A theory of rank for enveloping algebras. I

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1 Introduction

This is the first in a series of papers developing an algebraic analog of the theory of N-rank for unitary representations of classical Lie groups (see [How82, Li89]). Using the theory of reductive dual pairs, we introduce a family of ideals in the enveloping algebra of a complex classical Lie algebra $\mathfrak{g}$. These ideals, called *rank ideals*, are parametrized by complex classical Lie algebras $\mathfrak{g}'$ forming together with $\mathfrak{g}$ a reductive dual pair (**cf.** [How79]). More precisely, to each irreducible reductive dual pair of complex Lie algebras $(\mathfrak{g}, \mathfrak{g}') \subseteq \mathfrak{sp}(W)$ we attach a completely prime ideal in $\mathrm{U}(\mathfrak{g})$ (Definition 2.2.1). Roughly speaking, this ideal is the annihilator in $\mathrm{U}(\mathfrak{g})$ of the oscillator representation restricted to $\mathfrak{g}$. More generally, we introduce and study a correspondence between ideals in $\mathrm{U}(\mathfrak{g})$ and $\mathrm{U}(\mathfrak{g}')$ that is compatible with the *reductive dual pair correspondence* (see [How89b]). The rank ideal is just the minimal ideal of $\mathrm{U}(\mathfrak{g})$ that can appear in this algebraic correspondence. These considerations yield new information on the behavior of the annihilators of Harish-Chandra modules of real classical groups under the reductive dual pair correspondence. On the other hand, the algebraic theory of rank that we develop offers a new perspective on primitive, and more generally, prime ideals in enveloping algebras. One application is to the study of Goldie ranks of primitive ideals.

Organization of the series is as follows. In the present first part we define the rank ideals and give their detailed description for the case $\mathfrak{g} = \mathfrak{gl}_n$. In the second part we establish some properties of rank for primitive ideals in $\mathrm{U}(\mathfrak{gl}_n)$ that are very similar to the main results of the theory of N-rank. In the third part we study the general duality correspondence between ideals in $\mathrm{U}(\mathfrak{g})$ and $\mathrm{U}(\mathfrak{g}')$. In the projected fourth part we shall give a detailed description of rank ideals in the technically more demanding cases of orthogonal – symplectic reductive dual pairs. Applications of rank theory to the description of the reductive dual pair correspondence are left for a separate paper.

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2 General construction of rank ideals

Motivated by problems in the theory of automorphic forms (the classification of singular Siegel modular forms, theta correspondence), R. Howe introduced a notion of N-rank for unitary representations of a symplectic group over a local field, see [How82]. Later J.-S. Li generalized in [Li89] the definition of N-rank and the results of [How82] to other classical groups. He obtained a clean description for the "singular" part of the unitary dual of a classical group $G(\mathbb{F})$ over a local field $\mathbb{F}$. The main theorem of [Li89] describes a singular unitary representation of $G(\mathbb{F})$ as a quotient of the oscillator representation associated to a reductive dual pair $(G(\mathbb{F}), G'(\mathbb{F}))$ in the stable range with the larger member $G(\mathbb{F})$. In the real case the oscillator representation admits a neat algebraic formulation through the theory of *Weyl algebra*. We use this formulation to introduce a notion of rank for ideals in the enveloping algebra of a complex classical Lie algebra. This notion is an algebraic counterpart of the N-rank, and is also related to the *determinantal varieties* of matrices of bounded rank.

2.1 Reductive dual pairs of Lie algebras

Let W be a finite-dimensional symplectic complex vector space. Denote by $Sp(W)$ the isometry group of W and by $\mathfrak{sp}(W)$ its Lie algebra. We consider an algebraic counterpart of a notion of a reductive dual pair of subgroups of a symplectic group (see [How79]).

Definition 2.1.1. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie subalgebras of the Lie algebra $\mathfrak{sp}(W)$. Then $(\mathfrak{g}, \mathfrak{g}') \subseteq \mathfrak{sp}(W)$ is a *reductive dual pair* if the following two conditions hold:

(1) The Lie subalgebras $\mathfrak{g}$ and $\mathfrak{g}'$ are the mutual centralizers in $\mathfrak{sp}(W)$;

(2) The actions of $\mathfrak{g}$ and $\mathfrak{g}'$ on the vector space W are *reductive* (i.e. completely reducible).

In this setting we denote by G and G' the subgroups of $Sp(W)$ which are the pointwise stabilizers of $\mathfrak{g}'$ and $\mathfrak{g}$, respectively.

It follows immediately from the definitions that $\operatorname{Lie} G = \mathfrak{g}$, and $\operatorname{Lie} G' = \mathfrak{g}'$. Thus the groups G and G' determine the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$. As we will presently see, the group G or G' may be disconnected, namely, an orthogonal group. Thus the Lie algebra $\mathfrak{g}$ alone may not determine the group G. On the other hand, the relationship between the Lie algebra $\mathfrak{g}$ and the group G' is symmetric, due to the following proposition.

Proposition 2.1.2. *The subspace $\mathfrak{sp}(W)^{G'}$ of G'-invariants in $\mathfrak{sp}(W)$ is precisely the Lie algebra $\mathfrak{g}$. Likewise, $\mathfrak{sp}(W)^G = \mathfrak{g}'$.*

Proof. From the definition of G' we know that $\mathfrak{g} \subseteq \mathfrak{sp}(W)^{G'}$. On the other hand, since $\operatorname{Lie} G' = \mathfrak{g}'$, we see that $\mathfrak{sp}(W)^{G'} \subseteq \mathfrak{sp}(W)^{\mathfrak{g}'} = \mathfrak{g}$, the full centralizer of $\mathfrak{g}'$ in $\mathfrak{sp}(W)$ by the definition of a reductive dual pair. Thus $\mathfrak{sp}(W)^{G'} = \mathfrak{g}$. Reversing the roles of $\mathfrak{g}$ and $\mathfrak{g}'$ we see that $\mathfrak{sp}(W)^G = \mathfrak{g}'$. $\square$

From now on the notation $(\mathfrak{g}, \mathfrak{g}')$ or $(\mathfrak{g}, \mathfrak{g}') \subseteq \mathfrak{sp}(W)$ indicates that we are dealing with a reductive dual pair. It turns out that $\mathfrak{g}$ and $\mathfrak{g}'$ are necessarily reductive Lie algebras. Moreover, reductive dual pairs can be completely classified. One proceeds in exactly the same manner as in the case of reductive dual pairs of groups.[1] Here is an outline of the classification (**cf.** [How]).

First, note that reductive dual pairs are defined intrinsically in terms of the symplectic structure on W. Therefore, for any reductive dual pair $(\mathfrak{g}, \mathfrak{g}') \subseteq \mathfrak{sp}(W)$ and an isometry $T : W \to W_1$ of symplectic vector spaces, the images of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ under T also form a reductive dual pair. Such two reductive dual pairs are called *isomorphic*.

Next we introduce a *direct sum* of reductive dual pairs. Observe that if $(\mathfrak{g}_1, \mathfrak{g}_1{}') \subseteq \mathfrak{sp}(W_1)$ and $(\mathfrak{g}_2, \mathfrak{g}_2{}') \subseteq \mathfrak{sp}(W_2)$ are two reductive dual pairs (assume that W_1 and W_2 are at least one-dimensional), then letting $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, $\mathfrak{g}' = \mathfrak{g}_1{}' \oplus \mathfrak{g}_2{}'$, and $W = W_1 \oplus W_2$ we obtain a reductive dual pair $(\mathfrak{g}, \mathfrak{g}') \subseteq \mathfrak{sp}(W)$. Such reductive dual pairs are called reducible, the others irreducible. Thus a reductive dual pair is reducible if and only if both $\mathfrak{g}$ and $\mathfrak{g}'$ stabilize a nonzero proper symplectic subspace $W_1 \subsetneq W$, because then they also stabilize its orthogonal complement $W_2 = W_1^{\perp}$, and $(\mathfrak{g}, \mathfrak{g}')$ is a direct sum. Clearly, (ir)reducibility is preserved under isomorphisms.

A basic result in the theory of reductive dual pairs is that every reductive dual pair is isomorphic to a direct sum of irreducible reductive dual pairs, with the summands uniquely determined. We describe three families of irreducible reductive dual pairs.

Let U and V be vector spaces, $G = GL(U)$ and $G' = GL(V)$ the groups of linear automorphisms of U and V, and $\mathfrak{g} = \mathfrak{gl}(U)$ and $\mathfrak{g}' = \mathfrak{gl}(V)$ the corresponding Lie algebras. We let $W = \hom(U, V) \oplus \hom(V, U)$ and endow it with the symplectic form

$$\langle (S_1, T_1), (S_2, T_2) \rangle = \operatorname{tr}(S_1 T_2 - S_2 T_1), \quad S_i \in \hom(U, V), T_i \in \hom(V, U), i = 1, 2. \tag{2.1.3}$$

The groups $GL(U)$ and $GL(V)$ act on W by pre- and post-multiplication, preserving the symplectic form. The differentials of these actions are infinitesimally symplectic actions of

[1] It is not *a priori* clear that the groups G and G' are mutual centralizers, thus forming a reductive dual pair in the usual sense.

$\mathfrak{gl}(U)$ and $\mathfrak{gl}(V)$ on W. Then $(\mathfrak{g}, \mathfrak{g}')$ is an irreducible reductive dual pair in $\mathfrak{sp}(W)$. This is the *general linear* case, characterized by the condition that $\mathfrak{g}$ and $\mathfrak{g}'$ preserve a lagrangian subspace of W (such reductive dual pairs are called "type II", **cf.** [How79]). Up to an isomorphism, this reductive dual pair is specified by the dimensions of U and V.

Let U and V be two vector spaces endowed with nondegenerate bilinear forms, denoted $(\cdot,\cdot)$ and $(\cdot,\cdot)'$. We require either that U is a symplectic vector space and V is an orthogonal vector space (the *symplectic–orthogonal case*), or that U is an orthogonal vector space and V is a symplectic vector space (the *orthogonal–symplectic* case). Denote by G and G' the isometry groups of the forms on U and V, respectively, and by $\mathfrak{g}$ and $\mathfrak{g}'$ their Lie algebras. We let $W = \hom(U, V)$. Define the conjugate $* : \hom(U,V) \to \hom(V,U)$ by the condition $(Su, v)' = (u, S^*v)$ for any $u \in U, v \in V$. Endow the vector space W with a symplectic structure by the formula

$$\langle S_1, S_2 \rangle = \operatorname{tr}(S_1 S_2^*), \quad S_1, S_2 \in \hom(U, V). \tag{2.1.4}$$

As in the general linear case, the groups G and G' act on W by pre- and post-multiplication and preserve the symplectic form. Differentials of these actions define embeddings of $\mathfrak{g}$ and $\mathfrak{g}'$ into the symplectic Lie algebra $\mathfrak{sp}(W)$. Then $(\mathfrak{g}, \mathfrak{g}')$ is an irreducible reductive dual pair in $\mathfrak{sp}(W)$. In contrast with the general linear case above, there is no lagrangian subspace of W invariant under both $\mathfrak{g}$ and $\mathfrak{g}'$ (such reductive dual pairs are called "type I", **cf.** [How79]). Note that symplectic vector spaces are classified up to an isomorphism by their dimension, a positive even integer. Since the field $\mathbb{C}$ is algebraically closed, orthogonal vector spaces are also classified by their dimension, which may be any positive integer. The three families of reductive dual pairs just described admit explicit realizations in terms of matrices.

Theorem 2.1.5. *Let $(\mathfrak{g}, \mathfrak{g}') \subseteq \mathfrak{sp}(W)$ be a reductive dual pair.*

(1) *The reductive dual pair $(\mathfrak{g}, \mathfrak{g}')$ is isomorphic to a direct sum of irreducible reductive dual pairs. Moreover, the irreducible summands and their multiplicities are uniquely determined.*

(2) *Suppose that $(\mathfrak{g}, \mathfrak{g}')$ is irreducible. Then it is isomorphic to one of reductive dual pairs described above.*

2.2 Definition of rank ideals

Our next goal is to define rank ideals associated with reductive dual pairs of Lie algebras. Let $\mathcal{W}$ be the *Weyl algebra* of the symplectic vector space W, namely, $\mathcal{W}$ is the associative algebra with a unit generated by the vector space W with the defining relations $w_1 \cdot w_2 - w_2 \cdot w_1 = \langle w_1, w_2 \rangle \cdot 1$ for any $w_1, w_2 \in W$ (see [How95], Appendix 2 for a detailed discussion). The action of the group $Sp(W)$ on the vector space W extends canonically to an action on $\mathcal{W}$ by algebra automorphisms, the so-called *conjugation action.* Taking commutators, we can consider Weyl algebra as a Lie algebra. Then there is a canonical $Sp(W)$-equivariant embedding ψ of the Lie algebra $\mathfrak{sp}(W)$ into $\mathcal{W}$ compatible with the brackets (i.e., a Lie algebra homomorphism). Finally, the differential of the conjugation action of $Sp(W)$ on $\mathcal{W}$ is the map from $\mathfrak{sp}(W)$ to $\operatorname{End}\mathcal{W}$ which is the composition of $\psi : \mathfrak{sp}(W) \to \mathcal{W}$ with the adjoint action of $\mathcal{W}$.

Restricting the map $\psi : \mathfrak{sp}(W) \to \mathcal{W}$ to the Lie subalgebras $\mathfrak{g} \subseteq \mathfrak{sp}(W)$ and $\mathfrak{g}' \subseteq \mathfrak{sp}(W)$, we obtain two maps from Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ to the Weyl algebra that are compatible with the brackets. By the universal property of enveloping algebras, these maps uniquely extend to homomorphisms of the enveloping algebras of $\mathfrak{g}$ and $\mathfrak{g}'$ to the Weyl algebra $\mathcal{W}$. We denote these homomorphisms by

$$R : \mathrm{U}(\mathfrak{g}) \to \mathcal{W} \quad \text{and} \quad L : \mathrm{U}(\mathfrak{g}') \to \mathcal{W}.$$

Equivalently, one may first extend ψ to an associative algebra homomorphism from the enveloping algebra $\mathrm{U}(\mathfrak{sp}(W))$ to the Weyl algebra, still denoted ψ. Restriction of the resulting map to subalgebras $\mathrm{U}(\mathfrak{g})$ and $\mathrm{U}(\mathfrak{g}')$ of $\mathrm{U}(\mathfrak{sp}(W))$ yields the same maps R and L. Now we are ready to give our basic definition.

Definition 2.2.1. Let $(\mathfrak{g}, \mathfrak{g}')$ be an irreducible reductive dual pair. The corresponding *rank ideal* is an ideal in $\mathrm{U}(\mathfrak{g})$ which is the kernel of the homomorphism $R : \mathrm{U}(\mathfrak{g}) \to \mathcal{W}$.

Note that the image of the homomorphism R is contained in the subalgebra $\mathcal{W}^{G'}$ of G'-invariants in the Weyl algebra. This follows from the definition of the group G' and the $Sp(W)$-equivariance of the map ψ. We will see below that the homomorphism R is *onto* $\mathcal{W}^{G'}$ (Theorem 2.4.5). Note also that the map R is equivariant with respect to the natural actions of G on its enveloping algebra $\mathrm{U}(\mathfrak{g})$ and on the Weyl algebra.
The equivariance of the map ψ has another consequence, which is essential in applications of rank ideals to description of the reductive dual pair correspondence.

Proposition 2.2.2. *The rank ideals associated to two isomorphic reductive dual pairs coincide.*

We are now making explicit the definition of rank ideals in the general linear case.

Definition 2.2.3. For any $p, q \geq 1$, denote by $\mathsf{M}_{p,q}$ the complex vector space of $p \times q$ matrices. The *trace form* is the bilinear pairing $\mathsf{M}_{p,q} \oplus \mathsf{M}_{q,p} \to \mathbb{C}$ given by the formula

$$(A, B) \mapsto \mathrm{tr}(AB),$$

where $A \in \mathsf{M}_{p,q}, B \in \mathsf{M}_{q,p}$.

The trace form is a nondegenerate bilinear form, and we use it to identify the dual space $\mathsf{M}_{p,q}^*$ with $\mathsf{M}_{q,p}$.
Let $k, n \geq 1$, $V = \mathsf{M}_{k,n}$, $V^* = \mathsf{M}_{n,k}$, and $W = V \oplus V^* = \mathsf{M}_{k,n} \oplus \mathsf{M}_{n,k}$. Then W is a symplectic vector space, where the symplectic form is given by

$$\langle (X_1, Y_1), (X_2, Y_2) \rangle = \mathrm{tr}(X_1 Y_2 - X_2 Y_1), \quad X_1, X_2 \in \mathsf{M}_{k,n}, Y_1, Y_2 \in \mathsf{M}_{n,k}.$$

The vector spaces $V = \mathsf{M}_{k,n}$ and $V^* = \mathsf{M}_{n,k}$ are lagrangian subspaces of W. In this situation the Weyl algebra of W may be realized as the algebra of polynomial coefficient differential operators on V, denoted $\mathcal{PD}(V)$. Let $\mathcal{P}(\mathsf{M}_{k,n})$ be the algebra of polynomial functions on the vector space $V = \mathsf{M}_{k,n}$. Polynomial coefficient differential operators act on the polynomial functions. Thus the vector space $\mathcal{P}(\mathsf{M}_{k,n})$ acquires a structure of a module over the algebra $\mathcal{PD}(\mathsf{M}_{k,n})$.
Consider the following actions of the Lie algebras $\mathfrak{g} = \mathfrak{gl}_n = \mathsf{M}_{n,n}$ and $\mathfrak{g}' = \mathfrak{gl}_k = \mathsf{M}_{k,k}$ on W:

$$\mathsf{E} \cdot (X, Y) = (-X\mathsf{E}, \mathsf{E}Y), \quad \mathsf{E}' \cdot (X, Y) = (-\mathsf{E}'^t X, Y\mathsf{E}'^t), \tag{2.2.4}$$

where $\mathsf{E} \in \mathsf{M}_{n,n}, \mathsf{E}' \in \mathsf{M}_{k,k}, X \in \mathsf{M}_{k,n}, Y \in \mathsf{M}_{n,k}$. We claim that $(\mathfrak{g}, \mathfrak{g}')$ is an irreducible reductive dual pair.[2] In fact, $\mathfrak{g}$ and $\mathfrak{g}'$ stabilize the lagrangian subspaces $V = \mathsf{M}_{k,n}$ and $V^* = \mathsf{M}_{n,k}$, and $\mathfrak{g}$ and $\mathfrak{g}'$ sit inside the subalgebra $\mathfrak{gl}(V) \subset \mathfrak{sp}(W)$. This yields an explicit formula for the images of the standard generators of $\mathfrak{gl}_n$ under the map R.
Denote by $\mathsf{E}_{ij} \in \mathfrak{gl}_n$ the $n \times n$ matrix with the unit at the (i,j)th entry and zeros elsewhere. Using the normalized symbol map, we get the following expression:

$$R(\mathsf{E}_{ij}) = R^k(\mathsf{E}_{ij}) = \sum_{t=1}^{k} \frac{1}{2}(x_{ti}\partial_{tj} + \partial_{tj}x_{ti}) = \sum_{t=1}^{k} x_{ti}\partial_{tj} + \frac{k}{2}\delta_{ij}. \tag{2.2.5}$$

A similar formula holds for $L^n(\mathsf{E}'_{rs})$. The upper index k reflects the fact that in the definition of the map R we used the reductive dual pair $(\mathfrak{gl}_n, \mathfrak{gl}_k)$ with $\mathfrak{g}' = \mathfrak{gl}_k$.

Definition 2.2.6. The kth rank ideal in $\mathrm{U}(\mathfrak{gl}_n)$, denoted $\mathcal{J}_k$, is the kernel of the homomorphism $R^k : \mathrm{U}(\mathfrak{gl}_n) \to \mathcal{PD}(\mathsf{M}_{k,n})$ defined by Equation 2.2.5.

It follows from Theorem 3.2.4 below that for $k \geq n$, the rank ideal $\mathcal{J}_k$ is zero.
The definition of rank ideals can be naturally extended to the case $k = 0$. Let $\mathcal{P}(\mathsf{M}_{k,n})$ be the trivial one-dimensional module over $\mathfrak{gl}_n$ on which all generators of $\mathfrak{gl}_n$ act by 0, and let the rank ideal $\mathcal{J}_0$ be its annihilator. The ideal $\mathcal{J}_0$ is the so-called the *augmentation ideal* of $\mathrm{U}(\mathfrak{gl}_n)$.

Example 2.2.7. Let us consider the case $k = 1$. Denote the standard generators of $\mathcal{PD}(\mathbb{C}^n)$ by $x_1, \dots, x_n, \partial_1, \dots, \partial_n$. The commutation relations in $\mathcal{PD}(\mathbb{C}^n)$ are

$$[x_i, \partial_j] = \delta_{ij}, \quad [x_i, x_l] = [\partial_l, \partial_m] = 0.$$

The map R on the generators of $\mathfrak{gl}_n$ is given by

$$R(\mathsf{E}_{ij}) = \frac{1}{2}(x_i\partial_j + \partial_j x_i) = x_i\partial_j + \frac{1}{2}\delta_{ij}, \tag{2.2.8}$$

and the generator E'_{11} of $\mathfrak{gl}_1$ is mapped into $L(\mathsf{E}'_{11}) = \sum_{i=1}^{n} x_i\partial_i + \frac{n}{2} = \deg + \frac{n}{2}$, where deg is the degree operator, giving the homogeneity degree of homogeneous polynomials in $\mathcal{P}(\mathbb{C}^n)$.

Remark 2.2.9. The action of $\mathfrak{gl}_n$ defined by Equation 2.2.5 is a twist of the differential of the action of the group GL_n on the vector space of $k \times n$ matrices by right multiplication. Suppose that $k \leq n$, then the latter action has an open dense orbit consisting of matrices of maximal rank. Using the results of Borho and Brylinski ([BB82], Corollary 3.6), one can get a description of the kth rank ideal in $\mathrm{U}(\mathfrak{gl}_n)$ as an induced ideal.

We conclude this section by an elementary observation on the behavior of rank ideals in $\mathrm{U}(\mathfrak{gl}_n)$ under restriction. Let $m \leq n$. Consider a natural embedding of Lie algebras $\mathfrak{gl}_m \to \mathfrak{gl}_n$ given by adding to a $m \times m$ matrix $n - m$ zero columns on the right and $n - m$ zero rows on the bottom.

Proposition 2.2.10. *The intersection of the kth rank ideal in the algebra $\mathrm{U}(\mathfrak{gl}_n)$ with the subalgebra $\mathrm{U}(\mathfrak{gl}_m)$ is the kth rank ideal in the enveloping algebra $\mathrm{U}(\mathfrak{gl}_m)$.*

[2] The action of $\mathfrak{gl}_k$ defined here differs from the action described by Equation 2.1.3 by an automorphism $\mathsf{E}' \mapsto -\mathsf{E}'^t$. This has an effect of producing more natural formulas for infinitesimal character correspondence.

Proof. If in Equation 2.2.5 we restrict the range of indices to $1 \leq i, j \leq m$, we see that the composition of the embedding of $\mathfrak{gl}_m$ to $\mathfrak{gl}_n$ with the map R^k for $\mathfrak{gl}_n$ is the map R^k for $\mathfrak{gl}_m$. The same remains true on the level of enveloping algebras:

$$\mathrm{U}(\mathfrak{gl}_m) \to \mathrm{U}(\mathfrak{gl}_n) \xrightarrow{R^k} \mathcal{PD}(\mathsf{M}_{k,n}).$$

The map R^k for $\mathrm{U}(\mathfrak{gl}_m)$ is the composition of the two arrows, and the first map is injective. Therefore, the intersection of $\mathcal{J}_k = \operatorname{Ker} R^k \subseteq \mathrm{U}(\mathfrak{gl}_n)$ with the embedded $\mathrm{U}(\mathfrak{gl}_m)$ is the kth rank ideal in $\mathrm{U}(\mathfrak{gl}_m)$. $\square$

2.3 Filtered algebra

An important role in the study of enveloping algebras, their ideals, and their quotients is played by methods of filtered algebra. Here we collect the relevant definitions and a few useful facts about filtered vector spaces, algebras, and maps (**cf.** [NvO82, Bou72]).

Definition 2.3.1. A *filtered vector space* X is a vector space endowed with a nonnegative increasing filtration $\{\mathcal{F}^j X : j \geq 0\}$ by finite-dimensional linear subspaces, satisfying $X = \bigcup_{j\geq 0} \mathcal{F}^j X$. A filtered algebra $\mathcal{A}$ is a a filtered vector space which is an associative algebra, and the two structures are compatible in the following sense:

$$\mathcal{F}^0\mathcal{A} = \mathbb{C}\cdot 1, \text{ and } \mathcal{F}^i\mathcal{A}\cdot\mathcal{F}^j\mathcal{A} \subseteq \mathcal{F}^{i+j}\mathcal{A}.$$

Subspaces and quotients of a filtered vector space become filtered vector spaces in a natural way. Let $\mathrm{gr}X = \oplus_{j\geq 0}\mathcal{F}^j X/\mathcal{F}^{j-1}X$ be the associated graded space of a filtered vector space X. For any nonzero $x \in X$, its *degree* is the smallest d such that $x \in \mathcal{F}^d X$, and its *principal symbol* is the image of x in $\mathrm{gr}^d X = \mathcal{F}^d X/\mathcal{F}^{d-1}X$. Recall that a linear map $\phi : X \to Y$ between two filtered vector spaces (in particular, filtered algebras) is a *filtered map* if for all $j \geq 0$ we have $\phi(\mathcal{F}^j X) \subseteq \mathcal{F}^j Y$. If moreover for all $j \geq 0$ we have $\operatorname{Im}\phi \cap \mathcal{F}^j Y = \phi(\mathcal{F}^j X)$, then ϕ is called a *strict* map. Denote by $\mathrm{gr}\phi : \mathrm{gr}X \to \mathrm{gr}Y$ the associated graded map of a filtered map ϕ.
Here is the key property of strict maps (see [NvO82], Theorem D.III.3).

Proposition 2.3.2. *Suppose that $\phi : X \to Y$ is a filtered map of filtered vector spaces. The associated graded map $\mathrm{gr}\phi$ is surjective if and only if ϕ is surjective and strict, which is to say $\phi(\mathcal{F}^i U) = \mathcal{F}^i V$ for any $i \geq 0$. If this is the case, then $\operatorname{Ker}\mathrm{gr}\phi = \mathrm{gr}\operatorname{Ker}\phi$.*

Let X and Y be two filtered algebras and ϕ a filtered homomorphism, **i.e.** a filtered map preserving the algebra structure. Then $\mathrm{gr}\phi : \mathrm{gr}X \to \mathrm{gr}Y$ is also a homomorphism. Suppose that $\mathrm{gr}\phi$ is surjective. Then Proposition 2.3.2 yields surjectivity of ϕ and allows one to compute the associated graded of the ideal $\operatorname{Ker}\phi \subseteq X$ (which is an ideal in the algebra $\mathrm{gr}X$). As another application of Proposition 2.3.2, we derive a well-known criterion for generation of an ideal in a filtered algebra.

Proposition 2.3.3. *Let $\mathcal{A}$ be a filtered algebra, I a (left, right, or two-sided) ideal in $\mathcal{A}$. Suppose that $t_1, \ldots, t_N \in I$, and $\mathrm{gr}t_1, \ldots, \mathrm{gr}t_N$ generate the (left, right, or two-sided) ideal $\mathrm{gr}I$ in $\mathrm{gr}\mathcal{A}$. Then $t_1, \ldots, t_N$ generate I.*

Proof. The proof is identical in all three cases. For example, consider the case of a left ideal $I \subset \mathcal{A}$. Let ϕ be the natural map $\mathcal{A}t_1 \oplus \ldots \oplus \mathcal{A}t_N \to I$ coming from multiplication. Since $\mathrm{gr}t_1, \ldots, \mathrm{gr}t_N$ generate $\mathrm{gr}I$, the map $\mathrm{gr}\phi$ is surjective. Then by Proposition 2.3.2, ϕ is surjective, meaning that $t_1, \ldots, t_N$ generate I. □

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Its enveloping algebra $\mathrm{U}(\mathfrak{g})$ has a natural structure of a filtered algebra, where $\mathcal{F}^0\mathrm{U}(\mathfrak{g}) = \mathbb{C}, \mathcal{F}^1\mathrm{U}(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g}$, and $\mathcal{F}^j\mathrm{U}(\mathfrak{g})$ is the linear span of i-fold products for $i \leq j$ of elements of $\mathfrak{g}$. The algebra $\mathrm{U}(\mathfrak{g})$ is generated by the linear subspace $\mathcal{F}^1\mathrm{U}(\mathfrak{g})$ of elements of degree one, and $\mathcal{F}^j\mathrm{U}(\mathfrak{g})$ is the linear span of j-fold products of elements of $\mathcal{F}^1\mathrm{U}(\mathfrak{g})$. By the Poincaré–Birkhoff–Witt theorem, the associated graded algebra of $\mathrm{U}(\mathfrak{g})$ is the commutative algebra $\mathrm{S}(\mathfrak{g})$, the symmetric algebra of $\mathfrak{g}$.

Definition 2.3.4. An *almost commutative algebra* is a filtered algebra $\mathcal{A}$ such that $\mathrm{gr}\mathcal{A}$ is commutative and $\mathcal{A}$ is generated by its subspace $\mathcal{F}^1\mathcal{A}$.

This notion was introduced by M. Duflo, who proved that the class of almost commutative algebras coincides with the class of quotients of enveloping algebras of finite-dimensional Lie algebras.

2.4 Double Commutant Theorem

As a first step towards an explicit description of rank ideals, we shall compute their associated graded ideals. In this computation and in the construction of a duality correspondence between ideals in Part III, a crucial role is played by a variant of Howe's Double Commutant Theorem (Theorem 7 from [How89a]). To state and prove it, we have to use a few more properties of the Weyl algebra listed below. These properties are quite easy to verify (again, see [How95] for details).

(1) The Weyl algebra $\mathcal{W}$ is a filtered algebra. Here $\mathcal{F}^1\mathcal{W} = \mathbb{C} \oplus W$, and $\mathcal{W}^j$ is spanned by i-fold products for $i \leq j$ of elements of $W \subset \mathcal{W}$.

(2) The associated graded algebra $\mathrm{gr}\mathcal{W}$ is the graded commutative algebra $\mathrm{S}(W) = \oplus_{j\geq 0}\mathrm{S}^j(W)$, the symmetric algebra of the vector space W.

(3) The conjugation action of $Sp(W)$ preserves the filtration on $\mathcal{W}$, and under this action $\mathcal{W}$ decomposes into a direct sum of mutually non-isomorphic algebraic $Sp(W)$-modules $\mathcal{W}^j$ satisfying $\mathcal{W}^j \subseteq F^j\mathcal{W}$. Here $\mathcal{W}^0 = \mathbb{C}$, $\mathcal{W}^1 = W$, and $\mathcal{W}^2 = \psi(\mathfrak{sp}(W))$. In general, $\mathcal{W}^j$ is the linear span of w^j for $w \in W$.

(4) Let us choose a lagrangian subspace of W with a basis $\{x_1, \ldots, x_n\}$. The stabilizer of the flag determined by this basis is a Borel subgroup $B \subset Sp(W)$. Then x_1^j is a highest weight vector in $\mathcal{W}^j$ with highest weight $j\omega_1$. Here ω_1 is the first fundamental weight of $Sp(W)$, corresponding to the defining module W for $Sp(W)$.

(5) Restriction of the principal symbol map $\mathrm{gr} : \mathcal{W} \to \mathrm{S}(W)$ to the subspace $\mathcal{W}^j$ is an $Sp(W)$-equivariant linear isomorphism from $\mathcal{W}^j$ onto $\mathrm{S}^j(W)$ (which is an irreducible $Sp(W)$-module with highest weight $j\omega_1$). Using the direct sum decomposition $\mathcal{W} = \oplus_{j\geq 0}\mathcal{W}^j$, these isomorphisms combine into a linear $Sp(W)$-equivariant isomorphism from $\mathcal{W}$ to $\mathrm{S}(W)$, called the *normalized symbol map.*

The center of the symplectic group $Sp(W)$ is the cyclic group of order 2 generated by the transformation $\sigma : w \mapsto -w, w \in W$. This group fixes pointwise the subspaces $\mathcal{W}^j$ with j even, and acts by the sign character on the subspaces $\mathcal{W}^j$ with j odd.

Definition 2.4.1. The *even Weyl algebra* $\mathcal{A}$ is the subalgebra $\mathcal{W}^\sigma = \oplus_{i\geq 0}\mathcal{W}^{2i} \subset \mathcal{W}$. We endow it with the filtration $\mathcal{F}^j\mathcal{A} = \oplus_{0\leq i\leq j}\mathcal{W}^{2i}$.

Due to the properties above, the even Weyl algebra $\mathcal{A}$ is a filtered algebra. Note that the filtration on $\mathcal{A}$ that we have just defined is *different* from the filtration induced by the filtration on $\mathcal{W}$: if $a \in \mathcal{A} \subset \mathcal{W}$, the $\mathcal{A}$-degree of a is one half of the $\mathcal{W}$-degree of a. One concludes that the associated graded algebra $\mathrm{gr}_{\mathcal{A}}\mathcal{A}$ of $\mathcal{A}$ is the even symmetric algebra $\mathrm{S}(W)^\sigma = \oplus_{i\geq 0}\mathrm{S}^{2i}(W)$. (To avoid confusion, we denote the principal symbol map with respect to the filtration on $\mathcal{A}$ by $\mathrm{gr}_{\mathcal{A}}$.) In particular, $\mathrm{gr}_{\mathcal{A}}\mathcal{A}$ is a commutative algebra.

Proposition 2.4.2. *The algebra $\mathcal{A}$ is generated by its subspace $\mathcal{W}^2$. Moreover, the algebra $\mathrm{gr}\mathcal{A}$ is generated by its subspace $\mathrm{gr}\mathcal{W}^2$.*

Proof. Due to Proposition 2.3.2, the second statement implies the first. Nevertheless, to better reveal what is going on, we will establish both statements simultaneously.
The subspace $\mathcal{W}^2 \subset \mathcal{W}$ is invariant under the conjugation action of the symplectic group $Sp(W)$. Therefore, the subalgebra $\mathcal{A}'$ of the Weyl algebra generated by $\mathcal{W}^2$ is $Sp(W)$-invariant. On the other hand, $x_1^{2i} = (x_1^2)^i \in \mathcal{A}'$ is a highest weight vector in $\mathcal{W}^{2i}$ for the conjugation action. Therefore, $\mathcal{A}' \supset \mathcal{W}^{2i}$ for any $i \geq 0$. It follows that $\mathcal{A}' \supseteq \oplus_{i\geq 0}\mathcal{W}^{2i}$, hence $\mathcal{A}' = \mathcal{A}$. The same argument works for the associated graded situation and shows that the algebra generated by $\mathrm{gr}\mathcal{W}^2 = \mathrm{S}^2(W)$ is the full even symmetric algebra $\mathrm{gr}\mathcal{A}$. $\square$

Remark 2.4.3. By similar methods, one can show that $\mathcal{A}$ does not have a non-trivial $Sp(W)$-invariant multiplicatively closed linear subspace. Since the adjoint action of the subspace $\mathcal{W}^2$ on the Weyl algebra coincides with the differential of the conjugation action, any ideal of $\mathcal{A}$ would be automatically $Sp(W)$-invariant, and we obtain that $\mathcal{A}$ is a simple algebra.

We see that the filtered algebra $\mathcal{A}$ is an almost commutative algebra (Definition 2.3.4). One can be more precise: the algebra $\mathcal{A}$ is a quotient of the enveloping algebra of the Lie algebra $\mathcal{W}^2 = \mathfrak{sp}(W)$.

Proposition 2.4.4. *The even Weyl algebra $\mathcal{A} = \mathcal{W}^\sigma$ is precisely the image of the homomorphism $\psi : \mathrm{U}(\mathfrak{sp}(W)) \to \mathcal{W}$. Moreover, the filtration on $\mathcal{A}$ coincides with the image under the map ψ of the canonical filtration on $\mathrm{U}(\mathfrak{sp}(W))$, and ψ is a strict map.*

Proof. The enveloping algebra $\mathrm{U}(\mathfrak{sp}(W))$ is generated by its subspace $\mathfrak{sp}(W)$. By the properties above, the space $\mathfrak{sp}(W)$ is mapped by ψ onto $\mathcal{W}^2$, which by Proposition 2.4.2 generates the whole algebra $\mathcal{A}$. The filtration on $\mathcal{A}$ was set up exactly so as to make it the image under ψ of the natural filtration on $\mathrm{U}(\mathfrak{sp}(W))$. Moreover, we know that $\mathrm{gr}\psi$ is onto $\mathrm{gr}\mathcal{A}$, because $\mathrm{gr}\mathcal{A}$ is generated by $\mathrm{gr}\mathcal{W}^2 = \mathrm{gr}\psi(\mathfrak{sp}(W))$. By Proposition 2.3.2 we conclude that ψ is strict. $\square$

More generally, it turns out that for an arbitrary irreducible reductive dual pair, the algebra of G'-invariants on the even Weyl algebra is an almost commutative algebra, more precisely, the image under the map R of the enveloping algebra $\mathrm{U}(\mathfrak{g})$. (The situation of the proposition corresponds to a reductive dual pair $\mathfrak{g} = \mathfrak{sp}(W)$ and $\mathfrak{g}' = 0$, then $G = Sp(W)$ and $G' = \{1, \sigma\}$ is the center of $Sp(W)$.)

Theorem 2.4.5. *Let $(\mathfrak{g}, \mathfrak{g}') \subseteq \mathfrak{sp}(W)$ be an irreducible reductive dual pair. Then the map $R : \mathrm{U}(\mathfrak{g}) \to \mathcal{A}$ is a strict homomorphism of filtered algebras. The image of R is precisely the subalgebra $\mathcal{A}^{G'}$ of the G'-invariants in $\mathcal{A}$. It also coincides with with the algebra $\mathcal{W}^{G'}$ of G'-invariants in the Weyl algebra, and with the algebra $\mathcal{A}^{\mathfrak{g}'}$ of $\mathfrak{g}'$-invariants in $\mathcal{A}$.*

Proof. The map R is the restriction of the filtered homomorphism ψ to the subalgebra $\mathrm{U}(\mathfrak{g}) \subseteq \mathrm{U}(\mathfrak{sp}(W))$ with the induced filtration, hence R itself is a filtered map. This allows one to speak about the associated graded map $\mathrm{gr}R$. As in the proof of Proposition 2.4.4, we will show that R is strict and that the algebra $\mathcal{A}^{G'}$ is generated by its subspace $\mathcal{W}^2 \cap \mathcal{A}^{G'}$ by establishing the corresponding generation property on the level of associated graded spaces. In the place of Proposition 2.4.2 we shall use the following fact from Classical Invariant Theory.

Proposition 2.4.6. *The algebra $\mathrm{S}(W)^{G'}$ of G'-invariants on the symmetric algebra $\mathrm{S}(W)$ is generated by its subspace $\mathrm{S}^2(W)^{G'}$.*

This statement may be checked for each of the three families of irreducible reductive dual pairs described above using the First Fundamental Theorem of Classical Invariant Theory (see [How89a, How95]). We will discuss the general linear case below.
The image of the subspace $\mathfrak{g} \subset \mathrm{S}(\mathfrak{g}) = \mathrm{grU}(\mathfrak{g})$ under the associated graded map $\mathrm{gr}R : \mathrm{S}(\mathfrak{g}) \to \mathrm{gr}_{\mathcal{A}}\mathcal{A} = \mathrm{S}(W)^\sigma$ is $\mathrm{gr}_{\mathcal{A}}R(\mathfrak{g}) = \mathrm{gr}_{\mathcal{A}}(\mathcal{W}^2)^{G'} = \mathrm{S}^2(W)^{G'}$, where the first equality is by Proposition 2.1.2. Since the subspace $\mathfrak{g}$ generates the symmetric algebra $\mathrm{S}(\mathfrak{g})$, the image of the algebra $\mathrm{S}(\mathfrak{g}) = \mathrm{grU}(\mathfrak{g})$ under the homomorphism $\mathrm{gr}R$ is the algebra $\mathrm{S}(W)^{G'} = \mathrm{gr}_{\mathcal{A}}\mathcal{A}^{G'}$. By the criterion of Proposition 2.3.2, the homomorphism R is strict onto $\mathcal{A}^{G'}$. The situation is captured by the following commutative diagram:

$$\begin{array}{ccccccc} \mathrm{U}(\mathfrak{g}) & \xrightarrow{R} & \mathcal{A}^{G'} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{W} \\ \downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ \mathrm{S}(\mathfrak{g}) & \xrightarrow{\mathrm{gr}R} & \mathrm{gr}_{\mathcal{A}}\mathcal{A}^{G'} & \longrightarrow & \mathrm{S}(W)^\sigma & \longrightarrow & \mathrm{S}(W). \end{array} \tag{2.4.7}$$

Here we denoted by s the normalized symbol map for the Weyl algebra and its restrictions to $\mathcal{A}$ and $\mathcal{A}^{G'}$. Since $\mathcal{A} = \mathcal{W}^\sigma$ and $\sigma \in G'$, we see that $\mathcal{W}^{G'} = \mathcal{A}^{G'}$.
Slightly modifying the argument, one gets that the image of the map R also equals $\mathcal{A}^{\mathfrak{g}'}$. Instead of Proposition 2.4.6 one uses the fact that $\mathrm{S}(W)^{G'} = (\mathrm{S}(W)^\sigma)^{\mathfrak{g}'}$ is generated by $\mathrm{S}^2(W)^{\mathfrak{g}'} = \mathfrak{g}$. ☐

The argument that we gave proceeds along the lines of Howe's original proof. The paper [How89a] treats a more general case involving a reductive dual pair of Lie superalgebras, and the proof above easily extends to that case.
An important consequence of the theorem is the following Double Commutant Theorem.

Theorem 2.4.8. *The subalgebras $R(\mathrm{U}(\mathfrak{g}))$ and $L(\mathrm{U}(\mathfrak{g}'))$ are the mutual commutants in the algebra $\mathcal{A}$.*

Proof. The algebra $L(\mathrm{U}(\mathfrak{g}'))$ is generated by its subspace $L(\mathfrak{g}')$. Hence the commutant of $L(\mathrm{U}(\mathfrak{g}'))$ in $\mathcal{A}$ consists of the elements of $\mathcal{A}$ commuting with $L(\mathfrak{g}')$. Due to the equivariance of the map ψ, this commutant is $\mathcal{A}^{\mathfrak{g}'}$, which equals $R(\mathrm{U}(\mathfrak{g}))$ by Theorem 2.4.5. Likewise, the commutant of $R(\mathrm{U}(\mathfrak{g}))$ in $\mathcal{A}$ is $\mathcal{A}^{\mathfrak{g}'} = L(\mathrm{U}(\mathfrak{g}'))$. ☐

2.5 Relation with reductive dual pair correspondence

The definition of rank ideals originates in the theory of reductive dual pairs and local Howe duality over $\mathbb{R}$. We would like to explain how rank ideals arise in this context. The theory of reductive dual pairs is centered around the study of certain correspondences between representations of classical groups. These correspondences, called *dual pair correspondences*, are constructed by restricting the *oscillator representation* of a symplectic group to two groups forming a reductive dual pair. The theory makes sense for any local or finite field $\mathbb{F}$, but we consider the case $\mathbb{F} = \mathbb{R}$, the field of real numbers.

Let $W_{\mathbb{R}}$ be a real symplectic vector space. Denote by $Sp(W_{\mathbb{R}})$ the group of invertible linear transformations of $W_{\mathbb{R}}$ preserving the symplectic form. The oscillator representation ω is a projective representation of $Sp(W_{\mathbb{R}})$ that becomes a unitary linear representation of a suitable two-fold cover $Mp(W_{\mathbb{R}})$ of the symplectic group, the *metaplectic group*.

The oscillator representation can be described algebraically using the notion of the Weyl algebra of a symplectic vector space as follows. Denote by W the complexification of $W_{\mathbb{R}}$, it is a symplectic vector space over $\mathbb{C}$. Suppose that $W = V \oplus V^*$ is a lagrangian polarization of W, then the Weyl algebra $\mathcal{W}$ is isomorphic to the algebra $\mathcal{PD}(V)$ of the polynomial coefficient differential operators on V. Note that by definition, $\mathcal{PD}(V)$ is an algebra of operators on the vector space $\mathcal{P} = \mathcal{P}(V)$ of polynomial functions on V, so that $\mathcal{P}$ is a module over the Weyl algebra. The complexification of the Lie algebra of the group $Sp(W_{\mathbb{R}})$ is the symplectic Lie algebra $\mathfrak{sp}(W)$.

The embedding $\psi : \mathfrak{sp}(W) \to \mathcal{W}$ considered above is the infinitesimal version of the canonical homomorphism from $Sp(W_{\mathbb{R}})$ to the algebra of linear operators in the space of the oscillator representation. Since $\mathcal{P}$ is a module over $\mathcal{W}$, by restriction it becomes a module over the Lie algebra $\mathfrak{sp}(W)$. A choice of a polarization $W = V \oplus V^*$ satisfying a certain positivity condition corresponds to a choice of a maximal compact subgroup K in $Sp(W_{\mathbb{R}})$ (see [How]). When these choices are made and $\mathcal{W}$ is identified with $\mathcal{PD}(V)$, the vector space $\mathcal{P} = \mathcal{P}(V)$ becomes the so-called *Fock model* of ω. The complexified Lie algebra of K is the Lie algebra $\mathfrak{gl}(V)$, which is the stabilizer in $\mathfrak{sp}(W)$ of the lagrangian subspaces V and V^* in the direct sum decomposition $W = V \oplus V^*$.

Let $(G_{\mathbb{R}}, G_{\mathbb{R}}')$ be a reductive dual pair in a symplectic group $Sp(W_{\mathbb{R}})$ (**cf.** Definition 2.1.1). Denote by $\mathfrak{g}$ and $\mathfrak{g}'$ the complexified Lie algebras of the groups $G_{\mathbb{R}}$ and $G_{\mathbb{R}}'$. The Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ form a reductive dual pair in $\mathfrak{sp}(W)$. The groups $G_{\mathbb{R}}$ and $G_{\mathbb{R}}'$ commute, and we may form their product $G_{\mathbb{R}} \cdot G_{\mathbb{R}}'$, which is a subgroup of $Sp(W_{\mathbb{R}})$. The theory of reductive dual pairs considers the restriction of ω to $G_{\mathbb{R}} \cdot G_{\mathbb{R}}'$ and asks to describe the decomposition of this restriction into simple $G_{\mathbb{R}} \cdot G_{\mathbb{R}}'$-modules, in a sense which has to be made precise. Actually, one is forced to consider two-fold covers of the groups $G_{\mathbb{R}}$ and $G_{\mathbb{R}}'$. In the algebraic approach to the oscillator representation outlined above, one restricts the $\mathfrak{sp}(W)$-module $\mathcal{P} = \mathcal{P}(V)$ to $\mathfrak{g}$ and $\mathfrak{g}'$ and inquires which $\mathfrak{g} \oplus \mathfrak{g}'$-modules may appear as quotients $\mathcal{P}/\mathcal{N}$ for some $\mathfrak{g} \oplus \mathfrak{g}'$-invariant subspace $\mathcal{N}$. A full translation of the decomposition problem into algebraic language involves Harish-Chandra modules, see [How89b]. We will neglect a part of the structure and consider just modules over the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$.

Let us consider the quotient $\mathcal{P}/\mathcal{N}$ as a $\mathrm{U}(\mathfrak{g})$-module. Clearly, its annihilator $\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})}(\mathcal{P}/\mathcal{N})$ contains the ideal $\mathcal{J} = \mathrm{Ann}_{\mathrm{U}(\mathfrak{g})} \mathcal{P} \subseteq \mathrm{U}(\mathfrak{g})$. Tracing back the definitions, we see that $\mathcal{J}$ is what we have called the rank ideal associated to a reductive dual pair $(\mathfrak{g}, \mathfrak{g}')$. Thus $\mathcal{J}$ is a two-sided ideal in $\mathrm{U}(\mathfrak{g})$ which provides a lower bound for annihilators of modules over $\mathrm{U}(\mathfrak{g})$ occurring in the Howe duality for the reductive dual pair $(G_{\mathbb{R}}, G_{\mathbb{R}}')$. Although *a priori* the ideal $\mathcal{J}$ may have depended on the reductive dual pair $(G_{\mathbb{R}}, G_{\mathbb{R}}')$ of real groups, in fact $\mathcal{J}$ is entirely determined by the reductive dual pair of complex Lie algebras $(\mathfrak{g}, \mathfrak{g}')$ (**cf.** Proposition 2.2.2). More generally, the formulas

$$I \mapsto I' = I \cdot \mathcal{A} \cap L(\mathrm{U}(\mathfrak{g}')) \text{ and } I' \mapsto I = I' \cdot \mathcal{A} \cap R(\mathrm{U}(\mathfrak{g})) \tag{2.5.1}$$

define a pair of correspondences between ideals $I \subseteq R(\mathrm{U}(\mathfrak{g}))$ and $I' \subseteq L(\mathrm{U}(\mathfrak{g}'))$ with a similar lower bound property. These questions are treated in detail in Part III.

3 Rank ideals in $\mathrm{U}(\mathfrak{gl}_n)$

The main result of this section, Theorem 3.7.1, describes a natural system of generators of the kth rank ideal $\mathcal{J}_k \subseteq \mathrm{U}(\mathfrak{gl}_n)$. These generators, which we call *quantum minors*, are certain noncommutative determinants in the standard generators of the Lie algebra $\mathfrak{gl}_n$.

3.1 Determinantal varieties

Our purpose here is to review the determinantal varieties, paying particular attention to a description of a natural system of generators for their defining ideals.
Denote by $\mathsf{M}_{n,n}$ the vector space of $n \times n$ matrices. Consider the action of the group $H = GL_n \times GL_n$ on $\mathsf{M}_{n,n}$ by left and right multiplication:

$$A \mapsto hAg^{-1}, \quad g, h \in GL_n,\, A \in \mathsf{M}_{n,n}. \tag{3.1.1}$$

This gives rise to the rational representation of $GL_n \times GL_n$ on the space $\mathcal{P}(\mathsf{M}_{n,n})$ of polynomial functions on $\mathsf{M}_{n,n}$ given by the formula

$$(h, g) \cdot f(A) = f(h^{-1}Ag), \quad g, h \in GL_n,\, A \in \mathsf{M}_{n,n}, f \in \mathcal{P}(\mathsf{M}_{n,n}).$$

For any $1 \leq i, j \leq n$, denote by x_{ij} the linear functional on the vector space $\mathsf{M}_{n,n}$ which is the (i,j)-entry of an $n \times n$ matrix. Each x_{ij} may be viewed as a polynomial function on $\mathsf{M}_{n,n}$. We arrange these functions into an $n \times n$ matrix X over the algebra $\mathcal{P}(\mathsf{M}_{n,n})$ so that for all $1 \leq i, j \leq n$ the (i,j) entry X_{ij} of X is the polynomial function $x_{ij} \in \mathcal{P}(\mathsf{M}_{n,n})$.
Let A be an $n \times n$ matrix and $I = (i_0, i_1 \dots, i_l)$ and $J = (j_0, j_1, \dots, j_m)$ two sequences of indices between 1 and n. Then we denote by A_{IJ} the $(l+1) \times (m+1)$ matrix whose (r,s)-entry is $\mathsf{A}_{i_r j_s}$ $(0 \leq r \leq l,\, 0 \leq s \leq m)$. If I and J are *strictly increasing* sequences then A_{IJ} is a submatrix of A consisting of the entries of A in the rows $i_0 < i_1 < \dots < i_l$ and in the columns $j_0 < j_1 < \dots < j_m$. For arbitrary sequences I and J rows and columns of A_{IJ} may repeat or occur in a different order than in A.
Now suppose that I and J are two sequences of indices which have the same length $k+1$, not necessarily increasing. Denote by δ_{IJ} the function $\det \mathsf{X}_{IJ} \in \mathcal{P}(\mathsf{M}_{n,n})$. Extending the usual terminology, we call any δ_{IJ} a *minor of order* $k+1$ of the matrix X (with rows in I and columns in J). Let $\{\epsilon_1, \dots, \epsilon_n\}$ be the standard basis in $\mathbb{C}^n$. Denote by $\{\epsilon_1^*, \dots, \epsilon_n^*\}$ the dual basis in $\mathbb{C}^{n*}$. Define

$$v_I^* = \epsilon_{i_0}^* \wedge \dots \wedge \epsilon_{i_k}^*, \qquad v_J = \epsilon_{j_0} \wedge \dots \wedge \epsilon_{j_k}.$$

The set $\{v_I^* | 1 \leq i_0 < i_1 < \dots < i_k \leq n\}$ is a basis in $\wedge^{k+1}\mathbb{C}^{n*}$ and the set $\{v_J | 1 \leq j_0 < j_1 < \dots < j_k \leq n\}$ is a basis in $\wedge^{k+1}\mathbb{C}^n$. The module $\wedge^{k+1}\mathbb{C}^n$ is an irreducible module over GL_n. Its highest weight is a fundamental weight for GL_n, and is denoted ω_{k+1}.

Proposition 3.1.2. *The assignment*

$$v_I^* \otimes v_J \mapsto \delta_{IJ} \tag{3.1.3}$$

extends to a $GL_n \times GL_n$-equivariant linear isomorphism between the irreducible $GL_n \times GL_n$-module $\wedge^{k+1}\mathbb{C}^{n*} \otimes \wedge^{k+1}\mathbb{C}^n$ and the linear span of the minors δ_{IJ} of order $k+1$. The highest weight vector in the linear span of the minors is the lower left minor of size $k+1$, which we denote by δ^{k+1}.

Proof. Since determinant is skew-symmetric in rows and columns, minor δ_{IJ} is skew-symmetric in row indices I and column indices J. The determinant is a linear function of the rows of a matrix, and it is also a linear function of the columns. Identify the columns with elements of the standard basis $\{\epsilon_1, \dots, \epsilon_n\}$ in $\mathbb{C}^n$ and the rows with the elements of the dual basis in $\{\epsilon_1^*, \dots, \epsilon_n^*\}$ in $\mathbb{C}^{n*}$. The left and right GL_n-actions on $\mathsf{M}_{n,n}$ are described, respectively, as elementary row and column transformations of $n \times n$ matrices, and their effect on a minor δ_{IJ} of order $k+1$ is the same as on the corresponding $v_I^* \otimes v_J \in \wedge^{k+1}\mathbb{C}^{n*} \otimes \wedge^{k+1}\mathbb{C}^n$. It follows that the linear span of the minors of order $k+1$ is stable under the $GL_n \times GL_n$-action and moreover, that the assignment (3.1.3) extends to a $GL_n \times GL_n$-equivariant linear map. This map is surjective by definition, and injective because its kernel is a submodule of an irreducible $GL_n \times GL_n$-module, hence trivial.
The upper Borel subgroup of the left copy of GL_n, $GL_n \times 1 \subset GL_n \times GL_n$ acting on $\mathsf{M}_{n,n}$ by Equation 3.1.1 preserves the span of the last $k+1$ rows of the matrix A. Similarly, the upper Borel subgroup of the right copy of GL_n, $1 \times GL_n \subset GL_n \times GL_n$ preserves the span of the first $k+1$ columns. This implies that the minor δ^{k+1} is a common eigenvector of the two Borel subgroups. It is easy to verify directly that the weight of δ^{k+1} is $(\omega_{k+1}^*, \omega_{k+1})$. □

Let us view $\mathsf{M}_{n,n}$ as an affine algebraic variety with Zariski topology. By the reduction theory for matrices, the orbits of the $GL_n \times GL_n$-action on $\mathsf{M}_{n,n}$ are locally closed subvarieties of matrices of a fixed rank k, $0 \le k \le n$. Since the group $GL_n \times GL_n$ is connected, its orbits are irreducible algebraic varieties. Their closures are irreducible subvarieties of $\mathsf{M}_{n,n}$. Let R_k be the closure of the orbit consisting of matrices of rank k, the kth *determinantal variety.* For example, $R_0 = \{0\}$, corresponding to a unique closed orbit, and $R_n = \mathsf{M}_{n,n}$, corresponding to the open dense orbit $GL_n \subset \mathsf{M}_{n,n}$ of matrices of rank n. Denote by I_k the defining ideal of R_k.

Theorem 3.1.4. *Each determinantal variety R_k is the union of orbits of rank at most k. Let I'_k be the ideal in $\mathcal{P}(\mathsf{M}_{n,n})$ generated by the minors δ_{IJ} of order $k+1$. Then $I'_k = I_k$.*

Proof. From elementary linear algebra, R_k is the closed subvariety of $\mathsf{M}_{n,n}$ on which all minors δ_{IJ} of size $k+1$ vanish. The orbit of matrices of rank k consists of matrices in R_k for which there is a non-vanishing minor of size k. These facts imply the closure relation on the orbits and show that R_k consists of matrices of rank at most k. Moreover, we conclude that the closed subvariety defined by the ideal I'_k is precisely R_k. By Hilbert's Nullstellensatz, the ideal I_k is the radical of I'_k, and to prove the second statement of the theorem it remains to show that I'_k is prime.
The primeness of I'_k is best understood in the context of the theory of *multiplicity-free actions.* We sketch the argument, since we will need parts of it later in the discussion of inclusion relation between rank ideals (**cf.** [How95], Section 2.6 and [HU91], Sections 12–13). The action on $\mathsf{M}_{n,n}$ of the product of the upper Borel subgroups of GL_n has an open dense orbit,

consisting of matrices X for which $\delta^l(X) \neq 0$ for $1 \leq l \leq n$ (again, use the reduction theory for matrices). This implies that the action of $GL_n \times GL_n$ on $\mathsf{M}_{n,n}$ is multiplicity-free and imposes a strong restriction on the form of nonzero $GL_n \times GL_n$-invariant prime ideals in $\mathcal{P}(\mathsf{M}_{n,n})$. Namely, every such ideal must be generated by an irreducible $GL_n \times GL_n$-submodule of $\mathcal{P}(\mathsf{M}_{n,n})$ with highest weight (ω^*, ω), where ω is a fundamental weight for $\mathfrak{gl}_n$, and ω^* is its contragredient. By Proposition 3.1.2 the ideal I'_k satisfies this condition with $\omega = \omega_{k+1}$, and I'_k for $0 \leq k < n$, in fact, exhaust all ideals satisfying the condition. Thus every $GL_n \times GL_n$-invariant prime ideal of $\mathcal{P}(\mathsf{M}_{n,n})$ is contained among $\{I'_k : 0 \leq k < n\}$. Expanding a minor of order $k+1$ in a column, we see that $I'_k \supseteq I'_{k+1}$. On the other hand, ideals I_k are prime and $GL_n \times GL_n$-invariant, and I'_k is contained in I_k for any $0 \leq k < n$. From this we conclude that $I'_k = I_k$ for any $0 \leq k < n$. □

3.2 The associated graded ideals of rank ideals in $\mathrm{U}(\mathfrak{gl}_n)$

Here we prove that the associated graded ideal of the kth rank ideal in $\mathrm{U}(\mathfrak{gl}_n)$ is the kth determinantal ideal.

There is a chain of $GL_k \times GL_n$-equivariant identifications

$$\mathcal{PD}(\mathsf{M}_{k,n}) = \mathcal{P}(\mathsf{M}_{k,n} \oplus \mathsf{M}^*_{k,n}) = \mathcal{P}(\mathsf{M}_{k,n} \oplus \mathsf{M}_{n,k}) = \mathcal{P}(\mathsf{M}_{k,n}) \otimes \mathcal{P}(\mathsf{M}_{n,k}). \tag{3.2.1}$$

Here the first equality is the normalized symbol map, and the second equality is given by the trace form. For the generators of $\mathcal{P}(\mathsf{M}_{k,n})$ we get that

$$x_{ti} \mapsto x_{ti} \otimes 1, \qquad \partial_{uj} \mapsto 1 \otimes x_{ju}. \tag{3.2.2}$$

By the Poincaré–Birkhoff–Witt Theorem, the associated graded algebra of $\mathrm{U}(\mathfrak{gl}_n)$ is the symmetric algebra of $\mathsf{M}_{n,n}$. Using the trace form, we will identify $\mathrm{grU}(\mathfrak{gl}_n) = \mathrm{S}(\mathsf{M}_{n,n})$ with the algebra $\mathcal{P}(\mathsf{M}_{n,n})$ of polynomial functions on $\mathsf{M}_{n,n}$. On the elements of degree 1 the identification is given by

$$\mathrm{gr}\mathsf{E}_{ij} = x_{ji}. \tag{3.2.3}$$

Theorem 3.2.4. *For $k \leq n$ the associated graded $\mathrm{gr}\mathcal{J}_k \subseteq \mathrm{S}(\mathsf{M}_{n,n})$ of the kth rank ideal is identified via Equation 3.2.3 with the kth determinantal ideal of $\mathcal{P}(\mathsf{M}_{n,n})$. For $k \geq n$ the ideal $\mathcal{J}_k$ is zero.*

Proof. This follows from Theorem 2.4.5 in the case of the reductive dual pair $(\mathfrak{gl}_n, \mathfrak{gl}_k)$. Since R is a strict map, by Proposition 2.3.2 we have that $\mathrm{gr}\,\mathrm{Ker}\,R = \mathrm{Ker}\,\mathrm{gr}R$. Let us compute the right hand side explicitly. In the present case the commutative diagram (2.4.7) yields

$$\begin{array}{ccccc}
\mathrm{U}(\mathfrak{gl}_n) & \xrightarrow{R} & \mathcal{PD}(\mathsf{M}_{k,n})^{GL_k} & \longrightarrow & \mathcal{PD}(\mathsf{M}_{k,n}) \\
\big\downarrow & & \big\downarrow{\scriptstyle s} & & \big\downarrow{\scriptstyle s} \\
\mathcal{P}(\mathsf{M}_{n,n}) & \xrightarrow{\mu^*} & \mathcal{P}(\mathsf{M}_{k,n} \oplus \mathsf{M}_{n,k})^{GL_k} & \longrightarrow & \mathcal{P}(\mathsf{M}_{k,n}) \otimes \mathcal{P}(\mathsf{M}_{n,k}).
\end{array} \tag{3.2.5}$$

Equation 2.2.5 for $R^k(\mathsf{E}_{ij})$ along with the identifications (3.2.2) and (3.2.3), results in the following explicit formula for the map μ^*:

$$\mu^*(x_{ji}) = s(R(\mathsf{E}_{ij})) = \sum_{t=1}^{k} x_{ti}\partial_{tj} = \sum_{t=1}^{k} x_{ti} \otimes x_{jt}.$$

Thus the homomorphism μ^* of commutative algebras is dual to the morphism of affine algebraic varieties $\mu : \mathsf{M}_{n,k} \oplus \mathsf{M}_{k,n} \to \mathsf{M}_{n,n}$ given by multiplication of matrices. The surjectivity of μ^* that made its appearance in the proof of Theorem 2.4.5 and followed from Proposition 2.4.6 is synonymous to the geometric form of the First Fundamental Theorem of Classical Invariant Theory for the action of GL_k on $\mathsf{M}_{n,k} \oplus \mathsf{M}_{k,n}$, namely, that the algebra of polynomial invariants coincides with the pull-back of $\mathcal{P}(\mathsf{M}_{n,n})$ with respect to μ (**cf.** [How95], Equation (2.2.3.7)). The image of the morphism μ is the set of $n \times n$ matrices of rank at most $\min(k,n)$, which is $\mathsf{M}_{n,n}$ when $k \geq n$, and R_k, the kth determinantal variety, for $k < n$. Since $\operatorname{Ker}\mu^* \subseteq \mathcal{P}(\mathsf{M}_{n,n})$ is the defining ideal of $\operatorname{Im}\mu \subseteq \mathsf{M}_{n,n}$, $\operatorname{Ker}\mu^*$ equals the kth determinantal ideal I_k for $k < n$ and zero for $k \geq n$. $\square$

A parallel argument for the map $L : \mathrm{U}(\mathfrak{gl}_k) \to \mathcal{PD}(\mathsf{M}_{k,n})$ shows that L is injective for $k \leq n$ and $\operatorname{gr}\operatorname{Ker} L$ equals the nth determinantal ideal in $\mathcal{P}(\mathsf{M}_{k,k})$ for $k > n$.

3.3 Determinants with entries in a noncommutative algebra

Here we establish several properties of a noncommutative generalization of the determinant of square matrices. These properties will be used in the proof of Theorem 3.7.1, describing generators of the rank ideals in $\mathrm{U}(\mathfrak{gl}_n)$.
Consider an associative algebra $\mathcal{A}$. Since algebra $\mathcal{A}$ need not be commutative, there are several feasible analogs of the determinant. We define the determinant of a matrix by means of the full expansion, arranging the entries in each product in the order of their columns.

Definition 3.3.1. Let $\mathsf{A} = (\mathsf{A}_{ij})$ be a k by k square matrix with entries in $\mathcal{A}$. The *determinant* (more precisely, column determinant) of A is the element of $\mathcal{A}$ defined by the formula:

$$\det \mathsf{A} = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \overrightarrow{\prod}_{1 \leq s \leq k} \mathsf{A}_{\sigma(s)s}. \tag{3.3.2}$$

Here S_k is the permutation group on k symbols, $\operatorname{sgn}(\sigma)$ is the sign of a permutation σ, and the *ordered product* $\overrightarrow{\prod}_s$ is the product in which the factors are arranged from left to right in the order of increasing s.

For example,

$$\det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb. \tag{3.3.3}$$

Ordered products enjoy many properties of the ordinary products. For example, they can be split into several subproducts, provided that the order of indices is kept. Another useful property is the Leibniz Rule: for any $a, a_1, \dots, a_k \in \mathcal{A}$ we have

$$[a, \overrightarrow{\prod}_{1 \leq s \leq k} a_s] = \sum_{r=1}^{k} \left(\overrightarrow{\prod}_{s<r} a_s \right) [a, a_r] \overrightarrow{\prod}_{s>r} a_s.$$

In this formula and below, the ordered product over the empty set of indices is set to 1.
In the next three propositions we collect some properties of the column determinant. Throughout A is a k by k matrix with entries in $\mathcal{A}$.

Proposition 3.3.4 (Linearity).

(1) *The determinant is a multilinear function of the columns.*

(2) *The determinant is a skew-symmetric multilinear function of the rows.*

Proposition 3.3.5 (Expansion rules).

(1) *Decompose the sequence* $(1,\dots,k)$ *into* q *consecutive segments* $J_1\coprod\dots\coprod J_q$. *Split* A *accordingly into* q *vertical strips, so that the strip number* t *consists of consecutive columns of* A *with numbers* $\{s|s\in J_t\}$. *Denote by* $S_J=S_{J_1}\times\dots\times S_{J_q}$ *the subgroup of* S_k *stabilizing each* J_t. *Let* S^J *be the canonical system of left coset representatives of minimal length for the subgroup* $S_J\subseteq S_k$. *It consists of permutations in* S_k *preserving the order of elements in each* J_t. *Then*

$$\det\mathsf{A}=\sum_{\sigma\in S^J}\operatorname{sgn}(\sigma)\overrightarrow{\prod}_{1\le t\le q}\det(\mathsf{A}_{\sigma(J_t)J_t}). \tag{3.3.6}$$

In particular, the "expansion in a column" holds for the first or last column.

(2) *Suppose that all entries of* A *in column* r *commute with all entries of the next column* $r+1$. *Denote by* A' *be the matrix* A *with these two columns switched. Then* $\det\mathsf{A}'=-\det\mathsf{A}$.

(3) *Suppose that all* 2×2 *minors in two consecutive columns* r *and* $r+1$ *in* A *vanish. Then* $\det\mathsf{A}=0$.

Finally, we see how the determinant behaves under maps of algebras. If $\phi: A\to B$ is a map of associative algebras and A is a square matrix with entries in A, apply ϕ to A entrywise and denote the resulting matrix with entries in B by $\phi(\mathsf{A})$.

Proposition 3.3.7. *Suppose that* $\phi: A\to B$ *is a homomorphism of associative algebras. Then* $\phi(\det\mathsf{A})=\det\phi(\mathsf{A})$.

Proposition 3.3.8. *Let* $\mathcal{A}$ *be a filtered algebra and* $\mathrm{gr}:\mathcal{A}\to\mathrm{gr}\mathcal{A}$ *be the principal symbol map. Suppose that all entries of a matrix* A *have the same degree, and* $\det\mathrm{gr}\mathsf{A}\neq 0$. *Then* $\mathrm{gr}\det\mathsf{A}=\det\mathrm{gr}\mathsf{A}$.

Proof. Most of these facts are direct analogs of the corresponding properties of determinants over a commutative ring. They are proved in the usual way, interpreting all products as ordered products with the ordering determined by the columns. Thus Proposition 3.3.4 follows directly from the definition of the determinant (see Equation 3.3.2).
In order to prove Proposition 3.3.5, let us decompose S_k into the union of its left S_J-cosets. Completely expanding each factor in (3.3.6) and replacing each iterated ordered product $\overrightarrow{\prod}_{1\le t\le q}\left(\overrightarrow{\prod}_{s\in J_t}\right)$ with $\overrightarrow{\prod}_{1\le s\le q}$, we obtain the full expansion (3.3.2) of $\det\mathsf{A}$. Parts two and three of the proposition follow from the more general first statement. Indeed, let $q=3$, $J_1=[1,\dots,r-1]$, $J_2=[r,r+1]$ and $J_3=[r+2,\dots,k]$ (J_1 or J_3 may be empty). Expand $\det\mathsf{A}$ as in Equation 3.3.6. The assumption of the second part implies that 2×2 minors in the middle term of each triple product in the expansion are *de facto* commutative, hence skew-symmetric in the columns r and $r+1$. Thus the full determinant is skew-symmetric in these columns. The assumption of the third part implies that the analogous 2×2 minors in the middle terms of the products in the expansion vanish, so $\det\mathsf{A}=0$.

Proposition 3.3.7 is obvious from the definition of the determinant. To prove Proposition 3.3.8, note that under our assumptions all nonzero summands in the expansion (3.3.2) of $\det \mathrm{gr}\mathsf{A}$ have the same degree. Denote them by $a_1, \dots , a_n$. From the definition of the principal symbol it is clear that if $a_i \in \mathcal{A}$ all have the same degree and $\sum_{i=1}^n \mathrm{gr} a_i$ is not 0 then it coincides with $\mathrm{gr}(\sum_{i=1}^n a_i)$. In our case this means that $\det \mathrm{gr}\mathsf{A} = \sum_{i=1}^n \mathrm{gr} a_i = \mathrm{gr}(\sum_{i=1}^n a_i) = \mathrm{gr} \det \mathsf{A}$, as we claimed. ☐

We point out some distinctions between column determinants discussed here and ordinary determinants over a commutative ring. The column determinant is not skew-symmetric in columns, since switching columns changes the order of multiplication in the expansion (3.3.2). For the same reason, expansions in rows do not hold, and in general there is no connection between the determinant of a matrix and of its transpose.

3.4 Some computations in $\mathrm{U}(\mathfrak{gl}_n)$

Let us define

$$\mathsf{E}_{ij}(a) = \mathsf{E}_{ij} + a\delta_{ij} \cdot 1 \in \mathrm{U}(\mathfrak{gl}_n). \tag{3.4.1}$$

We claim that the map $\mathsf{E}_{ij} \mapsto \mathsf{E}_{ij}(a)$ extends to a $\mathfrak{gl}_n$-equivariant automorphism of the algebra $\mathrm{U}(\mathfrak{gl}_n)$. The inverse of this map is given by $\mathsf{E}_{ij} \mapsto \mathsf{E}_{ij}(-a)$. The $\mathfrak{gl}_n$-equivariance amounts to the following lemma.

Lemma 3.4.2. *For any $1 \le p, q, i, j \le n$ and $a \in \mathbb{C}$*

$$[\mathsf{E}_{pq}, \mathsf{E}_{ij}(a)] = \delta_{qi}\mathsf{E}_{pj}(a) - \delta_{jp}\mathsf{E}_{iq}(a). \tag{3.4.3}$$

Proof. For $a = 0$ this is the standard commutation relation in $\mathfrak{gl}_n$. For arbitrary a both sides are the same as for $a = 0$. ☐

Now let us check that the commutation relations in $\mathfrak{gl}_n$ are preserved under the map $\mathsf{E} \mapsto \mathsf{E}(a)$.

Lemma 3.4.4. *For any $1 \le p, q, i, j \le n$ and $a \in \mathbb{C}$*

$$[\mathsf{E}_{pq}(a), \mathsf{E}_{ij}(a)] = \delta_{qi}\mathsf{E}_{pj}(a) - \delta_{jp}\mathsf{E}_{iq}(a). \tag{3.4.5}$$

Proof. The expression in the left hand side is equal to $[\mathsf{E}_{pq}, \mathsf{E}_{ij}(a)]$, because $\mathsf{E}_{pq}(a) - \mathsf{E}_{pq} = a\delta_{pq} \cdot 1$ commutes with everything in $\mathrm{U}(\mathfrak{gl}_n)$. So the statement reduces to the previous lemma. ☐

Remark 3.4.6. In fact, the equivariance of the map $\mathsf{E} \mapsto \mathsf{E}(a)$ and compatibility with the commutation relations both follow from the fact that $\mathsf{E}_{ij} \mapsto a\delta_{ij}$ is a character of the Lie algebra $\mathfrak{gl}_n$.

The next proposition states a skew-symmetry property for certain 2×2 determinants with entries in $\mathrm{U}(\mathfrak{gl}_n)$. These determinants are quantum minors of order 2 (defined in the next section).

Proposition 3.4.7. *For any a, the following identity holds:*

$$\det \begin{vmatrix} \mathsf{E}_{ij}(a) & \mathsf{E}_{im}(a-1) \\ \mathsf{E}_{lj}(a) & \mathsf{E}_{lm}(a-1) \end{vmatrix} + \det \begin{vmatrix} \mathsf{E}_{im}(a) & \mathsf{E}_{ij}(a-1) \\ \mathsf{E}_{lm}(a) & \mathsf{E}_{lj}(a-1) \end{vmatrix} = 0. \tag{3.4.8}$$

Proof. This is a somewhat tedious computation that uses the commutation relation (3.4.3). The left hand side of Equation 3.4.8 is

$$(\mathsf{E}_{ij}(a)\mathsf{E}_{lm}(a-1) - \mathsf{E}_{lj}(a)\mathsf{E}_{im}(a-1)) + (\mathsf{E}_{im}(a)\mathsf{E}_{lj}(a-1) - \mathsf{E}_{lm}(a)\mathsf{E}_{ij}(a-1)),$$

or, changing the order of terms,

$$(\mathsf{E}_{ij}(a)\mathsf{E}_{lm}(a-1) - \mathsf{E}_{lm}(a)\mathsf{E}_{ij}(a-1)) \tag{3.4.9a}$$
$$+(\mathsf{E}_{im}(a)\mathsf{E}_{lj}(a-1) - \mathsf{E}_{lj}(a)\mathsf{E}_{im}(a-1)) \tag{3.4.9b}$$

Observe that (3.4.9b) may be obtained from (3.4.9a) by switching indices m and j, so that the left hand side of (3.4.8) is the symmetrization of (3.4.9a) in m and j. Expanding (3.4.9a), we see that it is equal to

$$\begin{aligned}
&\mathsf{E}_{ij}\mathsf{E}_{lm} + a\delta_{ij}\mathsf{E}_{lm} + (a-1)\delta_{lm}\mathsf{E}_{ij} + a(a-1)\delta_{ij}\delta_{lm}\\
&-\mathsf{E}_{lm}\mathsf{E}_{ij} - a\delta_{lm}\mathsf{E}_{ij} - (a-1)\delta_{ij}\mathsf{E}_{lm} - a(a-1)\delta_{lm}\delta_{ij}\\
&=[\mathsf{E}_{ij},\mathsf{E}_{lm}] + \delta_{ij}\mathsf{E}_{lm} - \delta_{lm}\mathsf{E}_{ij}\\
&=\delta_{jl}\mathsf{E}_{im} - \delta_{mi}\mathsf{E}_{lj} + \delta_{ij}\mathsf{E}_{lm} - \delta_{lm}\mathsf{E}_{ij}\\
&=(\delta_{lj}\mathsf{E}_{im} - \delta_{lm}\mathsf{E}_{ij}) + (\delta_{ij}\mathsf{E}_{lm} - \delta_{im}\mathsf{E}_{lj}).
\end{aligned}$$

The last expression is a sum of two terms that are skew-symmetric in m and j, hence its symmetrization in m and j is indeed zero. Thus the sum of the determinants in Equation 3.4.8 is zero. ☐

3.5 Quantum minors

We define certain elements of the enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$ which are noncommutative analogs of matrix minors. Their particular cases are involved in the formulation of the *Capelli identities* of Classical Invariant Theory (**cf.** [HU91, How89a, Wey39]).

Definition 3.5.1. Let $I = (i_0, i_1, \dots, i_k)$ and $J = (j_0, j_1, \dots, j_k)$ be two length $k+1$ sequences of indices between 1 and n, and $a \in \mathbb{C}$. The corresponding *quantum minor* of order $k+1$ is $\Delta_{I,J}(a) = \det|\mathsf{E}_{i_r j_s}(a + \frac{k}{2} - s)|_{0 \le r,s \le k}$. More explicitly, it is given by the following column determinant:

$$\Delta_{I,J}(a) = \det\begin{vmatrix} \mathsf{E}_{i_0j_0}(a+\frac{k}{2}) & \mathsf{E}_{i_0j_1}(a+\frac{k}{2}-1) & \dots & \mathsf{E}_{i_0j_k}(a-\frac{k}{2})\\ \mathsf{E}_{i_1j_0}(a+\frac{k}{2}) & \mathsf{E}_{i_1j_1}(a+\frac{k}{2}-1) & \dots & \mathsf{E}_{i_1j_k}(a-\frac{k}{2})\\ \vdots & \vdots & \ddots & \vdots\\ \mathsf{E}_{i_kj_0}(a+\frac{k}{2}) & \mathsf{E}_{i_kj_1}(a+\frac{k}{2}-1) & \dots & \mathsf{E}_{i_kj_k}(a-\frac{k}{2}) \end{vmatrix}. \tag{3.5.2}$$

We write $\Delta_{I,J}$ for $\Delta_{I,J}(0)$.

To avoid confusion, we often refer to conventional minors as *classical* minors. Here are some properties of quantum minors, reinforcing their similarity to classical minors.

Proposition 3.5.3. *Consider the set $Q_{k+1}(a)$ of quantum minors $\Delta_{I,J}(a)$ of order $k+1$ with fixed $a \in \mathbb{C}$ and varying I and J.*

(1) *Quantum minors $\Delta_{I,J}(a)$ are skew-symmetric with respect to row indices $i_r \in I$.*

(2) *Quantum minors $\Delta_{I,J}(a)$ are skew-symmetric with respect to column indices $j_s \in J$.*

(3) *The principal symbol of a quantum minor is the corresponding classical minor:* $\mathrm{gr}\Delta_{I,J}(a) = \det|\mathrm{gr}\mathsf{E}_{i_r j_s}|_{0\le r,s\le k}$, *which we denote by δ_{IJ}.*

Proof. Switching two row indices in a quantum minor is equivalent to switching the corresponding two rows in the determinant. Since the column determinant is skew-symmetric in rows (Proposition 3.3.4), a quantum minor is skew-symmetric with respect to row indices.

By Proposition 3.3.5, a quantum minor can be expanded in a pair of consecutive columns. The 2×2 determinants involved in this decomposition are quantum minors of order 2, which are skew-symmetric in column indices by Proposition 3.4.7. Therefore, quantum minors are skew-symmetric under switching the indices of two consecutive columns, $j_s \leftrightarrow j_{s+1}$. Since an arbitrary permutation is a product of such transpositions, quantum minors are skew-symmetric with respect to any permutation of column indices.

If either I or J contains repeating indices, both the quantum minor $\Delta_{I,J}(a)$ and the classical minor δ_{IJ} vanish because of the skew-symmetry. If both I and J consist of $k+1$ *distinct* terms, then the classical minor δ_{IJ} is a non-zero homogeneous polynomial of degree $k+1$. Moreover, all entries of the $(k+1)\times(k+1)$ matrix involved in the definition of a quantum minor have degree 1, and for any $i\in I, j\in J$ we have $\mathrm{gr}\mathsf{E}_{ij}(a) = \mathrm{gr}\mathsf{E}_{ij}$. By Proposition 3.3.8, we conclude that $\mathrm{gr}\Delta_{I,J}(a) = \det(\mathrm{gr}\mathsf{E}_{IJ})$. □

The proof of the proposition makes it clear that the skew-symmetry of $\Delta_{I,J}(a)$ with respect to the column indices is due to the fact that the "twists" $a+\frac{k}{2}, a+\frac{k}{2}-1, \dots, a-\frac{k}{2}$ of the columns $0,1,\dots,k$ in the definition of $\Delta_{I,J}(a)$ (see Equation 3.5.2) form an arithmetic progression with the difference -1.

Let us fix $a\in\mathbb{C}$ and consider the set $Q_{k+1}(a)$ of quantum minors of order $k+1$ with varying I and J as the set of components of a tensor in $\mathbb{C}^n$ with $k+1$ covariant and $k+1$ contravariant indices. Reinforcing the tensor interpretation, let us introduce the following notation.

Let $I=(i_0,i_1,\dots,i_k)$ is a sequence of indices of length $k+1$. Denote by $I\widehat{\imath_s}$ the sequence $(i_0,\dots,i_{s-1},i_{s+1},\dots,i_k)$, obtained from I by skipping the sth term of I and by $I(\widehat{\imath_r}q)$ the sequence $(i_0,\dots,i_{r-1},q,i_{r+1},\dots,i_k)$ obtained from I by replacing the rth term i_r with q. In both cases, the order of the other terms is preserved. The operations of skipping an index and replacing another index commute and may be combined in an obvious way.

Theorem 3.5.4. *For any $1\le p,q\le n$ and two sequences of indices I and J of length $k+1$, we have the following formula for the adjoint action of $\mathfrak{gl}_n$ on the linear span in $\mathrm{U}(\mathfrak{gl}_n)$ of quantum minors $\Delta_{I,J}(a)$ of order $k+1$:*

$$[\mathsf{E}_{pq}, \Delta_{I,J}(a)] = \sum_{r=0}^{k} \delta_{qi_r}\Delta_{I(\widehat{\imath_r}p),\,J}(a) - \sum_{r=0}^{k}\delta_{j_r p}\Delta_{I,\,J(\widehat{\jmath_r}q)}(a). \tag{3.5.5}$$

In particular, the linear span in $\mathrm{U}(\mathfrak{gl}_n)$ of the quantum minors of order $k+1$ is stable under the adjoint action of $\mathfrak{gl}_n$.

Proof. For $0\le s\le k$, let $a_s = a+\frac{k}{2}-s$. By the definition of the quantum minor, we have $\Delta_{I,J}(a) = \sum_{\sigma\in S_{k+1}} \mathrm{sgn}(\sigma) T_{I,J}(\sigma,a)$, where we set

$$T_{I,J}(\sigma,a) = \overrightarrow{\prod}_{0\le s\le k} \mathsf{E}_{i_{\sigma(s)}j_s}(a_s). \tag{3.5.6}$$

We claim that

$$[\mathsf{E}_{pq}, T_{I,J}(\sigma,a)] = \sum_{r=0}^{k} \delta_{qi_r} T_{I(\widehat{\imath}_r p),J}(\sigma,a) - \sum_{r=0}^{k} \delta_{j_r p} T_{I,J(\widehat{\jmath}_r q)}(\sigma,a). \tag{3.5.7}$$

Multiplying by $\operatorname{sgn}(\sigma)$ and summing over $\sigma \in S_{k+1}$ would yield Equation 3.5.5.
By the Leibniz rule,

$$[\mathsf{E}_{pq}, T_{I,J}(\sigma,a)] = \sum_{r=0}^{k} \left(\overrightarrow{\prod}_{s<r} \mathsf{E}_{i_{\sigma(s)}j_s}(a_s)\right) [\mathsf{E}_{pq}, \mathsf{E}_{i_{\sigma(r)}}(a_r)] \overrightarrow{\prod}_{s>r} \mathsf{E}_{i_{\sigma(s)}j_s}(a_s), \tag{3.5.8}$$

which by Equation (3.4.3) is equal to

$$\begin{aligned}
&\sum_{r=0}^{k} \left(\overrightarrow{\prod}_{s<r} \mathsf{E}_{i_{\sigma(s)}j_s}(a_s)\right) (\delta_{qi_{\sigma(r)}} \mathsf{E}_{pj_r}(a_r) - \delta_{j_r p} \mathsf{E}_{i_{\sigma(r)}q}(a_r)) \overrightarrow{\prod}_{s>r} \mathsf{E}_{i_{\sigma(s)}j_r}(a_s) \\
=&\sum_{r=0}^{k} \delta_{qi_{\sigma(r)}} \left(\overrightarrow{\prod}_{s<r} \mathsf{E}_{i_{\sigma(s)}j_s}(a_s)\right) \mathsf{E}_{pj_r}(a_r) \overrightarrow{\prod}_{s>r} \mathsf{E}_{i_{\sigma(s)}j_r}(a_s) \\
-&\sum_{r=0}^{k} \delta_{j_r p} \left(\overrightarrow{\prod}_{s<r} \mathsf{E}_{i_{\sigma(s)}j_s}(a_s)\right) \mathsf{E}_{i_{\sigma(r)}q}(a_r) \overrightarrow{\prod}_{s>r} \mathsf{E}_{i_{\sigma(s)}j_r}(a_s).
\end{aligned}$$

The second terms in the last expression and in the right hand side of (3.5.7) are the same. We claim that the first terms are also the same. Indeed, observe that

$$\delta_{qi_r} T_{I(\widehat{\imath}_r p),J}(\sigma,a) = \delta_{qi_r} \left(\overrightarrow{\prod}_{s<\sigma^{-1}(r)} \mathsf{E}_{i_{\sigma(s)}j_s}(a_s)\right) \mathsf{E}_{pj_{\sigma^{-1}(r)}}(a_{\sigma^{-1}(r)}) \overrightarrow{\prod}_{s>\sigma^{-1}(r)} \mathsf{E}_{i_{\sigma(s)}j_s}(a_s),$$

and the summation over r in (3.5.7) corresponds to the summation over $\sigma^{-1}(r)$ here. □

Note that the properties of quantum minors used in the proof of the theorem also hold for classical minors. Due to Proposition 3.5.3, the assignment $\Delta_{I,J}(a) \mapsto \delta_{IJ}$ uniquely extends to a linear isomorphism between the linear spans of the quantum minors in $Q_{k+1}(a)$ and of the classical minors of order $k+1$. We see that this isomorphism is equivariant with respect to the adjoint actions of $\mathfrak{gl}_n$ on $\mathrm{U}(\mathfrak{gl}_n)$ and on $\mathcal{P}(\mathsf{M}_{n,n})$.

Corollary 3.5.9. *The Capelli elements, defined as*

$$C_k(a) = \sum_{\{I:\,\mathrm{card}\, I=k\}} \Delta_{I,I}(a),$$

belong to the center of $\mathrm{U}(\mathfrak{gl}_n)$.

Proof. An element of $\mathrm{U}(\mathfrak{gl}_n)$ is in the center if and only if it is fixed by the adjoint action of $\mathfrak{gl}_n$. The theorem implies that there is a $\mathfrak{gl}_n$-equivariant isomorphism between the span of quantum minors $\Delta_{I,J}(a)$ of order k into the span of the corresponding classical minors, and the Capelli element $C_k(a)$ is the lift of $c_k = \sum_I \delta_{II}$. Under the adjoint action of $\mathfrak{gl}_n$ the minors δ_{II} transform as $v_I^* \otimes v_I$, where $\{v_I\}$ and $\{v_I^*\}$ are the standard dual bases in $\wedge^k\mathbb{C}^n$ and $\wedge^k\mathbb{C}^{n*}$, respectively (see Proposition 3.1.2). Two copies of the group GL_n act on $\wedge^k\mathbb{C}^n$ and $\wedge^k\mathbb{C}^{n*}$ in the mutually contragredient ways, and the adjoint action of GL_n is the restriction of this $GL_n \times GL_n$-action to the diagonal GL_n. Thus $\sum_I v_I^* \otimes v_I$ is invariant under the diagonal GL_n-action, and hence ad $\mathfrak{gl}_n$-invariant. Therefore, c_k and C_k are also ad $\mathfrak{gl}_n$-invariant. Moreover, a little thought further reveals that c_k is nothing but the coefficient of λ^{n-k} in the characteristic polynomial of $n \times n$ matrices (up to a sign), which is obviously invariant under the conjugation by GL_n. □

3.6 Behavior of rank ideals under tensor products

For any $a \in \mathbb{C}$ we have defined above an automorphism of the algebra $\mathrm{U}(\mathfrak{gl}_n)$ given on the generators $\mathsf{E}_{ij} \in \mathfrak{gl}_n$ by the formula $\mathsf{E}_{ij} \mapsto \mathsf{E}_{ij}(a)$. We call these automorphisms *twists*. Composing a twist with the map R leads to the following generalization of the kth rank ideal $\mathcal{J}_k \subseteq \mathrm{U}(\mathfrak{gl}_n)$. Consider a homomorphism $R^k(a) : \mathrm{U}(\mathfrak{gl}_n) \to \mathcal{PD}(\mathsf{M}_{k,n})$ given by the formula

$$R^k(a)(\mathsf{E}_{ij}) = R^k(\mathsf{E}_{ij}(a)) = \sum_{t=1}^{k} x_{ti}\partial_{tj} + (\frac{k}{2} + a)\delta_{ij} \tag{3.6.1}$$

(**cf.** Equation 2.2.5).

Definition 3.6.2. The ideal $\mathcal{J}_k(a) \subseteq \mathrm{U}(\mathfrak{gl}_n)$ is the kernel of the map $R^k(a) : \mathrm{U}(\mathfrak{gl}_n) \to \mathcal{PD}(\mathsf{M}_{k,n})$.

Denote by $\mathcal{P}(\mathsf{M}_{k,n})(a)$ the module $\mathcal{P}(\mathsf{M}_{k,n})$ with the twisted action of $\mathfrak{gl}_n$ that we have just defined. Since the algebra $\mathcal{PD}(\mathsf{M}_{k,n})$ acts *faithfully* on $\mathcal{P}(\mathsf{M}_{k,n})$, the ideal $\mathcal{J}_k(a)$ is the annihilator of the module $\mathcal{P}(\mathsf{M}_{k,n})(a)$.
Any enveloping algebra $\mathrm{U}(\mathfrak{g})$ is equipped with a coproduct $\delta : \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$. The coproduct is an algebra homomorphism defined on the generators $\mathsf{E} \in \mathfrak{g}$ by the formula $\delta(\mathsf{E}) = \mathsf{E} \otimes 1 + 1 \otimes \mathsf{E}$, and $\delta(1) = 1 \otimes 1$.

Theorem 3.6.3. *Let $k, l \geq 0$ and $a, b, c \in \mathbb{C}$ such that $a = b + c$. Then we have the following identity involving twisted rank ideals in $\mathrm{U}(\mathfrak{gl}_n)$*

$$\mathcal{J}_{k+l}(a) = \delta^{-1}(\mathcal{J}_k(b) \otimes \mathrm{U}(\mathfrak{gl}_n) + \mathrm{U}(\mathfrak{gl}_n) \otimes \mathcal{J}_l(c)).$$

Moreover, if M and N are $\mathrm{U}(\mathfrak{gl}_n)$-modules such that $\mathrm{Ann}\, M \supseteq \mathcal{J}_k(b)$ and $\mathrm{Ann}\, N \supseteq \mathcal{J}_l(c)$, then $\mathrm{Ann}(M \otimes N) \supseteq \mathcal{J}_{k+l}(a)$.

Proof. The idea behind the proof is to exploit the following Exponential Property of algebras of polynomial coefficient differential operators: for two vector spaces U and V, we have $\mathcal{PD}(U \oplus V) \simeq \mathcal{PD}(U) \otimes \mathcal{PD}(V)$. Let us show that the $\mathrm{U}(\mathfrak{gl}_n)$-module $\mathcal{P}(\mathsf{M}_{k+l,n})(a)$ is isomorphic to the the tensor product of the $\mathrm{U}(\mathfrak{gl}_n)$-modules $\mathcal{P}(\mathsf{M}_{k,n})(b)$ and $\mathcal{P}(\mathsf{M}_{l,n})(c)$. We have a direct sum decomposition $\mathsf{M}_{k+l,n} = \mathsf{M}_{k,n} \oplus \mathsf{M}_{l,n}$ of the vector space of $(k+l) \times n$ matrices. Concretely, split a $(k+l) \times n$ matrix into the upper $k \times n$ part and the lower $l \times n$ part. In the formula (3.6.1) for the action of E_{ij} on $\mathcal{P}(\mathsf{M}_{k+l,n})(a)$ the sum over $1 \leq t \leq k+l$ correspondingly splits into two parts, the sum over $1 \leq t \leq k$ and the sum over $k+1 \leq t \leq k+l$. Split also $a\delta_{ij}$ into $b\delta_{ij}$ and $c\delta_{ij}$ and add them to the first and second sum, respectively. We see that the action of any E_{ij} on the module $\mathcal{P}(\mathsf{M}_{k+l,n})(a)$ is the same as on the tensor product of $\mathcal{P}(\mathsf{M}_{k,n})(b)$ and $\mathcal{P}(\mathsf{M}_{l,n})(c)$.
Now we apply the following standard proposition describing the behavior of annihilators of modules under tensor products.

Proposition 3.6.4. *Suppose that M and N are $\mathrm{U}(\mathfrak{g})$-modules. Let $I = \operatorname{Ann} M$ and $J = \operatorname{Ann} N$ be their annihilators in $\mathrm{U}(\mathfrak{g})$. Then*

$$\operatorname{Ann}_{\mathrm{U}(\mathfrak{g})}(M \otimes N) = \delta^{-1}(I \otimes \mathrm{U}(\mathfrak{g}) + \mathrm{U}(\mathfrak{g}) \otimes J).$$

Proof. For the sake of completeness, we give the proof of the proposition. We need to show the following:

$$\operatorname{Ann}_{\mathrm{U}(\mathfrak{g})\otimes\mathrm{U}(\mathfrak{g})}(M \otimes N) = I \otimes \mathrm{U}(\mathfrak{g}) + \mathrm{U}(\mathfrak{g}) \otimes J. \tag{3.6.5}$$

This holds in a very general setup of countably-dimensional modules over algebras. Let us denote the first copy of $\mathrm{U}(\mathfrak{g})$ by B and the second $\mathrm{U}(\mathfrak{g})$ by C. Then we have a B-module M, a C-module N and maps $\phi : B \to \operatorname{End} M$ and $\psi : C \to \operatorname{End} N$, defining the module structures. We also have the tensor product of algebras $B \otimes C$ and a module $M \otimes N$ over it. Equation 3.6.5 in this notation becomes

$$\operatorname{Ker}(\phi \otimes \psi) = \operatorname{Ker} \phi \otimes C + B \otimes \operatorname{Ker} \psi.$$

The right inclusion $\supseteq$ is clear. We prove the opposite inclusion using the fact that the canonical map $\operatorname{End} M \otimes \operatorname{End} N \to \operatorname{End}(M \otimes N)$ is injective. Let us choose a linear complement B' to $\operatorname{Ker}\phi$ in B and a basis $\{b'_i\}$ in B'. Then any $x \in B\otimes C$ may be written as $\sum_i b'_i \otimes c'_i + \sum_j b''_j \otimes c''_j$, where b'_i are elements of the basis in B' and $b''_j \in \operatorname{Ker}\phi$. Suppose that $x \in \operatorname{Ker}\phi \otimes \psi$, then $0 = \phi\otimes\psi(x) = \sum \phi(b'_i)\otimes\psi(c'_i)$ (the second summand is annihilated by ϕ). By the construction, the restriction of ϕ to B' is injective, therefore all $\phi(b''_i)$ are linearly independent. Hence all $\psi(c'_j)$ must be 0, which means that $x \in B \otimes \operatorname{Ker}\psi + \operatorname{Ker}\phi \otimes C$, as claimed. ☐

Taking $\mathfrak{g} = \mathfrak{gl}_n$, $M = \mathcal{P}(\mathsf{M}_{k,n})(b)$, and $N = \mathcal{P}(\mathsf{M}_{l,n})(c)$, we obtain the first statement of the theorem. The second statement follows by an application of Proposition 3.6.4. ☐

Corollary 3.6.6. *For any k we have $\mathcal{J}_{k+1}(a) \subseteq \mathcal{J}_k(a + \frac{1}{2})$ and $\mathcal{J}_{k+1}(a) \subseteq \mathcal{J}_k(a - \frac{1}{2})$.*

Proof. It is enough to prove that $\mathcal{J}_1(\pm\frac{1}{2}) \subseteq \mathcal{J}_0$, for then we can apply the theorem with $l = 1, b = a$ and $c = \pm\frac{1}{2}$. By Equation 2.2.8, $R^1(\frac{1}{2})(\mathsf{E}_{ij}) = \partial_j x_i$ and $R^1(-\frac{1}{2})(\mathsf{E}_{ij}) = x_i\partial_j$. Consider the module $\mathbb{C}[x_1, \dots, x_n]$ over $\mathcal{PD}(\mathbb{C}^n)$. The subspace of polynomials without the constant term is invariant under $x_i\partial_j$ for any $1 \leq i, j \leq n$, so it is a submodule under the $R^1(-\frac{1}{2})$-action of $\mathfrak{gl}_n$. The quotient of the module of all polynomials by this submodule is the trivial one-dimensional module over $\mathrm{U}(\mathfrak{gl}_n)$, which has the annihilator $\mathcal{J}_0$. Hence $\mathcal{J}_0$ contains the annihilator in $\mathrm{U}(\mathfrak{gl}_n)$ of the module of all polynomials, which is $\mathcal{J}_1(-\frac{1}{2})$. To prove that $\mathcal{J}_0 \supseteq \mathcal{J}_1(\frac{1}{2})$, consider the module $\mathbb{C}[\partial_1, \dots, \partial_n]$ with the $R^1(\frac{1}{2})$-action of $\mathfrak{gl}_n$. Similarly to the above, the constant coefficient differential operators without the constant term form a $\mathfrak{gl}_n$-submodule of codimension one, and the quotient by it is the trivial one-dimensional module over $\mathfrak{gl}_n$, which has the annihilator $\mathcal{J}_0$. ☐

We see that unlike the determinantal ideals, the ideals $\mathcal{J}_k$ do not form a decreasing filtration. Nevertheless, we have $\mathcal{J}_{k+2} \subseteq \mathcal{J}_k$. The reason for this discrepancy between the rank ideals and the determinantal ideals is discussed below after the proof of Theorem 3.7.1.

3.7 Generators of the rank ideal $\mathcal{J}_k$

Theorem 3.7.1. *The set Q_{k+1} of the quantum minors $\Delta_{I,J}$ of order $k+1$ generates the rank k ideal $\mathcal{J}_k \subseteq \mathrm{U}(\mathfrak{gl}_n)$. More generally, the quantum minors $\Delta_{I,J}(a)$ of order $k+1$ generate the ideal $\mathcal{J}_k(a)$.*

Proof. Applying the "twist" $\mathsf{E}_{ij} \mapsto \mathsf{E}_{ij}(a)$, we see that for a given k the second statement of the theorem is equivalent to the first one. We will prove the theorem by induction in k. The thrust of the proof is to show that the quantum minors in Q_{k+1} belong to the rank ideal $\mathcal{J}_k$. The fact that the quantum minors, in fact, generate $\mathcal{J}_k$ would follow then by a standard argument with filtered algebras.
By definition of the rank ideal, $\Delta_{I,J} \in \mathcal{J}_k$ is equivalent to $R^k(\Delta_{I,J}) = 0$. For $k = 0$ the quantum minors of order 1 are simply the generators E_{ij} of $\mathfrak{gl}_n$. Clearly, they belong to the augmentation ideal $\mathcal{J}_0$ (and generate it). So suppose that $k \geq 1$. Our strategy for proving that $R^k(\Delta_{I,J}) = 0$ is as follows. For $k = 1$ we check the vanishing of the corresponding 2×2 determinants by a direct computation in $\mathcal{PD}(\mathbb{C}^n)$ (Proposition 3.7.2). Then we reduce the case $k \geq 2$ to $k = 1$. We use the same idea as in the proof of Theorem 3.6.3: we decompose the vector space $\mathsf{M}_{k,n}$ into a direct sum of $\mathfrak{gl}_n$-invariant vector subspaces, which leads to the decomposition of the $\mathfrak{gl}_n$-action on $\mathcal{P}(\mathsf{M}_{k,n})$ into the sum of commuting $\mathfrak{gl}_n$-actions. The determinant is a multilinear function of its columns, and we find a particular way to expand it (Proposition 3.7.4) so that every term of the expansion vanishes, due to properties of the determinant and the vanishing of 2×2 determinants for $k = 1$.
We now realize the strategy outlined above. First, let us consider the case $k = 1$ (see Example 2.2.7).

Proposition 3.7.2. *The quantum minors $\Delta_{I,J}$ of order 2 belong to the rank 1 ideal.*

Proof. We need to check that for any quantum minor $\Delta_{I,J}$ of order 2, we have $R(\Delta_{I,J}) = 0$. Let $I = (i,l)$, $J = (j,m)$, where $1 \leq i,j,l,m \leq n$. Since R is an algebra homomorphism, Proposition 3.3.7 tells us that $R(\Delta_{I,J})$ can be rewritten as a determinant with entries in $\mathcal{PD}(\mathbb{C}^n)$:

$$R(\Delta_{I,J}) = \det \begin{vmatrix} R(\mathsf{E}_{ij}(\frac{1}{2})) & R(\mathsf{E}_{im}(-\frac{1}{2})) \\ R(\mathsf{E}_{lj}(\frac{1}{2})) & R(\mathsf{E}_{lm}(-\frac{1}{2})) \end{vmatrix}. \tag{3.7.3}$$

Let us write down explicitly the entries of this matrix. We have $R(\mathsf{E}_{ij}(\frac{1}{2})) = (x_i\partial_j + \frac{1}{2}\delta_{ij}) + \frac{1}{2}\delta_{ij} = x_i\partial_j + \delta_{ij} = \partial_j x_i$, and $R(\mathsf{E}_{im}(-\frac{1}{2})) = (x_i\partial_m + \frac{1}{2}\delta_{im}) - \frac{1}{2}\delta_{im} = x_i\partial_m$. Likewise, $R(\mathsf{E}_{lj}(\frac{1}{2})) = \partial_j x_l$, and $R(\mathsf{E}_{lm}(-\frac{1}{2})) = x_l\partial_m$. Expanding the determinant (**cf.** Equation (3.3.3)), we get $(\partial_j x_i)(x_l\partial_m) - (\partial_j x_l)(x_i\partial_m) = 0$, which can be written as $\partial_j(x_i x_l - x_l x_i)\partial_m = 0$. Thus the determinant in Equation 3.7.3 is zero, and the proposition follows. ☐

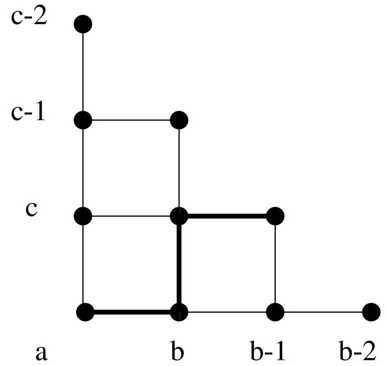
The following combinatorial construction is the heart of the proof. It will be applied to a quantum determinant $\Delta_{I,J}$ of size $k+1$ in order to show that $R^k(\Delta_{I,J}) = 0$.
Let $A = (a, a-1, \dots, a-k)$, $B = (b, b-1, \dots, b-k)$, $C = (c, c-1, \dots, c-k)$ be three "formal arithmetic progressions" with $a = b + c$. This means that $a - s$, $b - t$, $c - u$ $(0 \leq s,t,u \leq k)$ are expressions which behave according to the formal rules of arithmetic, namely, $a = b + c$, $a - 1 = (b-1) + c = b + (c-1)$, *etc.*, and in general, $a - s = (b - t) + (c - (s - t))$, where $0 \leq t \leq s \leq k$.

Proposition 3.7.4. *Keep the notation from the above. Let F be a multilinear function of $k+1$ arguments. We interpret the set of arguments of F as a sequence of length $k+1$. Then $F(a, a-1, \ldots, a-k)$ may be expanded using the multilinearity of F into a sum of $\sum_{t=0}^{k}\binom{k+1}{t} = 2^{k+1}$ terms of a special form. Each term in the expansion is the value of F on some shuffle of the initial segment of B of length t and the initial segment of C of length $k+1-t$ (this is a sequence of length $k+1$). The terms in the expansion correspond to paths of the length $k+1$ on the lattice $\mathbb{Z}^2$, connecting the point $(0,0)$ with a point on the line $x+y=k+1$.*

For example, for $k=0$ we have $F(a) = F(b) + F(c)$ and for $k=1$ we have

$$F(a, a-1) = F(b, a-1) + F(c, a-1) = (F(b, b-1) + F(b, c)) + (F(c, b) + F(c, c-1)).$$

Proof. Let $1 \le s \le k+1$. Consider the lattice $\mathbb{Z}^2$ on the Cartesian plane. Mark the x-axis with $b, b-1, \ldots, b-k$ and the y-axis with $c, c-1, \ldots, c-k$. Consider a path $P = (P_0, P_1, \ldots, P_s)$ of length s connecting the origin $P_0 = (0,0)$ with the point $P_s = (t, s-t)$, where $0 \le t \le s$. We construct a sequence of length $s+1$ whose ith term $(0 \le i \le s)$ is the mark on the x-axis below P_{i+1} if P_iP_{i+1} is horizontal and the mark on the y-axis to the left of P_{i+1} if P_iP_{i+1} is vertical. We extend this sequence to a sequence of length $k+1$ by adding the terms $a-(s+1), \ldots, a-k$ at the end. The sequence associated to a path P represents a term in the sth partial expansion of $F(a, a-1, \ldots, a-k)$.
The proposition is now proved by induction in s. Observe that a path of length $s \le k$ can be extended to a path of length $s+1$ in exactly two ways. If $P_s = (b-t, c-(s-t))$ then either $P_{s+1} = (b-t-1, c-(s-t))$ ("step right") or $P_{s+1} = (b-t, c-(s-t)-1)$ ("step up"). These correspond to expansions, respectively, $a-(s+1) = (b-t-1) + (c-(s-t))$ and $a-(s+1) = (b-t) + (c-(s-t)-1)$. The $(k+1)$st partial expansion is the full expansion of the proposition, involving 2^{k+1} terms.



The diagram illustrates the case $k=2$. The thick line gives the term $F(b, c, b-1)$ in the expansion of $F(a, a-1, a-2)$. $\square$

We are now in a position to prove that for any quantum minor $\Delta_{I,J}$ in Q_{k+1}, $R(\Delta_{I,J})$ vanishes. We argue by induction in k. The case $k=1$ was proved in Proposition 3.7.2. Let $k \ge 2$. Let us use a direct sum decomposition $\mathsf{M}_{k,n} = \mathsf{M}_{1,n} \oplus \mathsf{M}_{k-1,n}$ to decompose the representation $R^k(b+c)$ of $\mathfrak{gl}_n$ in the vector space $\mathcal{P}(\mathsf{M}_{k,n}) = \mathcal{P}(\mathsf{M}_{1,n}) \otimes \mathcal{P}(\mathsf{M}_{k-1,n})$ into the sum of commuting representations $R'(b)$ and $R''(c)$ on the factors of the tensor product (**cf.** Theorem 3.6.3).

Suppressing the indices in the notation, for any $b, c \in \mathbb{C}$ we have the following identity for the actions of a generator E_{ij} of $\mathfrak{gl}_n$:

$$R(b+c) = R'(b) + R''(c). \tag{3.7.5}$$

Note that R' is the map R^1 and R'' is the map R^{k-1} in the corresponding coordinates.
We may interpret the column determinant of a matrix as a multilinear function of the columns. As in the proof of Proposition 3.7.2 we see that $R(\Delta_{I,J})$ may be written as a column determinant with entries in $R(\mathrm{U}(\mathfrak{gl}_n)) \subseteq \mathcal{PD}(\mathsf{M}_{k,n})$. From the formula (3.5.2) for $\Delta_{I,J}$ we infer that the arguments of the determinant constitute a formal arithmetic progression $A = (\frac{k}{2}, \frac{k}{2}-1, \dots, -\frac{k}{2})$. This allows us to apply Proposition 3.7.4 to expand $R(\Delta_{I,J})$ into the sum of determinants whose arguments are columns of the form $R'(\frac{1}{2}-t)$ and $R''(\frac{k-1}{2}-u)$ with $t, u \geq 0$. We are going to use the properties of the determinant from Section 3.3 to show that all terms in this expansion vanish. Consider one term of the expansion. First, since the R'-action commutes with the R''-action, we can bring the columns containing R' to the left and the column containing R'' to the right, preserving the relative order of columns of the same "type" (possibly changing the sign of the determinant). If there are at least two columns of type R', then the first two columns in the new expression for the determinant would be $R'(\frac{1}{2})$ and $R'(-\frac{1}{2})$. All the 2×2 determinants in these two columns are of the form (3.7.3), and hence vanish by Proposition 3.7.2. By the third part of Proposition 3.3.5, the full determinant vanishes.
Consider one of the remaining terms of the expansion. Since it has at most one column of type R', there are at least $(k+1)-1 = k$ columns of type R'', starting from $R''(\frac{k-1}{2})$. These columns constitute a formal arithmetic progression $(\frac{k-1}{2}, \dots, -\frac{k-1}{2})$ of length $k-1$. Recall that the kernel of $R'' = R^{k-1}$ is the rank $k-1$ ideal. By the inductive assumption, every $k \times k$ determinant involving these columns vanishes. If there is a first column is of type R', expand the determinant in the first column. Otherwise, expand the determinant in the last column. In either case the determinant vanishes. Thus all terms in the expansion of $R(\Delta_{I,J})$ vanish. This concludes the proof of the fact that $R(\Delta_{I,J}) = 0$ and all quantum minors $\Delta_{I,J}$ of size $k+1$ belong to the kth rank ideal $\mathcal{J}_k$.
To finish the proof of the theorem, we apply the criterion of Proposition 2.3.3. Let us check that all conditions hold. We have just established that the quantum minors of order $k+1$ belong to the rank ideal $\mathcal{J}_k$. Their principal symbols are the classical minors δ_{IJ} of order $k+1$ (by Proposition 3.5.3). By Theorem 3.1.4, the classical minors of order $k+1$ generate the kth determinantal ideal I_k. We know from Theorem 3.2.4 that the ideal I_k the associated graded of the rank ideal $\mathcal{J}_k$. Thus we conclude that the quantum minors of order $k+1$ generate the rank ideal $\mathcal{J}_k \subseteq \mathrm{U}(\mathfrak{gl}_n)$. In fact, since the associated graded algebra of $\mathrm{U}(\mathfrak{gl}_n)$ is commutative, we see that the quantum minors even generate $\mathcal{J}_k$ as a *left* ideal. □

The last part of the proof of Theorem 3.7.1 makes clear the sense in which the quantum minors form a natural system of generators for the rank ideal $\mathcal{J}_k$: they "quantize" a natural system of generators for the associated graded of $\mathcal{J}_k$.
We note that the part of the proof using Proposition 3.7.4 has a natural interpretation in terms of the Hopf algebra structure on $\mathrm{U}(\mathfrak{gl}_n)$. Instead of using an "external" decomposition of R^k into R' and R'', we can consider the coproduct $\delta(\Delta_{I,J})$ of the quantum determinant. In fact, this can be done for an arbitrary Lie algebra $\mathfrak{g}$ and a character $\chi : \mathfrak{g} \to \mathbb{C}$. For any $X \in \mathfrak{g}, \alpha \in \mathbb{C}$ set $X(\alpha) = X + \alpha\chi(X)$. Now choose elements $X_{ij} \in \mathfrak{g}$ and let $\Delta'_{I,J}(\alpha) =$

$\det|X_{IJ}(\alpha - s)|$. Since the coproduct is the algebra homomorphism, it can be applied to $\Delta'_{I,J}(\alpha)$ entrywise (see Proposition 3.3.7), and $\delta(\Delta'_{I,J}(a))$ is thus expressed as a determinant with entries $\delta(X_{IJ}(\alpha - s))$. An application of Proposition 3.7.4 will expand this determinant into 2^{k+1} determinants with the columns of two types: $X(\beta - t)\otimes 1$ and $1 \otimes X(\gamma - u)$. Iterating the coproduct, we may obtain analogous identities for determinants with entries in $\mathrm{U}(\mathfrak{g})^{\otimes m}$. Theorem 3.7.1 clarifies why unlike the determinantal ideals, which satisfy $I_k \subseteq I_{k-1}$, the rank ideals only satisfy $\mathcal{J}_k \subseteq \mathcal{J}_{k-1}(\pm\frac{1}{2})$ (see Corollary 3.6.6). The reason for this discrepancy in the behavior of ideals can be traced back to the difference in the behavior of classical and quantum minors that generate them. As we saw in the proof of Theorem 3.1.4, the inclusion $I_k \subseteq I_{k-1}$ of the determinantal ideals relies on the row or column expansion of a minor of size k involving minors of size $k-1$. On the other hand, if we expand the generators of the rank ideal $\mathcal{J}_k$ in the first or last row, they will be expressed in terms of column determinants twisted by $\pm\frac{1}{2}$, which belong to $\mathcal{J}_{k-1}(\pm\frac{1}{2})$ rather than to $\mathcal{J}_{k-1}$. This gives rise to a "twisted" inclusion of the rank ideals in $\mathrm{U}(\mathfrak{gl}_n)$, see Corollary 3.6.6.

3.8 A localization property

In the theory of N-rank a prominent role played by the fact that an irreducible unitary representation of rank k remains irreducible when restricted to the kth maximal parabolic subgroup, see [How82]. The determinant expressions for generators of the kth rank ideal that we have found lead to an interesting analog of this property in the algebraic setting. Suppose that $2k \leq n$ (this is an algebraic counterpart of the condition that a reductive dual pair be in stable range). Let $\mathfrak{p}$ be the kth maximal parabolic subalgebra of $\mathfrak{gl}_n$, consisting of block upper triangular matrices with the blocks of orders k and $n-k$. We consider the images of the standard generators E_{ij} of $\mathfrak{gl}_n$ under R. Let $k+1 \leq i \leq n$ and $1 \leq j \leq k$. Set $i_0 = i$, $j_0 = j$, and $i_r = r$ for $1 \leq r \leq k$. Choose any $k+1 \leq j_1 < \ldots < j_k \leq n$, which is possible because of our restriction on k. Expanding the noncommutative determinant $\Delta_{I,J}$ in the first column, we get

$$\Delta_{I,J} = \mathsf{E}_{ij} \det|\mathsf{E}_{i_r j_s}(\frac{k}{2} - r)|_{1\leq r,s\leq k} + \sum_{a=1}^{k} \mathsf{E}_{i_a j}(\frac{k}{2}) \det|\mathsf{E}_{i_r j_s}(\frac{k}{2} - s)|_{1\leq s\leq k, 0\leq r\leq k, r\neq a}. \quad (3.8.1)$$

Due to our choice of I and J, everything in the right hand side except E_{ij} belongs to the enveloping algebra $\mathrm{U}(\mathfrak{p})$. Since the quantum determinant $\Delta_{I,J}$ of order $k+1$ is mapped by R^k to zero, we can express $R^k(\mathsf{E}_{ij})$ for $k+1 \leq i \leq n$ and $1 \leq j \leq k$ as a right quotient of the images under R^k of two elements of the enveloping algebra $\mathrm{U}(\mathfrak{p})$. This means that the algebra $\mathrm{U}(\mathfrak{gl}_n)/\mathcal{J}_k = R^k(\mathrm{U}(\mathfrak{gl}_n))$ is some noncommutative localization of its subalgebra $\mathrm{U}(\mathfrak{p})/\mathcal{J}_k \cap \mathrm{U}(\mathfrak{p}) = R^k(\mathrm{U}(\mathfrak{p}))$. Since these two algebras are noncommutative noetherian domains, they admit right quotient rings by Goldie's Theorem, and we obtain that these quotient rings coincide.

4 Rank of highest weight modules over $\mathfrak{gl}_n$

Let us call a two-sided ideal I in $\mathrm{U}(\mathfrak{gl}_n)$ an ideal of rank at most k if I contains the kth rank ideal $\mathcal{J}_k$. Primitive ideals satisfying this property will be classified in Part II. The classification uses a powerful theorem of M. Duflo claiming that every primitive ideal in the enveloping algebra of a complex reductive Lie algebra $\mathfrak{g}$ equals the annihilator of a simple highest weight module. This makes it natural to introduce and study the rank of highest weight modules, which is the objective of this section.

4.1 Characterization of rank in terms of the highest weight

We say that a $\mathrm{U}(\mathfrak{gl}_n)$-module M is a module of rank at most k if its annihilator $\operatorname{Ann} M$ is an ideal of rank at most k, **i.e.** contains $\mathcal{J}_k$. We will use the description of generators of the kth rank ideal $\mathcal{J}_k$ provided by Theorem 3.7.1 to characterize the simple highest weight modules over $\mathfrak{gl}_n$ of rank at most k.

Let us pick the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{gl}_n$ consisting of the diagonal matrices, and the Borel subalgebra $\mathfrak{b} \subseteq \mathfrak{gl}_n$ consisting of the upper triangular matrices. The nilradical $\mathfrak{n}$ of $\mathfrak{b}$ then consists of the upper triangular matrices with zeros on the diagonal. Thus the subalgebra $\mathfrak{n}$ is spanned by the generators $\mathsf{E}_{ij} \in \mathfrak{gl}_n$ with $1 \leq i < j \leq n$. Let $\{\epsilon_i | 1 \leq i \leq n\}$ be the basis in $\mathfrak{h}^*$ dual to the basis $\{\mathsf{E}_{ii} | 1 \leq i \leq n\}$ of the Cartan subalgebra. Define a symmetric bilinear form on $\mathfrak{h}^* = \mathbb{C}^n$ making ϵ an orthonormal basis and use it to identify $\mathfrak{h}^*$ with $\mathfrak{h}$. For a weight $\lambda \in \mathfrak{h}^*$, let $\lambda_1, \dots, \lambda_n$ be its components in the basis ϵ. We thus view weights of $\mathfrak{gl}_n$ as sequences of length n. With these conventions, the roots are $\{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq n\}$ The root subspace of $\epsilon_i - \epsilon_j$ is $\mathbb{C}\mathsf{E}_{ij}$. A root $\epsilon_i - \epsilon_j$ is positive for $i < j$ and negative for $i > j$. The half-sum of the positive roots is the following sequence:

$$\rho = \left(\frac{n-1}{2}, \frac{n-3}{2}, \dots, -\frac{n-1}{2} \right). \tag{4.1.1}$$

Denote by $L(\lambda)$ the simple highest weight module with highest weight $\mu = \lambda - \rho$. The relation between the entries of λ and μ is given by the formula $\lambda_i = \mu_i + \frac{n+1}{2} - i$. Denote by γ the the arithmetic progression of length $n-k$ with first term $\frac{n-k-1}{2}$ and difference -1:

$$\gamma = \left(\frac{n-k-1}{2}, \frac{n-k-3}{2}, \dots, -\frac{n-k-1}{2} \right).$$

The sequence γ numerically coincides with the half-sum of the positive roots ρ for the Lie algebra $\mathfrak{gl}_{n-k}$.

Theorem 4.1.2. *Let $\lambda = \mu + \rho$ be a weight of $\mathfrak{gl}_n$. Then the following are equivalent:*

(1) *The simple highest weight module $L(\lambda)$ is of rank at most k.*

(2) *The sequence λ contains γ as a subsequence.*

(3) *There exist a sequence of indices $j_0 = 0$, $1 \leq j_1 < j_2 < \dots < j_k$ such that for any $0 \leq s \leq k$, if $j_s < i < j_{s+1}$ then $\mu_i = -\frac{k}{2} + s$. Put differently, μ has the form*

$$\mu = (-\frac{k}{2}, \dots, -\frac{k}{2}, *, -\frac{k}{2} + 1, \dots, -\frac{k}{2} + 1, *, \dots, *, \frac{k}{2}, \dots, \frac{k}{2}), \tag{4.1.3}$$

where the stars represent the entries of μ with indices $j_1, \dots, j_k$.

Proof. Let us check the equivalence of the last two conditions. Denote by γ' the subsequence of $\lambda = \mu + \rho$ consisting of the first $n-k$ entries of λ with indices $i \notin \{j_1, \dots, j_k\}$. We claim that μ has the form (4.1.3) if and only if $\gamma' = \gamma$. Indeed, if $j_s < i < j_{s+1}$ then $\gamma'_{i-s} = \lambda_i$, hence $\gamma'_{i-s} - \gamma_{i-s} = \lambda_i - \gamma_{i-s} = (\mu_i + \frac{n+1}{2} - i) - (\frac{n+1-k}{2} - (i-s)) = \mu_i + \frac{k}{2} - s = \mu_i - (-\frac{k}{2} + s)$. The claim is now obvious.

It remains to show that the first condition of the theorem is equivalent to the second or third. We will prove that the first and third conditions are equivalent. Here is an outline of the proof. Proposition 4.1.4 below imposes restrictions on the highest weight μ of a highest weight module of rank at most k. These restrictions turn out to be precisely the third condition of the theorem. Conversely, for any μ of the form (4.1.3) we construct in Proposition 4.1.9 a module $\mathcal{P}$ over $\mathcal{PD}(\mathsf{M}_{k,n})$ and a vector $w \in \mathcal{P}$ which is a highest weight vector of weight μ with respect to the $\mathrm{U}(\mathfrak{gl}_n)$-action on $\mathcal{P}$. By virtue of the construction, the $\mathfrak{gl}_n$-module generated by w is a highest weight module of rank at most k with highest weight μ, demonstrating that the third condition of the theorem implies the first and completing the proof of the theorem.

To see when a highest weight $\mathfrak{gl}_n$-module M has rank at most k, let us analyze the effect of the generators of the rank k ideal on a highest weight vector of M. Our goal is to show that if the module M is of rank at most k then the highest weight has the special form (4.1.3).

Proposition 4.1.4. *Suppose that M is a highest weight module with the highest weight vector v of weight μ. Let $1 \le i_0 < i_1 < \ldots < i_k \le n$. Then the "principal" quantum minor $\Delta_{I,I}$ multiplies v by the scalar*

$$\prod_{s=0}^{k} (\mu_{i_s} + \frac{k}{2} - s). \tag{4.1.5}$$

Proof. To illustrate the idea behind the proof, let us consider the case $k = 1$ first. Since v is a highest weight vector of weight μ, it is annihilated by $\mathfrak{n}$ and is multiplied by $\mu(H)$ under the action of $H \in \mathfrak{h}$. With our conventions, $\mathfrak{n}$ consists of strictly upper-triangular matrices and $\mathfrak{h}$ consists of diagonal matrices. If $I = (i, l)$, $i < l$ then the principal quantum minor $\Delta_{I,I}$ is given by the formula

$$\Delta_{I,I} = \det \begin{vmatrix} \mathsf{E}_{ii}(\frac{1}{2}) & \mathsf{E}_{il}(-\frac{1}{2}) \\ \mathsf{E}_{li}(\frac{1}{2}) & \mathsf{E}_{ll}(-\frac{1}{2}) \end{vmatrix} = \mathsf{E}_{ii}(\frac{1}{2})\mathsf{E}_{ll}(-\frac{1}{2}) - \mathsf{E}_{li}(\frac{1}{2})\mathsf{E}_{il}(-\frac{1}{2}) \tag{4.1.6}$$

(**cf.** Equations 3.3.3, 3.5.2). Since $i < l$, $\mathsf{E}_{il}(-\frac{1}{2}) = \mathsf{E}_{il}$ is a strictly upper-triangular matrix, hence it annihilates v, and so the second term in the expansion (4.1.6) annihilates v. The first term in (4.1.6) is a product in $\mathrm{U}(\mathfrak{gl}_n)$ of diagonal matrices $\mathsf{E}_{ii}(\frac{1}{2})$ and $\mathsf{E}_{ll}(-\frac{1}{2}) \in \mathfrak{h} \subset \mathrm{U}(\mathfrak{gl}_n)$. Hence it multiplies v by $(\mu_i + \frac{1}{2})(\mu_l - \frac{1}{2})$. Note that for $k = 1$ the action of $\Delta_{I,I}$ on v reduces to the action of the first term of the expansion (4.1.6), which is easily computed. An analogous fact is true in general.

Now let k be arbitrary. Consider the full expansion of $\Delta_{I,I}$. We have

$$\Delta_{I,I} = \sum_{\sigma \in S_{k+1}} \mathrm{sgn}(\sigma) \overrightarrow{\prod}_{0 \le s \le k} \mathsf{E}_{i_{\sigma(s)} i_s}(\frac{k}{2} - s) \tag{4.1.7}$$

(**cf.** Equations 3.3.2, 3.5.2). We claim that for any $\sigma \neq id$, the product in the expansion (4.1.7) annihilates v. Indeed, since v is a highest weight vector, any $\mathsf{E} \in \mathfrak{h}$ multiplies v by a constant and $\mathsf{E} \in \mathfrak{n}$ annihilates v. Suppose that $\sigma \neq id$ and let r be the largest integer between 0 and k such that $\sigma(r) \neq r$. Since σ is a bijection and $\sigma(s) = s$ for $s > r$, we see that $\sigma(r) < r$, hence $i_{\sigma(r)} < i_r$. Therefore $\mathsf{E}_{i_{\sigma(s)} i_s} = \mathsf{E}_{i_s i_s} \in \mathfrak{h}$ for $s > r$ and $\mathsf{E}_{i_{\sigma(r)} i_r}(\frac{k}{2} - r) = \mathsf{E}_{i_{\sigma(r)} i_r} \in \mathfrak{n}$. Split the product in (4.1.7) into the product over $0 \le s < r$, the term $\mathsf{E}_{i_{\sigma(r)} i_r}(\frac{k}{2} - r)$ corresponding to

$s = r$, and the product over $r < s \leq k$. By what we have just said, the last product multiplies v by a scalar and the middle term annihilates v. Therefore, the sum in (4.1.7) reduces to one term with $\sigma = 1$, which multiplies v by $\prod_{s=0}^{k}(\mu_{i_s} + \frac{k}{2} - s)$. $\square$

In the setting of Proposition 4.1.4, suppose additionally that M is a module of rank at most k. Let us inductively define a sequence of indices $j_0 < j_1 < j_2 < \ldots < j_{k+1}$: $j_0 = 0$, $j_{s+1} = \min\{j > j_s | \mu_j \neq -\frac{k}{2} + s\}$. For any $1 \leq i \leq n$, if $j_s < i < j_{s+1}$ then $\mu_i = -\frac{k}{2} + s$. If $j_{k+1} > n$ then μ and the sequence J satisfy the third condition of the theorem. We will show that this is always the case, namely, that $j_{k+1} \leq n$ contradicts the assumption that M is of rank at most k. Indeed, if $j_{k+1} \leq n$ then $\prod_{s=0}^{k}(\mu_{j_{s+1}} + \frac{k}{2} - s) \neq 0$, because $\mu_{j_{s+1}} \neq -\frac{k}{2} + s$ for $0 \leq s \leq k$. Let I be the sequence $i_s = j_{s+1}$, $0 \leq s \leq k$, then by Proposition 4.1.4, $\Delta_{I,I} v \neq 0$. But Theorem 3.7.1 asserts that any quantum minor $\Delta_{I,J}$ of order $k+1$, and in particular $\Delta_{I,I}$, belongs to $\mathcal{J}_k$, hence acts on M by 0, which is the desired contradiction. Taking $M = L(\lambda)$, we see that if $L(\lambda)$ is of rank at most k then the third condition of the theorem holds.
We now proceed to construct a highest weight module over $\mathrm{U}(\mathfrak{gl}_n)$ with the highest weight μ for any weight μ satisfying the third condition of the theorem.

Notation 4.1.8. To simplify the notation, let us agree for the rest of the section that indices t and i are varying in the limits $1 \leq t \leq k, 1 \leq i \leq n$ and all direct sums and tensor products are taken over $1 \leq t \leq k, 1 \leq i \leq n$.

Recall that $\mathsf{M}_{k,n}$ is the vector space of $k \times n$ matrices and $\mathcal{PD}(\mathsf{M}_{k,n})$ is the algebra of polynomial coefficient differential operators on $\mathsf{M}_{k,n}$.

Proposition 4.1.9. *Let $1 \leq j_1 < j_2 < \ldots < j_k$, $a_t \in \mathbb{C}$. There exists a pair $(\mathcal{P}, w)$ consisting of a non-zero module $\mathcal{P}$ over the algebra $\mathcal{PD}(\mathsf{M}_{k,n})$ of polynomial coefficient differential operators on $\mathsf{M}_{k,n}$ and a vector $w \in \mathcal{P}$, with the following properties:*

(i) $x_{ti} w = 0$ *for* $1 \leq i < j_t$;
(ii) $\partial_{ti} w = 0$ *for* $i > j_t$;
(iii) $x_{ti}\partial_{ti} w = a_t w$ *for* $i = j_t$.

Proof. For $k = 1, n = 1$ this amounts to the following proposition.

Proposition 4.1.10. *Let $V = \mathbb{C}v$ be a one-dimensional vector space, $a \in \mathbb{C}$. There exists a pair $(\mathcal{P}, w)$ consisting of a module $\mathcal{P}$ over $\mathcal{PD}(V)$ and a vector $w \in \mathcal{P}$ satisfying any one of the following three conditions:*

(i) $\partial \cdot w = 0$;
(ii) $x \cdot w = 0$;
(iii) $x\partial \cdot w = aw$.

Proof. In the first case let $\mathcal{P} = \mathbb{C}[x]$ and $w = 1$. In the second case let $\mathcal{P} = \mathbb{C}[d]$ and $w = 1$. In the the third case let $\mathcal{P} = x^a\mathbb{C}[x, x^{-1}]$ and $w = x^a$. $\square$

Remark 4.1.11. If $a \in \mathbb{Z}$ then the module $x^a\mathbb{C}[x, x^{-1}] = \mathbb{C}[x, x^{-1}]$ is not irreducible. It has a unique nontrivial submodule $\mathbb{C}[x]$ and a unique irreducible quotient, which is isomorphic to $\mathbb{C}[\partial]$. Thus in the third case of the proposition, for $a \in \mathbb{Z}$ we can replace the module $x^a\mathbb{C}[x, x^{-1}]$ by an irreducible module $\mathcal{P}$ as follows. When $a \geq 0$, we let $\mathcal{P} = \mathbb{C}[x]$, and $x^a \in \mathcal{P} \subset \mathbb{C}[x, x^{-1}]$. When $a \leq -1$, x^a projects into a nonzero element proportional to ∂^{-a} of the quotient $\mathbb{C}[\partial]$ of $\mathbb{C}[x, x^{-1}]$.

To treat the case of arbitrary k and n, let us exploit the exponential property of Weyl algebras once more (**cf.** Theorem 3.6.3). Decompose the vector space $\mathsf{M}_{k,n}$ of $k \times n$ matrices into a direct sum of one-dimensional subspaces spanned by matrices with one non-zero entry, $\mathsf{M}_{k,n} = \oplus \mathbb{C} v_{ti}$. We have a corresponding tensor product decomposition

$$\mathcal{PD}(\mathsf{M}_{k,n}) = \otimes \mathcal{PD}(x_{ti}).$$

We define the module $\mathcal{P}$ and the vector w as tensor products, $\mathcal{P} = \otimes \mathcal{P}_{ti}$, $w = \otimes w_{ti}$, where $\mathcal{P}_{ti}$ is a module over $\mathcal{PD}(x_{ti})$ and $w_{ti} \in \mathcal{P}_{ti}$. For fixed t and i, precisely one of the conditions $1 \leq i < j_t$, $i > j_t$, $i = j_t$ holds, and we take a pair $(\mathcal{P}_{ti}, w_{ti})$ constructed in Proposition 4.1.10. $\square$

Let us use the homomorphism $R : \mathrm{U}(\mathfrak{gl}_n) \to \mathcal{PD}(\mathsf{M}_{k,n})$ to make a $\mathcal{PD}(\mathsf{M}_{k,n})$-module $\mathcal{P}$ into a $\mathrm{U}(\mathfrak{gl}_n)$-module. By definition, the $\mathrm{U}(\mathfrak{gl}_n)$-module obtained in this manner is of rank at most k. We claim that the vector $w \in \mathcal{P}$ constructed in Proposition 4.1.9 is in fact a highest weight vector for the $\mathfrak{gl}_n$-action. Indeed, suppose that $i < j$. Then for any t we have either $i < j_t$ or $j > j_t$. In the first case $x_{ti} w_{ti} = 0$, and so $x_{ti}\partial_{tj} w = \partial_{tj} x_{ti} w = 0$. In the second case $\partial_{tj} w_{tj} = 0$, and $x_{ti}\partial_{tj} w = 0$. Using Equation 2.2.5, we see that $R(\mathsf{E}_{ij})w = \sum_{t=1}^{k} x_{ti}\partial_{tj} w = 0$, because every term in the sum is 0, so w is annihilated by $\mathfrak{n} \subset \mathfrak{gl}_n$. Let us now compute the weight of w. First suppose that $i \notin \{j_1, \dots, j_k\}$ and choose s such that $j_s < i < j_{s+1}$. If $t \leq s$ then $i > j_s \geq j_t$, so $\partial_{ti} w_{ti} = 0$ and $x_{ti}\partial_{ti} w = 0$. If $t \geq s+1$ then $i < j_{s+1} \leq j_t$, so $x_{ti} w_{ti} = 0$ and $x_{ti}\partial_{ti} w_{ti} = (\partial_{ti} x_{ti} - 1)w = -w$. Therefore $R(\mathsf{E}_{ii})w = \sum_{t=1}^{k} x_{ti}\partial_{ti} w + \frac{k}{2} w = \sum_{t=1}^{s} x_{ti}\partial_{ti} w + \sum_{t=s+1}^{k} x_{ti}\partial_{ti} w + \frac{k}{2} w = 0 + (k-s)(-w) + \frac{k}{2} w = (-\frac{k}{2} + s)w$. When $i = j_s$, the same argument yields $R(\mathsf{E}_{ii})w = (a_s - \frac{k}{2} + s)w$. We conclude that the weight μ of w has the form (4.1.3) and conversely, setting $a_t = \mu_{j_t} + \frac{k}{2} - t$ we can get any μ of that form. Denote by M the $\mathrm{U}(\mathfrak{gl}_n)$-submodule of $\mathcal{P}$ generated by w. What we have established is that it is a highest weight module with the highest weight μ and $\operatorname{Ann} M \supseteq \mathcal{J}_k$, so $\operatorname{Ann} L(\mu + \rho) \supseteq \operatorname{Ann} M \supseteq \mathcal{J}_k$, and $L(\lambda) = L(\mu + \rho)$ is of rank at most k. This finishes the proof of the theorem. $\square$

4.2 Case of finite-dimensional modules

Let us use the results of the previous section to characterize the finite-dimensional $\mathfrak{gl}_n$-modules of rank at most k. Finite-dimensional representations of $\mathfrak{gl}_n$ admit the following combinatorial parametrization (see [How95]).

Let $\mu = (\mu_1, \dots, \mu_n)$ be an integral dominant weight for $\mathfrak{gl}_n$. Represent μ in the form

$$\mu_1 \geq \ldots \geq \mu_r \geq 0 \geq \ldots \geq 0 \geq \mu_{n-s+1} \geq \cdots \geq \mu_n,$$

so that $\mu_i = 0$ for $r < i \leq n - s$. We define the sequences $d = (d_1, \dots, d_r) = (\mu_1, \dots, \mu_r)$ and $e = (e_1, \dots, e_s) = (-\mu_n, \dots, -\mu_{n-s+1})$. Since μ is a dominant weight, we get that $d_1 \geq \ldots d_r \geq 0$ and $e_1 \geq \ldots e_s \geq 0$, so that d and e are partitions. Denote by D and E the Young diagrams corresponding to d and e. They satisfy the conditions

$$\operatorname{depth}(D) \leq r, \quad \operatorname{depth}(E) \leq s, \quad \text{and} \quad \operatorname{depth}(D) + \operatorname{depth}(E) \leq n. \tag{4.2.1}$$

(The depth of a Young diagram is the number of nonzero parts of the corresponding partition). Denote by $\rho^{D,E}$ the simple finite-dimensional highest weight module over $\mathfrak{gl}_n$ with highest weight $\mu = \mu(D, E)$. More generally, one can consider a finite-dimensional $\mathfrak{gl}_n$-module $\rho = \rho^{D,E} \otimes \det{}^{l}$ (here the "twist" l may be an arbitrary complex number).

Theorem 4.2.2. *Let ρ be a simple finite-dimensional module over $\mathfrak{gl}_n$. Then ρ is of rank at most k if and only if there exist $r, s \geq 0$ with $r + s = k$ and Young diagrams D and E with $depth(D) \leq r, depth(E) \leq s$ such that*

$$\rho = \rho^{D,E} \otimes \det{}^{\frac{r-s}{2}}. \tag{4.2.3}$$

Proof. Theorem 4.1.2 characterizes the highest weights of simple highest weight modules over $\mathfrak{gl}_n$ of rank at most k. We need to find which dominant weights satisfy the condition of the theorem. Suppose that μ is a dominant weight of the form (4.1.3). Then the sequence μ is weakly decreasing, and μ cannot contain two segments $J_t = (-\frac{k}{2} + t, \ldots, -\frac{k}{2} + t)$ and $J_r = (-\frac{k}{2} + r, \ldots, -\frac{k}{2} + r)$ with $t < r$. So μ contains exactly one such segment, let us say J_r. Then in the notation of Theorem 4.1.2, $j_t = t$ for $t \leq r$ and J_r has length at least $n - k$. Let $s = k - r$, then $-\frac{k}{2} + r = \frac{r-s}{2}$ and $r + (n - k) = n - s$. Denote $\mu' = \mu - \frac{r-s}{2}\omega_n$, where ω_n is the nth fundamental weight $(1, \ldots, 1)$. Then $\mu'_i = 0$ for $r + 1 \leq i \leq n - s$, and μ' is dominant, so by definition, the simple finite-dimensional module with highest weight μ' is $\rho^{D,E}$, and D and E satisfy the condition (4.2.1). Therefore $\rho = \rho^{D,E} \otimes \det^{\frac{r-s}{2}}$. □

By basic algebra, irreducible finite-dimensional representations of an associative algebra correspond bijectively to primitive ideals of finite codimension. Thus Theorem 4.2.2 describes those primitive ideals in $\mathrm{U}(\mathfrak{gl}_n)$ of rank at most k that have *finite codimension*. The general case will be studied in Part II.

Remark 4.2.4. Finite-dimensional representations of Theorem 4.2.2 arise in the dual pair correspondence for a reductive dual pair $(U_n, U_{r,s})$ (see [How83], [How95]). In fact, these representations (without a determinant twist) appear in Weyl's analysis of the representation of GL_n in the space of polynomial functions $\mathcal{P}(V)$ on the vector space $V = \mathbb{C}^r \otimes \mathbb{C}^n \oplus \mathbb{C}^s \otimes \mathbb{C}^{n*}$ (see [Wey39]). The new point of Theorem 4.2.2 is that the representations of form (4.2.3) *exhaust* all finite-dimensional representations of $\mathfrak{gl}_n$ of rank at most k.

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