General existence of minimal surfaces with prescribed flux II

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Introduction.

The present paper is a continuation of the papers [KUY1] and [KUY2] by the same authors. In the previous papers, we showed the existence of n*-end catenoids* ($n \geq 3$) in $\mathbf{R}^3$ for "almost all" flux data, without no upper restrictions of the number n of ends. However, the above result does not work for specified case, for example, the case that all the flux vector lies in the same plane. We say that such minimal surfaces are of *Type II*. In this paper, we shall show that the method as in [KUY2] works even for such specified case, and prove the general existence of n-end catenoids ($n \leq 8$) of Type II. Here we describe the details:

Consider a minimal immersion $x : M^2 \to \mathbf{R}^3$, whose induced Riemannian metric is complete and of finite total curvature. The conformal structure of M^2 is induced from the metric ds^2. So if we assume the surface is orientable, it can be considered as a Riemann surface and the immersion is a conformal map. Furthermore, since x is complete and of finite total curvature, there exists a compact Riemann surface $\overline{M}^2$ such that M^2 is biholomorphic to $\overline{M}^2 \setminus \{q_1, \dots, q_n\}$, where $q_1, \dots, q_n$ are a finite number of points on $\overline{M}^2$. The each point q_j is called an end of the surface. We set

$$\begin{aligned} g &:= \partial x^3/(\partial x^1 - \sqrt{-1}\,\partial x^2), \\ \omega &:= \partial x^1 - \sqrt{-1}\,\partial x^2. \end{aligned}$$

Then g is a meromorphic function and ω is a holomorphic 1-form. We have the relation

$$\partial x = \frac{1}{2}\left((1-g^2)\,\omega, \sqrt{-1}\,(1+g^2)\,\omega, 2g\omega\right).$$

This leads us to the well known Weierstrass representation

$$x = \operatorname{Re}\left(\int_{z_0}^{z}(1-g^2)\,\omega, \int_{z_0}^{z}\sqrt{-1}\,(1+g^2)\,\omega, \int_{z_0}^{z} 2g\omega\right),$$

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where z_0 is a fixed point on M^2. In this expression, the meromorphic function g is identified with the Gauss map by the stereographic projection from the north pole. Conversely, if M^2 is simply connected and

$$ds^2 = (1 + |g|^2)^2 \omega \cdot \overline{\omega}$$

is positive definite, x becomes a conformal minimal immersion, where "$\cdot$" means the symmetric tensor product. A degenerate point of the metric ds^2 is called a *branch point*. We now define a flux vector φ_j around each end q_j by taking the imaginary part of the above expression:

$$\varphi_j := -\operatorname{Im}\left(\oint_{\gamma_j} (1-g^2)\,\omega, \oint_{\gamma_j} \sqrt{-1}\,(1+g^2)\,\omega, \oint_{\gamma_j} 2g\omega\right) \qquad (j = 1, \dots, n),$$

where γ_j is a loop surrounding the end q_j from the left. There is an alternative definition of the flux vector: Let $n_j(s)$ be the conormal of $\gamma_j(s)$, where s is the length parameter. Then the flux vector v_j is expressed by

$$\varphi_j = \oint_{\gamma_j} n_j(s)\, ds \qquad (j = 1, \dots, n).$$

Now we denote by $\nu(q_j) := \lim_{z \to q_j} \nu(z)$ the limit of the Gauss map at the end. Then

- φ_j is proportional to the limit normal vector $\nu(q_j)$.
- The proportional constant $w_j := \varphi_j / 4\pi\nu(q_j)$ is called the *weight* at the end. It can take a negative value.

It seems quite natural that we restrict our attention to the special class of minimal surface, so-called n-end catenoid, which is a conformal minimal immersion $x : \mathbf{C} \cup \{\infty\} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$ whose ends are all asymptotic to catenoid ends. Recently, new examples of n-end catenoids have been found by [Kar], [L], [X], [R1], [R2], [Kat] and [UY].

For n-end catenoids, the weight is considered as the size of catenoid end relative to the standard catenoid. Now from the definition of the flux vector, we have

$$\sum_{j=1}^{n} \varphi_j = \sum_{j=1}^{n} 4\pi w_j \nu(q_j) = 0, \tag{0.1}$$

which is called the flux formula. The inverse problem of the flux formula is now stated as follows:

Problem. *For given unit vectors $v_1, \dots, v_n$ and given real numbers $a_1, \dots, a_n \neq 0$ satisfying the balancing condition $\sum_{j=1}^{n} a_j v_j = 0$, is there an n-end catenoid*

$$x : \mathbf{C} \cup \{\infty\} \setminus \{q_1, \dots, q_n\} \to \mathbf{R}^3$$

with $\nu(q_j) = v_j$ and $w_j = a_j$? (Is it unique?)

Now, we classify arrangements of $\boldsymbol{v} = \{v_1, \ldots, v_n\}$ to the following three types according as [KUY1]:

Type I $\quad \boldsymbol{v} = \{v_1, \ldots, v_n\}$ satisfies $\dim\langle v_1, \ldots, v_n\rangle = 1$,
Type II $\quad \boldsymbol{v} = \{v_1, \ldots, v_n\}$ satisfies $\dim\langle v_1, \ldots, v_n\rangle = 2$,
Type III $\quad \boldsymbol{v} = \{v_1, \ldots, v_n\}$ satisfies $\dim\langle v_1, \ldots, v_n\rangle = 3$.

We call a (branched) n-end catenoid is of Type I (resp. II, III), if $\boldsymbol{v}$ is of Type I (resp. II, III). The following facts are already known (e.g. [M], [B], [L], [Kat]):

(1) *There are no* 1*-end catenoids. Any* 2*-end catenoid is the catenoid.*

(2) *There are no* 3*-end catenoids of Type I. Consequently, any* 3*-end catenoid is of Type II. Moreover, for any unit vectors v_1, v_2, v_3 of Type II and nonzero real numbers a_1, a_2, a_3 satisfying $\sum_{j=1}^{3} a_j v_j = 0$, there exists an essentially unique* 3*-end catenoid $x : \mathbf{C} \setminus \{q_1, q_2, q_3\} \to \mathbf{R}^3$ which satisfies $\nu(q_j) = v_j$ and the weight $w(q_j) = a_j$ $(j = 1, 2, 3)$.*

From these results, for $n \leq 3$, the set of n-end catenoids is understood completely. In [KUY1] and [KUY2], we showed that *for almost all flux data $v_1, \ldots, v_n \in S^2(1)$ $(n \geq 4)$ (of Type III) and $a_1, \ldots, a_n \in \mathbf{R}\setminus\{0\}$ satisfying $\sum_{j=1}^{n} a_j v_j = 0$, there exists a (non-branched) n-end catenoid*

$$x : \mathbf{C} \cup \{\infty\} \setminus \{q_1, ..., q_n\} \to \mathbf{R}^3$$

with $\nu(q_j) = v_j$ and $w_j = a_j$. In the above statement, there is no upper restriction of n. We also remark that the uniqueness of the inverse problem is no longer true for $n \geq 4$. (See [KUY1].) Moreover, the "almost all" in the assertion is essential. In fact, there are several obstruction for the inverse problem. (See [KUY1] and also [P].) However, the class of n-end catenoids of Type II is not included in the above statement, because such flux data is not generic. In this paper, we shall prove the following

Theorem. *Suppose that $3 \leq n \leq 8$. For almost all flux data $v_1, \ldots, v_n \in S^2(1)$ of Type II and $a_1, \ldots, a_n \in \mathbf{R}\setminus\{0\}$ satisfying $\sum_{j=1}^{n} a_j v_j = 0$, there exists a (non-branched) n-end catenoid $x : \mathbf{C} \cup \{\infty\} \setminus \{q_1, \ldots, q_n\} \to \mathbf{R}^3$ with $\nu(q_j) = v_j$ and $w_j = a_j$.*

The method of the proof is based on that in [KUY2]. By virtue of it, the theorem is proved by finding a sampling point where the *flux map* attains non-vanishing Jacobian. Though an explicit proof is given only for $n \leq 8$, our numerical (not rigorous) calculations indicates that the Jacobian of the flux map at Jorge-Meeks' surface does not vanish at least $n \leq 13$. So it is expected that the theorem holds for any $n \geq 3$. To prove the theorem, we have to distinguish the cases of n into even and odd. In the case of odd n, the proof is rather easier, however in the case of n even, we need a computation of Pfaffians of certain skew-symmetric matrices instead of their determinants. In

Section 1, we reduce the problem to seeking a sampling point satisfying certain non-degeneracy condition. In Section 2, we prove the theorem. In Section 3, we give several new concrete examples of n-end catenoids of Type II.

For n-end catenoids of Type I, the general existence can not be expected. (See [KUY1].) Recently the moduli space of minimal surfaces with flat ends is investigated in [KS], where a construction of minimal surfaces is explained by using the term of spin structure of Riemann surfaces. It should be remarked that our approaches in [KUY1] and [KUY2] can also be interpreted in terms of spin structure. (See Remark 1.1.)

1 Outline of the proof.

We reduce the problem to solving a certain system of algebraic equations. Let $v_1, \ldots, v_n \in S^2(1)$ be a set of unit vectors, and set $p_j = \pi(v_j)$, where $\pi : S^2(1) \to \mathbf{C} \cup \{\infty\}$ is the stereographic projection. We may assume $p_1, \ldots, p_n \in \mathbf{C}$ by a suitable rotation of the surface.

Now we consider a system of equations (cf. [KUY1] and [KUY2]):

$$
\tag{1.1}
\begin{aligned}
b_j \sum_{k \neq j} b_k \frac{p_j - p_k}{q_j - q_k} &= a_j \\
b_j \sum_{k \neq j} b_k \frac{\overline{p_j} p_k + 1}{q_j - q_k} &= 0
\end{aligned}
\qquad (j = 1, \ldots, n).
$$

Here $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{a} = (a_1, \ldots, a_n)$ are flux data as given parameters, and $\boldsymbol{q} = (q_1, \ldots, q_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ are unknown parameters. If we find a solution q_j and b_j of the above equations, we get a branched n-end catenoid as follows: Set

$$
\tag{1.2}
P(z) := \sum_{j=1}^{n} p_j b_j \prod_{k \neq j} (z - q_k), \qquad Q(z) := \sum_{j=1}^{n} b_j \prod_{k \neq j} (z - q_k),
$$

and define Weierstrass data (g, ω) by

$$
\tag{1.3}
g := \frac{P(z)}{Q(z)}, \qquad \omega := -\left\{ \sum_{j=1}^{n} \frac{b_j}{z - q_j} \right\}^2 dz.
$$

Then the minimal immersion

$$
x = \operatorname{Re}\left(\int_{z_0}^{z} (1 - g^2)\, \omega, \int_{z_0}^{z} \sqrt{-1}\, (1 + g^2)\, \omega, \int_{z_0}^{z} 2g\omega \right)
$$

is a branched n-end catenoid with the flux data $(\boldsymbol{p}, \boldsymbol{a})$. Now $ds^2 > 0$ if and only if both of the following two conditions are satisfied:

- $P(z)$ and $Q(z)$ are mutually prime,

- $\mathrm{Max}(\deg P, \deg Q) = n-1$.

Remark 1.1 The constructin of n-end catenoids mentioned above is related to the spinor representation of minimal surfaces (cf. [KS]);

$$x = \mathrm{Re}\left(\int_{z_0}^{z} ({s_1}^2 - {s_2}^2), \int_{z_0}^{z} \sqrt{-1}\,({s_1}^2 + {s_2}^2), \int_{z_0}^{z} 2s_1\, s_2\right),$$

where (s_1, s_2) is a pair of meromorphic sectons of the half-canonical bundle on $\mathbf{C} \cup \{\infty\}$. In fact, s_1 and s_2 have the following explicit expressions in this case:

$$s_1 = \frac{Q(z)}{R(z)}\sqrt{-dz}, \qquad s_2 = \frac{P(z)}{R(z)}\sqrt{-dz},$$

where we set

$$R(z) = \prod_{k=1}^{n}(z - q_k).$$

We assume $n \geq 4$ hereafter. To prove our theorem, we may restrict our attention to the flux data of Type II. So we may assume

$$|p_j| = 1 \qquad (j = 1, ..., n). \tag{1.4}$$

This condition is rewritten as $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbf{C}^n_{II}$, where

$$\mathbf{C}^n_{II} := \{\boldsymbol{p} \in \mathbf{C}^n\,;\, |p_j| = 1\ (j = 1, ..., n)\}\,.$$

First, we fix our attention to the second equation of (1.1). Consider a skew-symmetric matrix

$$A_p := \left(\frac{p_j + p_k}{q_j - q_k}\right)_{j,k=1,\ldots,n}, \tag{1.5}$$

where the diagonal components are interpreted as 0. By (1.4), we have

$$\overline{p_j}p_k + 1 = \frac{p_k}{p_j} + 1 = \frac{p_j + p_k}{p_j}.$$

Thus we have

$$\mathrm{diag}(1/p_1, ..., 1/p_n)\cdot A_p = \left(\frac{\overline{p_j}p_k + 1}{q_j - q_k}\right)_{j,k=1,\ldots,n},$$

where $\mathrm{diag}(1/p_1, ..., 1/p_n)$ denotes the diagonal matrix with the diagonal components $1/p_1, ..., 1/p_n$. Then the vector ${}^t(b_1, \ldots, b_n)$ belongs to the kernel of the matrix A_p.

The case of odd n.

When n is odd, $\det A_p$ is identically zero. Moreover, it is reasonable to expect that the rank of the matrix A_p is generically $n-1$. (In fact, as shown in Lemma 2.1, Jorge-Meeks' $(2m+1)$-noid have such a property for any $m \geq 1$.) In this case, ${}^t(b_1, \dots, b_n)$ should be proportional to any column vector of the cofactor matrix $\widetilde{A}_p$ of A_p. (By the definition, $A_p \cdot \widetilde{A}_p = \widetilde{A}_p \cdot A_p = (\det A_p) I$ holds.) So we set

$$\boldsymbol{b}_p(\boldsymbol{q}) = {}^t\left(b_p^1(\boldsymbol{q}), \dots, b_p^n(\boldsymbol{q})\right) := \text{the first column of } \widetilde{A}_p(\boldsymbol{q}).$$

For fixed $\boldsymbol{p} := (p_1, \dots, p_n) \in \mathbf{C}_{II}$, define a rational map between two complex projective spaces

$$\mathcal{F}\ell_p = [f_p^1, \dots, f_p^n] : \mathbf{P}^{n-1} \to \mathbf{P}^{n-1}$$

by

$$f_p^j(q_1, \dots, q_n) := b_p^j(\boldsymbol{q}) \sum_{k \neq j} b_p^k(\boldsymbol{q}) \frac{p_j - p_k}{q_j - q_k} \qquad (j = 1, \dots, n). \tag{1.6}$$

The map $\mathcal{F}\ell_p$ is called a *flux map*. We set

$$f\ell_p^j(\boldsymbol{q}) := \Delta(\boldsymbol{q})^5 \cdot f_p^j(\boldsymbol{q}),$$

where $\Delta(\boldsymbol{q})$ is the difference product defined by

$$\Delta(q_1, \dots, q_n) := \prod_{j>k} (q_j - q_k). \tag{1.7}$$

It is easily seen that each $f\ell_p^j$ is a homogeneous polynomial in $q_1, \dots, q_n$ and $\mathcal{F}\ell_p$ has another expression

$$\mathcal{F}\ell_p = [f\ell_p^1, \dots, f\ell_p^n].$$

Since p_j is the stereographic image of v_j, the balancing condition (0.1) is rewritten as

$$\sum_{j=1}^n \frac{\overline{p_j}}{|p_j|^2+1} a_j = 0, \qquad \sum_{j=1}^n \frac{p_j}{|p_j|^2+1} a_j = 0. \tag{1.8}$$

We define a subspace $\mathcal{W}_p^{n-3}$ in $\mathbf{P}^{n-1}$ by

$$\mathcal{W}_p^{n-3} := \left\{ [a_1, \dots, a_n] \in \mathbf{P}^{n-1} ; \sum_{j=1}^n \frac{\overline{p_j}}{|p_j|^2+1} a_j = 0, \sum_{j=1}^n \frac{p_j}{|p_j|^2+1} a_j = 0 \right\}.$$

It should be remarked that the dimension of the space of flux data $\mathcal{W}_p^{n-3}$ is one up comparing the case of flux data of Type III. We set

$$J_p = \left(\frac{\partial f_p^k}{\partial q_j} \cdot f_p^n - f_p^k \cdot \frac{\partial f_p^n}{\partial q_j} \right)_{k,j=1,\dots,n-1}. \tag{1.9}$$

The following lemma holds.

Lemma 1.2 *Suppose that there exist $\boldsymbol{u}_0 \in \mathbf{C}^n_{II}$ $(n = 2m + 1)$ and a point $\boldsymbol{c} = [c_1, \dots, c_n] \in \mathbf{P}^{n-1}$ satisfying the following conditions*:

(1) $c_1, \dots, c_n$ *are all distinct*;

(2) *The rank of the matrix* $A_{u_0}(\boldsymbol{c})$ *is* $n-1$;

(3) *The rank of the matrix* $J_{u_0}(\boldsymbol{c})$ *is* $n-3$;

(4) *Two polynomials* $P(z)$ *and* $Q(z)$ *defined in* (1.2) *associated with the data* $(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{c}, \boldsymbol{u}_0)$ *and* $\boldsymbol{b} = \boldsymbol{b}_{u_0}(\boldsymbol{c})$ *are mutually prime and one of them has degree* $n-1$;

(5) $f^j_{u_0}(\boldsymbol{c}) \neq 0$ $(j = 1, \dots, n)$;

(6) $c_j \neq 0$ $(j = 1, \dots, n)$.

Then there exists an open dense subset $U \subset \mathbf{C}^n_{II}$ and an open dense subset Ω_p of the totally real set $\mathcal{W}_{\mathbf{R}} = \{[\boldsymbol{a}] \in \mathcal{W}^{n-3}_p\,;\ a_j \in \mathbf{R}\}$ such that, for any $\boldsymbol{p} \in U$ and $[\boldsymbol{a}] \in \Omega_p$, there exists an (non-branched) n-end catenoid with the flux data $(\boldsymbol{p}, \boldsymbol{a})$.

Proof. One can easily prove the lemma modifying the proof of Proposition 1.6 in [KUY2]. By condition (4), at least one $(n-3)$-submatrix S_{u_0} of J_{u_0} is of rank $n-3$. Let $1 \leq j_1 < j_2 < \cdots < j_{n-3} \leq n$ be the indices of the columns of the submatrix S_{u_0}, and $\{m_1, m_2, m_3\}$ their complement, namely $\{m_1, m_2, m_3\} = \{1, \dots, n\} \setminus \{j_1, \dots, j_{n-3}\}$. The position of the ends $\{q_1, \dots, q_n\}$ in $\mathbf{C} \cup \{\infty\}$ has the freedom of Möbius transformations. To kill these ambiguities, we may restrict the flux map into the following subspace of $\mathbf{P}^{n-1}$ containing the sampling point $\boldsymbol{c}$:

$$\mathcal{V}^{n-3} := \left\{[q_1, \dots, q_n] \in \mathbf{P}^{n-1}\,;\ c_{m_2} q_{m_1} - c_{m_1} q_{m_2} = 0,\ c_{m_3} q_{m_1} - c_{m_1} q_{m_3} = 0\right\}.$$

Now we define a homogeneous polynomial in $q_1, \dots, q_n$ by

$$H_p(\boldsymbol{q}) := \det\left(\Delta(\boldsymbol{q})^l S_p(\boldsymbol{q})\right) \cdot R_p(\boldsymbol{q}) \cdot \prod_{j=1}^{n} f\ell^j_p(\boldsymbol{q}) \cdot \prod_{k=1}^{n} q_k,$$

where l is chosen sufficiently large so that $\det(\Delta(\boldsymbol{q})^l S_p(\boldsymbol{q}))$ is a homogeneous polynomial in $q_1, \dots, q_n$, and R_p is the resultant of the two polynomials $P(z)$ and $Q(z)$ of degree $n-1$ defined by (1.2). (It can be easily shown that R_p is also a homogeneous polynomial with respect to $\boldsymbol{q}$. Or one may replace R_p by the resultant of $P(q_1 z)$ and $Q(q_1 z)$.) Then by the conditions (1)–(6), $c \in \mathcal{V}^{n-3}$ satisfies $H_{u_0}(c) \neq 0$.

Since $\det A_p = 0$ for any $[\boldsymbol{q}] \in \mathbf{P}^{n-1}$, we do not need to consider the subvariety X^{n-4} as in [KUY2]. So the arguments in the proof of Proposition 1.6 in [KUY2] can be easily modified in this case, only ignoring the parts concerning the differentials of $\det A_p$. (q. e. d.)

The case of even n.

When n is even, $\det A_p$ is not identically zero on $\mathbf{P}^{n-1}$. But the number of zero eigenvalues is even. So, we expect that the rank of the matrix A_p is generically $n-2$ (not $n-1$) for n-end catenoid of Type II. In this case, it is convenient to consider the Pfaffian $\mathrm{pf}\, A_p$ of the matrix A_p, which satisfies

$$\det A_p = (\mathrm{pf}\, A_p)^2. \tag{1.10}$$

Moreover, if we define a skew-symmetric matrix

$$\hat{A}_p = (b_{ij})\,, \qquad b_{ij} = (-1)^{i+j}\,\mathrm{pf}\, A_{ij} \quad \text{if } i < j, \tag{1.11}$$

where A_{ij} is a submatrix of A_p which is obtained after deleting i, j-th colummns and rows from A_p, it holds that

$$A_p \cdot \hat{A}_p = \hat{A}_p \cdot A_p = (\mathrm{pf}\, A_p) I. \tag{1.12}$$

We call $\hat{A}_p$ the *Pfaffian cofactor matrix* of A_p. In particular it is expressed as a homogeneous polynomial in components of A_p.

We set

$$\begin{aligned} \boldsymbol{b}_p^{(1)}(\boldsymbol{q}) &= {}^t\left(b_p^{(1),1}(\boldsymbol{q}), \ldots, b_p^{(1),n}(\boldsymbol{q})\right) := \text{the first column of } \hat{A}_p(\boldsymbol{q}), \\ \boldsymbol{b}_p^{(2)}(\boldsymbol{q}) &= {}^t\left(b_p^{(2),1}(\boldsymbol{q}), \ldots, b_p^{(2),n}(\boldsymbol{q})\right) := \text{the second column of } \hat{A}_p(\boldsymbol{q}). \end{aligned}$$

For a fixed $\boldsymbol{p} := (p_1, \ldots, p_n) \in \mathbf{C}^n_{II}$, define a rational map

$$\mathcal{F}\ell_p = [f_p^1, \ldots, f_p^n] : \mathbf{P}^{n-1} \times \mathbf{P}^1 \to \mathbf{P}^{n-1}$$

by

$$f_p^j(q_1, \ldots, q_n, s_1, s_2) := b_p^j(\boldsymbol{q}, \boldsymbol{s}) \sum_{k \neq j} b_p^k(\boldsymbol{q}, \boldsymbol{s}) \frac{p_j - p_k}{q_j - q_k} \quad (j = 1, \ldots, n), \tag{1.13}$$

where $[\boldsymbol{q}] = [q_1, \ldots, q_n] \in \mathbf{P}^{n-1}$, $[\boldsymbol{s}] = [s_1, s_2] \in \mathbf{P}^1$, and

$$\boldsymbol{b}_p([\boldsymbol{q}], [\boldsymbol{s}]) := s_1 b_p^{(1)}(\boldsymbol{q}) + s_2 b_p^{(2)}(\boldsymbol{q}).$$

We define an $(n-1)$-matrix J_p by

$$J_p := \left((f_p^n)^2 \left\{ \frac{\partial(\mathrm{pf}\, A_p)}{\partial q_n} \cdot \frac{\partial \mathring{f}_p^k}{\partial q_j} - \frac{\partial(\mathrm{pf}\, A_p)}{\partial q_j} \cdot \frac{\partial \mathring{f}_p^k}{\partial q_n} \right\} \right)_{k,j=1,\ldots,n-1},$$

where

$$\mathring{f}_p^j := \frac{f_p^j}{f_p^n} \qquad (j = 1, \ldots, n-1).$$

The matrix J_p has a direct expression

$$J_p = \left(\frac{\partial(\mathrm{pf}\, A_p)}{\partial q_n} \cdot \left\{ \frac{\partial f_p^k}{\partial q_j} \cdot f_p^n - f_p^k \cdot \frac{\partial f_p^n}{\partial q_j} \right\} - \frac{\partial(\mathrm{pf}\, A_p)}{\partial q_j} \cdot \left\{ \frac{\partial f_p^k}{\partial q_n} \cdot f_p^n - f_p^k \cdot \frac{\partial f_p^n}{\partial q_n} \right\} \right)_{k,j=1,\dots,n-1}. \tag{1.14}$$

Define an $(n-1) \times n$ matrix $\mathcal{J}_p$ as

- The first column of $\mathcal{J}_p$ is given by
$$\left(\frac{\partial(\mathrm{pf}\, A_p)}{\partial q_n} \left\{ \frac{\partial f_p^k}{\partial s_2} \cdot f_p^n - f_p^k \cdot \frac{\partial f_p^n}{\partial s_2} \right\} \right)_{k=1,\dots,n-1}.$$
- The second to n-th column are equal to J_p.

Lemma 1.3 *Suppose that there exist $\boldsymbol{u}_0 \in \mathbf{C}^n_{II}$ $(n = 2m)$ and a point $(\boldsymbol{c}, \boldsymbol{t}) = ([c_1, \dots, c_n], [t_1, t_2]) \in \mathbf{P}^{n-1} \times \mathbf{P}^1$ satisfying the following conditions*:

(1) $c_1, \dots, c_n$ *are all distinct*;

(2) *The rank of the matrix $A_{u_0}(\boldsymbol{c})$ is $n-2$*;

(3) $\partial(\mathrm{pf}\, A_{u_0})/\partial q_n$ *does not vanish at $\boldsymbol{q} = \boldsymbol{c}$*;

(4) *The rank of the matrix $\mathcal{J}_{u_0}(\boldsymbol{c}, \boldsymbol{t})$ is $n-3$*;

(5) *Two polynomials $P(z)$ and $Q(z)$ defined in* (1.2) *associated with the data $(\boldsymbol{q}, \boldsymbol{s}, \boldsymbol{p}) = (\boldsymbol{c}, \boldsymbol{t}, \boldsymbol{u}_0)$ and $\boldsymbol{b} = \boldsymbol{b}_{u_0}(\boldsymbol{c}, \boldsymbol{t})$ are mutually prime and one of them has degree $n-1$*;

(6) $f^j_{u_0}(\boldsymbol{c}, \boldsymbol{t}) \neq 0$ $(j = 1, \dots, n)$;

(7) $c_j \neq 0$ $(j = 1, \dots, n)$ *and* $t_j \neq 0$ $(j = 1, 2)$.

Then there exists an open dense subset $U \subset \mathbf{C}^n_{II}$ and an open dense subset Ω_p of the totally real set $\mathcal{W}_{\mathbf{R}} = \{[\boldsymbol{a}] \in \mathcal{W}_p^{n-3}\,;\, a_j \in \mathbf{R}\}$ such that, for any $\boldsymbol{p} \in U$ and $[\boldsymbol{a}] \in \Omega_p$, there exists an (non-branched) n-end catenoid with the flux data $(\boldsymbol{p}, \boldsymbol{a})$.

Proof. The lemma can also be obtained modifying the proof of Proposition 1.6 in [KUY2]. We restrict the flux map into the following subspace of $\mathbf{P}^{n-1} \times \mathbf{P}^1$ containing the sampling point c:

$$\mathcal{V}^{n-2} := \{[q_1, \dots, q_n] \in \mathbf{P}^{n-1}\,;\, c_{m_2} q_{m_1} - c_{m_1} q_{m_2} = 0,\ c_{m_3} q_{m_1} - c_{m_1} q_{m_3} = 0\} \times \mathbf{P}^1.$$

Now we define a homogeneous polynomial in $(q_1,\dots,q_n)$ and in (s_1,s_2) by

$$H_p(\boldsymbol{q},\boldsymbol{s}) := \Delta(\boldsymbol{q})^2\frac{\partial(\operatorname{pf} A_p)}{\partial q_n}(\boldsymbol{q})\cdot\det\Bigl(\Delta(\boldsymbol{q})^l S_p(\boldsymbol{q})\Bigr)\cdot R_p(\boldsymbol{q},\boldsymbol{s})\cdot\prod_{j=1}^{n} f\ell_p^j(\boldsymbol{q},\boldsymbol{s})\cdot\prod_{k=1}^{n} q_k\prod_{k=1}^{2} s_k.$$

Then by the conditions (1)–(7), $(c,t)\in\mathcal{V}^{n-2}$ satisfies $H_{u_0}(c,t)\neq 0$. The arguments in the proof of Proposition 1.6 in [KUY2] can be easily modified in this case, only replacing $\det A_p$ by $\operatorname{pf} A_p$ and X^{n-4} by X^{n-3}. (q. e. d.)

2 Computations of the rank at a sampling point.

In the previous section, we reduced our problem to finding a regular point of the flux map (Lemmas 1.2 and 1.3). In this section, we take Jorge-Meeks' n-noid [JM] as a sampling point, and show that it satisfies the assumptions of Lemma 1.2 or 1.3 for $n\leq 8$.

Let $n\geq 4$ be an integer. Jorge-Meeks' n-noid is an n-end catenoid of Type II with the data

$$p_j=q_j=q_j^0=\zeta^j,\qquad a^j=n-1,\qquad b^j=1\qquad (j=1,\dots,n), \tag{2.1}$$

where $\zeta:=\exp(2\pi\sqrt{-1}/n)$. For $\boldsymbol{p}$ in (2.1), the matrix defined in (1.5) is given by

$$A_p(\boldsymbol{q})=\left(\frac{\zeta^j+\zeta^k}{q_j-q_k}\right)_{j,k=1,\dots,n}, \tag{2.2}$$

where the diagonal components of $A_p(\boldsymbol{q})$ are zero. At the inital point

$$\boldsymbol{q}^0=\left(q_1^0,\dots,q_n^0\right)=(\zeta,\dots,\zeta^n),$$

we have

$$A_p(\boldsymbol{q}^0)=\begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \dots & \alpha_{n-1}\\ \alpha_{n-1} & 0 & \alpha_1 & \dots & \alpha_{n-2}\\ \alpha_{n-2} & \alpha_{n-1} & 0 & \dots & \alpha_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & 0\end{pmatrix},\qquad \text{where}\quad \alpha_k=\frac{1+\zeta^k}{1-\zeta^k}.$$

Since $\alpha_{n-j}=-\alpha_j$, $A_p(\boldsymbol{q}^0)$ is a skew-symmetric matrix.

Lemma 2.1 *The rank of $A_p(\boldsymbol{q}^0)$ is $n-1$ if n is odd, and $n-2$ if n is even.*

Proof. Since $A_p(\boldsymbol{q}^0)$ is a cyclic matrix, the eivengalues $\lambda_0,\dots,\lambda_{n-1}$ are given by

$$\lambda_l=\sum_{j=0}^{n-1}\zeta^{lj}\alpha_j=\begin{cases} 0 & (l=0)\\ -n+2l & (1\leq l\leq n-1)\end{cases},$$

where α_0 is considered as 0. Thus, if n is odd, only λ_0 is a zero-eigenvalue of $A_p(\boldsymbol{q}^0)$, and if n is even, zero-eigenvalues are λ_0 and $\lambda_{n/2}$. (q. e. d.)

In the rest of this section, we calculate the rank of the Jacobian matrix J_p in (1.9) for odd n and $\mathcal{J}_p$ for even n at the point $q_j = q_j^0$.

The case of odd n.

Lemma 2.2 *The derivaties of $\boldsymbol{b}_p(\boldsymbol{q})$ at $\boldsymbol{q} = \boldsymbol{q}^0$ are given by*

$$\frac{\partial}{\partial q_j}\begin{pmatrix} b_p^1 \\ b_p^2 \\ \vdots \\ b_p^n \end{pmatrix}(\boldsymbol{q}^0) = \beta\left\{\mathrm{Tr}\left(\frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0)\cdot Y\right) I - Y\cdot\frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0)\right\}\cdot\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

where

$$Y = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & A_1^{-1} & \\ 0 & & & \end{pmatrix}, \qquad A_1 = \begin{bmatrix} \textit{the square } (n-1)\textit{-submatrix} \\ \textit{of } A_p(\boldsymbol{q}^0) \textit{ excluded the first} \\ \textit{column and the first row.} \end{bmatrix},$$

and $\beta = \det A_1 \neq 0$.

Proof. Let $B(\boldsymbol{q}) = \widetilde{A}_p(\boldsymbol{q})$ be the cofactor matrix. It is easily checked that ${}^t(1, 1, \ldots, 1)$ is a zero-eigenvector of $A_p(\boldsymbol{q}^0)$ and ${}^tA_p(\boldsymbol{q}^0)$ simultaneously. Since rank of $A_p(\boldsymbol{q}^0)$ is $n-1$, we have

$$B(\boldsymbol{q}^0) = \beta\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix},$$

where $\beta = \det A_1 \neq 0$.

Take an $n \times n$ matrix X as

$$X = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

and let $A_t(\boldsymbol{q}) = A_p(\boldsymbol{q}) + tX$ for any real number t. Since

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \det A_t(\boldsymbol{q}^0) = \mathrm{Tr}\left(\frac{\partial A_t}{\partial t}(\boldsymbol{q}^0)\cdot B(\boldsymbol{q}^0)\right) = \mathrm{Tr}\left(X\cdot B(\boldsymbol{q}^0)\right) = \beta \neq 0,$$

$A_t(\boldsymbol{q})$ is non-singular for sufficiently small t and $\boldsymbol{q}$ which is sufficiently close to $\boldsymbol{q}^0$. Let $B_t(\boldsymbol{q})$ be the cofactor matrix of $A_t(\boldsymbol{q})$. Differentiating $A_t \cdot B_t = B_t \cdot A_t = (\det A_t)I$ with respect to q_j, we have

$$\begin{aligned} \frac{\partial B_t}{\partial q_j} &= A_t^{-1} \cdot \left(\frac{\partial \det A_t}{\partial q_j} I - \frac{\partial A_t}{\partial q_j} \cdot B_t \right) \\ &= \frac{1}{\det A_t} \left(B_t \cdot \frac{\partial \det A_t}{\partial q_j} I - B_t \cdot \frac{\partial A_t}{\partial q_j} \cdot B_t \right). \end{aligned}$$

On letting $t \to 0$, both the denominator and the numerator of the right-hand side of the above equation tending to zero. So we can apply de l'Hospital's rule. Thus, since $\det A_p(\boldsymbol{q}) \equiv 0$ and $\partial(\det A_t)/\partial t = \mathrm{Tr}(\partial A_t/\partial t) \cdot B_t$, we have

$$\begin{aligned} \frac{\partial B}{\partial q_j}(\boldsymbol{q}^0) = \frac{1}{\beta} \Bigg\{ & \mathrm{Tr}\left(\frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0) \cdot \check{Y} \right) B(\boldsymbol{q}^0) \\ & - \check{Y} \cdot \frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0) \cdot B(\boldsymbol{q}^0) - B(\boldsymbol{q}^0) \cdot \frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0) \cdot \check{Y} \Bigg\}, \end{aligned} \tag{2.3}$$

where $\check{Y} = \partial B_t(\boldsymbol{q}^0)/\partial t|_{t=0}$. Since

$$A_t(\boldsymbol{q}^0) = \begin{pmatrix} t & \alpha_1 & \alpha_2 & \dots & \alpha_{n-1} \\ \alpha_{n-1} & 0 & \alpha_1 & \dots & \alpha_{n-2} \\ \alpha_{n-2} & \alpha_{n-1} & 0 & \dots & \alpha_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & 0 \end{pmatrix},$$

the first row and the first column of the cofactor matrix $B_t(\boldsymbol{q}^0)$ do not depend on t, and one can easily see that

$$\check{Y} = \left. \frac{\partial B_t(\boldsymbol{q}^0)}{\partial t} \right|_{t=0} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \widetilde{A}_1 & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \beta A_1^{-1} & \\ 0 & & & \end{pmatrix}.$$

Substituting

$$\begin{pmatrix} b_p^1(\boldsymbol{q}) \\ \vdots \\ b_p^n(\boldsymbol{q}) \end{pmatrix} = [\text{the first column of } B(\boldsymbol{q})] = B(\boldsymbol{q}) \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

into (2.3), we have the conclusion, since $Y \cdot {}^t(1, \dots, 0) = 0$. (q. e. d.)

Let

$$\check{b}_p^j(\boldsymbol{q}) = \frac{1}{\beta} b_p^j(\boldsymbol{q}), \qquad \text{and} \qquad \check{f}_p^j(\boldsymbol{q}) = \frac{1}{\beta^2} f_p^j(\boldsymbol{q}).$$

Then

$$\frac{\partial}{\partial q_j}\begin{pmatrix} \check{b}_p^1 \\ \vdots \\ \check{b}_p^n \end{pmatrix}(\boldsymbol{q}^0) = \left\{ \mathrm{Tr}\left(\frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0)\cdot Y\right) I - Y \cdot \frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0)\right\} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

and differentiating (1.6), we have

$$\frac{1}{\beta^2}\frac{\partial f_p^j}{\partial q_l} = \frac{\partial \check{f}_p^j}{\partial q_l} = \sum_{k=1}^{n} \frac{\partial \check{b}_p^k}{\partial q_l} + (n-2)\frac{\partial \check{b}_p^j}{\partial q_l} + \begin{cases} -\dfrac{1}{2}\zeta^{-l}(n-1) & (\text{if } j = l) \\ \dfrac{1}{\zeta^j - \zeta^l} & (\text{if } j \neq l), \end{cases}$$

because $b_p^j(\boldsymbol{q}^0) = \beta$. Since $f^j(\boldsymbol{q}^0) = (n-1)\beta^2$, the Jacobian matrix (1.9) becomes

$$(2.4)\quad J_p = (n-1)\beta^4 \check{J}_p, \quad \text{where} \quad \check{J}_p = \left(\frac{\partial \check{f}_p^k}{\partial q_j}(\boldsymbol{q}^0) - \frac{\partial \check{f}_p^n}{\partial q_j}(\boldsymbol{q}^0)\right)_{k,j=1,\dots,n-1}.$$

Lemma 2.3 *If $n = 5$ or 7, the rank of $\check{J}_p$ is $n-3$.*

Proof. Let Δ_n be an $(n-3)$-matrix which consits of the first $(n-3)$ rows and $(n-3)$ columns of $\check{J}_p$.

By direct calculations, we have

$$|\det \Delta_5| = 20 \neq 0.$$

Hence the rank of J_p is $n-3$ if $n = 5$.

In the case $n = 7$, we can also make a direct calculation, and have $\det \Delta_7$ as a rational function of ζ. Since it holds that $1 + \zeta + \ldots + \zeta^6 = 0$, $\det \Delta_7$ is written as

$$\det \Delta_7 = \frac{a_{10} + a_{11}\zeta + a_{12}\zeta^2 + a_{13}\zeta^3 + a_{14}\zeta^4 + a_{15}\zeta^5}{a_{20} + a_{21}\zeta + a_{22}\zeta^2 + a_{23}\zeta^3 + a_{24}\zeta^4 + a_{25}\zeta^5},$$

where a_{ij}'s are integers whose values are apploximately 10^{440}. More precisely

$$\begin{cases} a_{10} \equiv 0, & a_{11} \equiv 4, & a_{12} \equiv 4, & a_{13} \equiv 1, & a_{14} \equiv 2, & a_{15} \equiv 3 \\ a_{20} \equiv 0, & a_{21} \equiv 3, & a_{22} \equiv 4, & a_{23} \equiv 0, & a_{24} \equiv 3, & a_{25} \equiv 4 \end{cases} \pmod 5$$

holds. Thus, $\det \Delta_7$ is not zero, and we have the conclusion. [1] (By a numerical calculation, we know that the value $|\det \Delta_7|$ is approximately 5773.) (q. e. d.)

Thus, by Lemma 1.2, we have proved the Theorem mentioned in Introduction for $n = 5$ and 7.

[1] To calculate $\det \Delta_n$ we use Mathematica Version 2.2, a system of symboic calculation

The case of even n

Let $B(\boldsymbol{q}) = \hat{A}_p(\boldsymbol{q})$ be the Pfaffian cofactor matrix and $\boldsymbol{b}_p(\boldsymbol{s}, \boldsymbol{q})$ as in the previous seciton. At $\boldsymbol{q} = \boldsymbol{q}^0$, one can easily see that ${}^t(0, 1, 0, 1, \ldots, 1)$ and ${}^t(1, 0, 1, \ldots, 0)$ are elements of the kernel of $A_p(\boldsymbol{q}^0)$. Thus, since rank of $A_p(\boldsymbol{q}^0) = n - 2$ and $B(\boldsymbol{q}^0)$ is skew-symmetric, we have

$$B(\boldsymbol{q}^0) = \beta \begin{pmatrix} 0 & -1 & 0 & -1 & \ldots & -1 \\ 1 & 0 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 0 & -1 & \ldots & -1 \\ 1 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 1 & 0 & \ldots & 0 \end{pmatrix},$$

where

$$(2.5) \qquad \beta = \operatorname{pf} A_{12} \neq 0, \quad A_{12} = \begin{bmatrix} \text{the square } (n-2)\text{-submatrix} \\ \text{of } A_p(\boldsymbol{q}^0) \text{ excluded the first} \\ \text{and second columns and the} \\ \text{first and second rows.} \end{bmatrix}.$$

On the other hand, at the initial point (Jorge-Meeks' n-noid) $b^1 = b^2 = \ldots = b^n$. Thus we can set

$$(2.6) \qquad s_1^0 = 1 \qquad \text{and} \qquad s_2^0 = -1.$$

Lemma 2.4 *The derivatives of* $\operatorname{pf} A_p$ *are given by*

$$\frac{\partial(\operatorname{pf} A_p)}{\partial q_j}(\boldsymbol{q}^0) = \zeta^{-j}(-1)^{j+1}\frac{1}{2}\beta\, m(m-1) \qquad (n = 2m).$$

In particular $\partial(\operatorname{pf} A_p)/\partial q_j \neq 0$.

Proof. In general,

$$\frac{\partial(\operatorname{pf} A)}{\partial q} = \frac{1}{2} \operatorname{Tr}\left(\frac{\partial A}{\partial q} \cdot \hat{A}\right)$$

holds, where $\hat{A}$ is the Pfaffian cofactor matrix of A. Using this, the conclusion is obtained by a direct calculation. (q. e. d.)

Lemma 2.5 *The derivatives of* $\boldsymbol{b}_p(\boldsymbol{q}, \boldsymbol{s})$ *at* $(\boldsymbol{q}, \boldsymbol{s}) = (\boldsymbol{q}^0, \boldsymbol{s}^0)$ *are given by*

$$\frac{\partial}{\partial q_j} \begin{pmatrix} b_p^1 \\ b_p^2 \\ \vdots \\ b_p^n \end{pmatrix} (\boldsymbol{q}^0, \boldsymbol{s}^0) = \beta \left\{ \frac{1}{2} \operatorname{Tr}\left(\frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0) \cdot Y\right) I - Y \cdot \frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0) \right\} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

where

$$Y = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & A_{12}^{-1} & \\ 0 & 0 & & & \end{pmatrix} \qquad (A_{12} \textit{ is as in } (2.5)). \tag{2.7}$$

Proof. Take an $n \times n$ skew-symmetric matrix X as

$$X = \begin{pmatrix} 0 & 1 & \dots & 0 \\ -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

and let $A_t(\boldsymbol{q}) = A_p(\boldsymbol{q}) + tX$ for any real number t. Since

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \operatorname{pf} A_t(\boldsymbol{q}^0) = \frac{1}{2} \operatorname{Tr}\left(\frac{\partial A_t}{\partial t}(\boldsymbol{q}^0) \cdot B(\boldsymbol{q}^0)\right) = \frac{1}{2} \operatorname{Tr}\left(X \cdot B(\boldsymbol{q}^0)\right) = \beta \neq 0,$$

$A_t(\boldsymbol{q})$ is non-singular for sufficiently small t and $\boldsymbol{q}$ which is sufficiently close to $\boldsymbol{q}^0$. Let $B_t(\boldsymbol{q})$ be the Pfaffian cofactor matrix of $A_t(\boldsymbol{q})$. Differentiating $A_t \cdot B_t = B_t \cdot A_t = (\operatorname{pf} A_t) I$ with respect to q_j, we have

$$\begin{aligned} \frac{\partial B_t}{\partial q_j} &= A_t^{-1} \cdot \left(\frac{\partial(\operatorname{pf} A_t)}{\partial q_j} I - \frac{\partial A_t}{\partial q_j} \cdot B_t\right) \\ &= \frac{1}{\operatorname{pf} A_t}\left(B_t \cdot \frac{\partial(\operatorname{pf} A_t)}{\partial q_j} I - B_t \cdot \frac{\partial A_t}{\partial q_j} \cdot B_t\right). \end{aligned}$$

On letting $t \to 0$, both the denominator and the numerator of the right-hand side of the above equation tend to zero. So we can apply de l'Hospital's rule. Thus we have

$$\begin{aligned} \frac{\partial B}{\partial q_j}(\boldsymbol{q}^0) = \frac{1}{\beta}\Bigg\{ & \frac{1}{2} \operatorname{Tr}\left(\frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0) \cdot \check{Y}\right) B(\boldsymbol{q}^0) \\ & + \frac{1}{2} \operatorname{Tr}\left(\frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0) \cdot B(\boldsymbol{q}^0)\right) \check{Y} \\ & - \check{Y} \cdot \frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0) \cdot B(\boldsymbol{q}^0) - B(\boldsymbol{q}^0) \cdot \frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0) \cdot \check{Y} \Bigg\}, \end{aligned} \tag{2.8}$$

where $\check{Y} = \partial B_t(\boldsymbol{q}^0)/\partial t|_{t=0}$. Since

$$A_t(\boldsymbol{q}^0) = \begin{pmatrix} 0 & \alpha_1 + t & \alpha_2 & \dots & \alpha_{n-1} \\ \alpha_{n-1} - t & 0 & \alpha_1 & \dots & \alpha_{n-2} \\ \alpha_{n-2} & \alpha_{n-1} & 0 & \dots & \alpha_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & 0 \end{pmatrix},$$

the first and second rows, and the first and second columns of the Pffafian cofactor matrix $B_t(\boldsymbol{q}^0)$ do not depend on t, and one can easily see that

$$\check{Y} = \left.\frac{\partial B_t(\boldsymbol{q}^0)}{\partial t}\right|_{t=0} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \hat{A}_{12} & \\ 0 & 0 & & & \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \beta A_{12}^{-1} & \\ 0 & 0 & & & \end{pmatrix},$$

where A_{12} is the matrix obtaind by excluding the first and second columns and the first and second rows from $A_p(\boldsymbol{q}^0)$.

Substituting

$$\begin{aligned} \begin{pmatrix} b_p^1(\boldsymbol{q}, \boldsymbol{s}^0) \\ b_p^2(\boldsymbol{q}, \boldsymbol{s}^0) \\ b_p^3(\boldsymbol{q}, \boldsymbol{s}^0) \\ \vdots \\ b_p^n(\boldsymbol{q}, \boldsymbol{s}^0) \end{pmatrix} &= \begin{bmatrix} s_1 \cdot (\text{the first column of } B(\boldsymbol{q})) + \\ s_2 \cdot (\text{the second column of } B(\boldsymbol{q})) \end{bmatrix}_{(s_1,s_2)=(1,-1)} \\ &= B(\boldsymbol{q}) \cdot {}^t(1, -1, 0, \dots, 0) \end{aligned}$$

into (2.8), we have the conclusion, since $Y \cdot {}^t(1, -1, 0, \dots, 0) = 0$. (q. e. d.)

Let

$$\check{b}_p^j(\boldsymbol{q}, \boldsymbol{s}) = \frac{1}{\beta} b_p^j(\boldsymbol{q}, \boldsymbol{s}), \qquad \text{and} \qquad \check{f}_p^j(\boldsymbol{q}, \boldsymbol{s}) = \frac{1}{\beta^2} f_p^j(\boldsymbol{q}, \boldsymbol{s}).$$

Then

$$(2.9) \quad \frac{\partial}{\partial q_j} \begin{pmatrix} \check{b}_p^1 \\ \check{b}_p^2 \\ \vdots \\ \check{b}_p^n \end{pmatrix} (\boldsymbol{q}^0, \boldsymbol{s}^0) = \left\{ \frac{1}{2} \operatorname{Tr} \left(\frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0) \cdot Y \right) I - Y \cdot \frac{\partial A_p}{\partial q_j}(\boldsymbol{q}^0) \right\} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

and differentiating (1.13), we have

$$\frac{1}{\beta^2} \frac{\partial f_p^j}{\partial q_l} = \frac{\partial \check{f}_p^j}{\partial q_l} = \sum_{k=1}^{n} \frac{\partial \check{b}_p^k}{\partial q_l} + (n-2) \frac{\partial \check{b}_p^j}{\partial q_l} + \begin{cases} -\dfrac{1}{2} \zeta^{-l}(n-1) & (\text{if } j = l) \\ \dfrac{1}{\zeta^j - \zeta^l} & (\text{if } j \neq l), \end{cases}$$

because $b_p^j(\boldsymbol{q}^0, \boldsymbol{s}^0) = \beta$. Since $f^j(q^0, s^0) = (n-1)\beta^2$ and by Lemma 2.4, the Jacobian matrix J_p in (1.14) becomes

$$J_p = \frac{1}{2} m(m-1)(2m-1) \beta^5 \check{J}_p \qquad (n = 2m),$$

where

$$\check{J}_p = \left(- \left\{ \frac{\partial \check{f}_p^k}{\partial q_j}(\boldsymbol{q}^0, \boldsymbol{s}^0) - \frac{\partial \check{f}_p^n}{\partial q_j}(\boldsymbol{q}^0, \boldsymbol{s}^0) \right\} \right.$$
$$\left. + (-1)^j \zeta^{-j} \left\{ \frac{\partial \check{f}_p^k}{\partial q_n}(q^0, s^0) - \frac{\partial \check{f}_p^n}{\partial q_n}(q^0, s^0) \right\} \right)_{k,j=1,\ldots,n-1}.$$

On the other hand, since

$$f^j(\boldsymbol{q}^0, \boldsymbol{s}) = \left(s_1 b_p^{(1),j}(\boldsymbol{q}^0) + s_2 b_p^{(2),j}(\boldsymbol{q}^0)\right) \sum_{k \neq j} \left(s_1 b_p^{(1),k}(\boldsymbol{q}^0) + s_2 b_p^{(2),k}(\boldsymbol{q}^0)\right)$$

and $b_p^{(2)}(\boldsymbol{q}^0) = \beta \cdot {}^t(0, -1, 0, \ldots, -1)$, we have

$$\frac{\partial f^j}{\partial s_2}(\boldsymbol{q}^0, \boldsymbol{s}^0) = \begin{cases} \left(-\frac{3}{2}n + 2\right) \beta^2 & \text{(if } j \text{ is even)} \\ \left(-\frac{n}{2}\right) \beta^2 & \text{(if } j \text{ is odd)} \end{cases}.$$

We set an $(n-1) \times n$ matrix $\check{\mathcal{J}}_p$ as follows:

- The first column is given by
$$\left(\frac{\partial \check{f}^k}{\partial s_2} - \frac{\partial \check{f}^n}{\partial s_2}\right)_{k=1,\ldots,n-1}.$$
- The second to n-th columns is equal to $\check{J}_p$

Thus by definition, the rank of $\mathcal{J}_p$ is equal to those of $\check{\mathcal{J}}_p$.

Lemma 2.6 *If $n = 4$, 6 or 8, the rank of $\check{\mathcal{J}}_p$ is $n - 3$.*

Proof. Let Δ_n be an $(n-3)$-matrix which consits of the first $(n-3)$ rows and $(n-3)$ columns of $\check{\mathcal{J}}_p$.

By direct calculations, we have

$$\begin{aligned} \det \Delta_4 &= 2 \\ \det \Delta_6 &= 2187 = 3^7 \\ \det \Delta_8 &= \sqrt{-1} \cdot 2^{15} \cdot 3^6 \cdot 5^2(\sqrt{2} - 2) \sim -3 \cdot 10^8 \sqrt{-1}. \end{aligned}$$

Thus we have the conclusion. [2] (q. e. d.)

Thus, by Lemma 2.6, we have proved the Theorem mentioned in Introduction for $n = 4$, 6 and 8.

Remark 2.7 *For calculations in this section, we must compute the inverse matrix of A_1 or A_{12}. For large n, it is difficult to do that by computer as symbolic calculations. However, we can calculate $|\det \Delta_n|$ numerically for large n. As a conclusion of such numerical calculations, we may say that the assertion of the theorem is true if $n \leq 13$.*

[2] To calculate $\det \Delta_n$ we use Mathematica Version 2.2, a system of symboic calculation

3 Examples

In [KUY1], [KUY2] and this paper, we showed general existence of n-end catenoids with prescribed flux of Type II and Type III. However, the following problems are still remaining:

- Are there futher obstructions for the inverse problem of the flux formula except for the ones listed in [KUY1], especially for Type II and III? (In fact, any obstruction for the existence of n-end catenoids of Type III have not been found yet.)
- Is the set of n-end catenoids connected?
- Can any n-end catenoids of Type II be deformed into n-end catenoids of Type III?

Before finding answers to these questions, maybe we should examine many concrete examples. In this section, we will observe two examples of deformations from Jorge-Meeks' $2m$-noid, one of them are related to the third question. First, we see a family of D_m-invariant $2m$-end catenoids of Type II.

Example 3.1 Let m be an integer greater than 1, and set $\zeta_m := \exp(2\pi\sqrt{-1}/m)$. Let p be a unit complex number such that $0 \le \arg p < \pi$. Then, for any flux data

$$\begin{cases} p_j = \pi(v_j) := \begin{cases} p{\zeta_m}^{j-1} & (j = 1, \dots, m), \\ p^{-1}{\zeta_m}^{j-1} & (j = m+1, \dots, 2m), \end{cases} \\ a_1 = \dots = a_{2m} = 1, \end{cases}$$

the equation (1.1) has a solution as follows:

$$\begin{cases} q_j = \begin{cases} q{\zeta_m}^{j-1} & (j = 1, \dots, m), \\ q^{-1}{\zeta_m}^{j-1} & (j = m+1, \dots, 2m), \end{cases} \\ b_j = \begin{cases} qr & (j = 1, \dots, m), \\ pr & (j = m+1, \dots, 2m), \end{cases} \end{cases}$$

where q is a unit complex number satisfying

$$q^{2-2m} = -p^2, \qquad q^{2m} \neq 1,$$

and r is given by

$$r := \sqrt{\frac{q^{2m} - 1}{q\{(m-1)p(q^{2m} - 1) + mq(p^2 q^{2m-2} - 1)\}}}.$$

Define an immersion

$$x_{(m,q)} : \mathbf{C} \setminus \{q, q\zeta_m, \dots, q{\zeta_m}^{m-1}, q^{-1}, q^{-1}\zeta_m, \dots, q^{-1}{\zeta_m}^{m-1}\} \to \mathbf{R}^3$$

by these data, or equivalently by

$$\begin{aligned} g(z) &:= \frac{(pq^m+q^{1-m})z^m-(p+q)}{z^{m-1}\{(p+q)z^m-(pq^m+q^{1-m})\}}, \\ \omega &:= -r^2\left[\frac{mz^{m-1}\{(p+q)z^m-(pq^m+q^{1-m})\}}{(z^m-q^m)(z^m-q^{-m})}\right]^2 dz. \end{aligned}$$

Then $x_{(m,q)}$ is a conformal minimal immersion satisfying

$$\begin{cases} g(q{\zeta_m}^{j-1})=p{\zeta_m}^{j-1}, \quad g(q^{-1}{\zeta_m}^{j-1})=p^{-1}{\zeta_m}^{j-1} \\ w(q{\zeta_m}^{j-1})=w(q^{-1}{\zeta_m}^{j-1})=1 \end{cases} \qquad (j=1,\dots,m),$$

where $w(q{\zeta_m}^{j-1})$ means the weight of the end $q{\zeta_m}^{j-1}$. It is easy to see that

(1) $x_{(m,q)}$ is branched if and only if $q^{2m}=-1$ and $p=-q$;

(2) In the other cases, $x_{(m,q)}$ is a D_m-invariant $2m$-end catenoid of Type II.

In particular,

(3) $x_{(m,\zeta_{4m})}$ is Jorge-Meeks' $2m$-noid.

In the case when $\mathrm{Re}(q/p)>0$, these surfaces are constructed also by Rossman [R1].

Clearly, for any unit complex number p such that $0\le \arg p\le \pi/2m$, complex numbers

$$\left\{q^{(l)}=p^{-1/(m-1)}\exp\left(\frac{2l-1}{2m-2}\pi\sqrt{-1}\right) : l=1,\dots,2m-2\right\}$$

satisfy the same equality $(q^{(l)})^{2-2m}=-p^2$. Hence there are $(2m-2)$-tuples of $2m$-end catenoids $x_{(m,q^{(1)})},\dots,x_{(m,q^{(2m-2)})}$ which have the same flux data. It can be easily observed that if $0<\arg p<\pi/2m$, then these $2m-2$ surfaces are not congruent to each other. When $\arg p=0$ or $\pi/2m$, some of the surfaces are branched or not realized, and some two of them are congruent to each other. However, if $\arg p=0$ (resp. $\pi/2m$), then $2[(m-1)/2]$ (resp. $2[m/2]-1$) surfaces of them are non-branched and not congruent to each other.

Next, we observe a typical example of a deformation of $2m$-end catenoids from Type II to Type III. Through this, we can see that Jorge-Meeks' 4-noid (resp. 6-noid) and the tetrahedrally (resp. octahedrally) symmetric one are included in the same connected component of the moduli of 4-end (resp. 6-end) catenoids. (The case of $n=4$ has been already observed in [KUY1].)

Example 3.2 Let m be an integer greater than 1, and set $\zeta_{2m} := \exp\,(2\pi\sqrt{-1}\,/2m)$, $\zeta_m := \zeta_{2m}^2$. Let p be a positive real number. Then there is a real number $c_m > 1$ depending only on m (e.g. $c_2 = (\sqrt{6}+\sqrt{2})/2$, $c_3 = 2$, $c_4 \approx 2.0503$) such that, if ${c_m}^{-1} \le p \le c_m$, then, for any flux data

$$\begin{cases} p_j = \sigma(v_j) := \begin{cases} p{\zeta_m}^{j-1} & (j = 1, \dots, m), \\ p^{-1}{\zeta_{2m}}^{2j-1} & (j = m+1, \dots, 2m), \end{cases} \\ a_1 = \cdots = a_{2m} = 1, \end{cases}$$

the equation (1.1) has a solution as follows:

$$\begin{cases} q_j = \begin{cases} q{\zeta_m}^{j-1} & (j = 1, \dots, m), \\ q^{-1}{\zeta_{2m}}^{2j-1} & (j = m+1, \dots, 2m), \end{cases} \\ b_j = \begin{cases} qr & (j = 1, \dots, m), \\ pr & (j = m+1, \dots, 2m), \end{cases} \end{cases}$$

where q is a non-zero real numbers satisfying

$$\frac{mq(q^{2m-2}-1)}{(m-1)(q^{2m}+1)} = \frac{p^2-1}{2p},$$

and r is given by

$$r := \sqrt{\frac{q^{2m}+1}{q\{(m-1)p(q^{2m}+1)+mq(p^2q^{2m-2}+1)\}}}.$$

Define an immersion

$$x_{(m,q)} : \mathbf{C} \setminus \{q, q\zeta_m, \dots, q{\zeta_m}^{m-1}, q^{-1}\zeta_{2m}, q^{-1}{\zeta_{2m}}^3, \dots, q^{-1}{\zeta_{2m}}^{2m-1}\} \to \mathbf{R}^3$$

by these data, or equivalently by

$$\begin{aligned} g(z) &:= \frac{(pq^m - q^{1-m})z^m + (p+q)}{z^{m-1}\{(p+q)z^m - (pq^m - q^{1-m})\}}, \\ \omega &:= -r^2 \left[\frac{mz^{m-1}\{(p+q)z^m - (pq^m - q^{1-m})\}}{(z^m - q^m)(z^m + q^{-m})}\right]^2 dz. \end{aligned}$$

Then $x_{(m,q)}$ is a branched conformal minimal immersion such that

$$\begin{cases} g(q{\zeta_m}^{j-1}) = p{\zeta_m}^{j-1}, \quad g(q^{-1}{\zeta_{2m}}^{2j-1}) = p^{-1}{\zeta_{2m}}^{2j-1} \\ w(q{\zeta_m}^{j-1}) = w(q^{-1}{\zeta_{2m}}^{2j-1}) = 1 \end{cases} \qquad (j = 1, \dots, m)$$

except for the case when $q = -\sqrt[m]{(m \pm \sqrt{2m-1})/(m-1)}$ (i.e. $r = \infty$). It is easy to see that

(1) $x_{(m,q)}$ is branched if and only if $q = -1$ $(p = 1)$;

(2) In the other cases, $x_{(m,q)}$ is a $2m$-end catenoid with the symmetry of a regular m-angular antiprism.

In particular, by a direct calculation ([Kat], [Xu]), we have

(3) $x_{(m,1)}$ is Jorge-Meeks' $2m$-noid;

(4) $x_{(2,(\sqrt{6}+\sqrt{2})/2)}$ is the tetrahedrally symmetric 4-end catenoid, and $x_{(3,(\sqrt{6}+\sqrt{2})/2)}$ is the octahedrally symmetric 6-end catenoid.

By (2), (3) and (4), we see that the family

$$\left\{x_{(2,q)}\,(\text{resp. } x_{(3,q)})\,;\,1 \le q \le (\sqrt{6}+\sqrt{2})/2\right\}$$

gives a deformation from Jorge-Meeks' 4-noid (resp. 6-noid) to the tetrahedrally (resp. octahedrally) symmetric one.

References

[B] E. L. Barbanel: *Complete minimal surface in* $\mathbf{R}^3$ *of low total curvature*, Ph.D Thesis, (1987) University of Massachusetts.

[JM] L. P. M. Jorge and W. H. Meeks, III: *The topology of complete minimal surfaces of finite total Gaussian curvature*, Topology, **22** (1983), 203–221.

[Kar] H. Karcher: *Construction of minimal surfaces*, Surveys in Geometry 1989/1990, University of Tokyo.

[Kat] S. Kato: *Construction of n-end catenoids with prescribed flux*, Kodai Math. J., **18** (1995), 86–98.

[KUY1] S. Kato, M. Umehara and K. Yamada: *An inverse problem of the flux for minimal surfaces*, preprint.

[KUY2] S. Kato, M. Umehara and K. Yamada: *General existence of minimal surface with prescribed flux*, preprint.

[KS] R. Kusner and N. Schmitt: *The spinor representation of minimal surfaces*, Preprint.

[L] F. J. Lopez: *The classification of complete minimal surfaces with total curvature greater than* -12π, Trans. Amer. Math. Soc., **334** (1992), 49–74.

[LR] F. J. Lopez and A. Ros: *On embedded complete minimal surfaces of genus zero*, J. Differential Geometry **33** (1991), 293–300.

[M] W. H. Meeks III: *The classification of complete minimal surfaces in* $\mathbf{R}^3$ *with total curvature greater than* -8π, Duke Math. J., **48** (1981), 523–535.

[P] J. Perez: *Balancing formulae for minimal surfaces in flat three-manifolds*, preprint.

[RT] H. Rosenberg and E. Toubiana: *Complete minimal surfaces and minimal herissons*, J. Differential Geometry, **28** (1988), 115–132.

[R1] W. Rossman: *Minimal surfaces in* $\mathbf{R}^3$ *with dihedral symmetry*, Tohoku Math. J., **47** (1995), 31–54.

[R2] W. Rossman: *On embeddedness of area-minimizing disks, and an application to constructing complete minimal surfaces*, preprint.

[UY] M. Umehara and K. Yamada: *Surfaces of constant mean curvature* c *in* $H^3(-c^2)$ *with prescribed hyperbolic Gauss map*, Math. Ann., **304** (1996), 203–224.

[X] Y. Xu: *Symmetric minimal surfaces in* $\mathbf{R}^3$, Pacific J. Math., **171** (1995), 275–296.

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