TRIVIALITY OF SCALAR LINEAR TYPE ISOTROPY SUBGROUP BY PASSING TO AN ALTERNATIVE CANONICAL FORM OF A HYPERSURFACE

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ABSTRACT. The Chern-Moser (CM) normal form of a real hypersurfaces in $\mathbb{C}^N$ can be obtained by considering automorphisms whose derivative act as the identity on the complex tangent space. However, the CM normal form is also invariant under a larger group (pseudo-unitary linear transformations) and it is the property that makes the CM normal form special. Without this additional restriction, various types of normal forms occur. One of them helps to give a simple proof of a (previously complicated) theorem about triviality of scalar linear type isotropy subgroup of a nonquadratic hypersurface. An example of an analogous nontrivial subgroup for a 2-codimensional CR surface in $\mathbb{C}^4$ is constructed.

1. Alternative canonical form

Let $M \in \mathbb{C}^{n+1}$ be a real-analytic hypersurface. For the coordinates in $\mathbb{C}^{n+1}$ we set $z = (z^1, ..., z^n)$, $\quad w = u + iv$. We assume that M has nondegenerate Levi form $\langle z, z\rangle$ at the origin.

Consider the space $\mathcal{F}$ of power series in $(z, \bar{z}, u)$ such that the series itself, its differential and the Hessian $\frac{\partial^2}{\partial z \partial \bar{z}}$ vanish at the origin. A hypersurface M can always be written in the form:

$$v = \langle z, z\rangle + H, \tag{1}$$

where $H(z, \bar{z}, u) \in \mathcal{F}$.

Any $H \in \mathcal{F}$ can be written as a sum $H = \sum_{k,l} H_{k,l}$ of homogeneous polynomials of degree k in z and l in $\bar{z}$ with analytic in u coefficients. By $\Pi_{(k_1,l_1),...,(k_s,l_s)}(H)$ we denote the sum $H_{k_1,l_1} + ... + H_{k_s,l_s}$. In $\mathcal{F}$ we consider the subspace

$$\mathcal{R} = \{2 \operatorname{Re} \chi((\langle z, z\rangle - v)|_{v=\langle z,z\rangle}\}, \tag{2}$$

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where χ runs over holomorphic vector fields of the form

$$\chi = g(z,w)\frac{\partial}{\partial w} + \sum_{j=1}^{n} f^j \frac{\partial}{\partial z^j}^{j},$$

here

$$f^j(0) = \frac{\partial f^j}{\partial z}|_0 = g(0) = \frac{\partial g}{\partial z}|_0 = \frac{\partial g}{\partial w}|_0 = 0.$$

Let $\kappa = \{(k,0),(k,1),(2,2),(3,2),(3,3)\}$ and Π_κ be a projection on the corresponding jet space. Π_κ is an isomorphism on $\mathcal{R}$ (see [CM]), so below we identify $\mathcal{R}$ and $\Pi_\kappa \mathcal{R}$. Let $\mathcal{N}$ be any direct complement of $\mathcal{R}$ in $\mathcal{F}$. The $\mathcal{R}$-component always can be eliminated from the equation (1) of M. The freedom in the choice of $\mathcal{N}$ space leads to various canonical forms of the equation. One of such canonical forms, constructed by Chern and Moser ([CM]), has some natural advantages compared to others. Here we introduce an alternative canonical form of a hypersurface to provide a simple proof of Beloshapka and Loboda ([Be],[Lo1]) theorem about the triviality of *scalar linear type* isotropy subgroup of a nonquadratic hypersurface M. The original proof of this theorem, based on Chern-Moser normal form is technically very hard.
It is convenient to consider two cases of the Levi form:

(3)

i.) $\langle z,z\rangle = 2\,\mathrm{Re}\, z^1\bar{z}^n + \sum_{\alpha=2}^{n-1}\varepsilon_\alpha |z^\alpha|^2$,
ii.) $\langle z,z\rangle = |z^1|^2 + \sum_{\alpha=2}^{n}\varepsilon_\alpha |z^\alpha|^2$,

here $\varepsilon_\alpha = \pm 1$.
Let $H_{k,l}$ be an arbitrary polynomial of the type (k,l). Each time in $H_{k,l}$ the product $z^1\bar{z}^n$ occurs in case (i), or, $z^1\bar{z}^1$ in case (ii), we replace this product by $\langle z,z\rangle - \langle z,z\rangle'$, where

$$(4)\qquad \langle z,z\rangle' = z^n\bar{z}^1 + \sum_{\alpha=2}^{n-1}\varepsilon_\alpha |z^\alpha|^2, \quad or, \quad \langle z,z\rangle' = \sum_{\alpha=2}^{n}\varepsilon_\alpha |z^\alpha|^2$$

in cases (i) and (ii) respectively. Thus, we obtain a decomposition of $H_{k,l}$ in the form

$$(5)\qquad H_{k,l} = \sum_m \sum_\alpha u^{m-\alpha}\langle z,z\rangle^\alpha Q_{k-\alpha,l-\alpha}(z,\bar{z})$$

where each monomial in $Q_{k-\alpha,l-\alpha}$ does not contain $z^1\bar{z}^n$, or $z^1\bar{z}^1$ respectively. Set $\nu = \langle z,z\rangle$ and write the equation of M in the form

$$v = \nu + \sum_{k,l} H_{k,l}(\nu, z, \bar{z}, u),$$

where polynomials $H_{k,l}$ are written in the form (5) and ν is considered as an independent variable.

By *canonical form* we call an equation of the type

$$v = \nu + \sum_{k,l \geq 2} N_{k,l}(\nu, z, \bar{z}, u), \tag{6}$$

where

$$\frac{\partial}{\partial \nu} N_{22} = \frac{\partial^2}{\partial \nu^2} N_{32} = \frac{\partial^3}{\partial \nu^3} N_{33} = 0.$$

The following is analogous to Chern-Moser theorem ([CM]):

Theorem 1. *i.) Any hypersurface can be locally biholomorphically transformed to a canonical form,*

ii.) Let $\Phi = (F(z,w), G(z,w))$ be a transformation to canonical form. Let J_Φ be the Jacobian of Φ:

$$J_\Phi = \begin{pmatrix} \lambda U & \lambda U a \\ 0 & \varepsilon \lambda^2 \end{pmatrix},$$

where $\varepsilon = \pm 1$, U is a pseudounitary matrix such that $\langle Uz, Uz \rangle = \varepsilon \langle z, z \rangle$, $a \in \mathbb{C}^n$, $\lambda \geq 0$, $r = \operatorname{Re} \frac{\varepsilon}{\lambda^2} \frac{\partial^2 G}{\partial w^2}|_0$.
Then the set of initial data $I_\Phi = \{U, a, \lambda, r\}$ determine Φ uniquely, and, conversely, any set of parameters $\{U, a, \lambda, r\}$ with the above properties corresponds to the transformation of M to some canonical form.

iii.) Any formal transformation that transforms M to a canonical form is automatically holomorphic.

Remark. For $n = 2$ and for quadrics $v = \langle z, z \rangle$ the described canonical form coincides with the Chern-Moser normal form.

By G_0 we denote the group of locally defined isotropic holomorphic automorphisms of M at 0. If M is written in canonical form then any $\Phi \in G_0$ can be considered as a transformation between canonical forms, so the set of initial data $\{U, a, \lambda, r\}$ uniquely determines Φ. If M is a quadric then all parameters of initial data are free, if M is nonquadratic, i.e. not locally equivalent to a quadric, Beloshapka and Loboda ([Be],[Lo1]) proved that matrix U uniquely determines other parameters $\{a, \lambda, r\}$. This result can be reformulated if we denote by $G_{0,\lambda\,\mathrm{id}}$ the subgroup of G_0 consisting of the automorphisms with $U = \mathrm{id}$ ($G_{0,\lambda\,\mathrm{id}}$ is called em scalar linear type isotropy subgroup) as follows:

Theorem 2. *If M is a nonquadratic hypersurface then* $G_{0,\lambda\,\mathrm{id}} = \{\mathrm{id}\}$.

Remark. Analogous theorem does not hold for higher codimensional CR surfaces as the example in the last section of the paper shows.

2. Triviality of conformal isotropies

We start with the proof of Theorem 1, which basically follows the proof of Chern-Moser theorem ([CM]).

Proof. We check that the described $\mathcal{N}$ space is complementary to $\mathcal{R}$. Expanding the right hand side of the expression for $\mathcal{R}$ we observe, that

$$\mathcal{R} = \{\sum_0^\infty (H_{k,0} + H_{0,k}) + \sum_2^\infty (H_{k,1} + H_{1,k}) + ({}^\tau z B_1(u)\bar{z}) +$$
$$\langle z,z\rangle({}^\tau z B_2(u)\bar{z}) + \langle z,z\rangle^2(\langle z, B_3(u)\rangle + \langle B_3(u), z\rangle) + b_4(u)\langle z,z\rangle^3\},$$

where $B_1(u)$ and $B_2(u)$ are hermitian matrices, $B_1(0) = 0$, $B_3(u)$ is an arbitrary vector in $\mathbb{C}^n$, $b_4(u)$ is a real-valued function.
It is clear that the subspace

$$\mathcal{N} = \{H \in \mathcal{F} | H_{k,0} = H_{k,1} = \frac{\partial}{\partial \nu} H_{22} = \frac{\partial^2}{\partial \nu^2} H_{32} = \frac{\partial^3}{\partial \nu^3} H_{33} = 0\}$$

is a direct compliment to $\mathcal{R}$ in $\mathcal{F}$. So we can follow Chern-Moser scheme to eliminate $\mathcal{R}$-component in the equation of M. By the "Convergence" part of ([CM]) the holomorphic transformations that do this job are parametrized by $\{U, a, \lambda, r\}$. By the "Formal theory" of ([CM]), any formal transformation with fixed initial data $\{U, a, \lambda, r\}$ that eliminates $\mathcal{R}$-compnent is unique, so, it has to be holomorphic. This completes the proof of Theorem 1. □

We continue with the proof of Theorem 2.

Proof. Consider $\Phi \in G_{0,\lambda\,\mathrm{id}}$. Since I_Φ determines Φ uniquely, it suffices to show that $I_\Phi = (\mathrm{id}, 0, 1, 0)$, which corresponds to the identity. Let $a \in I_\Phi$. We consider two cases:

(7)

i.) a is an isotropic vector for $\langle z,z\rangle$, i.e. $\langle a,a\rangle = 0$.
ii.) $\langle a,a\rangle = 1$ (without loss of generality this represents the situation $\langle a,a\rangle \neq 0$).

Consider a new basis $\{a, e_2, ..., e_n\}$ in $\mathbb{C}^n$ such that in the case (i) of (7)

$$\langle e_j, a\rangle = 0, \quad \langle e_j, e_k\rangle =^{+}_{-} \delta_{j,k};$$

in the the case (ii):

$$\langle e_n, a\rangle = 1, \quad \langle e_j, a\rangle = 0, \quad \langle e_j, e_n\rangle = 0, \quad \langle e_j, e_k\rangle =^{+}_{-} \delta_{j,k}, \quad j,k = 2, ..., n,$$

$\delta_{j,k}$ here is a Kronecker delta.
Let z^* be coordinates in $\mathbb{C}^n$ in the new basis and C be the corresponding n-matrix that determines the coordinate change in $\mathbb{C}^{n+1}$ via $z = Cz^*, w = w^*$
In (z^*, w^*) coordinates the Jacobian of Φ equals to

$$J_\Phi = \begin{pmatrix} C^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda\,\mathrm{id} & \lambda a \\ 0 & \lambda^2 \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda\,\mathrm{id} & \lambda C^{-1}a \\ 0 & \lambda^2 \end{pmatrix} = \begin{pmatrix} & \lambda \\ & 0 \\ \lambda\,\mathrm{id} & \vdots \\ & 0 \\ 0 & \lambda^2 \end{pmatrix}.$$

The equation of M takes the form

$$v = \langle z, z\rangle + N(z, \bar z, u),$$

where $\langle z, z\rangle$ has the form (i) and (ii) of (3) in the cases (i) and (ii) of (7) respectively. We introduce weight $\varkappa$ to the variables and to the coordinate differential operators in standard way:

$$\varkappa(w) = \varkappa(\bar w) = -\varkappa\left(\frac{\partial}{\partial w}\right) = -\varkappa\left(\frac{\partial}{\partial \bar w}\right) = 2;$$
$$\varkappa(z^\alpha) = \varkappa(\bar{z^\alpha}) = -\varkappa\left(\frac{\partial}{\partial z^\alpha}\right) = -\varkappa\left(\frac{\partial}{\partial \bar{z^\alpha}}\right) = 1.$$

end extend this weight to polynomials in $(z, \bar z, w, \bar w)$ by linearity and homogenuity. We expand an equation of M in a canonical form (6) as the sum of weighted homogeneous polynomials in $(z, \bar z, u)$:

$$v = \nu + \sum_{\gamma \geq 4} H_\gamma(\nu, z, \bar z, u) \tag{8}$$

Let γ_0 be the first value of γ such that $H_{\gamma_0} \neq 0$. As $\Phi \in G_{0,\lambda\,\mathrm{id}}$, there exists vector field

$$(9)\qquad \chi = \left(\ln\lambda z\frac{\partial}{\partial z} + 2\ln\lambda w\frac{\partial}{\partial w}\right) + \left(w\frac{\partial}{\partial z^1} + 2iz^\varepsilon\frac{\partial}{\partial z} + 2iz^\varepsilon w\frac{\partial}{\partial w}\right) + \left(rwz\frac{\partial}{\partial z} + rw^2\frac{\partial}{\partial w}\right) + \left(f_{\gamma_0}\frac{\partial}{\partial z} + 2ig_{\gamma_0+1}\frac{\partial}{\partial w}\right) + \dots,$$

such that $\Phi = \exp(\chi)$.
In the definition of χ we assume that

$$\frac{\partial}{\partial z} = \sum_\alpha \frac{\partial}{\partial z^\alpha};$$
$$z\frac{\partial}{\partial z} = \sum_\alpha z^\alpha\frac{\partial}{\partial z^\alpha};$$

here $\varepsilon = n$ or $\varepsilon = 1$ in the cases (i) or (ii) of (7) respectively. The collected terms in (9) represent the components χ_0, χ_1, χ_2 and χ_{γ_0-1} of χ of weights 0, 1, 2 and $\gamma_0 - 1$ respectively.
Since the iterations of Φ preserve the equation (8), it follows that χ satisfies the equation

$$(10)\qquad \mathcal{H} = 2\operatorname{Re}\chi\left(\langle z,z\rangle - v + \sum_{\gamma\geq\gamma_0} H_\gamma(\nu,z,\bar z,u)|_{v=\langle z,z\rangle+\sum_{\gamma\geq\gamma_0}H_\gamma(\nu,z,\bar z,u)}\right) \equiv 0$$

Considering component $\mathcal{H}_{\gamma_0}$ of weight γ_0 in (10) we obtain that

$$(\gamma_0 - 2)\ln\lambda H_{\gamma_0} = 0,$$

and since $\gamma_0 \geq 4$, it implies that $\lambda = 1$.
To obtain that $a = 0$ we consider the next, $\mathcal{H}_{\gamma_0+1}$, component. By *defect* of a polynomial in $(z, \bar z)$ we mean the difference in its degrees in holomorphic and antiholomorphic variables. Let $H_{\gamma_0,d,m}$ be the component of H_{γ_0} of maximal defect and minimal degree in (ν, u) (see (5)).
By $\rho \longrightarrow_{\alpha,\beta,\delta,\dots}$ we mean the contribution of an expression ρ into the component of the type $(\alpha, \beta, \delta, \dots)$, having weight α, defect β, degree δ in (ν, u), and other specifications denoted by the dots.
For simplicity in further computations we introduce the variables

$$\omega = u + i\nu, \quad \bar\omega = u - i\nu$$

Thus, the degree of a polynomial in (u,ν) equals to the degree in $(\omega,\bar\omega)$. So, the expression (5) for $H_{\gamma_0,d,m}$ takes the form

$$H_{\gamma_0,d,m} = \sum_m \sum_{\alpha=0}^{m} \omega^{m-\alpha}\bar{\omega}^{\alpha} Q_{m-\alpha}(z,\bar{z}) \tag{11}$$

Consider at first the case (i) in (7). Collecting in $\mathcal{H}$ the terms of type $(\gamma_0+1, d+1, m)$ we obtain:

$$\begin{aligned}\mathcal{H}_{\gamma_0+1,d+1,m} = \{ &\left(iz^n \left(2z\frac{\partial}{\partial z} + \omega\frac{\partial}{\partial w}\right) + \bar{\omega}\frac{\partial}{\partial \bar{z}^1}\right) H_{\gamma_0,d,m} - 2iz^n H_{\gamma_0,d,m} + \\ &2\,\mathrm{Re}\,(\langle f_{\gamma_0}, z\rangle - g_{\gamma_0+1}|_{v=\nu})\} \longrightarrow_{\gamma_0+1,d+1,m}\end{aligned} \tag{12}$$

Remark. The differential operator in the latter formula is applied to the $H_{\gamma_0,d,m}$ only, the term of minimal degree in $(\omega, \bar{\omega})$, because neither of

$$2iz^n\frac{\partial}{\partial z}, \quad iz^n\omega\frac{\partial}{\partial w}, \quad \bar{\omega}\frac{\partial}{\partial \bar{z}^1} \tag{13}$$

decreases the degree in $(\omega, \bar{\omega})$. Moreover, operators $iz^n\omega\frac{\partial}{\partial w}$, $\bar{\omega}\frac{\partial}{\partial \bar{z}^1}$ increase the degree in $(\omega, \bar{\omega})$ unless applied to $\omega^{m-\alpha}\bar{\omega}^{\alpha}$ in (11).

Exact computation shows that in terms of the contribution to $(\gamma_0+1, d+1, m)$ type component, the operators (13) can be substituted respectively by

$$iz^n \left(2h\,\mathrm{id} + (\omega - \bar{\omega})\left(\frac{\partial}{\partial \omega} - \frac{\partial}{\partial \bar{\omega}}\right)\right), \quad iz^n\omega\left(\frac{\partial}{\partial \omega} + \frac{\partial}{\partial \bar{\omega}}\right), \quad iz^n\bar{\omega}\left(\frac{\partial}{\partial \omega} - \frac{\partial}{\partial \bar{\omega}}\right) \tag{14}$$

Making the sum of the operators in (14) we obtain a diagonal operator D:

$$2iz^n D = 2iz^n \left(h\,\mathrm{id} + \omega\frac{\partial}{\partial \omega}\right) \tag{15}$$

So far, in (12) we obtain:

$$\begin{aligned}\mathcal{H}_{\gamma_0+1,d+1,m} = 2iz^n &\sum_{\alpha=0}^{m} (m-\alpha+h-1)\omega^{m-\alpha}\bar{\omega}^{\alpha} Q_{m-\alpha}(z,\bar{z}) + \\ &(2\,\mathrm{Re}\,\chi_{\gamma_0-1}(v-\nu)|_{v=\nu} \longrightarrow_{\gamma_0+1,d+1,m})\end{aligned} \tag{16}$$

Suppose that $\chi_{\gamma_0-1} \neq 0$. Then the last term in (16), i.e. $2\,\mathrm{Re}\,\chi_{\gamma_0-1}(v-\nu)|_{v=\nu} \longrightarrow_{\gamma_0+1,d+1,m}$ contains a nontrivial $\mathcal{R}$ component. So, $DH_{\gamma_0,d,m}$ must contain $\mathcal{R}$ terms as well. It is easy to check that two operators that compose $2iz^nD$, namely, $2iz^n\frac{\partial}{\partial z}$/quad and $iz^n\omega\frac{\partial}{\partial w}$/quad preserve $\mathcal{N}$ space. So, the $\mathcal{R}$ component in $DH_{\gamma_0,d,m}$ may arise only from

$$\bar{\omega}\frac{\partial}{\partial \bar{z}^1}H_{\gamma_0,d,m} \longrightarrow_{\gamma_0+1,d+1,m} \tag{17}$$

Degree m in $(\omega,\bar{\omega})$ does not increase if $\bar{\omega}\frac{\partial}{\partial \bar{z}^1}$ is applied only to ν (and not to $Q_{m-\alpha}(z,\bar{z})$).

What sort of $\mathcal{R}$ component may arise?

Term $\mathcal{R}_{2,1}$ can not appear in (17) from $\bar{\omega}\frac{\partial}{\partial \bar{z}^1}H_{2,2}$ because, by (6), $\frac{\partial}{\partial \nu}H_{2,2}=0$.

Analogously, $\mathcal{R}_{3,2}=\langle z,b\rangle\nu^2$ can not arise from (16) because $\frac{\partial^3}{\partial \nu^3}H_{3,3}=0$.

Term $\mathcal{R}_{3,1}$ may arise from $u^{m-1}\nu Q_{2,1}(z,\bar{z})$ in $H_{3,2}$ as

$$\bar{\omega}\frac{\partial}{\partial \bar{z}^1}(u^{m-1}\nu Q_{2,1}) \longrightarrow_{\mathcal{R}} u^m z^n Q_{2,1} = u^m z^n \langle Q_{2,0}, z\rangle$$

This implies for χ_{γ_0-1} that for compensation,

$$\chi_{\gamma_0-1} = -\omega^m z^n Q_{2,0}\frac{\partial}{\partial z}.$$

Therefore, by (16) and (10)

$$\mathcal{H}_{\gamma_0+1,2,m} = -2i\omega^m\langle z^n Q_{2,0}, z\rangle z^n \sum_{\alpha=0}^{m}(m-\alpha+2-1)\omega^{m-\alpha}\bar{\omega}^\alpha Q_{m-\alpha}(z,\bar{z}) = 0$$

Thus,

$$\mathcal{H}_{\gamma_0,2,m} = \frac{1}{2i(m+1)}\omega^m\langle Q_{2,0}, z\rangle$$

But in usual coordinates the latter expression contains a $(3,1)$ term

$$\frac{1}{2i(m+1)}u^m\langle Q_{2,0}, z\rangle,$$

which does not lie in the $\mathcal{N}$ space. Contradiction. So, $\mathcal{R}_{3,1}$ can not arise as well.

Term $\mathcal{R}_{k,1}$, $k\geq 4$ can arise from $u^{m-2}\nu^2 Q_{k-2,0}$ or from $u^{m-1}\nu Q_{k-1,1}(z,\bar{z})$. The second option is considered as above.

Consider the first option:

$$\bar{\omega}\frac{\partial}{\partial \bar{z}^1}(u^{m-2}\nu Q_{k-2,0}) \longrightarrow_{\mathcal{R}} u^m z^n Q_{2,1} = 2u^{m-1}\nu z^n Q_{k-2,0}$$

It follows that for compensation

$$\chi_{\gamma_0 - 1} = -2\omega^{m-1}\nu z^n Q_{k-2,0} z\frac{\partial}{\partial z}$$

and so,

$$2\,\mathrm{Re}\,\chi_{\gamma_0-1}(v-\nu)|_{v=\nu} = i\omega^{m-1}(\omega-\bar\omega)z^n Q_{k-2,0} = i\omega^m z^n Q_{k-2,0} - i\omega^{m-1}\bar\omega z^n Q_{k-2,0}.$$

Equations (10) and (16) imply that

$$\mathcal{H}_{\gamma_0+1,k-1,m} =$$

$$i\omega^m z^n Q_{k-2,0} - i\omega^{m-1}\bar\omega z^n Q_{k-2,0} + 2iz^n \sum_{\alpha=0}^{m}(m-\alpha+k-3)\omega^{m-\alpha}\bar\omega^\alpha Q_{m-\alpha}(z,\bar z) \equiv 0.$$

Thus,

$$\mathcal{H}_{\gamma_0,k-2,m} = -\frac{i}{m+k-3}\omega^m Q_{k-2,0} + \frac{i}{m+k-4}\omega^{m-1}\bar\omega Q_{k-2,0} = Q_{k-2,0}\left(-\frac{i}{m+k-3}\omega^m + \frac{i}{m+k-4}\omega^{m-1}\bar\omega\right).$$

The latter does not lie in the $\mathcal{N}$ space since it contains a $(k-1,0)$ term. Contradiction. Suppose now that $\chi_{\gamma_0-1} = 0$. Then (10) implies that

$$DH_{\gamma_0,d,m} = 2iz^n \sum_{\alpha=0}^{m}(m-\alpha+h-1)\omega^{m-\alpha}\bar\omega^\alpha Q_{m-\alpha} = 0$$

This can happen only in two cases

i.) $h=1, \alpha = m$,
ii.) $h=0, \alpha = m-1$.

In (i) case

$$H_{\gamma_0,d,m} = \omega^m Q_{1,0},$$

which contains $u^m Q_{1,0} \in \mathcal{R}$; in (ii) case

$$H_{\gamma_0,d,m} = \omega^{m-1}\bar\omega Q_{1,0},$$

which contains $u^m Q_{1,0} \in \mathcal{R}$ as well. Contradiction. Hence, $a = 0$ in the set of initial data of Φ in case (i) (7).
The case (ii) in (7) is similar: instead of of $H_{\gamma_0,d,m}$ we take

$$H_{\gamma_0,d,m,d_{\bar 1}} = \sum_m \sum_{\alpha=0}^{m} \omega^{m-\alpha}\bar\omega^\alpha \bar z_1^{d_{\bar 1}} Q_{m-\alpha-d_{\bar 1}}(z,\bar z),$$

the component of $H_{\gamma_0,d,m}$ of minimal degree $d_{\bar 1}$ in $\bar z^1$.
If $d_{\bar 1} = 0$ then the proof goes literally as above. Let $d_{\bar 1} \neq 0$ then operators

$$2iz^1\frac{\partial}{\partial z}, \quad iz^1\omega\frac{\partial}{\partial w}, \quad \bar{\omega}\frac{\partial}{\partial \bar{z}^1}$$

being applied to $\omega^{m-\alpha}\bar{\omega}^{\alpha}$, contribute to the components of degrees m and $m+1$ in $(\omega, \bar{\omega})$. We are interested in the contribution to the minimal dgree m:

$$\mathcal{H}_{\gamma_0+1,d+1,m,d_{\mathrm{I}}-1} = 2i\frac{\langle z,z\rangle'}{\bar{z}^1}(D-\mathrm{id})H_{\gamma_0,d,m,d_{\mathrm{I}}}+ \\ \left(2\operatorname{Re}\chi_{\gamma_0-1}(v-\nu)|_{v=\nu} \longrightarrow_{\gamma_0+1,d+1,m,d_{\mathrm{I}}-1}\right) = \\ 2i\langle z,z\rangle'\sum_{\alpha=0}^{m}(m-\alpha+h-1)\omega^{m-\alpha}\bar{\omega}^{\alpha}\bar{z}^{1\,d_{\mathrm{I}}-1}Q_{m-\alpha-d_{\mathrm{I}}}(z,\bar{z})+ \\ \left(2\operatorname{Re}\chi_{\gamma_0-1}(v-\nu)|_{v=\nu} \longrightarrow_{\gamma_0+1,d+1,m,d_{\mathrm{I}}-1}\right)$$

(see (4) and (15) for $\langle z,z\rangle'$ and D). The latter expression does not vanish for the same reasons as above. This proves that $a=0$ in both cases (i) and (ii) of (7), and so $\chi_1 = 0$.

The remaining part is to show that $r=0$. Consider

$$\chi_2 = rwz\frac{\partial}{\partial z} + rw^2\frac{\partial}{\partial w}.$$

It surely preserves the $\mathcal{N}$ space, so it implies that $\chi = \chi_2$. But it is clear that $2\operatorname{Re}\chi_2$ doesn't annulate the equation of M (it suffices to consider the component of H_{γ_0,d,m,d_u} of maximal degree d_u in u and to observe that χ_2 on this component is just a multiplication times u). □

3. Counterexample for 2-codimensional CR surfaces

Let $M \in \mathbb{C}^4$ be a real-analytic Levi-nondegenerate CR surface of codimension 2. Set $z=(z^1,z^2)$, $w=(w^1=u^1+iv^1,\ w^2=u^2+iv^2)$ for coordinates in $\mathbb{C}^4$. Up to nondegenerate linear transformation in (z,w) the Levi form of M may have one of three types, namely, elliptic, hyperbolic and parabolic (see [Lo2]).

Isotropy group G_0M is defined like for hypersurfaces. Set

$$G_{0,\mathrm{id}}M = \{\Phi \in G_0M : d\Phi(0)|_{T_0^{\mathbb{C}}M} = \{\mathrm{id}\}\}.$$

If M is a quadratic surface, i.e. it is locally equivalent CR quadric, its automorphism group, and, in particular, $G_{0,\mathrm{id}}M$ are described in ([ES]). If M is nonquadratic the description of G_0M remains an open question. On the contrary to the Theorem 2 even for nonquadratic M the subgroup $G_{0,\mathrm{id}}M$ may be nontrivial as the example below shows:

Let M be a surface with the parabolic type of the Levi form, defined by the equation

$$\begin{aligned} v^1 &= |z^1|^2 \\ v^2 &= 2\,\mathrm{Re}\, z^1\bar{z}^2 + |z^1|^8. \end{aligned} \tag{18}$$

To check that M is nonquadratic one may estimate the dimension of the automorphism group.
Let χ be the vector field

$$\chi = i\left(2(z^1)^2\frac{\partial}{\partial z^2} + 2z^1w^1\frac{\partial}{\partial w^2} - iw^1\frac{\partial}{\partial z^2}\right) \tag{19}$$

It is easy to check that χ and M satisfy the identity:

$$2\,\mathrm{Re}\,\chi\begin{pmatrix} |z^1|^2 - \frac{1}{2i}(w^1 - \bar{w}^1) \\ 2\,\mathrm{Re}\, z^1\bar{z}^2 - \frac{1}{2i}(w^2 - \bar{w}^2) + |z^1|^8 \end{pmatrix}|_M \equiv 0.$$

Therefore, Φ generates a 1-parameter subgroup in $G_{0,\mathrm{id}}M$, namely, $\Phi^t = \exp(t\chi)$, since

$$d\Phi^t(0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The 1-parameter subgroup Φ^t consists of polynomial transformations

$$\begin{aligned} z^1 &\mapsto z^1 \\ z^2 &\mapsto z^2 + tw^1 + 2it(z^1)^2 \\ w^1 &\mapsto w^1 \\ w^2 &\mapsto w^2 + 2itw^1z^1 \end{aligned}$$

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