High genus GW invariants of blowups and a vanishing theorem

Jianxun Hu[1]

Department of Mathematics, Zhongshan University,
Guangzhou, 510275, P. R. China
and
Max-Planck-Institute fur Mathematics,
Vivatsgasse 7, 53111 Bonn, Germany

Abstract: In this paper, using the gluing formula of Gromov-Witten invariants for symplectic cutting developed by Li and Ruan, we established a relation between high genus Gromov-Witten invariants of a symplectic manifold M and its blowups at a smooth point and also proved a vanishing theorem.

Keywords: Gluing formula, Blowup, Log GW-invariant, vanishing theorem.

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1 Introduction

Over the last few years, many mathematicians contributed their efforts to establish the mathematical foundation of the theory of quantum cohomology or GW(Gromov-Witten) invariants. In 1995, Ruan-Tian [R1, RT] first established for the semipositive symplectic manifolds. Recently the semipositivity condition has been removed by many suthors. Now the focus is on the calculations and applications. Many Fano manifolds were computed. We think that it is important to study the change of GW invariants under surgery.

Let M be a compact symplectic manifold of dimension $2n$, $\tilde{M}$ be the blowup of M. Denote by $p : \tilde{M} \longrightarrow M$ the natural projection. In addition to the two motivations to study the GW invariants of bowups, which we gave in [H1], some recent research indicated that there is a deep amazing relation between quantum cohomology and birational geometry. The quantum minimal model conjecture [R2, R3] lead to attempt to find quantum cohomology of a minimal model without knowing minimal model. This problem requires a thorough

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understanding of blowup type formula of GW invariants and quantum cohomology.

According to McDuff [M] the blowup operation in symplectic geometry amounts to a removal of an open symplectic ball followed by a collapse of some boundary directions. Lerman [L] gave a generalization of blowup construction, "the symplectic cut".

The author [H1, H2] carefully studied the blowup formula of GW invariants. There we conjectured Theorem 1.2 in [H1] is true for any genus. But we did not know how to overcome the genus condition $g \leq 1$. In this paper, we first generalized Li-Ruan gluing formula to **Log GW invariants**, then establish the result for any genus. Our first theorem is the following

Theorem 1: Suppose that $A \in H_2(M)$, $\alpha_1, \cdots, \alpha_m \in H^*(M)$. Then

$$\Psi^M_{(A,g)}(\alpha_1, \cdots, \alpha_m) = \Psi^{\tilde{M}}_{(p!(A),g)}(p^*\alpha_1, \cdots, p^*\alpha_m)$$

where $p!(A) = PDp^*PD(A)$.

To state our vanishing theorem, we first need to fix some notations. We call a cohomology class $\alpha \in H^*(\tilde{M})$ non-exceptional class if α is the pullback of some class in $H^*(M)$ by the natural projection $p : \tilde{M} \longrightarrow M$. We will call a class $\alpha \in H^*(\tilde{M})$ exceptional class if $\alpha \in H^*(E)$ and $\deg \alpha > 0$. Here E is the exceptional divisor. Roughly speaking, our vanishing theorem says that the GW invariant will vanish if there are "not too many" exceptional cohomology classes in the invariants.

Since the GW invariants are multilinear in the cohomology classes, we will for computational purpose only consider invariants where the cohomology classes are either non-exceptional or exceptional. We have

Theorem 2: Let $A \in H_2(M)$, $A \neq 0$, $e \in H_2(E)$ be the class of a line in the exceptional divisor E. Suppose that $\alpha_i \in H^*(\tilde{M})$, $1 \leq i \leq m$, is either nonexceptional or exceptional. Denote by $W(\alpha)$ the sum of degrees of all exceptional classes among α_i. Then

$$\Psi^{\tilde{M}}_{p!(A)+re}(\alpha_1, \cdots, \alpha_m) = 0,$$

if $0 < W(\alpha) \leq 2(r+1)(n-1)$ and $r > 0$.

Gathman [G] proved the same result for projective variety and used the condition $0 < W(\alpha) < 2(r+1)(n-1)$. Our theorem is also true for any symplectic manifolds and weaken Gathman's conditions.

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2 Review of Gluing Formula

The proofs of our results depend strongly on the gluing formula for symplectic cutting developed by Li and Ruan [LR]. Let $\overline{M}^+$, $\overline{M}^-$ be the symplectic cuttings of M along $N \subset M$. Here N is the common boundary of $M^{\pm}$ with a S^1-action and a generic level set of a local periodic hamiltonian function. The quotient $Z = N/S^1$ is the famous symplectic reduction, see [LR], p9-10 for details. Li and Ruan defined **Log GW invariants** $\Psi^{(\overline{M}^{\pm},Z)}$ of $\overline{M}^{\pm}$ relative Z by counting the number of log stable holomorphic maps intersecting Z at finitely many points with prescribed tangency. In this section, we want to extend Li and Ruan's gluing formula to the case with more ends. Li and Ruan have pointed out that their proof works in this case, see Remark 5.8 of [LR]. Therefore, here we only state the results.

Let Z_i, $i = 1, \cdots, l$, be real codimension two symplectic submanifolds of M. By a result of Guillemian and Sternberg [GS], the symplectic structure of each Z_i is modeled on a neighborhood of Z in either $\overline{M}^+$ or $\overline{M}^-$.

Let $K = \{0, \cdots, 0|k_1^{(1)}, \cdots, k_{\nu_1}^{(1)}| \cdots |k_1^{(l)}, \cdots, k_{\nu_l}^{(l)}\}$ with m-many zeros be a set of nonnegative integers with $k_j^i > 0$. Denote by $\mathcal{M}_A(g, Z_1, \cdots, Z_l, K)$ the space of genus g pseudo-holomorphic maps u, which represent the homology class A, such that u has marked points $(x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+\nu_1}, \cdots,$ $x_{m+\sum_{k=1}^{l}\nu_k})$ with the property u is tangent to Z_i at $x_{m+\sum_{k=1}^{i-1}\nu_k+j}$ with order k_j^i and $f(x_j) \notin Z_i$ for $0 \le j \le m$ and $\sum_{j=1}^{\nu_i} k_j^{(i)} = Z_i^*(A)$, where Z_i^* is the Poincare dual to Z_i. We compactify $\mathcal{M}_A(g, Z_1, \cdots, Z_l, K)$ by $\overline{\mathcal{M}}_A(g, Z_1, \cdots, Z_l, K)$, the space of log stable maps. We have evaluation maps

$$e_k : \overline{\mathcal{M}}_A(g, Z_1, \cdots, Z_l, K) \longrightarrow M,$$

for marked points $x_1, \cdots, x_m$ and

$$e_j^i : \overline{\mathcal{M}}_A(g, z_1, \cdots, Z_l, K) \longrightarrow Z_i, \quad i = 1, \cdots, l, \quad j = 1, \cdots, \nu_i,$$

for marked points $x_{m+\sum_{k=1}^{i-1}\nu_k+j}$. Roughly the Log GW invariants may be defined as

$$\Psi^{(M,Z_1,\cdots,Z_l)}_{(A,g,K)}(\alpha_1, \cdots, \alpha_m, \beta_j^i) = \int_{\overline{\mathcal{M}}_A(g,Z_1,\cdots,Z_l,K)} \Pi_{k=1}^m e_k^* \alpha_k \wedge \Pi_{i,j}(e_j^i)^* \beta_j^i,$$

for $\alpha_i \in H^*(M)$, $\beta_j^i \in H^*(Z_i)$. To be precise, we need to construct a virtual neighborhood of $\overline{\mathcal{M}}_A(g, Z_1, \cdots, Z_l, K)$. Then we take the integrand over the normalization of the virtual neighborhood.

Suppose Z is a common real codimension two symplectic submanifold of $\overline{M}^{\pm}$. Next we will state the gluing formula of Log GW invariants with many ends. Clearly other ends do not influence our gluing when we are gluing along some end Z. Therefore, to simplify the notations, we just drop data from other ends when we write the gluing formula for some end Z

Given $u^{\pm} \in \mathcal{M}_{A^{\pm}}(\overline{M}^{\pm}, Z, Z_1^{\pm}, \cdots, Z_{l^{\pm}}^{\pm}, K^{\pm})$ where $K^{\pm}$ have $m^{\pm}$-many zeros respectively and $m = m^+ + m^-$. We consider the linearization of $\overline{\partial}$-operator

$$D_{u^{\pm}} = D\partial_J(u^{\pm}) : C^{\infty}((u^{\pm})^*TM^{\pm}) \longrightarrow \Omega^{0,1}((u^{\pm})^*TM^{\pm}).$$

If we choose a proper weighted Sobolev norm, we can define the Fredholm index $IndD_{u^{\pm}}$. see section 3.4 of [LR] for details. We also can glue u^+ and u^- along the end Z to obtain a pseudo-holomorphic map u in M. Suppose that $[u] = A$. Then

Proposition 2.1:([LR], Theorem 3.23)

$$IndD_{u^+} + IndD_{u^-} = 2(n-1)\nu + 2C_1(A) + 2(n-3)(1-g) + 2m,$$

where C_1 is the first Chern class of M.

Suppose that the homology classes of u^+, u^-, u are A^+, A^-, A respectively. If $(u^{+'}, u^{-'})$ is another representative and glue to u', by Lemma 2.11 of [LR], we have $[u'] = [u]$. Denote by $C = \{A^+, g^+, K^+; A^-, g^-, K^-\}$ the gluing component. The gluing formula counted the contribution of the gluing components to Log GW invariant of M. Denote by Ψ_C the contribution of C.

Choose a homology bass $\{\beta_b\}$ of $H^*(Z, R)$. Let (δ_{ab}) be its intersection matrix. There is a map

$$\pi : M \longrightarrow \overline{M}^+ \cup_Z \overline{M}^-,$$

where $\overline{M}^+ \cup_Z \overline{M}^-$ is the union of $\overline{M}^{\pm}$ along Z. For the gluing component $C = \{A^+, g^+, K^+; A^-, g^-, K^-\}$ where $K^{\pm} = (0, \cdots, 0|k_1, \cdots, k_{\nu}|(k_j^i)^{\pm})$ with $m^{\pm}$-many zeros, we have

Proposition 2.2: Let $\alpha_i^{\pm}$ be differential forms with $\deg \alpha_i^+ = \deg \alpha_i^-$ even. Suppose that $\alpha_i^+|_Z = \alpha_i^-|_Z$ and $\alpha_i^+ \cup_Z \alpha_i^- \in H^*(\overline{M}^+ \cup_Z \overline{M}^-, R)$. Let $\alpha_i = \pi(\alpha_i^+ \cup_Z \alpha_i^-)$. We have the following gluing formula

$$\begin{aligned} \Psi_C^{(M, Z_i^+, Z_k^-)}(\alpha_1, \cdots, \alpha_m|(\beta_j^i)^+|(\beta_j^k)^-) &= |K| \sum_{I,J} \Psi_{(A^+, g^+, K^+)}^{(\overline{M}^+, Z, Z_i^+)}(\alpha_i^+, \beta_I, (\beta_j^i)^+) \\ &\times \delta^{I,J} \Psi_{(A^-, g^-, K^-)}^{(\overline{M}^-, Z, Z_k^-)}(\alpha_i^-, \beta_J, (\beta_j^k)^-), \end{aligned}$$

where we associate $\beta_i \delta^{ij} \beta_j$ to every intersection with Z and put $|K| = k_1 \cdots k_{\nu}$, $\delta^{I,J} = \delta^{i_1, j_1} \cdots \delta^{i_{\nu}, j_{\nu}}$. Denote by $\Psi_{(A^{\pm}, g^{\pm}, K^{\pm})}^{(\overline{M}^{\pm}, Z, Z_i^{\pm})}(\alpha_i^{\pm}, \beta_I, (\beta_j^i)^{\pm})$ the product of Log GW invariants corresponding to each component.

This Proposition is an easy extension of Theorem 5.8 of [LR] to Log GW invariant. Their proof works in this case. Therefore we omit the proof.

Proposition 2.3: For $C=\{A^{\pm},g^{\pm},(0,\cdots,0)\}$, we have

$$\Psi_C(\alpha_i^{\pm})=\Psi^{(\overline{M}^{\pm},Z)}_{(A^{\pm},g^{\pm},(0,\cdots,0))}(\alpha_i^{\pm}).$$

3 High genus GW invariant

In this section, we will give the proof of our Theorem 1 about the blowup formula of high genus GW invariants. Theorem 1.3 in [H1] shows our theorem is true for the case $n\leq 3$. Therefore, in this section, we assume $n>3$.

Proof of Theorem 1: Let P_0 be the blown-up point. We perform the symplectic cutting for M at P_0 as in [H1]. We have

$$\overline{M}^{+}=\mathbf{P}^n,\quad \overline{M}^{-}=\tilde{M}.$$

Similar to [H1], we first consider the contribution of each component to the GW invariant. Therefore, we consider the component

$$C=\{A^+,g^+,m^+,K^+;A^-,g^-,m^-,K^-\}.$$

We will use the same convention as [H1]. Assume $u^{\pm}:\Sigma^{\pm}\longrightarrow\overline{M}^{\pm}$ may have many connected components $u_i^{\pm}:\Sigma_i^{\pm}\longrightarrow\overline{M}^{\pm}$, $i=1,2,\cdots,l^{\pm}$. Suppose $\Sigma_i^{\pm}$ has arithemetic genus $g_i^{\pm}$, $g^{\pm}=\sum g_i^{\pm}$ with $m_i^{\pm}$ marked points and $m^{\pm}=\sum m_i^{\pm}$. The same calculation as in [H1] shows

$$\begin{aligned}
IndD_{u^+}&=2(3-n)(g^+-l^+)+2n\sum k_i+2\nu+2m^+,\\
IndD_{u^-}&=2C_1(A)+2(3-n)(g-g^++l^+-1)+2(n-2)\nu-2n\sum k_i.
\end{aligned}$$

Moreover, $\Psi_C=0$ except $m^-=m$. Without loss of generality, we assume

$$\sum\deg\alpha_i=2C_1(A)+2(3-n)(g-1)+2m.$$

Now we want to prove the following claim.

Claim: Either $\Psi_C=0$ or $\Psi_C=\Psi_{C'}(\alpha_i^-)$ for $C'=\{A^-,g,m\}$.

First of all, we consider the case $g^+=0$. Then we have

$$\begin{aligned}
\sum\deg\alpha_i^-&=2C_1(A)+2(3-n)(g-1)+2m\\
&>2C_1(A)+2(3-n)(g+l^+-1)-2\sum k_i-2\nu\\
&\quad+2(n-1)(\nu-\sum k_i)+2m\\
&=IndD_{u^-}+2m.
\end{aligned}$$

Therefore, by the definition of Log GW invariants, we have for any $\beta_b \in H^*(Z)$

$$\Psi^{(\overline{M}^-,Z)}_{(A^-,g^-,m,K^-)}(\alpha_i^-,\beta_b)=0.$$

Therefore $\Psi_C=0$ if $g^+=0$.

Next we consider the component C with $g^+>0$. By the gluing formula, we have

$$\Psi_C=\sum_{I,J}|K|\Psi^{(\overline{M}^-,Z)}_{(A^-,g^-,K^-)}(\alpha_1^-,\cdots,\alpha_m^-,\beta_I)\delta^{I,J}\Psi^{(\overline{M}^+,Z)}_{(A^+,g^+,K^+)}(\beta_J).$$

If $\Psi^{(\overline{M}^+,Z)}_{(A^+,g^+,K^+)}(\beta_J)=0$, then we are done. Otherwise, we perform the symplectic cutting for $\overline{M}^+$ at P_0. Then the same arguement as above shows

$$\begin{aligned}\Psi_C &= \sum_{I,J}|K|\Psi^{(\overline{M}^-,Z)}_{(A^-,g^-,K^-)}(\alpha_1^-,\cdots,\alpha_m^-,\beta_I)\delta^{I,J}\\ &\times \sum_{i,j}|K_1|\Psi^{(\overline{(\overline{M}^+)}^-,Z,Z_1)}_{((A^+)^-,g_1^-,K_1^-)}(\beta_J,\beta_i)\delta^{i,j}\Psi^{(\overline{(\overline{M}^+)}^+,Z_1)}_{((A^+)^+,g_1^+,K_1^+)}(\beta_j)\\ &= \sum_{i,j}|K_1|\Psi^{(\overline{M}^-,Z_1)}_{(A_2^-,g_2^-,K_1^-)}(\alpha_1^-,\cdots,\alpha_m^-,\beta_i)\delta^{i,j}\Psi^{(\overline{M}^+,Z_1)}_{((A^+)^+,g_1^+,K_1^+)}(\beta_j)\\ &= \Psi_{C'},\quad \text{for } C'=\{A_2^-,g_2^-,m,K_1^-;(A^+)^+,g_1^+,K_1^+\}\end{aligned}$$

where $0<g_1^+<g^+$. In the second equality we use the gluing formula and the fact that A_2^- is the gluing class of A^- and $(A^+)^-$, and the identifications $\overline{M}^+\cong\overline{\overline{M}^+}^+\cong\mathbf{P}^n$ and the fact that the symplectic gluing of $\overline{M}^-$ and $(\overline{M^+})^-$ also recovers $\overline{M}^-$. Therefore, we may reduce the contribution of C to that of another component $C'=\{A_2^-,g_2^-,m,K_1^-;(A^+)^+,g_1^+,K_1^+\}$ with $g_1^+<g^+$. By the finiteness of the genus g^+, after finite steps, we may reduce the component C to either the case $g^+=0$ or $C'=\{A^-,g,m\}$. So we have proved our claim.

Therefore we have

$$\Psi^M_{(A,g,m)}(\alpha_1,\cdots,\alpha_m)=\Psi^{(\overline{M}^-,Z)}_{(A^-,g,m)}(\alpha_1^-,\cdots,\alpha_m^-).$$

The remain proof of the theorem is similar to that of Theorem 1.2 in [H1] and the above argument. Therefore, we omit it here and proved our theorem.

4 A vanishing theorem

In this section, we are concerned with a vanishing theorem for GW invariants of blowups of a symplectic manifold at a smooth point. In fact, we will give the proof of Theorem 2.

Proof of Theorem 2: We perform the symplectic cutting for $\tilde{M}$ along the exceptional divisor E as in [H1, H2, LR]. Since the normal bundle N_E of E in $\tilde{M}$ is $\mathcal{O}(-1)$, we may consider it as a bundle with fiber $(C, -\sqrt{-1}dz \wedge d\bar{z})$. Furthermore, we may consider N_E over E with the symplectic form

$$\tilde{\omega}_E = \tilde{\omega}|_E + (-\sqrt{-1}dz \wedge d\bar{z}),$$

where $\tilde{\omega}|_E$ is the restriction of the symplectic form $\tilde{\omega}$ on $\tilde{M}$ to E, z is the coordinates in the fiber. The hamiltonian is

$$H(x, z) = |z|^2 - \varepsilon$$

and the S^1-action is given by

$$e^{i\theta}(x, z) = (x, e^{i\theta}z).$$

Consider the symplectic vector bundle $N_E \oplus \mathcal{O}$ with symplectic form $\tilde{\omega}_E + (-\sqrt{-1}dw \wedge d\bar{w})$ and the momentum map $\mu(x, z, w) = H(x, z) + |\omega|^2$ arising from the action of S^1 on $N_E \oplus \mathcal{O}$. As in [H1], the manifold $\tilde{M}^+ := \{(x, z)|H(x, z) < 0\}$ embeds as an open dense submanifold into the reduced space

$$\overline{\tilde{M}}_E^+ := \{(x, z, w)||z|^2 + |w|^2 = \varepsilon\}/S^1$$

and the difference $\overline{\tilde{M}}_E^+ - \tilde{M}_E$ is symplectomorphic to the reduced space $H^{-1}(0)/S^1$. A similar procedure defines

$$\overline{\tilde{M}}_E^- := \{(x, z, w)||z|^2 - |w|^2 = \varepsilon\}/S^1.$$

It is easy to see that the symplectic manifold $H^{-1}(0)/S^1$ is embedded on both $\overline{\tilde{M}}_E^+$ and $\overline{\tilde{M}}_E^-$ as a real codimension two symplectic submanifold but with opposite normal bundles. So the symplectic gluing of $\overline{\tilde{M}}_E^+$ and $\overline{\tilde{M}}_E^-$ along the reduced space $H^{-1}(0)/S^1$ recovers the neighborhood $\mathcal{N}_\delta(E)$, i. e. the normal bundle $\mathcal{O}(-1)$.

We define $\overline{\tilde{M}}^+ := \overline{\tilde{M}}_E^+$ and $\overline{\tilde{M}}^- := (\tilde{M} - \mathcal{N}_\delta(E)) \cup \overline{\tilde{M}}_E^-$. From the above description, we know the symplectic gluing of $\overline{\tilde{M}}^+$ and $\overline{\tilde{M}}^-$ recovers the original manifold $\tilde{M}$. Lerman [L] called the operation that produces $\overline{\tilde{M}}^+$ and $\overline{\tilde{M}}^-$ symplectic cutting.

According to [H1, L, LR], we have

$$\overline{\tilde{M}}^+ = \mathbf{P}(\mathcal{O}(-1) \oplus \mathcal{O}), \quad \overline{\tilde{M}}^- \cong \tilde{M}.$$

Without loss of geneality, we may assume that $\alpha_1, \cdots, \alpha_{m^+}$ are exceptional classes and $\alpha_{m^++1}, \cdots, \alpha_m$ support away from E. Furthermore, we may assume $\alpha_i^- = 0$, $1 \leq i \leq m^+$, and $\alpha_i^+ = 0$, $m^+ + 1 \leq i \leq m$, if we choose a sufficiently small $\delta > 0$. Define $m^- = m - m^+$.

We first consider the contribution of each component of the GW invariants. By the connectness of stable peudo-holomorphic curve and support reason, we only need to consider the following component

$$C = \{(p!(A) + re)^+ . K^+; (p!(A) + re)^-, K^-\}.$$

where $K^\pm = (0, \cdots, 0, k_1, \cdots, k_\nu)$ with $m^\pm$-many zeros and $\nu > 0$. From Proposition 2.1, we have

$$IndD_{u^+} + IndD_{u^-} = 2(n-1)\nu + 2C_1(A) + 2r(n-1) + 2n - 6 + 2m.$$

where C_1 is the first Chern class of M.

According to our convention, $u^\pm : \Sigma^\pm \longrightarrow \overline{\tilde{M}}^\pm$ may have many connected components $u_i^\pm : \Sigma_i^\pm \longrightarrow \overline{\tilde{M}}^\pm$, $i = 1, 2, \cdots, l^\pm$. Then we have

$$\begin{aligned} IndD_{u^+} &= \sum_{i=1}^{l^+} IndD_{u_i^+} \\ &= 2\sum_{i=1}^{l^+} C_1([u_i^+]) + 2(n-3)l^+ + 2\nu - 2\sum k_i + 2m^+, \end{aligned}$$

where C_1 is the first Chern class of $\overline{\tilde{M}}^+$.

Now we want to calculate $IndD_{u_i^+}$. Observe that we obtained $\overline{\tilde{M}}^+$ from M by performing the symplectic cutting twice. We also note that $\overline{\tilde{M}}^+$ is independent of the order of these two symplectic cuttings. Therefore, If we commute the order of these two symplectic cuttings, it is easy to see $\mathbf{P}(\mathcal{O}(-1) \oplus \mathcal{O}) \cong \tilde{\mathbf{P}}^n$. By Mori's cone theory, we have $[u_i^+] = a(L - e) + be$, $a \geq 0$,$b \geq 0$, where L is the class of a line in $\tilde{\mathbf{P}}^n$ with $L \cdot E = 1$ and e is the class of a line in the exceptional divisor E. Let H be the infinite section in $\mathbf{P}(\mathcal{O}(-1) \oplus \mathcal{O})$. Since $H \cdot [u_i^+] = \sum k_j$, where summation runs over the ends of u_i^+. So we have $a = \sum k_j$. Since $(p!(A) + re) \cdot E = -r$, then $E \cdot [u_i^+] = 2a - b = -r$. Therefore, $b = 2\sum k_j + r$, i. e. $[u_i^+] = \sum k_j L + (\sum k_j + r)e$. A simple index calculation shows

$$\begin{aligned} C_1[u_i^+] &= ((n+1)H - (n-1)E) \cdot [u_i^+] \\ &= ((n+1)H - (n-1)E)(\sum k_j L + (\sum k_j + r)e) \\ &= (n+1)\sum k_j + r(n-1). \end{aligned}$$

Therefore, we have

$$\begin{aligned}
IndD_{u^+} &= 2(n+1)\sum k_i + 2r(n-1)l^+ + 2(n-3)l^+ + 2\nu - 2\sum k_i \\
&= 2n\sum k_i + 2r(n-1)l^+ + 2(n-3)l^+ + 2\nu \\
IndD_{u^-} &= 2C_1(A) + 2r(n-1)(1-l^+) + 2(n-3)(1-l^+) \\
&\quad -2n\sum k_i - 2(n-2)\nu + 2m^+
\end{aligned}$$

where C_1 is the first Chern class of M.

Without loss of generality, we assume

$$\sum \deg \alpha_i = 2C_1(A) + 2r(n-1) + 2(n-3) + 2m.$$

If $0 < W(\alpha) \leq 2(r+1)(n-1)$, then

$$\begin{aligned}
\sum \deg \alpha_i^- &= 2C_1(A) + 2r(n-1) + 2(n-3) + 2m - W(\alpha) \\
&\geq 2C_1(A) + 2r(n-1) + 2(n-3) + 2m - 2(r+1)(n-1) \\
&= 2C_1(A) + 2r(n-1) + 2(n-3) + 2m \\
&\quad -[2r(n-1) + 2(n-3) + 4] \\
&\geq 2C_1(A) + 2r(n-1) + 2(n-3) + 2m \\
&\quad -2r(n-1)l^+ - 2(n-3)l^+ - 2(n-2)(\sum k_i - \nu) - 4\sum k_i \\
&> 2C_1(A) + 2r(n-1)(1-l^+) + 2(n-3)(1-l^+) \\
&\quad -2n\sum k_i - 2(n-2)\nu + 2m^- \\
&= IndD_{u^-}.
\end{aligned}$$

Therefore, for any $\beta_b \in H^*(Z)$

$$\Psi^{(\overline{\tilde{M}}^-, Z)}_{((p!(A)+re)^-, K^-)}(\alpha_{m^++1}, \cdots, \alpha^-_m, \beta_b) = 0.$$

By the gluing formula (Proposition 2.2), we proved the theorem.

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Email: stsjxhu@zsu.edu.cn