{-j-4-2n}$$

$$+ \quad c'_{\ell}\, \varepsilon_{\nu+1}^{-j-3-2n} \Big\{ \| D_{b}^{\nu *} N_{b}^{\nu} D_{b} f^{\nu} \|_{(\ell'-2n), U_{r_\nu}(f^{\nu})}$$

$$+ \; \delta_{\nu}^{-1} \| D_{b}^{\nu *} N_{b}^{\nu} D_{b} f^{\nu} \|_{(\ell'-2n-1), U_{r_\nu}(f^{\nu})}$$

$$+ \ldots .$$

$$+ \; \delta_{\nu}^{-(\ell-2n)} \| D_{b}^{\nu *} N_{b}^{\nu} D_{b} f^{\nu} \|_{(o'), U_{r_\nu}(f^{\nu})}$$

On the other hand ,

Lemma 7.3.2 .

$$\| D_{b}^{\nu *} N_{b}^{\nu} D_{b} f^{\nu} \|_{(\ell'-2n), U_{r_\nu}(f^{\nu})} \leqq \quad c''_{\ell} \| D_{b} f^{\nu} \|_{(\ell'), U_{r_\nu}(f^{\nu})}$$

Proof .

$$\| D_{b}^{\nu *} N_{b}^{\nu} D_{b} f^{\nu} \|_{(\ell'-2n), U_{r_\nu}(f^{\nu})} \leqq \quad c_{\ell} \Big\{ \; \| D_{b} f^{\nu} \|_{(\ell'-2n), U_{r_\nu}(f^{\nu})}$$

$$+ \quad j^{(1)}(f^{\nu} - f^{o}) \, \| D_{b} f^{\nu} \|_{(\ell'-2n-1), U_{r_\nu}(f^{\nu})}$$

$$+ \; \ldots .$$

$$+ \quad j^{(\ell-2n)}(f^{\nu} - f^{o}) \, \| D_{b} f \|_{(o'), U_{r_\nu}(f^{\nu})} \Big\}$$

$$\leqq C_\ell \{ \| D_b f^{\nu} \|_{(\ell-2n), U_{r_\nu}(f^\nu)} + C_\ell \| f^\nu - f^o \|_{(\ell), U_{r_\nu}(f^\nu)} \| D_b f^\nu \|_{(\ell-2n), U_{r_\nu}(f^\nu)}$$

We set a constant C satisfying

$$\| f^\nu - f^o \|_{(\acute{\ell}), U_{r_\nu}(f^\nu)} \leqq C \quad .$$

Then we have our lemma . Q.E.D.

By Lemma 7.3.2. , we have

$$\| D_b f^{\nu+1} \|_{(\acute{\ell}+j), U_{r_\nu - \delta_\nu}(f^\nu)}$$

$$\leqq \varepsilon_\nu^{-j-4-2n} + C'_\ell C''_\ell C \varepsilon_{\nu+1}^{-j-3-2n} \{ (1+ \delta_\nu^{-1} + \cdot\cdot + \delta_\nu^{-(\ell-2n)}) \| D_b f^\nu \|_{(\acute{\ell}), U_{r_\nu}(f^\nu)}$$

$$\leqq \varepsilon_\nu^{-j-4-2n} + \tilde{C}_\ell \varepsilon_{\nu+1}^{-j-3-2n} \varepsilon_\nu^{-1} P_\nu$$

$$\leqq (1/2) \varepsilon_{\nu+1}^{-j-4-2n} + \tilde{C}_\ell \varepsilon_{\nu+1}^{-j-4-2n} \varepsilon_\nu^{(1/10)\ell}$$

So if we choose r_o sufficiently small ,

$$0 < \tilde{C}_\ell \varepsilon_\nu^{(1/10)\ell} = \tilde{C}_\ell r_o < 1/2 \quad .$$

Hence

$$\| D_b f^{\nu+1} \|_{(\acute{\ell}+j), U_{r_\nu - \delta_\nu}(f^\nu)} \leqq \varepsilon_{\nu+1}^{-j-4-2n} \quad .$$

Therefore we have our proposition . Q.E.D.

$(\acute{\ell}+j),U_{r_\nu-\delta_\nu}(f^\nu)$

For $\|f^{\nu+1}-f^{o}\|_{(\acute{\ell}+j),U_{r_\nu-\delta_\nu}(f^\nu)}$, we have

Proposition 7.3.3.

$$\| f^{\nu+1} - f^{o} \|_{(\acute{\ell}+j),U_{r_\nu-\delta_\nu}(f^\nu)} \leqq \varepsilon_{\nu+1}^{-j-2n-3}$$

Proof . We show this by induction on ν . Like Proposition 7.3.1 , the case $\nu = 0$ is trivial . We assume the case ν namely ,

$$\| f^{\nu} - f^{o} \|_{(\acute{\ell}+j),U_{r_{\nu-1}-\delta_{\nu-1}}(f^{\nu-1})} \leqq \varepsilon_{\nu}^{-j-2n-3} .$$

With this , we show the case $\nu+1$. By the definition of $f^{\nu+1}$,

$$f^{\nu+1} = f^{\nu} - M_{\nu+1}\chi_\nu D_b^{\nu *} N_b^{\nu} D_b f^{\nu} .$$

Hence

$$\begin{aligned}
\| f^{\nu+1} - f^{\nu} \|_{(\acute{\ell}+j),U_{r_\nu-\delta_\nu}(f^\nu)} &= \| M_{\nu+1}\chi_\nu D_b^{\nu *} N_b^{\nu} D_b f^{\nu} \|_{(\acute{\ell}+j),U_{r_\nu-\delta_\nu}(f} \\
&\leqq \| M_{\nu+1}\chi_\nu D_b^{\nu *} N_b^{\nu} D_b f^{\nu} \|_{(\acute{\ell}+j),U_{r_o}(f^{o})} \\
&\leqq C_{\ell}\varepsilon_{\nu+1}^{-j-2n} \| \chi_\nu D_b^{\nu *} N_b^{\nu} D_b f^{\nu} \|_{(\acute{\ell}-2n),U_{r_o}(}
\end{aligned}$$

$$\leq C_\ell \varepsilon_{\nu+1} \{ \| D_b^{\nu*} N_b^\nu D_b f^\nu \|_{(\ell-2n), U_{r_\nu}(f^\nu)}$$

$$+ \quad \delta_\nu^{-1} \| D_b^{\nu*} N_b^\nu D_b f^\nu \|'_{(\ell-2n), U_{r_\nu}(f^\nu)}$$

$$+ \dots$$

$$+ \quad \delta^{-(\ell-2n)} \| D_b^{\nu*} N_b^\nu D_b f^\nu \|'_{(o), U_{r_\nu}(f^\nu)}$$

$$\leq C'_\ell \varepsilon_{\nu+1}^{-j-2n-1} \, 2 \, \varepsilon_\nu^{-1} C''_\ell \quad \| D_b f^\nu \|'_{(\ell), U_{r_\nu}(f^\nu)}$$

$\leq \varepsilon_{\nu+1}^{-j-2n-2}$ (by Lemma 7.5.2 and if we choose r_o sufficiently small) .

So we have

$$\| f^{\nu+1} - f^o \|'_{(\ell+j), U_{r_\nu - \delta_\nu}(f^\nu)}$$

$$\leq \| f^{\nu+1} - f^\nu + f^\nu - f^o \|'_{(\ell+j), U_{r_\nu - \delta_\nu}(f^\nu)}$$

$$\leq \| f^{\nu+1} - f^\nu \|'_{(\ell+j), U_{r_\nu - \delta_\nu}(f^\nu)} + \| f^\nu - f^o \|'_{(\ell+j), U_{r_\nu - \delta_\nu}(f^\nu)}$$

$$\leq \varepsilon_{\nu+1}^{-j-2n-2} + \| f^\nu - f^o \|'_{(\ell+j), U_{r_{\nu-1} - \delta_{\nu-1}}(f^{\nu-1})}$$

$$\leq \varepsilon_{\nu+1}^{-j-2n-2} + \varepsilon_\nu^{-j-2n-3} \quad \text{(by induction)}$$

$$\leq (1/2)\, \varepsilon_{\nu+1}^{-j-2n-3} + (1/2)\, \varepsilon_\nu^{-j-2n-3}$$

$$\leq \varepsilon_{\nu+1}^{+j-2n-3}$$

So we have our theorem . Q.E.D.

7.4. $\sup\limits_{p \in U_{r_o}(f^{\nu+1})} | b(f^{\text{[illegible]}})$ [illegible] $(f^{\text{[illegible]}} - f^{\text{[illegible]}}) | < C_{\ell}(f^{\text{[illegible]}})$

For this , it suffices to estimate

$$\| f^{\nu+1} - f^o \|_{(\acute{\ell}+2n+3), U_{r_o}(f^o)} \quad .$$

While

$$f^{\nu+1} - f^o = f^{\nu+1} - f^{\nu} + f^{\nu} - f^{\nu-1} + \ldots\ldots\ldots + f^1 - f^o \quad ,$$

we have

$$\begin{aligned}
\| f^{\nu+1} - f^o \|_{(\acute{\ell}+2n+3), U_{r_o}(f^o)} \leq\ & \| f^{\nu+1} - f^{\nu} \|_{(\acute{\ell}+2n+3), U_{r_o}(f^o)} \\
& + \ldots\ldots \\
& + \| f^1 - f^o \|_{(\acute{\ell}+2n+3), U_{r_o}(f^o)} \\
\leq\ & \| M_{\nu+1} \chi_{\nu} D_b^{\nu *} N_b^{\nu} D_b f^{\nu} \|_{(\acute{\ell}+2n+3), U_{r_o}(f^o)} \\
& + \| M_{\nu} \chi_{\nu-1} D_b^{\nu-1 *} N_b^{\nu-1} D_b f^{\nu-1} \|_{(\acute{\ell}+2n+3), U_{r_o}(f^o)} \\
& + \ldots\ldots \\
& + \| M_1 \chi_o D_b^{*} N_b D_b f^o \|_{(\acute{\ell}+2n+3), U_{r_o}(f^o)}
\end{aligned}$$

$$\leqq C_\ell \Big\{ \varepsilon_{\nu+1}^{-2-4n} \| \chi_\nu D_b^{\nu *} N_b^{\nu} D_b f^{\nu} \|_{(\ell-2n), U_{r_o}(f^o)}$$

$$+ \; \varepsilon_{\nu}^{-2-4n} \| \chi_{\nu-1} D_b^{\nu-1 *} N_b^{\nu-1} D_b f^{\nu-1} \|_{(\ell-2n), U_{r_o}(f^o)}$$

$$+ \ldots\ldots$$

$$+ \; \varepsilon_{1}^{-2-4n} \| \chi_o D_b^{*} N_b D_b f^{o} \|_{(\ell-2n), U_{r_o}(f^o)} \Big\}$$

$$\leqq C'_\ell \Big\{ \varepsilon_{\nu+1}^{-3-4n} \| D_b^{\nu *} N_b^{\nu} D_b f^{\nu} \|_{(\ell-2n), U_{r_\nu}(f^\nu)}$$

$$+ \; \varepsilon_{\nu}^{-3-4n} \| D_b^{\nu-1 *} N_b^{\nu-1} D_b f^{\nu-1} \|_{(\ell-2n), U_{r_{\nu-1}}(f^{\nu-1})}$$

$$+ \ldots\ldots$$

$$+ \; \varepsilon_{1}^{-3-4n} \| D_b^{*} N_b D_b f^{o} \|_{(\ell-2n), U_{r_o}(f^o)} \Big\}$$

$$\leqq C''_\ell \Big\{ \varepsilon_{\nu+1}^{-3-4n} \| D_b f^{\nu} \|_{(\ell), U_{r_\nu}(f^\nu)}$$

$$+ \; \varepsilon_{\nu}^{-3-4n} \| D_b f^{\nu} \|_{(\ell), U_{r_{\nu-1}}(f^{\nu-1})}$$

$$+ \ldots\ldots$$

$$+ \; \varepsilon_{1}^{-3-4n} \| D_b f^{\nu} \|_{(\ell), U_{r_o}(f^o)} \Big\} \quad (\text{ by Lemma 7.3.2 })$$

$$\leqq C'''_\ell \Big\{ \varepsilon_{\nu+1}^{-3-4n} \; (1/2C^{*}_\ell) \; \varepsilon_{\nu}^{(1/10)\ell} + \ldots\ldots + \varepsilon_{1}^{-3-4n} (1/2C \;) \varepsilon_{o}^{(1/10)} \Big\}$$

So if $(1/10)\ell > (3/2)(3 + 4n) + 1$, we have

$$\| f^{\nu+1} - f^{0} \|_{(\ell+2n+3),U_{r_0}(f^0)} \leq C''_{\ell}\{\varepsilon_{\nu} + \varepsilon_{\nu-1} + \dots + \varepsilon_1\}$$

$$\leq 2C'''_{\ell}\,\varepsilon_1$$

$$= 2C'''_{\ell}\, r_0^{p} \quad .$$

Hence if r_0 is chosen sufficiently small, we have our estimate . Q.E.D.

We first show

Proposition 7.5.1.

$$U_{r_{\nu+1}}(f^{\nu+1}) \subset U_{r_\nu - \delta_\nu}(f^\nu)$$

Proof . We recall $f^{\nu+1}|_C = f^o$ and $f^\nu|_C = f^o$.
Therefore , because of $r_{\nu+1} = r_\nu - 2\delta_\nu$, there is a point of $U_{r_{\nu+1}}(f^{\nu+1})$, which is included in $U_{r_\nu-\delta_\nu}(f^\nu)$. We show that every point of $U_{r_{\nu+1}}(f^{\nu+1})$ is of $U_{r_\nu-\delta_\nu}(f^\nu)$.
We assume that there is a point of $U_{r_{\nu+1}}(f^{\nu+1})$, which is not included in $U_{r_\nu-\delta_\nu}(f^\nu)$. Then there is a point p satisfying

$$t \cdot f^{\nu+1}(p) = r_{\nu+1}$$

and

$$t \cdot f^\nu(p) = r_\nu - \delta_\nu \text{ (so } p \in U_{r_\nu}(f^\nu) \text{)} .$$

Therefore

$$t \cdot f^{\nu+1}(p) - t \cdot f^\nu(p) = \mathrm{Re} \Big\{ (f_n^{\nu+1}(p) - f_n^\nu(p)) \cdot$$

$$(1 + (\lambda/2) (f_n^{\nu+1}(p) + f_n^\nu(p)) \Big\}$$

$$-\delta_\nu = \mathrm{Re}\left\{\left(f_n^{\nu+1}(p) - f_n^{\nu}(p)\right)\left(1 + (\lambda/2)\left(f_n^{\nu+1}(p) + f_n^{\nu}(p)\right)\right)\right\}$$

So

$$\delta_\nu \leqq \sup_{p \in U_{r_\nu - \delta_\nu}(f^\nu)} \left| f_n^{\nu+1}(p) - f_n^{\nu}(p)\right| \left|(1 + (\lambda/2)\left(|f_n^{\nu+1}(p)| + |f_n^{\nu}(p)|\right)\right|$$

On the other hand, for $\| f^{\nu+1}\|'_{(2n),U_{r_\nu-\delta_\nu}(f^\nu)}$, by the definition of $f^{\nu+1}$,

$$\| f^{\nu+1}\|'_{(2n),U_{r_\nu-\delta_\nu}(f^\nu)} \leqq \| f^0 \|'_{(2n),U_{r_\nu-\delta_\nu}(f^\nu)} + \varepsilon_{\nu+1}^{-2n}\left\{\| f^\nu - f^0\|'_{(0),U_{r_\nu}(f^\nu)} + \| D_b f^\nu\|'_{(0),U_{r_\nu}(f^\nu)}\right\}$$

And by the Sobolev lemma,

$$\sup_{p \in U_{r_\nu-\delta_\nu}(f^\nu)} | f_n^{\nu+1}(p) - f_n^{\nu}(p)| \leqq C \| f^{\nu+1} - f^\nu \|'_{(2n),U_{r_\nu-\delta_\nu}(f^\nu)} .$$

Hence

$$\delta_\nu \leq c\,\varepsilon_{\nu+1}^{\mu}\varepsilon_{\nu+1}^{-2n} .$$

But if we choose r_0 sufficiently small, this inequality is absurd. Hence we have our proposition. Q.E.D.

Now we see the estimate for $\| D_b f^{\nu+1} \|'_{(\ell), U_{r_{\nu+1}}(f^{\nu+1})}$.
We recall the definition of $f^{\nu+1}$.

$$f^{\nu+1} = f^{\nu} - M_{\nu+1}\chi_\nu D_b^{\nu *} N_b^{\nu} D_b f^{\nu}$$

We show

Theorem 7.5.2.

$$\| D_b f^{\nu+1} \|'_{(\ell), U_{r_\nu - \delta_\nu}(f^\nu)} \leqq c_\ell^{\#} (\cdot\, \varepsilon_{\nu+1}^{-2n-3} p_\nu^2 + \varepsilon_{\nu+1}^{\ell} \varepsilon_\nu^{-\ell-3n-5})$$

Proof .

$$\begin{aligned} D_b f^{\nu+1} &= D_b f^{\nu} - D_b(\chi_\nu D_b^{\nu *} N_b^{\nu} D_b f^{\nu}) + D_b R_{\nu+1}\chi_\nu D_b^{\nu *} N_b^{\nu} D_b f^{\nu} \\ &= D_b f^{\nu} - D_b^{\nu}(\chi_\nu D_b^{\nu *} N_b^{\nu} D_b f^{\nu}) + (D_b^{\nu} - D_b)(\chi_\nu D_b^{\nu *} N_b^{\nu} D_b f^{\nu}) \\ &\quad + D_b R_{\nu+1}\chi_\nu D_b^{\nu *} N_b^{\nu} D_b f^{\nu} \end{aligned}$$

On $U_{r_\nu - \delta_\nu}(f^\nu)$,

$$\begin{aligned} D_b f^{\nu+1} &= D_b f^{\nu} - D_b^{\nu} D_b^{\nu *} N_b^{\nu} D_b f^{\nu} + (D_b^{\nu} - D_b) D_b^{\nu *} N_b^{\nu} D_b f^{\nu} \\ &\quad + D_b R_{\nu+1}\chi_\nu D_b^{\nu *} N_b^{\nu} D_b f^{\nu} \\ &= D_b^{\nu *} D_b^{\nu} N_b^{\nu} D_b f^{\nu} + (D_b^{\nu} - D_b) D_b^{\nu *} N_b^{\nu} D_b f^{\nu} \\ &\quad + D_b R_{\nu+1}\chi_\nu D_b^{\nu *} N_b^{\nu} D_b f^{\nu} \quad . \end{aligned}$$

Hence

$$\| D_b f^{\nu+1} \|_{(\acute{\ell}), U_{r_\nu} - \delta_\nu}(f^\nu) \leqq \| D_b^{\nu*} D_b^\nu N_b^\nu D_b f^\nu \|_{(\acute{\ell}), U_{r_\nu} - \delta_\nu}(f^\nu)$$

$$+ \| (D_b^\nu - D_b) D_b^{\nu*} N_b^\nu D_b f^\nu \|_{(\acute{\ell}), U_{r_\nu} - \delta_\nu}(f^\nu)$$

$$+ \| D_b R_{\nu+1} \chi_\nu D_b^{\nu*} N_b^\nu D_b f^\nu \|_{(\acute{\ell}), U_{r_\nu} - \delta_\nu}(f^\nu)$$

First we estimate the term

$$\| D_b R_{\nu+1} \chi_\nu D_b^{\nu*} N_b^\nu D_b f^\nu \|_{(\acute{\ell}), U_{r_\nu} - \delta_\nu}(f^\nu)$$

$$\| D_b R_{\nu+1} \chi_\nu D_b^{\nu*} N_b^\nu D_b f^\nu \|_{(\acute{\ell}), U_{r_\nu} - \delta_\nu}(f^\nu)$$

$$\leqq \quad C_\ell \| R_{\nu+1} \chi_\nu D_b^{\nu*} N_b^\nu D_b f^\nu \|_{(\acute{\ell}+1), U_{r_\nu} - \delta_\nu}(f^\nu)$$

$$\leqq \quad C_\ell \| R_{\nu+1} \chi_\nu D_b^{\nu*} N_b^\nu D_b f^\nu \|_{(\acute{\ell}+1), U_{r_\nu}}(f^\nu)$$

$$\leqq \quad C'_\ell \varepsilon_{\nu+1}^{\ell-6n-1} \| \chi_\nu D_b^{\nu*} N_b^\nu D_b f^\nu \|_{(\acute{\ell}+1+\ell-6n-1), U_{r_\nu}}(f^\nu)$$

$$\leqq \quad C''_\ell \varepsilon_{\nu+1}^{\ell-6n-1} \Big\{ \| D_b^{\nu*} N_b^\nu D_b f^\nu \|_{(\acute{\ell}+1+\ell-6n-1), U_{r_\nu}}(f^\nu)$$

$$+ \quad \delta_\nu^{-1} \| D_b^{\nu*} N_b^\nu D_b f^\nu \|_{(\acute{\ell}+\ell-6n-1), U_{r_\nu}}(f^\nu)$$

$$+ \ldots\ldots$$

$$+ \quad \delta_\nu^{-k} \| D_b^{\nu*} N_b^\nu D_b f^\nu \|_{(\acute{\ell}+1-k+\ell-6n-1), U_{r_\nu}}(f^\nu)$$

$$+ \quad \delta_\nu^{-(2\ell-4n)} \| D_b^{\nu*} N_b^\nu D_b f^\nu \|_{(\acute{0}), U_{r_\nu}}(f^\nu) \Big\}$$

$$\| D_b R_{\nu+1} \chi_\nu D_b^{\nu *} N_b^\nu D_b f^\nu \|_{(\acute{\ell}), U_{r_\nu} - \delta_\nu}(f^\nu)$$

$$\leqq C''_\ell \varepsilon_{\nu+1}^{\ell-6n-1} \varepsilon_\nu^{-1} \| D_b^{\nu *} N_b^\nu D_b f^\nu \|_{(\acute{\ell}+1+\ell-6n-1), U_{r_\nu}}(f^\nu)$$

(by $\delta_\nu = \varepsilon_\nu^{(1/10)\ell}$)

For $\| D_b^{\nu *} N_b^\nu D_b f^\nu \|_{(2\acute{\ell}-6n), U_{r_\nu}}(f^\nu)$, we have Lemma 7.5.3.

$$\| D_b^{\nu *} N_b^\nu D_b f^\nu \|_{(2\acute{\ell}-6n), U_{r_\nu}}(f^\nu) \leqq \varepsilon_\nu^{-(\ell-4n+1)}$$

Proof .

$$\| D_b^{\nu *} N_b^\nu D_b f^\nu \|_{(2\acute{\ell}-6n), U_{r_\nu}}(f^\nu) \leqq c \Big\{ \| D_b f^\nu \|_{(2\acute{\ell}-6n), U_{r_\nu}}(f^\nu)$$

$$+ c^{(1)}(f^\nu - \psi) \| D_b f^\nu \|_{(2\acute{\ell}-6n-1), U_{r_\nu}}$$

$$+ c^{(2\ell-6n)}(f^\nu - \psi) \| D_b f^\nu \|_{(\acute{0}), U_{r_\nu}}(f^\nu)$$

where

$$c^{(k)}(f^\nu - \psi)$$

is a linear combination of

$j^{\alpha_1}(f^{\nu}-\psi)\ldots\ldots j^{\alpha_s}(f^{\nu}-\psi)$,

where $\alpha_1,\ldots\ldots,\alpha_s$ are integers satisfying

$$\Sigma_i\ \alpha_i = k\ .$$

So by the Sobolev lemma , $c^{(k)}(f^{\nu}-\psi)$ can be estimated by

$$\| f^{\nu}-\psi \|'_{(k+2n),U_{r_\nu}(f^{\nu})}$$

Hence , if $k \leqq \ell-2n$,

$$c^{(k)}(f^{\nu}-\psi)\| D_b f \|'_{(2\ell-6n-k),U_{r_\nu}(f^{\nu})}$$

$$\leqq \| f^{\nu} - \psi\|'_{(k+2n),U_{r_\nu}(f^{\nu})}\ \| D_b f^{\nu}\|'_{(\ell+\ell-6n-k),U_{r_\nu}(f^{\nu})}$$

$$\leqq \| f^{\nu} - \psi\|'_{(\ell)}\ \| D_b f^{\nu}\|'_{(\ell+\ell-6n-k),U_{r_\nu}(f^{\nu})}$$

$$\leqq\ \varepsilon_{\nu}^{-(\ell-6n-k)} \qquad (\ k < \ell-6n\ )$$

(by the assumption for the ν -case) .

If $\ell-2n < k \leqq 2\ell-6n$,

$$c^{(k)}(f^{\nu}-\psi)\| D_b f^{\nu}\|'_{(2\ell-6n-k),U_{r_\nu}(f^{\nu})}$$

$$\leqq \| f^{\nu} - \psi\|'_{(k+2n),U_{r_\nu}(f^{\nu})}\ \| D_b f^{\nu}\|'_{(\ell-4n),U_{r_\nu}(f^{\nu})}$$

$$\leqq \varepsilon_\nu^{-(\ell-4n-1)}$$

So

$$\| D_b^{\nu *} N_b^\nu D_b f^\nu \|_{(2\ell-6n), U_{r_\nu}(f^\nu)} \leqq \varepsilon_\nu^{-(\ell-4n-1)} \qquad \text{Q.E.D.}$$

Hence

$$\| D_b R_{\nu+1} \chi_\nu D_b^{\nu *} N_b^\nu D_b f^\nu \|_{(\ell), U_{r_\nu - \delta_\nu}(f^\nu)}$$

$$\leqq C_\ell'' \varepsilon_{\nu+1}^{\ell-6n-1} \varepsilon_\nu^{-(\ell-4n-1)}$$

For $\| D_b R_{\nu+1} \chi_\nu (f^\nu - f^o) \|_{(\ell), U_{r_\nu - \delta_\nu}(f^\nu)}$,

$$\| D_b R_{\nu+1} \chi_\nu (f^\nu - f^o) \|_{(\ell), U_{r_\nu - \delta_\nu}(f^\nu)} \leqq C_\ell \| R_{\nu+1} \chi_\nu (f^\nu - f^o) \|_{(\ell+1), U_{r_\nu}(f^\nu)}$$

$$\leqq C_\ell \varepsilon_{\nu+1}^{\ell-6n-1} \| \chi_\nu (f^\nu - f^o) \|_{(2\ell-6n), U_{r_\nu}(f^\nu)}$$

$$\leqq C_\ell \varepsilon_{\nu+1}^{\ell-6n-1} \Big\{ \| (f^\nu - f^o) \|_{(2\ell-6n), U_{r_\nu}(f^\nu)}$$

$$+ \delta_\nu^{-1} \| (f^\nu - f^o) \|_{(2\ell-6n-1), U_{r_\nu}(f)}$$

$$+ \dots$$

$$+ \delta_\nu^{-(2\ell-6n)} \| (f^\nu - f^o) \|_{(0), U_{r_\nu}(f^\nu)} \Big\}$$

$$\geqq c_{\ell}\varepsilon_{\nu+1} \quad \varepsilon_{\nu} \quad \| I - I \|_{(\ell+\ell-6n)', U_{r_\nu}(f^\nu)}$$

$$\leqq c_{\ell}' \varepsilon_{\nu+1}^{\ell-6n-1} \; \varepsilon_{\nu}^{-1} \; \varepsilon_{\nu}^{-(\ell-6n)+1}$$

$$\leqq c_{\ell}' \varepsilon_{\nu+1}^{\ell-6n-1} \; \varepsilon_{\nu}^{-(\ell-6n)}$$

Next we treat with

$$\| (D_b^\nu - D_b) D_b^{\nu *} N_b^\nu D_b f^\nu \|_{(\ell)', U_{r_\nu - \delta_\nu}'(f^\nu)} \quad .$$

For this we show

Lemma 7.5.4.

$$\| (D_b^\nu - D_b) v \|_{(\ell)', U_{r_\nu - \delta_\nu}(f^\nu)}$$

$$\leqq \; c_\ell \| (D_b f^\nu) j^{(1)}(f^\nu) \|_{(\ell)', U_{r_\nu - \delta_\nu}(f^\nu)} \; \| v \|_{(\ell+1)', U_{r_\nu - \delta_\nu}(f^\nu)}$$

From the definition of D_b^ν , D_b , namely ,

$$D_b^\nu v \; (X) \; = \; (X + \alpha(f^\nu)(X)) v$$

$$D_b v(X) \; = \; Xv \quad \text{for} \quad X \in T_b'' \quad ,$$

where $\alpha(f^\nu)$ is defined by

$$(X + \alpha(f^\nu)(X)) \; f \; = 0 \quad \text{for} \quad X \in T_b'' \quad ,$$

Lemma 7.5.4 is obvious .

So

$$\| D_b^{\nu *} D_b^{\nu} N_b D_b f^{\nu} \|_{(\acute{\ell}), U_{r_\nu} - \delta_\nu}(f^{\nu})$$

$$= \| N_b^{\nu} D_b^{\nu *}(D_b^{\nu} - D_b) D_b f^{\nu} \|_{(\acute{\ell}), U_{r_\nu} - \delta_\nu}(f^{\nu})$$

$$\leq \| \chi_\nu N_b^{\nu} D_b^{\nu *}(D_b^{\nu} - D_b) D_b f^{\nu} \|_{(\acute{\ell}), U_{r_\nu}}(f^{\nu})$$

$$\leq \| M_{\nu+1} \chi_\nu N_b^{\nu} D_b^{\nu *}(D_b^{\nu} - D_b) D_b f^{\nu} \|_{(\acute{\ell}), U_{r_\nu}}(f^{\nu})$$

$$+ \| R_{\nu+1} \chi_\nu N_b^{\nu} D_b^{\nu *}(D_b^{\nu} - D_b) D_b f^{\nu} \|_{(\acute{\ell}), U_{r_\nu}}(f^{\nu})$$

$$\leq C_\ell \, \varepsilon_{\nu+1}^{-2n-1} \| \chi_\nu N_b^{\nu} D_b^{\nu *}(D_b^{\nu} - D_b) D_b f^{\nu} \|_{(\acute{\ell}-2n-1), U_{r_\nu}}(f^{\nu})$$

$$+ C'_\ell \, \varepsilon_{\nu+1}^{\ell-6n} \| \chi_\nu N_b^{\nu} D_b^{\nu *}(D_b^{\nu} - D_b) D_b f^{\nu} \|_{(2\acute{\ell}-6n), U_{r_\nu}}(f^{\nu})$$

$$\leq C''_\ell \, \varepsilon_{\nu+1}^{-2n-1} \Big\{ \| N_b^{\nu} D_b^{\nu *}(D_b^{\nu} - D_b) D_b f^{\nu} \|_{(\acute{\ell}-2n-1), U_{r_\nu}}(f^{\nu})$$

$$+ \delta_\nu^{-1} \| N_b^{\nu} D_b^{\nu *}(D_b^{\nu} - D_b) D_b f^{\nu} \|_{(\acute{\ell}-2n-2), U_{r_\nu}}(f^{\nu})$$

$$+ \cdots\cdots$$

$$+ \delta_\nu^{-(\ell-2n-1)} \| N_b^{\nu} D_b^{\nu *}(D_b^{\nu} - D_b) D_b f^{\nu} \|_{(\acute{0}), U_{r_\nu}}(f^{\nu}) \Big\}$$

$$+ C'''_\ell \, \varepsilon_{\nu+1}^{\ell-6n} \Big\{ \| N_b^{\nu} D_b^{\nu *}(D_b^{\nu} - D_b) D_b f^{\nu} \|_{(2\acute{\ell}-6n), U_{r_\nu}}(f^{\nu})$$

$$+ \quad \delta_\nu^{-1} \| N_b^\nu D_b^{\nu *}(D_b^\nu - D_b) D_b f^\nu \|_{(2\ell-6n-1)', U_{r_\nu}(f^\nu)}$$

$$+ \ldots .$$

$$+ \quad \delta_\nu^{-(2\ell-6n)} \| N_b^\nu D_b^{\nu *}(D_b^\nu - D_b) D_b f^\nu \|_{(0)', U_{r_\nu}(f^\nu)}$$

$$\leqq C_\ell'' \, \varepsilon_{\nu+1}^{-2n-1} \, \varepsilon_\nu^{-2} \, \| N_b^\nu D_b^{\nu *}(D_b^\nu - D_b) D_b f^\nu \|_{(\ell-2n-1)', U_{r_\nu}(f^\nu)}$$

$$+ \quad C_\ell''' \, \varepsilon_{\nu+1}^{\ell-6n} \, \varepsilon_\nu^{-2} \, \| N_b^\nu D_b^{\nu *}(D_b^\nu - D_b) D_b f^\nu \|_{(2\ell-6n)', U_{r_\nu}(f^\nu)}$$

For $\| N_b^\nu D_b^{\nu *}(D_b^\nu - D_b) D_b f^\nu \|_{(\ell-2n-1)', U_{r_\nu}(f^\nu)}$, and

$\| N_b^\nu D_b^{\nu *}(D_b^\nu - D_b) D_b f^\nu \|_{(2\ell-6n)', U_{r_\nu}(f^\nu)}$, we have

Lemma 7.5.5.

$$\| N_b^\nu D_b^{\nu *}(D_b^\nu - D_b) D_b f^\nu \|_{(\ell-2n-1)', U_{r_\nu}(f^\nu)}$$

$\leqq \quad C_\ell'' \, \varepsilon_\nu^{-1} p_\nu^2$, where C_ℓ'' is independent of f^ν .

Proof . $\| N_b^\nu D_b^{\nu *}(D_b^\nu - D_b) D_b f^\nu \|_{(\dot{\ell}-2n-1), U_{r_\nu}(f^\nu)}$

$$\leqq c_\ell \Big\{ \| (D_b^\nu - D_b) D_b f^\nu \|_{(\dot{\ell}-2n), U_{r_\nu}(f^\nu)}$$

$$+ \; c^{(1)}(f^\nu - \psi) \| (D_b^\nu - D_b) D_b f^\nu \|_{(\dot{\ell}-2n-1), U_{r_\nu}(f^\nu)} \; + \cdots\cdots$$

$$+ \; c^{(\ell-2n-1)}(f^\nu - \psi) \| (D_b^\nu - D_b) D_b f^\nu \|_{(\dot{1}), U_{r_\nu}(f^\nu)} \Big\}$$

$$\leqq c'_\ell \, \varepsilon_\nu^{-1} \| (D_b^\nu - D_b) D_b f^\nu \|_{(\dot{\ell}-2n), U_{r_\nu}(f^\nu)} \; c^{(\ell-2n-1)}(f^\nu - \psi)$$

$$\leqq c'_\ell \, \varepsilon_\nu^{-1} \| f^\nu - \psi \|_{(\ddot{\ell}), U_{r_\nu}(f^\nu)} \| D_b f^\nu \|^2_{(\dot{\ell}), U_{r_\nu}(f^\nu)}$$

$$\leqq c''_\ell \, \varepsilon_\nu^{-1} \, p_\nu^2$$

Q.E.D.

Lemma 7.5.6.

$$\| N_b^\nu D_b^{\nu *}(D_b^\nu - D_b) D_b f^\nu \|_{(2\dot{\ell}-6n), U_{r_\nu}(f^\nu)}$$

$\leqq c'''_\ell \; \varepsilon_\nu^{-1} \; \varepsilon_\nu^{-(\ell-6n+1)}$, where c'''_ℓ is independent of f^ν .

Proof . $\| N_b^\nu D_b^{\nu *}(D_b^\nu - D_b) D_b f^\nu \|_{(2\dot{\ell}-6n), U_{r_\nu}(f^\nu)}$

$$\lesssim \| D_b^{\nu *}(D_b^\nu - D_b) D_b f^\nu \|_{(2\dot{\ell}-6n), U_{r_\nu}(f^\nu)}$$

$$+ \; c^{(1)}(f^\nu - \psi) \| D_b^{\nu *}(D_b^\nu - D_b) D_b f^\nu \|_{(2\dot{\ell}-6n-1), U_{r_\nu}(f^\nu)}$$

$+ \cdots\cdots$

$$+ \; c^{(2\ell-6n)}(f^\nu - \psi) \| D_b^{\nu *}(D_b^\nu - D_b)D_b f^\nu \|'_{(0),U_r}(f^\nu)$$

$$\leqq \; \| (D_b^\nu - D_b)D_b f^\nu \|'_{(2\ell-6n+1),U_{r_\nu}}(f^\nu)$$

$$+ \; c^{(1)}(f^\nu - \psi) \| (D_b^\nu - D_b)D_b f^\nu \|'_{(2\ell-6n),U_{r_\nu}}(f^\nu)$$

$$+ \cdots\cdots$$

$$+ \; c^{(2\ell-6n)}(f^\nu - \psi) \| (D_b^\nu - D_b)D_b f \|'_{(1),U_{r_\nu}}(f^\nu)$$

On the other hand ,

$$L^{(2\ell-6n+1)}((D_b^\nu - D_b)D_b f^\nu) = \sum_{\substack{\alpha,\beta \\ \alpha+\beta=2\ell-6n+1}} L^\alpha(D_b f^\nu) L^\beta(D_b f^\nu) \quad ,$$

where L^γ is a differential operator of order γ .

Since $\alpha+\beta=2\ell-6n+1$, α or β is less than $\ell-4n$. And so ,

$$\| L^\alpha(D_b f^\nu) L^\beta(D_b f^\nu) \|'_{(0),U_{r_\nu}}(f^\nu) \leqq \sup | L^\beta(D_b f^\nu) | \| L^\alpha(D_b f^\nu) \|'_{(0),U_{r_\nu}}(f^\nu)$$

(here we assume $\beta \leqq \ell - 4n$)

$$\leqq C_\ell \| D_b f^\nu \|'_{(\ell),U_{r_\nu}}(f^\nu) \| D_b f^\nu \|'_{(2\ell-6n+1),U_{r_\nu}}(f^\nu)$$

Hence

$$\| (D_b^\nu - D_b) D_b f^\nu \|_{(2\ell-6n+1)', U_{r_\nu}(f^\nu)}$$

$$\leq \; c_\ell \varepsilon_\nu^{-1} \Big\{ \| D_b f^\nu \|_{(\ell)', U_{r_\nu}(f^\nu)} \| D_b f^\nu \|_{(2\ell-6n+1)', U_{r_\nu}(f^\nu)}$$

$$+ \; \| D_b f^\nu \|_{(\ell)', U_{r_\nu}(f^\nu)} \| D_b f^\nu \|_{(2\ell-6n+1)', U_{r_\nu}(f^\nu)} \Big\}$$

$$\leq \; c'_\ell \varepsilon_\nu^{-1} \; \varepsilon_\nu^{-(\ell-6n+1)} \qquad \text{Q.E.D.}$$

So

$$\| D_b^{\nu *} D_b^\nu N_b^\nu D_b f^\nu \|_{(\ell)', U_{r_\nu - \delta_\nu}(f^\nu)}$$

$$\leq \; c''_\ell \varepsilon_{\nu+1}^{-2n-1} \varepsilon_\nu^{-2} \varepsilon_\nu^{-1} p_\nu^2 \; + \; c'''_\ell \varepsilon_{\nu+1}^{\ell-6n} \varepsilon_\nu^{-2} \varepsilon_\nu^{-1} \varepsilon_\nu^{-(\ell-6n+1)}$$

Therefore

$$\| D_b f^{\nu+1} \|_{(\ell)', U_{r_\nu - \delta_\nu}(f^\nu)}$$

$$\leq \; c_\ell \Big\{ \varepsilon_{\nu+1}^{-2n-3} p_\nu^2 \; + \; \varepsilon_{\nu+1}^{\ell} \varepsilon_\nu^{-\ell-9n-3+6n-2} \Big\}$$

$$= \; c_\ell \Big\{ \varepsilon_{\nu+1}^{-2n-3} p_\nu^2 \; + \; \varepsilon_{\nu+1}^{\ell} \varepsilon_\nu^{-\ell-3n-5} \Big\} \qquad \text{Q.E.D.}$$

Let f^o be an embedding of M into C^n satisfying

$$(A_1) \quad o(f^o) \in \Gamma(M, {}^o\bar{T}'' \otimes ({}^oT'')^*) \quad ,$$

$$(A_2) \quad o(f^o)(p_o) = 0 \quad ,$$

$$(A_3) \quad (1/\widetilde{b}(f^o)^{2\ell})Df^o \text{ is bounded near } p_o \quad ,$$

where

$$\widetilde{b}(f^o) = \sqrt{\sum_{j=1}^{n-1} |Y_j t_{f^o}|^2} \quad .$$

Let f be a solution associated with f^o , which is established in Chapter 7 , namely

$$\sup_{U_r(f)} |\widetilde{b}(f^o)^{-2} j^{(1)}(f^o - f)| < c_{f^o}$$

and

$$D_b f = 0 \quad \text{along } t_f$$

where $U_r(f) = \{p \ ; \ p \in M \ , \ t_f(p) < r\}$. From now on , we use the abbreviation $U_r = U_r(f)$. And we set

$$C = \{q \ ; \ q \in M \ , \ \widetilde{b}(f^o)(q) = 0\} \quad .$$

For the C^∞-function $t_f = \mathrm{Re}\, h \circ f$, we construct

$$Y^{o} = \sum_{i=1}^{n-1} ((\bar{Y}_i t_f)/\tilde{b}(f))Y_i \quad , \quad W_i = Y_i - ((Y_i t_f)/b(f))Y \quad ,$$

where $b(f) = \sqrt{\sum_{j=1}^{n-1} |Y_j t_f|^2}$, and set

$$X^{f} = \sqrt{-1}\,\tilde{b}(f)S + \bar{\alpha}_f Y^{o} - \alpha_f \bar{Y}^{o} \quad ,$$

where α_f is a C^{∞}-function on $U_r - C$ defined by

$$\sqrt{-1}\,\tilde{b}(f)S(h\,f) + \bar{\alpha}_f Y^{o}(h\circ f) - \alpha_f \bar{Y}^{o}(h\circ f) = 0$$

We note that α_f is of $C^{2\ell-2}$ on U_r . In fact , we have

$$\alpha_f - (k_f\bar{\alpha}_f + \ell_f) = 0 \quad ,$$

where $k_f = (Y^{o}(h\circ f)/\bar{Y}^{o}(h\circ f))$, $\ell_f = (\sqrt{-1}\,\tilde{b}(f)S(h\circ f)/\bar{Y}^{o}(h\circ f))$

We claim that k_f and ℓ_f are of $C^{2\ell-2}$ and sufficiently small . If these are proved , we have that α_f is of $C^{2\ell-2}$ on U_r . In fact ,

$$x + iy - (u+iv)(x-iy) = w + iz \quad ,$$

where $\alpha_f = x+iy$, $k_f = u+iv$, $\ell_f = w+iz$.

Then ,

$$(1-u)x + (-v)y = m_1$$

$$(-v)x + (1+u)y = m_2 \quad .$$

So

$$x = ((m_1(1+u)+m_2 v)/(1-u^2-v^2))$$

$$y = ((m_1 v+m_2(1-u))/(1-u^2-v^2)) \quad .$$

Hence it suffices to show that k_f and ℓ_f are of $C^{2\ell-2}$.
First we see that k_f is of $C^{2\ell-2}$ on U_r .
By $t_f = 2\mathrm{Re}\ h \circ f$,

$$\bar{Y}^o(h \circ f) = \bar{Y}^o(\ (h \circ f) + (h \circ f)\ ) - \bar{Y}^o(h \circ f)$$

$$= \bar{Y}^o t_f - \bar{Y}^o(h \circ f)$$

$$= \tilde{b}(f\ ) - \sum_{i=1}^{n-1} ((Y_i t_f\ )/\tilde{b}(f\ ))(Y_i(h \circ f)) \quad .$$

By (A_3) and the property of the solution f ,

$(1/\tilde{b}(f\ ))Y_i(h \circ f)$ is of $C^{2\ell-2}$ on U_r and vanishes on C .

Hence

$$((Y^o(h \circ f)/(\tilde{b}(f\ ) - \bar{Y}^o(h \circ f)))$$

$$((\sum_{i=1}^{n-1} (\bar{Y}_i t_f\ /\ \tilde{b}(f\ ))Y_i(h \circ f\ ))/(\tilde{b}(f\ ) - \sum_{i=1}^{n-1} (\bar{Y}_i t_f\ /\ \tilde{b}(f\ ))Y_i(h \circ f]$$

$=((\sum_{i=1}^{n-1}(\bar{Y}_i t_f/\widehat{b}(f))\widehat{b}(f)^{-1}(Y_i(h\circ f))/(1-(\sum_{i=1}^{n-1}(\bar{Y}_i t_f/\widehat{b}(f))\widehat{b}(f)^{-1}(Y_i(h\circ f))$

is of $C^{2\ell-2}$. Hence k_f is of $C^{2\ell-2}$. The proof for ℓ_f is the same . So we omit this . In this section , we adopt the Kuranishi's notation , for example Θ_{-1} , and Θ_0 , Θ_1 (see (2.16) in (3)) with respect to the C^∞-embedding f established in Chapter 7 in this paper .

Then our main theorem is stated as follows .

Main Theorem 8.1. For a sufficiently small $r > 0$, we have the following estimate ; let u be an element of $\Gamma(U_r - C, ({}^{o}T''))$ satisfying

(i) Du , $D\,u$, $\widetilde{b}(f)^{-1}u$ are of L^2 , and $W_j u$ is also ,

(ii) $u(Y^{o}) = 0$ on $bU_r - C$.

Then ,

$$\| Du \|^2_{(o),U_r} + \| D^{*}u \|^2_{(o),U_r} \geqq c\Big\{ \| (Y^{o} + \widetilde{b}(f)^{-1}\alpha_f)u \|^2_{(o),U_r}$$

$$+ \sum_j ((n-2)/(n-3)) \| W_j u \|^2_{(o),U_r}$$

$$+ \sum_j (1/(n-2)) \| \bar{W}_j u \|^2_{(o),U_r}$$

$$+ (n-3) \| (\alpha_f / \widetilde{b}(f))u \|^2_{(o),U_r} \Big\}$$

In order to prove our main theorem , we make preparations .

Proposition 8.2 .

$$[W_i, \bar{W}_j] = -(\sqrt{-1}/\tilde{b}(f))C_S(W_i, W_j)X^f + \Sigma_\ell \Theta_0 W_\ell + \Sigma_\ell \Theta_0 \bar{W}_\ell \quad ,$$

where $C_S(W_i, W_j)$ means the Levi-form with respect to the vector bundle decomoposition

$$CTM = {}^{o}T'' + {}^{o}\bar{T}'' + CS \quad .$$

Proof . Let $CTM(f)$ be a vector bundle on $U_r - C$ defined by

$$\{X' \; ; \; X' \in CTM \; , \; X' t_f = 0 \; , \; X'(h \circ f) = 0 \} \quad .$$

Then , obviously ,

$$\dim_C CTM(f) = 2n-3 \quad ,$$

and

$$W_i \quad , \quad X^f \quad \text{are of} \quad CTM(f) \quad .$$

Furthermore $\bar{W}_i$ are of $CTM(f)$. In fact

$$\bar{W}_i t_f = \bar{W}_i((1/2)(\, h \circ f \; + \; h \circ f\,)$$

$$= (1/2)\, \bar{W}_i(h \circ f) \quad (\text{ because } \quad W_i f_\alpha = 0 \; , \; f = (f_1, \ldots\ldots, f_n)\,)$$

Since t_f is a real valued function ,

$$\bar{W}_i \, t_f = 0 \quad .$$

So

$$\bar{W}_i (h \circ f) = 0 \quad .$$

Hence $CTM(f)$ is generated by

$$\{ W_i , \bar{W}_i , X^f \}$$

because the dimension of the space generated by W_i , $\bar{W}_i$, X^f is $2n-3$.
Furthermore

$$[W_i , \bar{W}_j]$$

is of $CTM(f)$. Hence there are C^∞-functions c_{ij} , $a_{\ell,(i,j)}$, $b_{\ell,(i,j)}$ satisfying

$$(8.1) \quad [W_i , \bar{W}_j] = c_{ij} (\sqrt{-1} \, \tilde{b}(f) S + \bar{\alpha}_f Y^o - \alpha_f \bar{Y}^o)$$
$$+ \sum_\ell a_{\ell,(i,j)} W_\ell + \sum_\ell b_{\ell,(i,j)} \bar{W}_\ell \quad .$$

By comparing S-term with respect to the C^∞-vector bundle decomposition

$$CTM = {}^o T'' + {}^o \bar{T}'' + S \quad ,$$

we have

$$C_S(W_i, W_j) = c_{ij}\sqrt{-1}\,\widetilde{b}(f) \quad .$$

Next we determine $a_{\ell,(i,j)}$ and $b_{\ell,(i,j)}$. However the proof of Proposition 2.3.2 is valid to our case . So we have our proposition .

Q.E.D.

$$8.3.1)\quad [W_i, W_j] = \widetilde{b}(f)^{-1}\alpha_f\,(Y_i t_f / \widetilde{b}(f))W_j$$

$$- \widetilde{b}(f)^{-1}\alpha_f\,(Y_j t_f / \widetilde{b}(f))W_i + \textstyle\sum_\ell \Theta_o W_\ell$$

$$8.3.2)\quad [Y^o, W_i] = -\widetilde{b}(f)^{-1}(W_i\widetilde{b}(f))Y^o - \widetilde{b}(f)^{-1}\alpha_f W_i + \textstyle\sum_\ell \Theta_o W_\ell$$

$$8.3.3)\quad [Y^o, \bar{W}_i] = -\widetilde{b}(f)^{-1}(\bar{W}_i\widetilde{b}(f))Y^o + \textstyle\sum_\ell \Theta_o W_\ell + \sum_\ell \Theta_o \bar{W}_\ell + \Theta_o Y^o + \Theta_o \bar{Y}^o$$

$$8.3.4)\quad [Y^o, \bar{Y}^o] = \widetilde{b}(f)^{-1}(Y^o\widetilde{b}(f))\bar{Y}^o - \widetilde{b}(f)^{-1}(\bar{Y}^o\widetilde{b}(f))Y^o + \widetilde{b}(f)^{-1}X^f$$

$$+\textstyle\sum_\ell \Theta_o W_\ell + \sum_\ell \Theta_o \bar{W}_\ell$$

$$8.3.5)\quad [W_i, X^f] = \widetilde{b}(f)^{-1}(\,W_i\widetilde{b}(f) + \widetilde{b}(f)\Theta_o\,)X^f + |\alpha_f|^2\, b(f)^{-1}W_i$$

$$+\textstyle\sum_\ell \Theta_o W_\ell + \sum_\ell \Theta_o \bar{W}_\ell$$

$$8.3.6)\quad [Y^o, X^f] = \widetilde{b}(f)^{-1}(Y^o\widetilde{b}(f))X^f + \Theta_o X^f - \widetilde{b}(f)^{-1}(X^o\widetilde{b}(f))Y^o$$

$$+\textstyle\sum_\ell \Theta_o W_\ell + \sum_\ell \Theta_o \bar{W}_\ell$$

$$8.3.7)\quad [\bar{Y}^o, X^f] = -\widetilde{b}(f)^{-1}(Y^o\widetilde{b}(f))X^f + \Theta_o X^f + \widetilde{b}(f)^{-1}(X^f\widetilde{b}(f))Y^o$$

$$+\textstyle\sum_\ell \Theta_o W_\ell + \sum_\ell \Theta_o \bar{W}_\ell$$

The method in Chapter 2 and Chapter 3 is valid to our case . So we omit this .

For u in $\Gamma(\bar{U}_r-C, ({}^{\rho}T'')^*)$, we set

$$\|u\|^2 = \sum_{i=1}^{n-1}\int_{U_r} u(W_i)u(\bar{W}_i)dv + \int_{U_r} u(Y^o)\bar{u}(Y^o)dv ,$$

where dv means the volume element defined by the Levi-metric and we assume that r is chosen sufficiently small . Similarly , for an element of $\Gamma(\bar{U}_r-C, \Lambda^2({}^{\rho}T'')^*)$, we set

$$\|u\|^2 = \sum_{i<j} \int_{U_r} u(W_i,W_j)\bar{u}(W_i,W_j)dv + \sum_{i=1}^{n-1}\int_{U_r} u(Y^o,W_i)\bar{u}(Y^o,W_i)dv$$

And for u of $\Gamma(\bar{U}_r-C,1)$, i.e., a C^{∞}-function on $\bar{U}_r-C$,

$$\|u\|^2 = \int_{U_r} f\cdot\bar{f}\, dv .$$

Then we have

Lemma 8.4 . For u in $\Gamma(\bar{U}_r-C, ({}^{\rho}T'')^*)$,

$$W_k^*u = -\bar{W}_k u + ((n-2)/\tilde{b}(f)^2)(\bar{Y}_k t_f)\bar{\alpha}_f u + \Theta_0 u ,$$

where W_k^* denotes the formal adjoint operator of W_k .

Proof . For u , v in $\Gamma(\bar{U}_r-C, ({}^{\rho}T'')^*)$, which have a compact support in U_r-C ,

$$(W_k u , v) = (Y_k u - ((Y_k t_f/\tilde{b}(f))\sum_{i=1}^{n-1}((\bar{Y}_i t_f/\tilde{b}(f))Y_i u , v)$$

$$= (u , -\bar{Y}_k v) + (u, \Theta_{-1}v) + \sum_{i=1}^{n-1}(Y_i u, ((-\bar{Y}_k t_f)(Y_i t_f)/\tilde{b}(f)^2)$$

$$= (u , - \bar{Y}_k v + \Theta_{-1} v)$$

$$+ (u , \sum_{i=1}^{n-1} \bar{Y}_i (((\bar{Y}_k t_f)(Y_i t_f)/\widetilde{b}(f)^2)v) + \Theta_{-1}((\bar{Y}_k t_f)(Y_i t_f)/\widetilde{b}(f)$$

By a simple calculation ,

$$\bar{Y}_i(((\bar{Y}_k t_f)(Y_i t_f)/\widetilde{b}(f)^2)v) = \bar{Y}_i((\bar{Y}_k t_f)(Y_i t_f)/\widetilde{b}(f)^2)v + ((\bar{Y}_k t_f)(Y_i t_f)/\widetilde{b}(f)^2)\bar{Y}$$

$$= ((\bar{Y}_k t_f)\bar{\widetilde{\alpha}}_f/\widetilde{b}(f)^2)v - ((|Y_i t_f|^2)(\bar{Y}_k t_f)\widetilde{\alpha}_f/\widetilde{b}(f)$$

$$+ \Theta_0 v$$

$$+ ((\bar{Y}_k t_f)(Y_i t_f)/ b(f)^2)\bar{Y}_i v \quad .$$

Therefore

$$(W_k u,v) = (u , - \bar{W}_k v + ((\bar{Y}_k t_f)/\widetilde{b}(f)^2)(n-2)\widetilde{\alpha}_f v + \Theta_0 v) \quad .$$

So we have our lemma . Q.E.D.

Next, for the formal adjoint operator Y^{o*}, we have

Lemma 8.5. For u in $\Gamma(\bar{U}_r - C, (\rho T'')^*)$,

$$Y^{o*}u = -\bar{Y}^o u - \sum_k ((\bar{Y}_k Y_k t_f / \widetilde{b}(f))u$$

$$+ \sum_k (Y_k t_f / \widetilde{b}(f))(\bar{Y}_k \widetilde{b}(f)/\widetilde{b}(f))u + \Theta_o u$$

or

$$= -\bar{Y}^o u - ((2n-3)/2\widetilde{b}(f))\bar{\alpha}_f u + \Theta_o u ,$$

where Y^{o*} denotes the formal adjoint operator of Y^o.

Proof. For u, v in $\Gamma(\bar{U}_r - C, (\rho T'')^*)$, which have a compact support in $U_r - C$,

$$(Y^o u, v) = (\sum_{k=1}^{n-1} ((\bar{Y}_k t_f / \widetilde{b}(f))(Y_k u)), v)$$

$$= (u, \sum_{k=1}^{n-1} -\bar{Y}_k(((Y_k t_f)/\widetilde{b}(f))v) + \Theta_{-1} v)$$

$$= (u, \sum_{k=1}^{n-1} -\bar{Y}_k((Y_k t_f)/\widetilde{b}(f))v -$$

$$\sum_{k=1}^{n-1} ((Y_k t_f)/\widetilde{b}(f))\bar{Y}_k v + \Theta_{-1} v)$$

$$= (u, -\bar{Y}^o v - \sum_{k=1}^{n-1} (((\bar{Y}_k Y_k t_f)\widetilde{b}(f) - (Y_k t_f)(\bar{Y}_k \widetilde{b}(f)))/\widetilde{b}(f)^2)v)$$

$$+ \Theta_{-1} v)$$

$$= (u \, , \; - \bar{Y}^0 v - \sum_{k=1}^{n-1} (\bar{Y}_k Y_k t_f / \widetilde{b}(f)) v \; + \sum_{k=1}^{n-1} (Y_k t_f / \widetilde{b}(f)) (\bar{Y}_k \widetilde{b}(f) / \widetilde{b}(f)) v$$

$$+ \; \Theta_{-1} v \,)$$

And

$$(Y_k t_f / \widetilde{b}(f)) (\bar{Y}_k \widetilde{b}(f) / \widetilde{b}(f)) = ((Y_k t_f)(\bar{Y}_k \widetilde{b}(f)^2) / 2\widetilde{b}(f)^3)$$

$$= (Y_k t_f / 2\widetilde{b}(f)^3) \sum_{\ell=1}^{n-1} \left\{ (\bar{Y}_k Y_\ell t_f)(\bar{Y}_\ell t_f) + (\bar{Y}_\ell t_f)(\bar{Y}_k \bar{Y}_\ell t_f) \right\}$$

$$= (Y_k t_f / 2b(f)^3)(\bar{\alpha}_f (\bar{Y}_k t_f) \; + \; b(f)^2 \Theta_0 \,)$$

$$= \bar{\alpha}_f / 2b(f)^3) \, | \, Y_k t_f \, |^2 \; + \; \Theta_0$$

So we have our lemma . Q.E.D.

And we have

Lemma 8.6. For u in $\Gamma(\bar{U}_r - C, (\rho T'')^*)$,

$$D^* u \; = \; - \sum_{k=1}^{n-1} \bar{W}_k u_k \; - \; (\bar{Y}^0 u_0 \; + \; ((2n-3)/2b(f)) \bar{\alpha}_f u_0) \; + \; \Theta_0 u$$

or

$$= \sum_{k=1}^{n-1} W_k^* u_k \; + \; Y^{0*} u_0 \; + \; \Theta_0 u \quad ,$$

where $\Theta_0 u$ means $\sum_{k=1}^{n-1} a_k u_k + a u_0$, where a_k and a are of Θ_0 , $u_k = u(W_k)$, $u_0 = u(Y^0)$, and from now on we use these notations .

Proof . By the definition of D^* ,

$$D^*u = -\sum_{i=1}^{n-1} \bar{Y}_i u(Y_i) + \Theta_{-1}u \quad .$$

We will rewrite this in terms of u_i , u_o , W_i , Y^o . Namely , as

$$Y_i = W_i + (Y_i t_f / \widetilde{b(f)}) Y^o \quad ,$$

we have

$$D^*u = -\sum_{i=1}^{n-1} \overline{(W_i + (Y_i t_f / b(f))Y^o)}\, u(W_i + (Y_i t_f / \widetilde{b(f)})Y^o) + \Theta_{-1}u$$

$$= -\sum_{i=1}^{n-1} (\bar{W}_i + (\bar{Y}_i t_f / \widetilde{b(f)})\bar{Y}^o)(u_i - (Y_i t_f / \widetilde{b(f)})u_o) + \Theta_{-1}u$$

$$= -\sum_{i=1}^{n-1} \bar{W}_i u_i - \sum_{i=1}^{n-1} (\bar{Y}_i t_f / \widetilde{b(f)})\bar{Y}^o u_i$$

$$- \sum_{i=1}^{n-1} \bar{W}_i ((Y_i t_f / \widetilde{b(f)})u_o) - \sum_{i=1}^{n-1} (\bar{Y}_i t_f / \widetilde{b(f)})\bar{Y}^o ((Y_i t_f / \widetilde{b(f)})u_o)$$

$$+ \Theta_{-1}u$$

$$= -\sum_{i=1}^{n-1} \bar{W}_i u_i - \bar{Y}^o u_o$$

$$- \sum_{i=1}^{n-1} (\bar{Y}_i t_f / \widetilde{b(f)})\bar{Y}^o u_i - \sum_{i=1}^{n-1} \bar{W}_i ((Y_i t_f / \widetilde{b(f)})u_o)$$

$$- \sum_{i=1}^{n-1} (\bar{Y}_i t_f / \widetilde{b(f)})(\bar{Y}^o (Y_i t_f / \widetilde{b(f)}))u_o + \Theta_{-1}u$$

First , we can neglect

$$-\sum_{i=1}^{n-1} (\bar{Y}_i t_f / \widetilde{b}(f))\bar{Y}^o u_i \quad .$$

We see this . By the relation , $\sum_{i=1}^{n-1} (\bar{Y}_i t_f)u_i = 0$, we have

$$\sum_{i=1}^{n-1} (1/\widetilde{b}(f))(\bar{Y}^o(\bar{Y}_i t_f))u_i + \sum_{i=1}^{n-1} (\bar{Y}_i t_f/\widetilde{b}(f))(\bar{Y}^o u_i) = 0 \quad .$$

While

$$\bar{Y}^o(\bar{Y}_i t_f) = \sum_{\ell=1}^{n-1} (Y_\ell t_f/\widetilde{b}(f))\bar{Y}_\ell \bar{Y}_i t_f \quad .$$

So $\sum_{i=1}^{n-1} (1/\widetilde{b}(f))(\bar{Y}^o(\bar{Y}_i t_f))u_i$ is of Θ_o . Therefore $-\sum_{i=1}^{n-1} \bar{Y}^o u_i$ is of Θ_o . So

$$\overset{*}{D}u = -\sum_{i=1}^{n-1} \bar{W}_i u_i - \bar{Y}^o u_o - \sum_{i=1}^{n-1} \bar{W}_i((Y_i t_f/\widetilde{b}(f))u_o)$$

$$-\sum_{i=1}^{n-1} (\bar{Y}_i t_f/\widetilde{b}(f))(Y^o(Y_i t_f/\ b(f)))u_o + \Theta_o u_o \quad .$$

Furthermore

$$\sum_{i=1}^{n-1} \bar{W}_i((Y_i t_f/\widetilde{b}(f))u_o) = \sum_{i=1}^{n-1} (\bar{W}_i(Y_i t_f/\widetilde{b}(f)))u_o + \sum_{i=1}^{n-1}(Y_i t_f/\widetilde{b}(f))\bar{W}_i u_o$$

$$= \sum_{i=1}^{n-1} (\bar{W}_i(Y_i t_f/\ b(f))u_o$$

because of $\sum_{i=1}^{n-1} (\bar{Y}_i t_f/\widetilde{b}(f))W_i = 0$.

So

$$D^{*}u = -\sum_{i=1}^{n-1} \bar{W}_i u_i - \bar{Y}^{o} u_o - \sum_{i=1}^{n-1} (\bar{W}_i (Y_i t_f / \tilde{b}(f))) u_o$$

$$- \sum_{i=1}^{n-1} (\bar{Y}_i t_f / b(f)) \bar{Y}^{o} (Y_i t_f / \tilde{b}(f)) u_o + \Theta_o u .$$

We must compute

$$\sum_{i=1}^{n-1} \bar{W}_i (Y_i t_f / \tilde{b}(f))$$

and

$$\sum_{i=1}^{n-1} (\bar{Y}_i t_f / \tilde{b}(f)) \bar{Y}^{o} (Y_i t_f / \tilde{b}(f)) .$$

For $\sum_{i=1}^{n-1} \bar{W}_i (Y_i t_f / \tilde{b}(f))$,

$$\sum_{i=1}^{n-1} \bar{W}_i (Y_i t_f / \tilde{b}(f)) = \sum_{i=1}^{n-1} (((\bar{W}_i (Y_i t_f)) \tilde{b}(f) - (Y_i t_f) \bar{W}_i \tilde{b}(f)) / \tilde{b}(f)^2)$$

$$= \sum_{i=1}^{n-1} (\bar{W}_i (Y_i t_f) / \tilde{b}(f)) \quad (\text{by } \sum_{i=1}^{n-1} (\bar{Y}_i t_f) W_i = 0)$$

$$= ((n-2)\bar{\alpha}_f / \tilde{b}(f)) + \Theta_o .$$

For $\sum_{i=1}^{n-1} (\bar{Y}_i t_f / \tilde{b}(f)) \bar{Y}^{o} (Y_i t_f / \tilde{b}(f))$,

$$\sum_{i=1}^{n-1} (\bar{Y}_i t_f / \tilde{b}(f)) \bar{Y}^{o} (Y_i t_f / \tilde{b}(f))$$

$$= \sum_{i=1}^{n-1} \sum_{\ell=1}^{n-1} (\bar{Y}_i t_f / \tilde{b}(f))(Y_\ell t_f / \tilde{b}(f)) \bar{Y}_\ell (Y_i t_f / \tilde{b}(f))$$

$$= \sum_{i,\ell} (((\bar{Y}_i t_f)(Y_\ell t_f))/\tilde{b}(f)^2)(((\bar{Y}_\ell Y_i t_f)\tilde{b}(f) - (Y_i t_f)(\bar{Y}_\ell \tilde{b}(f))/\tilde{b}(f)^2)$$

$$= (1/\tilde{b}(f))\bar{\alpha}_f - (1/2\tilde{b}(f))\bar{\alpha}_f + \Theta_0$$

$$= (1/2\tilde{b}(f))\bar{\alpha}_f + \Theta_0$$

So we have our lemma . Q.E.D.

Because of $\bar{U}_r - C$ being non-compact , we must introduce a function χ . Namely pick a real valued C^∞-function $g(s)$ in a real variable s with support in $\{s ; s \geqq (1/2)\}$, which is equal to 1 on $\{s ; s \geqq 1\}$. Set

$$\chi = g(e\,\tilde{b}(f)^2) \quad .$$

Then we have

Lemma 8.7. There is a constant k satisfying

$$|[Y^0, \chi]u| \leqq |(k/\tilde{b}(f))u|$$

and

$$|[\bar{Y}^0, \chi]u| \leqq |(k/\tilde{b}(f))u| ,$$

where k is independent of e .

Furthermore we have

$$[W_i, \mathcal{X}]u = \Theta_0 u$$

and

$$[\bar{W}_i, \mathcal{X}]u = \Theta_0 u .$$

Proof . We only show

$$[W_i, \mathcal{X}]u = \Theta_0 u$$

and

$$|[Y^0, \mathcal{X}]u| \leqq |(k/\widetilde{b}(f))u| .$$

The other cases are proved by the same method . So we omit those . For $[W_i, \mathcal{X}]u$,

$$\begin{aligned}(W_i g(e\widetilde{b}(f)^2)u &= (W_i\widetilde{b}(f)^2)eg'(e\widetilde{b}(f)^2)u \\ &= ((W_i\widetilde{b}(f)^2)\widetilde{b}(f)^{-2})(e\widetilde{b}(f)^2 g'(e\widetilde{b}(f)^2)u .\end{aligned}$$

By the same reason as in (3) , $e\widetilde{b}(f)^2 g'(e\widetilde{b}(f)^2)$ can be estimated by a constant independent of e . Furthermore

$$\begin{aligned}(W_i\widetilde{b}(f)^2)\widetilde{b}(f)^{-2} &= (1/\widetilde{b}(f)^2)\sum_k\left\{(W_i\bar{Y}_k t_f)(Y_k t_f)+(\bar{Y}_k t_f)(W_i Y_k t_f)\right\} \\ &= (1/b(f)^2)\sum_k\left\{\sum_\ell Q_{\ell i}(X_\ell\bar{Y}_k t_f)(Y_k t_f)+(\bar{Y}_k t_f)\sum_\ell Q_{\ell i}(Y_\ell Y_k)\right. \\ &= (1/\widetilde{b}(f)^2)\left\{\sum_k Q_{ki}\alpha_f(Y_k t_f) + b(f)^2\Theta_0\right\} \\ &= \Theta_0 .\end{aligned}$$

For $[Y^{o},\chi]u$, we have

$$(Y^{o}g(e\widetilde{b}(f)^{2}))u = (Y^{o}\widetilde{b}(f)^{2})eg'(e\widetilde{b}(f)^{2})u$$

$$= ((Y^{o}\widetilde{b}(f)^{2})/\widetilde{b}(f)^{2})e\widetilde{b}(f)^{2}g'(e\widetilde{b}(f)^{2})u \quad .$$

So ,

$$((Y^{o}\widetilde{b}(f)^{2})/\widetilde{b}(f)^{2}) = (1/\widetilde{b}(f)^{2})\sum_{k}\left\{(Y^{o}\bar{Y}_{k}t_{f})(Y_{k}t_{f})+(\bar{Y}_{k}t_{f})(Y^{o}Y_{k}t_{f})\right\}$$

$$= (1/\widetilde{b}(f)^{2})\sum_{k}\left\{(1/\widetilde{b}(f))|Y_{k}t_{f}|^{2}\alpha_{f}+\Theta_{o}b(f)^{2}\right\}$$

(because $Y^{o}\bar{Y}_{k}t_{f} = \sum(\bar{Y}_{\ell}t_{f}/\widetilde{b}(f))Y_{\ell}\bar{Y}_{k}t_{f} = (\bar{Y}_{k}t_{f}/\widetilde{b}(f))\alpha_{f} + \Theta_{o}b(f)$

$$= (1/\widetilde{b}(f))\alpha_{f} + \Theta_{o} \quad .$$

So we have our lemma . Q.E.D.

With these preparations , we show Theorem 8.1 .

The proof of Main Theorem 8.1 . For u of $\Gamma(U_{r}-C,(\rho T'')^{*})$ satisfying

$$u(Y^{o}) = 0 \quad \text{on} \quad bU_{r} - C$$

and

$$(1/\widetilde{b}(f))u \quad \text{of} \quad L^{2} \quad ,$$

$W_{1}u$, $Y^{o}u$, Du , $D^{*}u$ are of L^{2} ,

$$Du(W_i,W_j) = W_iu_j - W_ju_i - u([W_i,W_j])$$

$$= W_iu_j - W_ju_i - \widetilde{b}(f)^{-1}\alpha_f(Y_it_f/\widetilde{b}(f))u_j + \widetilde{b}(f)^{-1}\alpha_f(Y_jt_f/\widetilde{b}(f))u_i + \Theta_0 u \quad (\text{ by } (8.3.1)\ )$$

$$Du(Y^0,W_i) = Y^0u_i - W_iu_0 - u([Y^0,W_i])$$

$$= Y^0u_i - W_iu_0 + \widetilde{b}(f)^{-1}(W_i\widetilde{b}(f))u_0 + \widetilde{b}(f)^{-1}\alpha_f u_i$$

$$+ \Theta_0 u \quad (\text{ by } (8.3.2)\ )$$

$$D^*u = -\sum_{k=1}^{n-1} W_k^*u_k + Y^{0*}u_0 + \Theta_0 u \quad (\text{ by Lemma } 8.6\ )$$

Hence we have

$$\| Du \|^2 + \| D^*u \|^2 = \sum_{i<j} \| (W_iu_j - W_ju_i + \Theta_0 u) \|^2$$

$$+ \sum_i \| (Y^0u_i - W_iu_0 + \widetilde{b}(f)^{-1}(W_i\widetilde{b}(f))u_0 + \widetilde{b}(f)^{-1}\alpha_f u_i) \|^2$$

$$+ \| (\sum_{k=1}^{n-1} W_k^*u_k + Y^{0*}u_0 + \Theta_0 u) \|^2$$

In order to compute this , we must introduce the following notations .

$$\| u \|'^2 = \sum_{i,j} (\| \chi W_ju_i \|^2 + \| \chi \bar{W}_ju_i \|^2 + \| \chi Y^0u_i \|^2)$$

$$+ \sum_i (\| \chi W_iu_0 \|^2 + \| \chi \bar{W}_ju_0 \|^2 + \| \chi Y^0u_0 \|^2 + \| \chi \bar{Y}^0u_0 \|^2)$$

$$+ \sum_j \| (\chi \widetilde{b}(f))u_j \|^2 + \| (\chi \widetilde{b}(f))u_0 \|^2$$

Then , there is a large constant K , which doesn't depend on e , satisfying ; for any $\varepsilon > 0$

$$8.1)\quad \|\chi Du\|^2 + \|\chi D^* u\|^2 + \varepsilon\|\chi u\|^{\cdot 2} + (K/\varepsilon)\|\chi \Theta_o u\|^2$$

$$\geqq \sum_{i\leqq j}\|\chi(W_i u_j - W_j u_i - \widetilde{b}(f)^{-1}\alpha_f(Y_i t_f/\widetilde{b}(f))u_j + \widetilde{b}(f)^{-1}\alpha_f(Y_j t_f/\widetilde{b}(f))u_i\|^2$$

$$+ \sum_i \|\chi(Y^o u_i - W_i u_o + \widetilde{b}(f)^{-1}\alpha_f u_i)\|^2$$

$$+ \|\chi(\sum_{k=1}^{n-1} W_k^* u_k + Y^{o*} u_o)\|^2 ,\quad \text{where}$$

$u_i = u(W_i)$, $u_o = u(Y^o)$. For the first term of the right hand side ,

$$8.2)\quad \sum_{i\leqq j}\|\chi(W_i u_j - W_j u_i - \widetilde{b}(f)^{-1}\alpha_f(Y_i t_f/\widetilde{b}(f))u_j + b(f)^{-1}\alpha_f(Y_j t_f/\widetilde{b}(f))u_i\|^2$$

$$= \sum_{i\leqq j}\Big\{\|\chi(W_i u_j - W_j u_i)\|^2 - 2\mathrm{Re}(\chi W_i u_j,\ \widetilde{b}(f)^{-1}\alpha_f(Y_i t_f/\widetilde{b}(f))u_j)$$

$$+ 2\mathrm{Re}(\chi W_j u_i,\ \widetilde{b}(f)^{-1}\alpha_f(Y_i t_f/\widetilde{b}(f))u_j)$$

$$+ 2\mathrm{Re}(\chi W_i u_j,\ \widetilde{b}(f)^{-1}\alpha_f(Y_j t_f/\widetilde{b}(f))u_i)$$

$$- 2\mathrm{Re}(\chi W_j u_i,\ \widetilde{b}(f)^{-1}\alpha_f(Y_j t_f/\widetilde{b}(f))u_i)$$

$$+ \|\widetilde{b}(f)^{-1}\alpha_f(Y_i t_f/\widetilde{b}(f))u_j - \widetilde{b}(f)^{-1}\alpha_f(Y_j t_f/\widetilde{b}(f))u_i\|^2\Big\}$$

$$= \sum_{i\leq j} \| \chi(W_i u_j - W_j u_i) \|^2$$

$$+ \sum_{i,j} 2\mathrm{Re}\, (\chi W_j u_i \,, \chi \widetilde{b}(f)^{-1} \alpha_f (Y_i t_f / \widetilde{b}(f)) u_j \,)$$

$$- \sum_{i,j} 2\mathrm{Re}\, (\chi W_j u_i \,, \chi \widetilde{b}(f)^{-1} \alpha_f (Y_j t_f / \widetilde{b}(f)) u_i \,)$$

$$+ \sum_j \| \widetilde{b}(f)^{-1} \alpha_f u_j \|^2$$

On the other hand ,

$$\sum_i (\chi W_j u_i \,, \chi \widetilde{b}(f)^{-1} \alpha_f (Y_i t_f / \widetilde{b}(f)) u_j \,)$$

$$= \sum_i (\chi (\bar{Y}_i t_f) W_j u_i \,, \chi \widetilde{b}(f)^{-2} \alpha_f u_j \,) \quad .$$

Since

$$\sum_i (\bar{Y}_i t_f) u_i = 0 \quad ,$$

$$\sum_i (\chi (\bar{Y}_i t_f) W_j u_i \,, \chi \widetilde{b}(f)^{-2} \alpha_f u_j \,)$$

$$= - \sum_i (\chi (W_j \bar{Y}_i t_f) u_i \,, \chi \widetilde{b}(f)^{-2} \alpha_f u_j \,)$$

$$= - \sum_i (\chi \alpha_f (\delta_{ij} - (Y_j t_f / \widetilde{b}(f))(\bar{Y}_i t_f / \widetilde{b}(f)) + \widetilde{b}(f) \Theta_0) u_i \,, \widetilde{b}(f)^{-2} \alpha_f u_j)$$

$$= \sum_i (\chi \Theta_0 u_i \,, \chi \widetilde{b}(f)^{-1} \alpha_f u_j \,)$$

Hence

$$\sum_{i,j} 2\mathrm{Re}\,(\chi W_j u_i \,,\chi \widetilde{b}(f)^{-1}\alpha_f(Y_i t_f/\widetilde{b}(f))u_j\,)$$

$$= -\sum_i (\chi(\alpha_f/\widetilde{b}(f))u_i, \chi(\alpha_f/\widetilde{b}(f))u_i\,)$$

$$+ \sum_{i,j} (\chi \Theta_o u_i \,, \chi(\alpha_f/\widetilde{b}(f))u_j\,)$$

And

$$\sum_{i,j} 2\mathrm{Re}\,(\chi W_j u_i \,,\chi \widetilde{b}(f)^{-1}\alpha_f(Y_j t_f/\widetilde{b}(f))u_i\,)$$

$$= 0 \quad (\text{ by } \sum_j (\bar{Y}_j t_f) W_j = 0\,) \quad .$$

Hence 8.2) becomes

$$8.3)\ \sum_{i\leq j} \|\chi(W_i u_j - W_j u_i)\|^2 \ -2\sum_i \|\chi(\alpha_f/\widetilde{b}(f))u_i\|^2$$

$$+ \sum_i \|\chi(\alpha_f/\widetilde{b}(f))u_i\|^2 \ + \sum_{i,j}(\chi\Theta_o u_i \,,\chi(\alpha_f/\widetilde{b}(f))u_j\,)$$

$$= \sum_{i\leq j} \|\chi(W_i u_j - W_j u_i)\|^2 - \sum_i \|\chi(\alpha_f/\widetilde{b}(f))u_i\|^2$$

$$+ \sum_{i,j}(\chi\Theta_o u_i \,,\chi(\alpha_f/\widetilde{b}(f))u_j\,)$$

Furthermore

$$8.4)\ \Sigma_i \| \chi(Y^o u_i - W_i u_o + \hat{b}(f)^{-1}\alpha_f u_i)\|^2$$

$$= \Sigma_i \Big\{ \|\chi(Y^o u_i - W_i u_o)\|^2 + 2\mathrm{Re}\,(\chi(Y^o u_i - W_i u_o), \chi\hat{b}(f)^{-1}\alpha_f u_i)$$

$$+ \|\chi\hat{b}(f)^{-1}\alpha_f u_i\|^2 \Big\}$$

Hence by 8.3) and 8.4) , 8.1) becomes

$$8.5)\ \|\chi D u\|^2 + \|\chi D^* u\|^2 + \varepsilon\|\chi u\|^2 + (K/\varepsilon)\|\chi \Theta_o u\|^2$$

$$\geqq \Sigma_{i\leqq j} \|\chi(W_i u_j - W_j u_i)\|^2 + \Sigma_i \|\chi(Y^o u_i - W_i u_o)\|^2$$

$$+ \Sigma_i\, 2\mathrm{Re}\,(\chi(Y^o u_i - W_i u_o), \chi\hat{b}(f)^{-1}\alpha_f u_i)$$

$$+ \|\chi(\textstyle\sum_{k=1}^{n-1} W_k^* u_k)\|^2 + \Sigma_k\, 2\mathrm{Re}\,(\chi W_k^* u_k, \chi Y^{o*} u_o)$$

$$+ \|\chi Y^{o*} u_o\|^2$$

$$= \Big\{ \Sigma_{i\leqq j} \|\chi(W_i u_j - W_j u_i)\|^2 + \Sigma_i\, 2\mathrm{Re}\,(\chi Y^o u_i, \chi\hat{b}(f)^{-1}\alpha_f u_i)$$

$$+ \|\chi(\textstyle\sum_{k=1}^{n-1} W_k u_k)\|^2 + \Sigma_i \|\chi Y^o u_i\|^2 \Big\}$$

$$+ \Big\{ \Sigma_i \|\chi W_i u_o\|^2 + \|\chi Y^{o*} u_o\|^2 \Big\} - \Sigma_i\, 2\mathrm{Re}\,(\chi W_i u_o, \chi\hat{b}(f)^{-1}\alpha_f u_i)$$

$$- \Sigma_i\, 2\mathrm{Re}\,(\chi Y^o u_i, \chi W_i u_o) + \Sigma_k\, 2\mathrm{Re}\,(\chi W_k^* u_k, \chi Y^{o*} u_o)$$

We manipulate

$$- \Sigma_i\, 2\mathrm{Re}\,(\chi Y^o u_i, \chi W_i u_o) + \Sigma_k\, 2\mathrm{Re}\,(\chi W_k^* u_k, \chi Y^{o*} u_o)$$

For this , we have

Lemma 8.8 . For any $\varepsilon>0$, there is a constant K satisfying

$$\lim_{e\to+\infty} \left| \sum_\ell 2\mathrm{Re}\,(\chi W_\ell^* u_\ell , \chi Y^{o*} u_o) - \sum_\ell 2\mathrm{Re}\,(\chi Y^o u_\ell , \chi W_\ell u_o) \right|$$

$$\leq \varepsilon \|\chi u\|^{\cdot 2} + (K/\varepsilon)\|\chi \Theta_o u\|^2 \quad \text{for } u \text{ satisfying (i)}$$

and (ii) .

Proof . We show , by a direct computation , i.e., in integral by parts . Namely , we have

$$2\mathrm{Re}\,(\chi W_\ell^* u_\ell , \chi Y^{o*} u_o)$$

$$= 2\mathrm{Re}\,(\chi W_\ell^* u_\ell , Y^{o*}(\chi u_o)) + 2\mathrm{Re}\,(\chi W_\ell^* u_\ell , [\chi, Y^{o*}] u_o) .$$

First , we see

$$\lim_{e\to+\infty} 2\mathrm{Re}\,(\chi W_\ell^* u_\ell , [\chi , Y^{o*}] u_o) = 0 .$$

Because

$$\begin{aligned} 2\mathrm{Re}\,(\chi W_\ell^* u_\ell , [\chi , Y^{o*}] u_o) &= 2\mathrm{Re}\,([\chi , W_\ell^*] u_\ell , [\chi , Y^{o*}] u_o) \\ &\quad + 2\mathrm{Re}\,(W_\ell^*(\chi u_\ell), [\chi, Y^{o*}] u_o) \\ &= 2\mathrm{Re}\,([\chi , W_\ell^*] u_\ell , [\chi , Y^{o*}] u_o) \\ &\quad + 2\mathrm{Re}\,(\chi u_\ell , W_\ell([\chi, Y^{o*}] u_o)) \end{aligned}$$

$$= 2\mathrm{Re}\left([\chi, W_\ell^*]u_\ell \;, [\chi, Y^{o*}]u_o \right)$$

$$+ 2\mathrm{Re}\left(\chi u_\ell \;, (W_\ell[\chi, Y^{o*}])u_o \right)$$

$$+ 2\mathrm{Re}\left(\chi u_\ell \;, [\chi, Y^{o*}] W_\ell u_o \right) .$$

By Lemma 8.7 with $W_\ell u$, $\tilde{b}(f)^{-1}u$ being of L^2 ,

$$2\mathrm{Re}\left([\chi, W_\ell^*]u_\ell \;, [\chi, Y^{o*}]u_o \right)$$

and

$$2\mathrm{Re}\left(\chi u_\ell \;, [\chi, Y^{o*}] W_\ell u_o \right)$$

converge to zero as $e \to +\infty$. Furthermore by a direct computation , we have

$$|W_\ell(Y^{o*}\chi)| \leqq |(k/\tilde{b}(f)^2)u_o| .$$

So

$$2\mathrm{Re}\left(\chi u_\ell \;, (W_\ell[\chi, Y^{o*}])u_o \right)$$

converges to zero as $e \to +\infty$. In integral by parts ,

$$2\mathrm{Re}(\chi W_\ell^* u_\ell \;, Y^{o*}(\chi u_o)) = 2\mathrm{Re}(Y^o(\chi W_\ell^* u_\ell) \;, \chi u_o)$$

$$= 2\mathrm{Re}((Y^o\chi)W_\ell^* u \;, \chi u_o) + 2\mathrm{Re}(\chi Y^o W_\ell^* u_\ell \;, \chi u_o)$$

Similarly, because of

$$2\mathrm{Re}\,(\,(Y^0\chi)W_\ell^* u_\ell\,,\chi u_0\,) = 2\mathrm{Re}(\,[(Y^0\chi),W_\ell^*]u_\ell\,,\chi u_0\,)$$

$$+\,2\mathrm{Re}(\,W_\ell^*((Y^0\chi)u_\ell),\ \ u_0\,)$$

$$= 2\mathrm{Re}(\,[Y^0\chi\,,W_\ell^*]u\ \ ,\chi u_0\,)$$

$$+\,2\mathrm{Re}(\,(Y^0\chi)u_\ell\,,\ W_\ell(\chi u_0)\,)\quad,$$

we have

$$\lim_{e\to+\infty} 2\mathrm{Re}\,(\,(Y^0\chi)W_\ell^* u_\ell\,,\chi u_0\,) = 0\quad.$$

Furthermore

$$2\mathrm{Re}\,(\chi Y^0 W_\ell^* u_\ell\,,\chi u_0\,) = 2\mathrm{Re}\,(\chi\, W_\ell^* Y^0 u_\ell\,,\chi\, u_0\,)$$

$$+\,2\mathrm{Re}\,(\chi[Y^0,W_\ell^*]u_\ell\,,\chi u_0\,)\quad.$$

By Lemma 8.4 ,

$$[Y^0,W_\ell^*]\ u_\ell = [Y^0,\ -\bar{W}_\ell + ((n-2)/\widehat{b}(f)^2)(\bar{Y}_\ell t_f)\bar{\alpha}_f + \Theta_0]\ u_\ell\quad,$$

$$= [Y^0,\ -\bar{W}_\ell]\ u_\ell + [Y^0,((n-2)(\bar{Y}_\ell t_f)\bar{\alpha}_f/\widehat{b}(f)^2]u_\ell + \Theta_1\ u$$

$$= [Y^0,\ -\bar{W}_\ell]\ u_\ell + ((n-2)^2(\bar{Y}_\ell t_f)\bar{\alpha}_f/\widehat{b}(f)^2)u\ \ +\ \Theta_1\ u$$

$$= [Y^o , - \bar{W}_\ell]u_\ell + \ominus_1 u \quad (\text{by } \Sigma_\ell (\bar{Y}_\ell t_f)u_\ell = 0)$$

$$= \Sigma_k \ominus_o W_k u + \Sigma_k \ominus_o \bar{W}_k u + \ominus_o Y^o u + \ominus_o \bar{Y}^o u$$

$$+ \ominus_1 u$$

Therefore

$$2\mathrm{Re}(\chi [Y^o, W_\ell^*] u_\ell, \chi u_o)$$

$$= 2\mathrm{Re}(\chi(\Sigma_k \ominus_o W_k u_\ell + \Sigma_k \ominus_o \bar{W}_k u_\ell + \ominus_o Y^o u_\ell + \ominus_o \bar{Y}^o u_\ell , \quad u_o)$$

$$+ \quad 2\mathrm{Re} (\chi \ominus_1 u_\ell , \chi u_o)$$. So this can be estimated by

$$\varepsilon \|\chi u \|'^2 + (K/\varepsilon) \| \ominus_1 u \|^2 .$$

Now we see that $2\mathrm{Re}(\chi W_\ell^* Y^o u_\ell , \chi u_o) - 2\mathrm{Re}(\chi Y^o u_\ell , \chi W_\ell u_o)$ converges to 0 as $e \to +\infty$. However, by

$$2\mathrm{Re}(\chi W_\ell^* Y^o u , \chi u_o) = 2\mathrm{Re}(W_\ell^* (\chi Y^o u_\ell), \chi u_o) + 2\mathrm{Re}([\chi, W_\ell^*] Y^o u_\ell , \chi u_o) ,$$

obviously we have

$$\lim_{e \to +\infty} 2\mathrm{Re}([\chi, W_\ell^*] Y^o u_\ell , \chi u_o) = 0 .$$

Furthermore

$$2\mathrm{Re}(W_\ell(\chi Y^o u_\ell), \chi u_o) = 2\mathrm{Re}(\chi Y^o u_\ell, W_\ell(\chi u_o))$$

$$= 2\mathrm{Re}(\chi Y^o u_\ell, (W_\ell \chi) u_o) + 2\mathrm{Re}(\chi Y^o u_\ell, \chi W_\ell u_o)$$

And we already know

$$\lim_{e \to +\infty} 2\mathrm{Re}(\chi Y^o u_\ell, (W_\ell \chi) u_o) = 0 .$$

So we have our lemma . Q.E.D.

Henceforth in the process of integral by parts , we omit the term which includes $\bar{W}_i\chi$, $W_i\chi$, $Y^o\chi$, $\bar{Y}^o\chi$. As we know in the proof of Lemma 8.8 , for example ,

$$(\chi W_i u_j, \chi W_i u_j) = ([\chi, W_i]u_j, \chi W_i u_j) + (W_i(\chi u_j), \chi W_i u_j)$$

$$= ([\chi, W_i]u_j, \chi W_i u_j) + (\chi u_j, W_i^*(\chi W_i u_j))$$

$$= ([\chi, W_i]u_j, \chi W_i u_j) + (\chi u_j, (W_i^*\chi) W_i u_j) + (\chi u_j, \chi W_i^* W_i u_j) .$$

In this equality , we proved

$$\lim_{e \to +\infty} ([\chi, W_i]u_j, \chi W_i u_j) = 0$$

and

$$\lim_{e \to +\infty} (\chi u_j, (W_i^*\chi) W_i u_j) = 0 .$$

Hence , for brevity we will write as follows .

$$(\chi W_i u_j , \chi W_i u_j) = (\chi u_j , \chi W_i^* W_i u_j) + A(\chi) ,$$

where $\lim_{e \to +\infty} A(\chi) = 0$. And of course many $A(\chi)$'s appear and may differ . However in order to avoid unnecessary complications , we adopt this notation . By this lemma , we have

$$
\begin{aligned}
8.5)\ & \|\chi Du\|^2 + \|\chi D^* u\|^2 + \varepsilon\|\chi u\|'^2 + (K/\varepsilon)\|\chi \textcircled{H}_o u\|^2 \\
& \geq \Big\{ \Sigma_{i \leq j} \|\chi(W_i u_j - W_j u_i)\|^2 + \|\chi(\Sigma_{k=1}^{n-1} W_k^* u_k)\|^2 \\
& \quad + \Sigma_i \|\chi Y^o u_i\|^2 \Big\} + \Sigma_i 2\mathrm{Re}\, (\chi Y^o u_i , \chi \widetilde{b}(f)^{-1} \alpha_f u_i) \\
& \quad + \Big\{ \Sigma_i \|\chi W_i u_o\|^2 + \|\chi Y^{o*} u_o\|^2 \Big\} \\
& \quad - \Sigma_i 2\mathrm{Re}\, (\chi W_i u_o , \chi \widetilde{b}(f)^{-1} \alpha_f u_i) + A(\chi) ,
\end{aligned}
$$

where $\lim_{e \to +\infty} A(\chi) = 0$.

In order to prove the main theorem , we show

$$
\begin{aligned}
(\mathrm{I})\ & \Sigma_{i \leq j} \|\chi(W_i u_j - W_j u_i)\|^2 + \|\chi(\Sigma_{k=1}^{n-1} W_k^* u_k)\|^2 \\
& + \varepsilon\|\chi u\|'^2 + (K/\varepsilon)\|\chi \textcircled{H}_o u\|^2
\end{aligned}
$$

$$\geqq ((n-3)/(n-2)) \sum_{i,j} \|\chi W_i u_j\|^2 + (1/(n-2)) \sum_{i,j} \|\chi \bar{W}_i u_j\|^2$$

$$+ (n-2) \sum_i \|\chi (\alpha_f/\widetilde{b}(f)) u_i\|^2 + A(\chi), \text{ where } \lim_{e \to +\infty} A(\chi) = 0,$$

(II) $$\sum_i \|\chi W_i u_o\|^2 + \|\chi Y^{o*} u_o\|^2 + \varepsilon \|\chi u\|_{\cdot}^2 + (K/\varepsilon)\|\chi \Theta_o u\|^2$$

$$\geqq ((n-3)/(n-2)) \sum_i \|\chi W_i u_o\|^2 + (1/(n-2)) \sum_i \|\chi \bar{W}_i u_o\|^2$$

$$+ (1/(4n-3))\|\chi \bar{Y}^o u_o\|^2 + ((n-2)/2) \|(\bar{\alpha}_f/2b(f))\chi u_o\|^2$$

$$+ (1/2) \|\chi Y^o u_o\|^2 + A(\chi), \text{ where } \lim_{e \to +\infty} A(\chi) = 0$$

If the estimate (I) is proved, then 8.5) becomes

8.6) $$\|\chi Du\|^2 + \|\chi D^* u\|^2 + \varepsilon \|\chi u\|_{\cdot}^2 + (K/\varepsilon)\|\chi \Theta_o u\|^2$$

$$\geq ((n-3)/(n-2)) \sum_{i,j} \|\chi W_i u_j\|^2 + (1/(n-2)) \sum_{i;j} \|\chi \bar{W}_i u_j\|^2$$

$$+ (n-2) \sum_i \|\chi (\alpha_f/\widetilde{b}(f)) u_i\|^2 + \sum_i \|\chi Y^o u_i\|^2$$

$$+ \sum_i 2\mathrm{Re}\, (\chi Y^o u_i, \chi \widetilde{b}(f)^{-1} \alpha_f u_i)$$

$$+ \left\{ \sum_i \|\chi W_i u_o\|^2 + \|\chi Y^{o*} u_o\|^2 \right\}$$

$$- \sum_i 2\mathrm{Re}\, (\chi W_i u_o, \chi \widetilde{b}(f)^{-1} \alpha_f u_i)$$

$$+ A(\chi), \text{ where } \lim_{e \to +\infty} A(\chi) = 0 .$$

Namely ,

$$8.6)' \quad \|\chi D u\|^2 + \|\chi D^* u\|^2 + \varepsilon\|\chi u\|^{\cdot 2} + (K/\varepsilon)\|\chi \Theta_o u\|^2$$

$$\geq ((n-3)/(n-2)) \sum_{i,j} \|\chi W_i u_j\|^2 + (1/(n-2)) \sum_{i,j} \|\chi \bar{W}_i u_j\|^2$$

$$+ \quad (n-3) \sum_i \|\chi (\alpha_f / \widehat{b}(f)) u_i\|^2$$

$$+ \quad \sum_i \|\chi (Y^o u_i + (\alpha_f / \widehat{b}(f)) u_i)\|^2$$

$$+ \quad \left\{ \sum_i \|\chi W_i u_o\|^2 + \|\chi Y^{\alpha *} u_o\|^2 \right\}$$

$$- \quad \sum_i 2\mathrm{Re}\, (\chi W_i u_o , \chi \widehat{b}(f)^{-1} \alpha_f u_i) + A(\chi)$$

$$= \quad ((n-3)/(n-2)) \sum_{i,j} \|\chi W_i u_j\|^2 + (1/(n-2)) \sum_{i,j} \|\chi \bar{W}_i u_j\|^2$$

$$+ \quad (n-4) \sum_i \|\chi (\alpha_f / \widehat{b}(f)) u_i\|^2$$

$$+ \quad \sum_i \|\chi (Y^o u_i + (\alpha_f / \widehat{b}(f)) u_i\|^2$$

$$+ \quad \sum_i \|\chi (W_i u_o - (\alpha_f / \widehat{b}(f)) u_i\|^2$$

$$+ \quad \|\chi Y^{\alpha *} u_o\|^2 + A(\chi) , \quad \text{where} \quad \lim_{e \to +\infty} A(\chi) = 0 .$$

Hence , if $n \geq 4$, then

Hence , if $n \geq 4$, then

8.7) $\|\chi Du\|^2 + \|\chi D^* u\|^2 + \varepsilon\|\chi u\|'^2 + (K/\varepsilon)\|\chi \Theta_o u\|^2$

$\geqq ((n-3)/(n-2)) \sum_{i,j} \|\chi W_i u_j\|^2 + (1/(n-2)) \sum_{i,j} \|\chi \bar{W}_i u_j\|^2$

$+ A(\chi)$, where $\lim_{e \to +\infty} A(\chi) = 0$.

By 8.7) with (I) , we can estimate $(\alpha_f / \hat{b}(f))u_i$ as follows .

8.8) $\|\chi Du\|^2 + \|\chi D^* u\|^2 + \varepsilon\|\chi u\|'^2 + (K/\varepsilon)\|\chi \Theta_o u\|^2$

$\geqq c \sum_i \|\chi (\alpha_f / \hat{b}(f))u_i\|^2 + A(\chi)$, where $\lim_{e \to +\infty} A(\chi) = 0$.

Hence with 8.6)′, we can estimate $Y^o u_i$ as follows .

8.9) $\|\chi Du\|^2 + \|\chi D^* u\|^2 + \varepsilon\|\chi u\|'^2 + (K/\varepsilon)\|\chi \Theta_o u\|^2$

$\geqq c \sum_i \|\chi Y^o u_i\|^2 + A(\chi)$.

For u_o , by 8.6)′ and the estimates for u_i , we have

8.10) $\|\chi Du\|^2 + \|\chi D^* u\|^2 + \varepsilon\|\chi u\|'^2 + (K/\varepsilon)\|\chi \Theta_o u\|^2$

$\geqq c \sum_i \|\chi W_i u_o\|^2 + \|\chi Y^{o*} u_o\|^2 + A(\chi)$, where $\lim_{e \to +\infty} A(\chi) =$

Therefore if the estimate (II) is proved , our proof for u_o is complete . So we must show (I) and (II) .

The proof of (I)

$$\sum_{i\leq j} \|\chi(W_i u_j - W_j u_i)\|^2 + \|\chi(\sum_{k=1}^{n-1} W_k^* u_k)\|^2$$

$$= \sum_{i\leq j} \|\chi(W_i u_j - W_j u_i)\|^2 + \|\chi(-\sum_{k=1}^{n-1} \bar{W}_k u_k)\|^2 + B(u)$$

(by Lemma 8.4) , where $B(u)$ means the term which can be estimated by $\varepsilon\|\chi u\|'^2 + (K/\varepsilon)\|\chi\Theta_0 u\|^2$. Henceforth we use this notation (of course many $B(u)$'s appear and may differ) . With this notation the above becomes

$$\sum_{i,j}\|\chi W_i u_j\|^2 + \mathrm{Re}\sum_{i,j}\{(\chi\bar{W}_i u_i, \chi\bar{W}_j u_j) - (\chi W_j u_i, \chi W_i u_j)\}$$

$$+ \quad B(u) \ .$$

And

$$\sum_{i,j} (\chi\bar{W}_i u_i, \chi\bar{W}_j u_j)$$

$$= \sum_{i,j} (\chi\bar{W}_j^*\bar{W}_i u_i, \chi u_j) + A(\chi)$$

$$= \sum_{i,j} (\chi(-W_j\bar{W}_i u_i + ((n-2)/\widetilde{b}(f)^2)(Y_j t_f)\alpha_f(\bar{W}_i u_i)), \chi u_j)$$

$$+ A(\chi) + B(u)$$

$$= \sum_{i,j} (\chi[\bar{W}_i, W_j] u_i, \chi u_j) - (\chi\bar{W}_i W_j u_i, \chi u_j)$$

$$+ (\chi((n-2)/\widetilde{b}(f)^2)(Y_j t_f)\alpha_f(\bar{W}_i u_i), \chi u_j) + A(\chi) + B(u)$$

$$= \Sigma_{i,j} (\chi[\bar{W}_i, W_j]u_i, \chi u_j) - (\chi \bar{W}_i W_j u_i, \chi u_j)$$

$$+ A(\chi) + B(u) \quad (\text{ by } \Sigma_j (\bar{Y}_j t_f) u_j = 0) \ .$$

On the other hand ,

$$\Sigma_{i,j} - (\chi \bar{W}_i W_j u_i, \chi u_j) + \varepsilon \|\chi u\|^2 + (K/\varepsilon)\|\chi \Theta_0 u\|^2$$

$$= \Sigma_{i,j} (\chi W_j u_i, \chi(W_i u_j - ((n-2)/\sigma(f)^2)(Y_i t_f)\alpha_f u_j))$$

$$+ \Sigma_{i,j} (\chi W_j u_i, \chi \Theta_0 u) + A(\chi) + B(u)$$

$$= \Sigma_{i,j} (\chi W_j u_i, \chi W_j u_i) - \Sigma_{i,j} (\chi W_j u_i, ((n-2)/\sigma(f)^2)(Y_i t_f)\alpha_f u_j))$$

$$+ A(\chi) + B(u)$$

Furthermore

$$- \Sigma_{i,j} (\chi W_j u_i, ((n-2)/\sigma(f)^2)(Y_i t_f) \ _f u_j)$$

$$= \Sigma_{i,j} (\chi u_i, (\bar{W}_j - ((n-2)/\sigma(f)^2)(\bar{Y}_j t_f)\alpha_f + \Theta_0)((n-2)/\sigma(f)^2)((Y_i t_f)\alpha_f$$

$$+ A(\chi)$$

$$= \Sigma_{i,j} (\chi u_i, \bar{W}_j((n-2)/\sigma(f)^2)(Y_i t_f)\alpha_f u_j)) + A(\chi) + B(u)$$

$$= \sum_{i,j} (\chi u_i, \chi(\bar{W}_j(Y_i t_f))((n-2)/\tilde{b}(f)^2)\alpha_f u_j) + A(\chi)$$

$$+ B(u) \quad (\text{ by } \sum_i (\bar{Y}_i t_f) W_i = 0) .$$

On the other hand

$$\bar{W}_j(Y_i t_f) = \bar{Y}_j Y_i t_f - (\bar{Y}_j t_f/b(f)) \sum_{\ell=1}^{n-1} (Y_\ell t_f/\tilde{b}(f)) \bar{Y}_\ell Y_i t_f$$

$$= (\delta_{ji} - (\bar{Y}_j t_f/\tilde{b}(f)) \sum_{\ell=1}^{n-1} (Y_\ell t_f/b(f)) \delta_{\ell i}) \bar{\alpha}_f + \tilde{b}(f) \Theta_0$$

$$= (\delta_{ji} - ((\bar{Y}_j t_f)(Y_i t_f)/\tilde{b}(f)^2)) \alpha_f + b(f) \Theta_0 .$$

Hence

$$\sum_{i,j} (\chi u_i, \chi(\bar{W}_j(Y_i t_f))((n-2)/\tilde{b}(f)^2)\alpha_f u_j) + \varepsilon \|\chi u\|'^2 + (K/\varepsilon) \|\chi \Theta_0 u\|^2$$

$$\geqq \sum_{i,j} (\chi u_i, \chi \delta_{ji} \alpha_f \bar{\alpha}_f ((n-2)/\tilde{b}(f)^2) u_j)$$

$$= (n-2) \sum_i \| \chi(\alpha_f/b(f)) u_i \|^2$$

So we have

$$\sum_{i \leq j} \| \chi (W_i u_j - W_j u_i) \|^2 + \|\chi(- \sum_{k=1}^{n-1} \bar{W}_k u_k)\|^2 + \varepsilon \|\chi u\|'^2 + (K/\varepsilon) \|\chi \Theta_0 u \|^2$$

$$\geqq \sum_{i,j} \|\chi W_i u_j \|^2 - \operatorname{Re} \sum_{i,j} (\chi [W_j, \bar{W}_i] u_i, \chi u_j)$$

$$+ (n-2) \sum_i \| \chi(\alpha_f/\tilde{b}(f)) u_i \|^2 + A(\chi) .$$

By Proposition 8.2 ,

$$-\mathrm{Re}\ \sum_{i,j}(\chi[W_j,\bar{W}_i]u_i,\chi u_j)+\varepsilon\|\chi u\|'^2+(K/\varepsilon)\|\chi\Theta_0 u\|^2$$

$$\geq -\mathrm{Re}\ \sum_{i,j}(\chi b(f)^{-1}(\delta_{i,j}-((\bar{Y}_i t_f)(Y_j t_f)/\hat{b}(f)^2)X^f u_i,\chi u_j)$$

$$= -\mathrm{Re}\ \sum_{i,j}(\chi b(f)^{-1}\delta_{i,j}X^f u_i,\chi u_j)$$

$$= -\mathrm{Re}\ \sum_i(\chi b(f)^{-1}X^f u_i,\chi u_i)$$

Namely we have

$$\sum_{i\leq j}\|\chi(W_i u_j-W_j u_i)\|^2+\|\chi(-\sum_{k=1}^{n-1}\bar{W}_k u_k)\|^2$$

$$+\varepsilon\|\chi u\|'^2+(K/\varepsilon)\|\chi\Theta_0 u\|^2$$

$$=\sum_{i,j}\|\chi W_i u_j\|^2-\sum_i \mathrm{Re}(\chi b(f)^{-1}X^f u_i,\chi u_j)$$

$$+\varepsilon\|\chi u\|'^2+(K/\varepsilon)\|\chi\Theta_0 u\|^2$$

$$=((n-3)/(n-2))\sum_{i,j}\|\chi W_i u_j\|^2$$

$$+(1/(n-2))\sum_{i,j}(\|\chi\bar{W}_i u_j\|^2-\mathrm{Re}(\chi b(f)^{-1}(n-2)X^f u_i,\chi u_j))$$

$$+(n-2)\sum_i\|(\chi/b(f))\alpha_f^i u_i\|^2 .$$

On the other hand we have

Lemma 8.9 . Let v be a C^∞-function on $\bar{U}_r - C$.. Then

$$\Sigma_j \|\chi W_j v\|^2 + \varepsilon\|\chi v\|'^2 + (K/\varepsilon)\|\chi \Theta_0 v\|^2$$

$$\geqq \Sigma_j \|\chi \bar{W}_j v\|^2 + \mathrm{Re}(\chi \tilde{b}(f)^{-1}(n-2)X^f v, \chi v) + A(\chi) .$$

Proof . $\Sigma_j \|\chi W_j v\|^2 + \varepsilon\|\chi v\|'^2 + (K/\varepsilon)\|\chi \Theta_0 v\|^2$

$$= \Sigma_j (\chi W_j^* W_j v, \chi v) + \varepsilon\|\chi v\|'^2 + (K/\varepsilon)\|\chi \Theta_0 v\|^2 + A(\chi)$$

$$= \Sigma_j (\chi(-\bar{W}_j + ((n-2)/\tilde{b}(f)^2)(\bar{Y}_j t_f)\bar{\alpha}_f) W_j v, \chi v)$$

$$+ \varepsilon\|\chi v\|'^2 + (K/\varepsilon)\|\chi \Theta_0 v\|^2 + A(\chi)$$

$$= \Sigma_j (\chi(-\bar{W}_j W_j v), \chi v) + \varepsilon\|\chi v\|'^2 + (K/\varepsilon)\|\chi \Theta_0 v\|^2 + A(\chi)$$

(by $\Sigma_j (\bar{Y}_j t_f) W_j = 0$)

$$= \Sigma_j (\chi[W_j, \bar{W}_j]v, \chi v) + \Sigma_j (\chi(-W_j \bar{W}_j v), \chi v)$$

$$+ \varepsilon\|\chi v\|'^2 + (K/\varepsilon)\|\chi \Theta_0 v\|^2 + A(\chi)$$

$$\geqq \mathrm{Re}(\chi b(f)^{-1}(n-2)X^f v, \chi v) + \Sigma_j (\chi \bar{W}_j v, \chi \bar{W}_j v) + A(\chi)$$

So we have our lemma . Q.E.D.

By this lemma , we have

$$\Sigma_{i\leq j} \|\chi(W_i u_j - W_j u_i)\|^2 + \|\chi u - \Sigma_{k=1}^{n-1} \bar{W}_k u_k)\|^2$$

$$+ \varepsilon\|\chi u\|'^2 + (K/\varepsilon)\|\chi \Theta_o u\|^2$$

$$\geqq ((n-3)/(n-2)) \Sigma_{i,j} \|\chi W_i u_j\|^2 + (1/(n-2)) \Sigma_{i,j} \|\chi \bar{W}_i u_j\|^2$$

$$+ (n-2) \Sigma_i \|\chi(\alpha_f/b(f))u_i\|^2 + A(\chi) \quad .$$

Therefore we have (I) . Next we proceed to the proof of (II) .

The proof of (II) . By Lemma 8.9 , we have

$$\Sigma_j \|\chi W_j u_o\|^2 + \varepsilon\|\chi u_o\|'^2 + (K/\varepsilon)\|\chi \Theta_o u_o\|'^2$$

$$\geqq \Sigma_j \|\chi \bar{W}_j u_o\|^2 + \mathrm{Re}(\chi \widetilde{b}(f)^{-1} X^f u_o , \chi u_o) \quad .$$

And by Lemma 8.5

$$Y^{o*} = - \bar{Y}^o - ((2n-3)/2\widehat{b}(f))\bar{\alpha}_f + \Theta_o \quad .$$

Therefore

$$\|\chi Y^{o*} u_o\|^2 + \varepsilon\|\chi u_o\|'^2 + (K/\varepsilon)\|\chi \Theta_o u_o\|^2$$

$$\geqq \|\chi(\bar{Y}^o + ((2n-3)/2\widetilde{b}(f))\bar{\alpha}_f) u_o\|^2$$

$$= \|\chi \bar{Y}^o u_o\|^2 + 2\mathrm{Re}(\chi \bar{Y}^o u_o , \chi((2n-3)/2b(f))\bar{\alpha}_f u_o)$$

$$+ \|\chi((2n-3)/2\widehat{b}(f))\bar{\alpha}_f u_o\|^2$$

On the other hand ,

$$\|\chi Y^{o*}u_o\|^2 + \varepsilon\|\chi u_o\|^{\cdot 2} + (K/\varepsilon)\|\chi \Theta_o u_o\|^2$$

$$= (\chi Y^o Y^{o*}u_o , \chi u_o) + A(\chi) + \varepsilon\|\chi u_o\|^{\cdot 2} + (K/\varepsilon)\|\chi\Theta_o u_o\|^2$$

$$= (\chi Y^{o*}Y^o u_o , \chi u_o) + (\chi[Y^o, Y^{o*}] u_o , \chi u_o)$$

$$+ A(\chi) + \varepsilon\|\chi u_o\|^{\cdot 2} + (K/\varepsilon)\|\chi\Theta_o u_o\|^2$$

$$= (\chi Y^o u_o , \chi Y^o u_o) + (\chi[Y^o, -\bar{Y}^o - ((2n-3)/2\widehat{b}(f))\bar{\alpha}_f + \Theta_o]u_o , \chi u_o)$$

$$+ A(\chi) + \varepsilon\|\chi u_o\|^{\cdot 2} + (K/\varepsilon)\|\chi\Theta_o u_o\|^2$$

$$\geq \|\chi Y^o u_o\|^2 - \mathrm{Re}(\chi[Y^o, \bar{Y}^o] u_o , \chi u_o)$$

$$+ \mathrm{Re}(\chi((2n-3)/2\widehat{b}(f)^2)\bar{\alpha}_f(Y^o\widehat{b}(f))u_o , \chi u_o) + (\chi\Theta_1 u_o , \chi u_o)$$

$$+ A(\chi) + B(u)$$

$$\geq \|\chi Y^o u_o\|^2 + ((2n-3)/4)(\chi(\alpha_f/\widehat{b}(f))u_o , \chi(\alpha_f/b(f))u_o)$$

$$- \mathrm{Re}(\chi[Y^o, \bar{Y}^o]u_o , \chi u_o) + A(\chi) + B(u)$$

By (8.3.4) , this becomes

$$\|\chi Y^o u_o\|^2 + ((2n-3)/4)\|\chi(\alpha_f/\widehat{b}(f))u_o\|^2 - \mathrm{Re}(\chi\widehat{b}(f)^{-1}[Y^o, \widehat{b}(f)]\bar{Y}^o u_o , \chi u$$

$$+ \mathrm{Re}(\chi\widehat{b}(f)^{-1}[\bar{Y}^o, \widehat{b}(f)]Y^o u_o , \chi u_o) - \mathrm{Re}(\chi\widehat{b}(f)^{-1}X^f u_o , \chi u_o)$$

$$+ A(\chi) + B(u) .$$

As $Y^{o}b(f) = \alpha_f/2 + b(f)\Theta_o$,

$$(1/2(2n-3))\|\chi Y^{o}{*}u_o\|^2 + \|\chi Y^{o}{*}u_o\|^2 + \varepsilon\|\chi u_o\|^{\cdot 2} + (K/\varepsilon)\|\chi\Theta_o u_o\|$$

$$\geqq (1/2(2n-3))\|\chi\bar{Y}^{o}u_o\|^2 + (1/2(2n-3))\|\chi((2n-3)/2\widehat{b}(f))\bar{\alpha}_f u_o\|^2$$

$$+ \|\chi Y^{o}u_o\|^2 + ((2n-3)/4)\|\chi(\alpha_f'/\widehat{b}(f))u_o\|^2$$

$$+ \mathrm{Re}(\chi\widehat{b}(f)^{-1}[\bar{Y}^{o},b(f)]\,Y^{o}u_o , \chi u_o)$$

$$- \mathrm{Re}(\chi\widehat{b}(f)^{-1}X^{f}u_o , \chi u_o) + A(\chi)$$

So we have

$$((4n-5)(n-2)/2(2n-3))\|\chi Y^{o}{*}u_o\|^2 + 2\textstyle\sum_j\|\chi W_j u_o\|^2$$

$$\geqq ((n-2)/2(2n-3))\|\chi\bar{Y}^{o}u_o\|^2 + ((n-2)(2n-3)/2)\|\chi(\alpha_f/2b(f))u_o\|^2$$

$$+ ((n-2)/2)\|\chi Y^{o}u_o\| + \textstyle\sum_j\|\chi\bar{W}_j u_o\|^2 + \textstyle\sum_j\|\chi W_j u_o\|^2 + A(\chi) .$$

Therefore we have (II) .

By (I) and (II) , we have

$$\|\chi Du\|^2 + \|\chi D{*}u\|^2 + \varepsilon\|\chi u\|^{\cdot 2} + (K/\varepsilon)\|\chi\Theta_o u\|^2$$

$\geqq C\|\chi u\|^{\cdot 2} + A(\chi)$, where C is a positive constant independent of ε and r . So , let ε be $(1/2)C$. Then ,

$$\|\chi Du\|^2 + \|\chi D{*}u\|^2 + (K/\varepsilon)\|\chi\Theta_o u\|^2 \geqq (1/2)C\|\chi u\|^{\cdot 2} + A(\chi) .$$

Furthermore if we choose r sufficiently small, we can assume

$$(2C/\varepsilon)\|\chi \textcircled{H}_0 u\|^2 \leq (1/4)C \|(\alpha/\widetilde{b}(f))u\|^2 .$$

So we have

$$\|\chi Du\|^2 + \|\chi D^*u\|^2 \geqq (1/4)C\|\chi u\|^{\cdot 2} + A(\alpha) .$$

If we let e to $+\infty$, then we have our theorem. Q.E.D.

By the main theorem, we have an L^2-solution for D. Namely, for an element v of $\Gamma(\bar{U}_r - C, ({}^\circ T'')^*)$ satisfying

(i) Dv, D^*v, $\widetilde{b}(f)^{-1}v$, $Y^o v$, $W_j v$ are of L^2,

there is an L^2-element u satisfying

$$Du = v ,$$

where $W_j u$, $\bar{W}_j u$, $Y^o u$, $\bar{Y}^o u$, $W_i W_j u$, $W_i \bar{W}_j u$, $\bar{W}_i W_j u$, $\bar{W}_i \bar{W}_j u$, $(1/\widetilde{b}(f)^2)u$ are of L^2. And if v is of C^∞, then u is also of C^∞ on U_r by the interior regularity theorem. By using this, we show the local embedding theorem. For the embedding f of C^k-class, established in Chapter 7, i.e., (namely f satisfies that f is of C^k and of C^∞ on $U_r(f) - C$, and

(ii) $D_b f = 0$ along t_f

and

(iii) $(1/\widetilde{b}(f)^{\ell})Df$ is of L^2 .

Now we consider the differential equation

$$Du = Df$$

where f is the above C^{∞}-embedding of a neighborhood $U_r(f)$ satisfying (ii) and (iii) . And such a solution D^*NDf exists because of the standard argument . And the Kuranishi's estimate , namely in his notation ((4)) ,

$$\| D^*NDf \|_{\langle a-1, \ell \rangle}$$

depends continuously on $\| Df \|_{\langle a+1, \ell \rangle}$. We set $a = 1 - \ell$. Then

$$\operatorname{Sup} | j^{(1)} (D^*NDf) | \leq \| D^*NDf \|_{\langle -\ell , \ell \rangle} .$$

By (iii) , we have

$$f - D^*NDf$$

is a CR-embedding of $(M, {}^{\rho}T'')$.

REFERENCES

(1) Akahori,T. : Intrinsic formula for Kuranishi's $\bar{\partial}_b^{\phi}$. Publ. RIMS., Kyoto Univ., 14, 615-641(1978)

(2) Akahori,T. : The New Neumann Operator Associated with Deformations of Strongly Pseudo Convex Domains and Its Application to Deformation Theorey . Invent.math.68,317-352(1982)

(3) Kuranishi,M. : Strongly pseudo convex CR-structure over small balls , Part I. An a priori estimate , Ann.of Math. 115(1982), 451-500

(4) Kuranishi,M. : Strongly pseudo convex CR-structure over small balls , Part II , A regularity theorem , Ann.of Math. 116(1982), 669-732

(5) Kuranishi,M, : Strongly pseudo convex CR-structure over small balls , Part III , An embedding theorem , Ann.of Math. 116(1982), 249-330