THE MODULI SPACE OF FLAT CONNECTIONS ON SURFACES WITH BOUNDARY

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ABSTRACT. A symplectic form is described for the space of connections on a surface with boundary, with boundary holonomies in given conjugacy classes. The induced symplectic structure on the moduli space of flat connections is studied.

1. INTRODUCTION

In this report we present certain results concerning a symplectic structure on the moduli space of flat connections on compact oriented surfaces with boundary. Some of the results are proven here; detailed proofs of the other results are in a longer article under preparation ([21]).

We work with a compact, connected, oriented 2-manifold Σ with boundary components $\partial_1\Sigma, ..., \partial_r\Sigma$, and a Lie group G. We assume that G is compact, connected, semisimple, and we choose an Ad-invariant metric $\langle\cdot,\cdot\rangle_{\underline{g}}$ on its Lie algebra $\underline{g}$ (several of the results are valid under weaker hypotheses on G). Let Θ_i, for $i = 1, ..., r$, be conjugacy classes in G. Thus $\Theta = \Theta_1 \times \cdots \times \Theta_r$ is a conjugacy class in G^r. Let $\mathcal{A}(\Theta)$ be the set of all connections on a principal G–bundle over Σ having holonomy around $\partial_i\Sigma$ in Θ_i, for each $i = 1, ..., r$. Denote by $\mathcal{G}$ the group of gauge transformations, and by $\mathcal{A}^0(\Theta)$ the subset of $\mathcal{A}(\Theta)$ consisting of flat connections. We have then the moduli space $\mathcal{M}^0(\Theta) = \mathcal{A}^0(\Theta)/\mathcal{G}$ of flat connections with boundary holonomies in Θ. In this article we

- construct a 2–form Ω_Θ on $\mathcal{A}(\Theta)$ and demonstrate that it is symplectic
- show that the curvature function is the moment map of the action of $\mathcal{G}$ on $\mathcal{A}(\Theta)$
- discuss the symplectic form induced by Ω_Θ on a (subspace of) the moduli space of flat connections
- describe the quantum Yang-Mills measure and give explicit formulas for loop expectation values

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- describe the relationship between the symplectic volume measure on the moduli $\mathcal{M}^0(\Theta)$ and the classical limit of the quantum Yang-Mills measure.

The moduli space of flat connections over surfaces with boundary/punctures has been studied from several points of view in the papers [2],[3], [8],[5],[6], [7],[9],[12, 13], [14, 15], [17, 18], [22, 23]. (This is only a sample of the literature, not an exhaustive list. I thank J. Huebschmann for bringing the papers [14, 15] to my attention). Interest in the moduli space of flat connections arises also from theoretical physics where it appears in Chern-Simons theory and (2+1)–dimensional gravity.

2. A 2-form for connections over a circle

Let $\mathcal{A}_C$ be the space of connections on a principal G–bundle $\pi_C : P_C \to C$ over an oriented circle C. Denote by $\mathcal{G}_C$ the set of all smooth bundle equivalences of π_C. Then $\mathcal{G}_C$ is a group under composition and acts on $\mathcal{A}_C$ by pullbacks.

We shall construct two 2–forms Ω^1 and Ω^2 on $\mathcal{A}_C$ which are $\mathcal{G}_C$–invariant. We shall define Ω_Θ^C to be the sum $\Omega^1 + \Omega^2$.

Let $\omega \in \mathcal{A}_C$ and $A, B \in T_\omega\mathcal{A}_C$. Thus A, B are G–equivariant $\underline{g}$–valued 1–forms on P which vanish on vertical vectors.

Let Ω^1 be the 2–form on $\mathcal{A}_C$ given by:

$$\Omega^1_\omega(A, B) \stackrel{\text{def}}{=} \frac{1}{2}\int_0^1\int_0^1 \epsilon_{st}\langle A\big(\tilde{c}'_\omega(s)\big), B\big(\tilde{c}'_\omega(t)\big)\rangle_{\underline{g}}\, dsdt \tag{2.1}$$

where

$$\epsilon_{st} = \begin{cases} 1 & \text{if } s \le t \\ -1 & \text{if } s > t \end{cases}$$

and $\tilde{c}_\omega : [0, 1] \to P_C$ is any ω–horizontal path whose projection $\pi_C \circ \tilde{c}_\omega : [0, 1] \to C$ traces out the circle C once positively. Since $\tilde{c}_\omega$ is unique up to global shifts by elements of G, the equivariance of A, B and the Ad-invariance of $\langle\cdot,\cdot\rangle_{\underline{g}}$ implies that $\Omega^1_\omega(A, B)$ is independent of the choice of $\tilde{c}_\omega$. Moreover, $\Omega^1_\omega(A, B)$ is invariant under smooth reparametrizations of C. It is also readily verified that Ω^1 is $\mathcal{G}_C$–invariant.

We recall quickly some standard observations about bundles. If $\phi \in \mathcal{G}_C$, and $p \in P_C$, define $\hat{\phi}(p)$ to be the element of G for which $\phi(p) = p\hat{\phi}(p)$; then it is a readily verified standard fact that $\hat{\phi}$ is a smooth map $P_C \to G$ satisfying $\hat{\phi}(pg) = g^{-1}\hat{\phi}(p)g$ for all $(p, g) \in P_C \times G$, and the association $\phi \mapsto \hat{\phi}$ is an isomorphism which identifies $\mathcal{G}_C$ with the group (under pointwise multiplication) of all smooth maps $\psi : P_C \to G$ which satisfy $\psi(pg) = g^{-1}\psi(p)g$ for all $p \in P_C$ and $g \in G$. The action of $\mathcal{G}_C$ on $\mathcal{A}_C$, in this version, takes the form: $(\psi, \omega) \mapsto \mathrm{Ad}\,(\psi^{-1})\omega + \psi^{-1}d\psi$.

The Lie algebra $L\mathcal{G}_C$ can then be identified with the set of smooth $\mathrm{Ad}\,(G)$–equivariant maps $P_C \to \underline{g}$, which is indeed a Lie algebra under pointwise operations. If $H \in L\mathcal{G}_C$ then (see also (3.4) below) there is induced a "vector field" $\tilde{H}$ on $\mathcal{A}_C$ whose value at $\omega \in \mathcal{A}_C$ is $\tilde{H}_\omega \in T_\omega\mathcal{A}_C$ defined by

$$\begin{aligned}\tilde{H}_\omega(p) &= \frac{d}{dt}\Big|_{t=0}\left[\mathrm{Ad}\left(\exp(-tH(p)\right)\omega_p + \exp(-tH(p))d\exp(tH(p))\right]\\ &= dH_p + [\omega_p, H_p]\end{aligned}$$

i.e.

$$\tilde{H}_\omega = D^\omega H \stackrel{\text{def}}{=} dH + [\omega, H] \tag{2.2}$$

Lemma 2.1. *For any $\omega \in \mathcal{A}_C$, $A \in T_\omega\mathcal{A}_C$ and $H \in L\mathcal{G}_C$,*

(2.3)
$$\Omega^1_\omega(A, \tilde{H}(\omega)) = -\int_0^1 \langle A\big(\tilde{c}'_\omega(u), H\big(\tilde{c}_\omega(u)\big)\rangle_{\underline{g}}\, du - \frac{1}{2}\langle\alpha(1), \big(Ad\,(c_\omega^{-1}) + 1\big)H(\tilde{c}_\omega(0))\rangle_{\underline{g}}$$

where

$$\alpha(1) = -\int_0^1 A\big(\tilde{c}'_\omega(t)\big)\, dt \tag{2.4}$$

and $\tilde{c}_\omega$ is a ω–horizontal curve projecting to c as before.

Proof. Let

$$\alpha(t) = -\int_0^t A\big(\tilde{c}'_\omega(u)\big)\, du$$

and

$$\beta(t) = -\int_0^t B\big(\tilde{c}'_\omega(u)\big)\, du$$

where $B = \tilde{H}(\omega)$. Then

$$\begin{aligned}\Omega^1_\omega(A, B) &= \frac{1}{2}\left[\int_0^1 \langle\alpha'(s), \beta(1) - \beta(s)\rangle_{\underline{g}}\, ds - \int_0^1 \langle\alpha'(s), \beta(s)\rangle_{\underline{g}}\, ds\right]\\ &= -\int_0^1 \langle\alpha'(s), \beta(s)\rangle_{\underline{g}}\, ds + \frac{1}{2}\langle\alpha(1), \beta(1)\rangle_{\underline{g}}\end{aligned}$$

Now since $\tilde{c}_\omega$ is ω–horizontal, $B = \tilde{H}$ and equation (2.2) imply that

$$B\big(\tilde{c}'_\omega(t)\big) = \big(H \circ \tilde{c}_\omega\big)'(t)$$

and so

$$\beta(t) = H(\tilde{c}_\omega(0)) - H(\tilde{c}_\omega(t)) \tag{2.5}$$

So

$$\begin{aligned}
\Omega^1_\omega(A,B) &= \int_0^1 \langle \alpha'(s), H\big(\tilde{c}_\omega(s)\big)\rangle_{\underline{g}}\, ds - \langle \alpha(1), H\big(\tilde{c}_\omega(0)\big)\rangle_{\underline{g}} + \frac{1}{2}\langle \alpha(1), H\big(\tilde{c}_\omega(0)\big) - H\big(\tilde{c}_\omega(1)\big)\rangle_{\underline{g}} \\
&= -\int_0^1 \langle A\big(\tilde{c}'_\omega(u)\big), H\big(\tilde{c}'_\omega(u)\big)\rangle_{\underline{g}}\, du - \frac{1}{2}\langle \alpha(1), H\big(\tilde{c}_\omega(0)\big) + H\big(\tilde{c}_\omega(1)\big)\rangle_{\underline{g}} \\
&= -\int_0^1 \langle A\big(\tilde{c}'_\omega(u)\big), H\big(\tilde{c}'_\omega(u)\big)\rangle_{\underline{g}}\, du - \frac{1}{2}\langle \alpha(1), \big(\mathrm{Ad}\,(c_\omega^{-1}) + 1\big) H\big(\tilde{c}_\omega(0)\big)\rangle_{\underline{g}}
\end{aligned}$$

where $c_\omega \in G$ is the holonomy of ω around C as given by $\tilde{c}_\omega(1) = \tilde{c}_\omega(0)c_\omega$. □

It is useful to recall a standard formula ([19, Equation (3.6g)]) concerning variation of parallel-transports. Let $c : [0,1] \to \Sigma$ be a piecewise smooth path, and fix an initial point u on the fiber over $c(0)$. If η is any connection on $\pi : P \to \Sigma$ denote by $\tilde{c}_\eta$ the η–horizontal path, initiating at u, and projecting by π onto c. Let $s \mapsto \omega_s \in \mathcal{A}$ be such that $(s,p) \mapsto \omega_s(p)$ is smooth, with $\omega_0 = \omega$ and let $Y = \frac{d}{ds}|_{s=0}\omega_s$. Define $h(s,t) \in G$ by requiring that $\tilde{c}_{\omega_s}(t) = \tilde{c}_\omega(t)h(s,t)$. Then

$$\frac{\partial}{\partial s}\Big|_{s=0} h(s,t) = -\int_{\tilde{c}_\omega|[0,t]} Y \tag{2.6}$$

Let $h(C;\omega)$ denote the holonomy (with initial point u) of the connection ω around a loop C. Then, as a special case of the above formmula we have:

$$h(C;\omega)^{-1}\frac{d}{ds}\Big|_{s=0} h(C;\omega_s) = -\int_{\tilde{C}_\omega} Y \tag{2.7}$$

Now consider another 2–form Ω^2_C on $\mathcal{A}_C$: for $A, B \in T_\omega\mathcal{A}_C$, we set

(2.8)

$$\Omega^2_C(A,B) \stackrel{\text{def}}{=} \frac{1}{2}\Big\langle \big(\mathrm{Ad}\,(c_\omega^{-1}) - 1\big)^{-1}\alpha(1), \big(\mathrm{Ad}\,(c_\omega^{-1}) - \mathrm{Ad}\,(c_\omega)\big)\big(\mathrm{Ad}\,(c_\omega^{-1}) - 1\big)^{-1}\beta(1)\Big\rangle_{\underline{g}}$$

where $\tilde{c}_\omega$ is a ω–horizontal curve projecting to c, $c_\omega \in G$ is the holonomy specified by

$$\tilde{c}_\omega(1) = \tilde{c}_\omega(0)c_\omega \tag{2.9}$$

and

$$\alpha(1) = -\int_0^1 A\big(\tilde{c}'_\omega(t)\big)\, dt = \frac{d}{ds}\Big|_{s=0} c_\omega^{-1}c_{\omega_s} \tag{2.10}$$

where $s \mapsto \omega_s \in \mathcal{A}$ is any map, with $\omega_0 = \omega$, such that $(s,p) \mapsto \omega_s(p)$ is smooth and $\frac{d}{ds}|_{s=0}\omega_s(p) = A(p)$, and β is defined similarly for B.

We are using the convention that, for $x \in G$, $\bigl(\mathrm{Ad}\,(x^{-1})-1\bigr)^{-1}$ is 0 on Z_x (the Lie algebra of the centralizer Z_x of x) and so $\bigl(\mathrm{Ad}\,(x^{-1})-1\bigr)^{-1}$ maps $\underline{g}$ onto $LZ_x^{\perp} \subset \underline{g}$.

Lemma 2.2. *If $\omega \in \mathcal{A}_C$ and $H \in L\mathcal{G}_C$, $A \in T_\omega \mathcal{A}_C$,*

$$\Omega_C^2(A,\tilde{H}) = \frac{1}{2}\bigl\langle \alpha(1), \bigl(Ad\, c_\omega^{-1}+1\bigr)\mathrm{pr}H\bigl(\tilde{c}_\omega(0)\bigr)\bigr\rangle_{\underline{g}} \tag{2.11}$$

where $\mathrm{pr}: \underline{g} \to LZ_{c_\omega}^{\perp}$ *is the orthogonal projection.*

Proof. The expression for $\beta(t)$ in (2.5) gives

$$\begin{aligned}\beta(1) &= H\bigl(\tilde{c}_\omega(0)\bigr) - H\bigl(\tilde{c}_\omega(1)\bigr)\\ &= \bigl(1-\mathrm{Ad}\,c_\omega^{-1}\bigr)H\bigl(\tilde{c}_\omega(0)\bigr)\end{aligned}$$

So, from the definition (2.8) of Ω_C^2 we have:

$$\begin{aligned}\Omega_C^2(A,\tilde{H}) &= \frac{1}{2}\Bigl\langle \bigl(\mathrm{Ad}\,c_\omega^{-1}-1\bigr)^{-1}\alpha(1), \bigl(\mathrm{Ad}\,c_\omega^{-1}-\mathrm{Ad}\,c_\omega\bigr)\bigl(\mathrm{Ad}\,c_\omega^{-1}-1\bigr)^{-1}\bigl(1-\mathrm{Ad}\,c_\omega^{-1}\bigr)H\bigl(\tilde{c}_\omega(0)\bigr)\Bigr\rangle_{\underline{g}}\\ &= \frac{1}{2}\Bigl\langle \bigl(\mathrm{Ad}\,c_\omega^{-1}-1\bigr)^{-1}\alpha(1), \bigl(\mathrm{Ad}\,c_\omega-1\bigr)\bigl(1+\mathrm{Ad}\,c_\omega^{-1}\bigr)\mathrm{pr}H\bigl(\tilde{c}_\omega(0)\bigr)\Bigr\rangle_{\underline{g}}\\ &= \frac{1}{2}\bigl\langle \alpha(1), \bigl(\mathrm{Ad}\,c_\omega^{-1}+1\bigr)\mathrm{pr}H\bigl(\tilde{c}_\omega(0)\bigr)\bigr\rangle_{\underline{g}}\end{aligned}$$

□

Let Θ be a conjugacy class in G and let $\mathcal{A}_C(\Theta)$ be the set of all $\omega \in \mathcal{A}$ whose holonomy around C is in Θ:

$$\mathcal{A}_C(\Theta) \stackrel{\text{def}}{=} \{\omega \in \mathcal{A}_C : \text{ holonomy of } \omega \text{ around } C \text{ is in } \Theta\} \tag{2.12}$$

Let $T_\omega\mathcal{A}_C(\Theta)$ be the set of all $A \in T_\omega\mathcal{A}_C$ such that there is a map $t \mapsto \omega_t \in \mathcal{A}_C(\Theta)$, with $\omega_0 = \omega$, and $(t,p) \mapsto \omega_t(p)$ smooth, satisfying $\frac{d}{dt}\big|_{t=0}\omega_t = A$ pointwise.

We introduce the 2–form Ω_C^Θ on $\mathcal{A}_C(\Theta)$ given by

$$\Omega_C^\Theta \stackrel{\text{def}}{=} \Omega_C^1 + \Omega_C^2 \tag{2.13}$$

Proposition 2.3. *If $\omega \in \mathcal{A}_C(\Theta)$, $A \in T_\omega\mathcal{A}_C(\Theta)$, and $H \in L\mathcal{G}_C$, then*

$$\Omega_C^\Theta(A,\tilde{H}_\omega) = -\int_0^1 \bigl\langle A\bigl(\tilde{c}'_\omega(t)\bigr), H\bigl(\tilde{c}_\omega(t)\bigr)\bigr\rangle_{\underline{g}}\, dt \tag{2.14}$$

Proof. In view of Lemmas 2.2 and 2.1, we need only check that $\alpha(1) \in LZ_{c_\omega}^{\perp}$, where $\alpha(t) = -\int_0^t A\bigl(\tilde{c}'_\omega(s)\bigr)\,ds$. Now by equation (2.10),

$$\alpha(1) = \frac{d}{ds}\Big|_{s=0} c_\omega^{-1}c_{\omega_s}$$

where $\frac{d}{ds}|_{s=0}\omega_s = A$. Since $A \in T_\omega \mathcal{A}_C(\Theta)$, we can take $\omega_s \in \mathcal{A}_C(\Theta)$. Then each c_{ω_s} belongs to Θ. Thus $\alpha(1) \in (\mathrm{Ad}\, c_\omega^{-1} - 1)\,(\underline{g}) = LZ_{c_\omega}^{\perp}$. $\square$

For a 2–form η on $\mathcal{A}_C$ define the exterior derivative $d\eta$ to be the 3–form:

$$d\eta_\omega(A,B,C) = \delta_C\big[\eta(A,B)\big] + \text{cyclic permutation} \tag{2.15}$$

where, on the right, δ_C is the "directional derivative" of functions on $\mathcal{A}_C$. The following Lemma is essentially taken from [12, Remark 4.7]. We use the same letter C with two obviously different meanings.

Lemma 2.4. *Let $\omega \in \mathcal{A}_C$, and $A, B, C \in T_\omega\mathcal{A}_C$. Then*

$$d\Omega_C^1(A,B,C) = \frac{1}{2}\,\langle[\alpha(1),\beta(1)],\gamma(1)\rangle_{\underline{g}} \tag{2.16}$$

where $\alpha(1) = -\int_0^1 A\big(\tilde{c}_\omega'(t)\big)\,dt$ and similarly for β, B and γ, C.

Proof. Let $\alpha(t) = -\int_0^t A\big(\tilde{c}_\omega'(s)\big)\,ds$, and define $\beta(t)$ and $\gamma(t)$ similarly with respect to B and C, respectively. Then computation shows that

$$\delta_C\big(\alpha'(t)\big) = [\alpha'(t),\gamma(t)]$$

(proof: Let $s \mapsto \omega_s$ be such that $(s,p) \mapsto \omega_s(p)$ is smooth and $\frac{d}{ds}|_{s=0}\omega_s = C$, and let $h(s,t) \in G$ be such that $\tilde{c}_{\omega_s}(t) = \tilde{c}_\omega(t)h(s,t)$. Since A vanishes on vertical vectors, i.e. vectors tangent to fibers of the bundle π, and $A(vg) = \mathrm{Ad}\,(g^{-1})A(v)$ in obvious notation, it follows that $A(\tilde{c}_{\omega_s}'(t)) = \mathrm{Ad}\,\big(h(s,t)^{-1}\big)A(\tilde{c}_\omega'(t))$. Differentiating this with respect to s at $s=0$ gives, upon using (2.6), $[A(\tilde{c}_\omega'(t)),\gamma(t)]$.)

So

$$\begin{aligned}
\delta_C[\Omega_\omega^1(A,B)] &= \frac{1}{2}\int_0^1\int_0^1 \epsilon_{st}\langle[\alpha'(s),\beta'(t)],\gamma(t)-\gamma(s)\rangle_{\underline{g}}\,dsdt \\
&= \frac{1}{2}\int_0^1\int_0^1\int_s^t \epsilon_{st}\langle[\alpha'(s),\beta'(t)],\gamma'(u)\rangle_{\underline{g}}\,dudsdt \\
&= \frac{1}{2}\int_{0\le s\le u\le t\le 1}\langle[\alpha'(s),\beta'(t)],\gamma'(u)\rangle_{\underline{g}}\,dudsdt + \frac{1}{2}\int_{0\le t\le u\le s\le 1}[\cdots]
\end{aligned}$$

Adding up the cyclic permutations we obtain equation (2.16). $\square$

If Θ is a conjugacy class in G, let ν be the 2–form on Θ given by

$$\nu(cA,cB) \overset{\text{def}}{=} \frac{1}{2}\langle\big(\mathrm{Ad}\,c^{-1}-1\big)^{-1}A,\big(\mathrm{Ad}\,c^{-1}-\mathrm{Ad}\,c\big)\big(\mathrm{Ad}\,c^{-1}-1\big)^{-1}B\rangle_{\underline{g}} \tag{2.17}$$

Then a fairly straightforward calculation ([13, Proposition 3.9]) shows that

$$d\nu(cA, cB, cC) = -\frac{1}{2}\langle [A,B], C\rangle_{\underline{g}} \tag{2.18}$$

Consider the map

$$h : \mathcal{A}_C(\Theta) \to G : \omega \mapsto h(\omega) = \text{holonomy of } \omega \text{ around } C \tag{2.19}$$

with some chosen fixed initial point for determining holonomy. Then

$$h(\omega)^{-1} dh_\omega(A) = -\int_0^1 A\big(\tilde{c}'_\omega(t)\big)\, dt = \alpha(1) \tag{2.20}$$

Combining this with the definition of ν and the expression for Ω_C^2 in (2.8) we conclude that

$$h^*\nu = \Omega_C^2 | \mathcal{A}(\Theta) \tag{2.21}$$

So it is reasonable to believe that $d\big(\Omega_C^2|\mathcal{A}(\Theta)\big) = h^* d\nu$, i.e.

$$d\Omega_C^2(A,B,C) = -\frac{1}{2}\langle [\alpha(1), \beta(1)], \gamma(1)\rangle_{\underline{g}} \tag{2.22}$$

when $A, B, C \in T_\omega \mathcal{A}_C(\Theta)$.

Recall that

$$\Omega_C^\Theta = (\Omega_C^1 + \Omega_C^2)|\mathcal{A}_C(\Theta) \tag{2.23}$$

So, by (2.16) and (2.22) we have

$$d\Omega_C^\Theta = 0 \tag{2.24}$$

3. The symplectic structure for the space $\mathcal{A}(\Theta)$

We return now to the bundle $\pi : P \to \Sigma$. We have the space $\mathcal{A}$ of all connections on π and the group $\mathcal{G}$ of all automorphisms of π, acting on $\mathcal{A}$ by pull backs.

There is a standard 2–form Ω on $\mathcal{A}$, introduced by Atiyah and Bott in [1], defined as follows. If $A, B \in T_\omega\mathcal{A}$, we can form a well-defined 2–form $\langle A \wedge B\rangle$ on Σ: its value on vectors $X, Y \in T_x\Sigma$ is

$$\langle A \wedge B\rangle(X,Y) = \langle A(X'), B(Y')\rangle_{\underline{g}} - \langle A(Y'), B(X')\rangle_{\underline{g}}$$

where X' and Y' are vectors in $T_{x'}P$ for some point $x' \in \pi^{-1}(x)$ such that $\pi_* X' = X$ and $\pi_* Y' = Y$. The 2–form Ω is specified by the formula

$$\Omega(A,B) = \int_\Sigma \langle A \wedge B\rangle$$

For any conjugacy class $\Theta = \Theta_1 \times \cdots \Theta_r$ in G^r, there is a restriction map

$$\rho_i : \mathcal{A}(\Theta) \to \mathcal{A}_{\partial_i \Sigma}(\Theta_i)$$

We can finally define the two-form Ω_Θ on $\mathcal{A}(\Theta)$:

$$\Omega_\Theta \stackrel{\text{def}}{=} \Omega - \sum_{i=1}^{r} \rho_i^* \Omega_{\partial_i \Sigma}^{\Theta_i} \tag{3.1}$$

Since Ω is a constant 2–form on $\mathcal{A}$, we have $d\Omega = 0$. On the other hand, we know from (2.23) that $d\Omega_{\partial_i \Sigma}^{\Theta_i}$ is 0. So it is reasonable to conclude that

$$d\Omega_\Theta = 0, \text{ i.e. } \Omega_\Theta \text{ is closed.}$$

Next consider any non-zero $A \in T_\omega \mathcal{A}(\Theta)$. Define a $\underline{g}$–valued 1–form $*A$ on P as follows. For any $p \in P$ choose $e_1, e_2 \in T_p P$ which project to a positively oriented orthonormal basis $(\pi_* e_1, \pi_* e_2)$ of $T_{\pi(p)}\Sigma$; define the 1–form $*A$ at p by requiring that it vanish on the kernel of π_* and satisfy $*A(e_2) = A(e_1)$ and $*A(e_1) = -A(e_1)$. Now let $\phi : \Sigma \to [0, 1]$ be a smooth function, equal to zero near $\partial\Sigma$ and equal to 1 in a non-empty open set over which A is not identically equal to zero. Let $\psi = \phi \circ \pi$. For any real number t, the connection $\omega + t\psi(*A)$ is equal to ω over $\partial\Sigma$, and so $\psi(*A) \in T_\omega \mathcal{A}(\Theta)$. Moreover, $\Omega_\Theta(A, \psi(*A)) = \Omega(A, \psi(*A))$ is strictly positive. We conclude that

$$\Omega_\Theta \text{ is non-degenerate.}$$

So

$$\Omega_\Theta \text{ is symplectic on } \mathcal{A}(\Theta)$$

The action of the group $\mathcal{G}$ on $\mathcal{A}(\Theta)$ preserves the 2–form Ω_Θ. We shall determine the moment map of this action.

We can view $L\mathcal{G}^*$ as the space of smooth $\underline{g}$–valued Ad-equivariant 2–forms on P which vanish on vertical vectors. The pairing of $L\mathcal{G}$ and $L\mathcal{G}^*$ is by integration over Σ.

Consider the curvature map

$$J : \mathcal{A} \to L\mathcal{G}^* : \omega \mapsto F^\omega \stackrel{\text{def}}{=} d\omega + \frac{1}{2}[\omega, \omega] \tag{3.2}$$

where $[\omega, \omega]$ is the $\underline{g}$–valued two-form whose value on (X, Y) is $[\omega(X), \omega(Y)] - [\omega(Y), \omega(X)] = 2[\omega(X), \omega(Y)]$. We take its derivative to be

$$dJ_\omega : T_\omega \mathcal{A} \to L\mathcal{G}^* : A \mapsto D^\omega A \stackrel{\text{def}}{=} dA + [\omega, A] \tag{3.3}$$

Let

$$\gamma_\omega : \mathcal{G} \to \mathcal{A} : \phi \mapsto \phi^* \omega$$

be the orbit map through $\omega \in \mathcal{A}$. Then, as essentially already noted in (2.2),

$$\gamma'_\omega H = dH + [\omega, H] \tag{3.4}$$

Proposition 3.1. *Let $\omega \in \mathcal{A}(\Theta)$. Then for any $X \in T_\omega \mathcal{A}(\Theta)$ and $H \in L(\mathcal{G})$*

$$\Omega_\Theta(A, \gamma'_\omega H) = {}_{L\mathcal{G}^*}\langle dJ_\omega(A), H\rangle_{L\mathcal{G}} \tag{3.5}$$

In particular, if $\omega \in \mathcal{A}^0(\Theta)$ and $A \in T_\omega \mathcal{A}^0(\Theta)$ then $\Omega_\Theta(A, \gamma'_\omega(H))$ is 0.

Proof. Since $\partial\Sigma \neq \emptyset$, the bundle P is trivial. So, using a global section, we can take $\mathcal{A}$ to be the space of all smooth $\underline{g}$–valued 1–forms over Σ. Then

$$\begin{aligned}
\Omega(A, \gamma'_\omega H) &= \int_\Sigma \langle A \wedge \big(dH + [\omega, H]\big)\rangle \\
&= \int_\Sigma \{\langle dA, H\rangle - d\langle A, H\rangle + \langle[\omega, A] \wedge H\rangle\} \\
&\overset{\text{Stokes}}{=} \int_\Sigma \langle D^\omega A, H\rangle - \int_{\partial\Sigma} \langle A, H\rangle
\end{aligned}$$

On the other hand, by Proposition 2.3,

$$\sum_{i=1}^r \rho_i^* \Omega^{\Theta_i}_{\partial_i\Sigma}(A, \gamma'_\omega H) = -\int_{\partial\Sigma} \langle A, H\rangle$$

So

$$\Omega_\Theta(A, \gamma'_\omega H) = \int_\Sigma \langle D^\omega A, H\rangle = {}_{L\mathcal{G}^*}\langle dJ_\omega(A), H\rangle_{L\mathcal{G}}$$

□

Thus the quotient $J^{-1}(0)/\mathcal{G}$ is the moduli space of flat connections $\mathcal{A}^0(\Theta)/\mathcal{G}$, where $\mathcal{A}^0(\Theta) = \{\omega \in \mathcal{A}(\Theta) : \omega \text{ is flat}\}$.

4. The symplectic structure for the moduli space of flat connections

There are piecewise smooth loops

$$A_1, B_1, ..., A_g, B_g, C_1, ..., C_r$$

in Σ, all based at some point $o \in \Sigma$, such that they generate $\pi_1(\Sigma, o)$ subject to the relation

$$C_r \dots C_1 \bar{B}_g \bar{A}_g B_g A_g \dots \bar{B}_1 \bar{A}_1 B_1 A_1 = I, \text{the identity in } \pi_1(\Sigma, o)$$

Each C_i is of the form $\bar{L}_i \cdot \partial_i\Sigma \cdot L_i$ for some segment L_i running from o to $\partial_i\Sigma$. Consider the holonomy map

(4.1)
$$\mathcal{H} : \mathcal{A} \to G^{2g+r} : \omega \mapsto \big(h(A_1;\omega), h(B_1;\omega), ..., h(A_g;\omega), h(B_g;\omega), h(C_1;\omega), ..., h(C_r;\omega)\big)$$

It takes $\mathcal{A}(\Theta)$ to $G^{2g} \times \Theta$, and $\mathcal{A}^0(\Theta)$ to $\Pi^{-1}(e) \subset G^{2g} \times \Theta$, where $\Pi : G^{2g} \times \Theta \to G$ is given by:

$$\Pi(a_1, b_1, ..., a_g, b_g, c_1, ..., c_r) = c_r \ldots c_1 b_g^{-1} a_g^{-1} b_g a_g \ldots b_1^{-1} a_1^{-1} b_1 a_1 \tag{4.2}$$

The group $\mathcal{G}$ acts on $\mathcal{A}^0(\Theta)$ and the group G acts on $\Pi^{-1}(e)$. We have the homomorphism $\mathcal{G} \to G : \phi \mapsto \tilde{\phi}(u)$, where $\tilde{\phi}(u) \in G$ is given by $\phi(u) = u\tilde{\phi}(u)$.

Theorem 4.1. *The map $\mathcal{H}$ has the following properties:*
1. *$\mathcal{H} : \mathcal{A}^0(\Theta) \to \Pi^{-1}(e)$ is surjective and $\mathcal{G} - G$–equivariant*
2. *$\mathcal{H}$ induces a bijection $\mathcal{A}^0(\Theta)/\mathcal{G} \to \Pi^{-1}(e)/G$*
3. *$\mathcal{H}'(\omega)A$ is tangent to the G–orbit through $\mathcal{H}(\omega)$ if and only $A \in \gamma'_\omega(\mathcal{G})$*
4. *if $\tilde{A}$ is tangent to a smooth path in $\Pi^{-1}(e)$ at $\mathcal{H}(\omega)$ then $\tilde{A} = \mathcal{H}'(\omega)A$ for some $A \in T_\omega\mathcal{A}(\Theta)$.*

The proof is along the lines of Corollary 4.2(a), (b) in [19] (where the case $\partial\Sigma = \emptyset$ is covered).

As a consequence we have:

Corollary 4.2. *There is a unique 2–form $\tilde{\Omega}_{o,\Theta}$ on $\Pi^{-1}(e) \subset G^{2g} \times \Theta$ such that*

$$\mathcal{H}^*\tilde{\Omega}_{o,\Theta} = \Omega_\Theta \tag{4.3}$$

The space $\Pi^{-1}(e)$ is, in general, not a submanifold of $G^{2g} \times \Theta$. In the corollary above, by 2-form we understand (for the moment) only a function on pairs of vectors tangent (at a common point) to smooth paths in $G^{2g} \times \Theta$ lying entirely on $\Pi^{-1}(e)$.

5. Explicit Formula for $\tilde{\Omega}_{o,\Theta}$

We shall write a typical element $\alpha = G^{2g+r}$ as $\alpha = (\{\alpha_j\}_{j\in J}, \alpha_{4g+1}, ..., \alpha_{4g+r})$, where J is the indexing set

$$J \stackrel{\text{def}}{=} \{1, 2, 5, 6, ..., 4g-3, 4g-2\}$$

Explicitly,

$$\alpha = (\alpha_1, \alpha_2, \alpha_5, \alpha_6, ..., \alpha_{4g-3}, \alpha_{4g-2}, \alpha_{4g+1}, ..., \alpha_{4g+r})$$

Then we set

$$\alpha_3 = \alpha_1^{-1}, \alpha_4 = \alpha_2^{-1}, ..., \alpha_{4g-1} = \alpha_{4g-3}^{-1}, \alpha_{4g} = \alpha_{4g-2}^{-1}$$

i.e. we set

$$\alpha_{j+2} \stackrel{\text{def}}{=} \alpha_j^{-1} \qquad \text{for all } j \in \mathrm{J}$$

Next, we write the components of a tangent vector $\alpha H \in T_\alpha(G^{2g} \times G^r)$ as $\alpha_j H_j$, with j running over $J \cup \{4g+1, ..., 4g+r\}$. Then we set

$$H_{j+2} = -\mathrm{Ad}\,(\alpha_j) H_j$$

for $j \in J$.

Theorem 5.1. *Let $\alpha \in \Pi^{-1}(e) \subset G^{2g} \times \Theta$ and $\alpha H^{(1)}, \alpha H^{(2)} \in T_\alpha(G^{2g} \times \Theta)$ two vectors tangent, at α, to smooth paths in $G^{2g} \times \Theta$ lying entirely on $\Pi^{-1}(e)$. Then*

$$\begin{aligned}
&\tilde{\Omega}_{o,\Theta}(\alpha H^{(1)}, \alpha H^{(2)}) \\
&\qquad = \frac{1}{2} \sum_{i,j=1}^{4g+r} \epsilon_{ij} \langle f_{i-1}^{-1} H_i^{(1)}, f_{j-1}^{-1} H_j^{(2)} \rangle_{\underline{g}} \\
&\qquad + \frac{1}{2} \sum_{k=1}^{r} \langle \big(Ad\, c_k^{-1} - 1\big)^{-1} C_k^{(1)}, \big(Ad\, c_k - Ad\, c_k^{-1}\big)\big(Ad\, c_k^{-1} - 1\big) C_k^{(2)} \rangle_{\underline{g}}
\end{aligned}$$

where we have taken $\alpha = (\{\alpha_j\}_{j \in J}, c_1, ..., c_r)$ and set $C_k^{(i)} = H_{4g+k}^{(i)}$, and

$$\epsilon_{ij} = \begin{cases} 1 & \text{if } i < j \\ 0 & \text{if } i = j \\ -1 & \text{if } i > j \end{cases} \tag{5.1}$$

Sketch Proof. The proof uses the following strategy:

- consider two smooth paths $[0,1] \to G^{2g} \times \Theta : t \mapsto \alpha^{(i)}(t)$, for $i = 1, 2$, lying entirely on $\Pi^{-1}(e)$, which are tangent at $\alpha = \alpha^{(i)}(0)$ to the vectors $\alpha H^{(1)}$ and $\alpha H^{(2)}$, respectively;
- construct two paths $t \mapsto \omega_t^{(i)}$, with $i = 1, 2$, in $\mathcal{A}^0(\Theta)$ such that $\mathcal{H}(\omega_t^{(i)}) = \alpha_t^{(i)}$;
- evaluate $\Omega_\Theta(A^1, A^2)$, where A^i is the vector in $T_\omega \mathcal{A}$ tangent to the path $t \mapsto \omega_t^{(i)}$ at $t = 0$.

For the second step mentioned above, the construction of flat connections with specified holonomies is described in detail for the case of surfaces with no boundary in [19]. □

6. Symplectics and Volume of the Moduli Space

Recall from (4.2) the map $\Pi : G^{2g} \times \Theta \to G$. Using the explicit formula for $\tilde{\Omega}_{o,\Theta}$ mentioned above, we can prove:

Theorem 6.1. *Let Θ be a conjugacy class in G^r such that:* (i) *e is a regular value of $\Pi : G^{2g} \times \Theta \to G$ and* (ii) *there is a non-empty open subset $\Pi^{-1}(e)^0$ of $\Pi^{-1}(e)$ such that the isotropy group of the conjugacy action of G on $\Pi^{-1}(e)$ is $Z(G)$ at every point of $\Pi^{-1}(e)^0$. Then on the manifold $\Pi^{-1}(e)^0/G$:*

- *$\tilde{\Omega}_{o,\Theta}$ induces a 2–form $\overline{\Omega}_\Theta$ on $\mathcal{M}^0(\Theta)^0 = \Pi^{-1}(e)^0/G$*
- *the 2–form $\overline{\Omega}_\Theta$ is symplectic*

Sketch Proof. The fact that $\tilde{\Omega}_{o,\Theta}$ induces a 2–form on $\Pi^{-1}(e)^0/G$ is proven by checking directly that $\tilde{\Omega}_{o,\Theta}(A,B)$ is 0 whenever A (or B) is tangent to the G–orbit.

The fact that $\overline{\Omega}_\Theta$ is *closed* follows by straightforward calculation: we calculate $d\tilde{\Omega}_{o,\Theta}(A,B,C)$ on vectors A, B, C tangent to (smooth paths lying entirely on) $\Pi^{-1}(e)$ and this works out to 0 (very similar calculations have been done in [10, 12]).

The fact that $\overline{\Omega}_\Theta$ is *non-degenerate* is proved by computing a determinant which is a non-zero multiple of the determinant of the matrix for $\overline{\Omega}_\Theta$. This determinant turns out to be non-zero (this strategy has been used earlier in [11, 17]). □

Theorem 6.2. *Assume that the genus of the surface Σ is greater than 1. Then there is a set $\mathcal{D}$ of full measure in G^r such that for any conjugacy class Θ containing a point of $\mathcal{D}$ the conditions of Theorem 6.1 hold. Moreover, the volume of $\mathcal{M}(\Theta)^0$ relative to the symplectic structure $\overline{\Omega}_\Theta$ is*

$$\#Z(G)\frac{\mathrm{vol}(G)^{2g+r-2}}{\mathrm{vol}(T)^r}\prod_{i=1}^{r}\sqrt{F_i}\sum_{\alpha}\frac{\chi_\alpha(c_1)\ldots\chi_\alpha(c_r)}{(\dim\alpha)^{2g+r-2}} \tag{6.1}$$

where $(c_1, \ldots, c_r) \in \Theta$, α runs over the irreducible representations of G, T is any maximal torus in G, and $F_i = \det\left(1 - Ad\,c_i^{-1}\right)|L\big(Z(c_i)\big)^\perp$, where $L\big(Z(c_i)\big)^\perp$ is the orthogonal complement of the Lie algebra of the centralizer $Z(c_i)$ of c_i.

The volume formula for $\mathrm{vol}_{\overline{\Omega}_\Theta}\left(\mathcal{M}^0(\Theta)^0\right)$ agrees with that given by Witten in [22, equation (4.116)].

The proof of Theorem 6.1 uses the heat kernel method introduced by Witten.

7. The Quantum Theory and its Classical Limit

In this section we assume that Σ is a compact connected Riemannian 2–manifold with boundary $\partial\Sigma$ having $r \geq 1$ components $\partial_1\Sigma, ..., \partial_r\Sigma$. Note that we are not assuming that Σ is orientable or that G is semisimple. Fix basepoints on each boundary component $\partial_i\Sigma$. Let $\Theta_1, ..., \Theta_r$ be conjugacy classes in G, $\Theta = \Theta_1 \times \cdots \times \Theta_r$, and $\mathcal{A}(\Theta)$ the set of all $\omega \in \mathcal{A}$ for which $\big(h(\partial_1\Sigma;\omega), ..., h(\partial_r\Sigma;\omega)\big) \in \Theta$, where $h(\partial_i\Sigma;\omega)$ is the holonomy of ω around the loop $\partial_i\Sigma$. We are interested in a probability measure μ_t^Θ (here $t > 0$) on $\mathcal{A}(\Theta)$ given by the formal expression

$$d\mu_t^\Theta(\omega) = \frac{1}{Z_t(\Theta)} e^{-\frac{S_{\mathrm{YM}}(\omega)}{t}} \prod_{i=1}^r \delta_{\Theta_i}\big(h(\partial_i\Sigma;\omega)\big)\,[\mathcal{D}\omega] \tag{7.1}$$

where $S_{\mathrm{YM}} : \mathcal{A} \to \mathbf{R} : \omega \mapsto \frac{1}{2}\int_\Sigma |F^\omega|^2_{\underline{g}}\, d\sigma$ (F^ω is the curvature of ω and $d\sigma$ is the Riemannian area-measure on Σ) is the Yang-Mills action, $[\mathcal{D}\omega]$ is the informal "Lebesgue measure" on $\mathcal{A}$ pushed down to $\mathcal{A}/\mathcal{G}$, δ_{Θ_i} a "delta-function", and $Z_t(\Theta)$ a "normalizing constant".

There is a rigorously meaningful measure μ_t^Θ which corresponds, in a reasonable sense, to the expression (7.1).

Let $q : D \to \Sigma$ be the realization of Σ as a quotient of the unit disk D obtained by identifying certain pairs of arcs on ∂D. Let D_L and D_U be, respectively, the lower and upper (closed) halves of D, and let H_L be a Hilbert-Schmidt closure of $L^2(D_L, d\sigma)$ and H_R that of $L^2(D_R, d\sigma)$, where $d\sigma$ is the area measure induced by q from that on Σ.

Let

$$S = H_L \times H_R \times G^k \times \Theta \tag{7.2}$$

where $k = 2g$ (genus of Σ) if Σ is oriented, and if Σ is unorientable then k is specified by the requirement that $\pi_1(\Sigma)$ is generated by loops $A_1, ..., A_k, C_1, ..., C_r$ subject to the relation

$$C_r \cdots C_1 \cdot A_k^2 \cdots A_1^2 = I$$

Then

- there is a probability measure μ_t^Θ defined on the Borel σ–algebra of S which can be constructed using the formal expression (7.1)
- for each "good" loop κ on the surface Σ, based at o, there is a μ_t^Θ–almost-everywhere defined G–valued random variable $\omega \mapsto h(\kappa;\omega)$, which is the extension of the usual notion of holonomy; the random variable $h(\kappa;\omega)$ is defined by interpreting the usual differential equation of parallel transport as a Stratonovich stochastic differential equation (this idea is due to L. Gross)

- let γ be the Gaussian measure on $H_L \times H_R$, and λ the unit-mass measure on $G^k \times \Theta$ invariant under the action of G (translation on G^k and conjugation on Θ); then μ_t^Θ is defined by conditioning the product measure $\gamma \times \lambda$ so that the holonomies around the generating loops of $\pi_1(\Sigma, o)$ satisfy the geometrically imposed relationship between curvature and holonomies. Thus, in particular, the measure μ_t^Θ is concentrated on an infinite-dimensional "submanifold" of S.

This measure has been constructed in [17] for the case $r = 1$, i.e. when $\partial\Sigma$ has one connected component. The construction in the case where $\partial\Sigma$ has more than one component is exactly similar.

A curve on Σ will be called "good" if it is made up of the projection on Σ of radial and cross-radial segments on D. A "good triangulation" of Σ is a triangulation which is obtained by projecting by q onto Σ a triangulation of D consisting of radial and cross-radial segments. Let $o = q(O)$, where O is the center of the disk D.

Let $(t, x) \mapsto Q_t(x)$ be the heat kernel function on G. This is a C^∞–function of $(t, x) \in (0, \infty) \times G$ satisfying

- the heat equation $\frac{\partial Q_t(x)}{\partial t} = \frac{1}{2}\Delta_G Q_t(x)$, where Δ_G is the Laplacian on G,
- and the initial condition $\lim_{t\to 0} \int_G f(x)Q_t(x)\,dx = f(e)$, where e is the identity in G, for every continuous function f on G.

Then we have the following result on expectation values with respect to μ_t^Θ:

Theorem 7.1. *Let $\kappa_1, ..., \kappa_n$ be loops on Σ, all based at o, composed of oriented 1–simplices of a good triangulation $\mathcal{T}$ of Σ. Let f be a bounded measurable function on G^n. Then, for any conjugacy class Θ in G^r,*

$$
\begin{aligned}
&\int f\big(h(\kappa_1;\omega), ..., h(\kappa_n;\omega)\big)\, d\mu_t^\Theta(\omega) \\
&\qquad = \frac{1}{N_t(\Theta)} \int_{G^m\times\Theta} f\big(x(\kappa_1), ..., x(\kappa_n)\big) \prod_{i=1}^{r} \delta\big(x(\partial_i\Sigma)c_i^{-1}\big)\cdot \\
&\qquad\qquad \cdot \prod_{j=1}^{m} Q_{t|\Delta_j|}\big(x(\partial\Delta_j)\big)\, dx_{e_1}...dx_{e_N}\, d_\Theta c \qquad (7.3)
\end{aligned}
$$

where $\{\Delta_1, ..., \Delta_m\}$ are the oriented 2–simplices of the triangulation, $e_1, \bar{e}_1, ..., e_M, \bar{e}_M$ the edges, $|\Delta_j|$ is the area of Δ_j, and $d_\Theta c$ is the unit-mass conjugation-invariant measure on $\Theta \subset G^r$. The notation $x(\kappa)$ means $x(b_n)\ldots x(b_1)$, where $\kappa = b_n \ldots b_1$, with $b_1, ..., b_n$ oriented 1–simplices. The presence of the delta-functions has to be interpreted as follows: for each $\partial_i\Sigma$, for any one particular edge e_i^* on $\partial_i\Sigma$, $x(e_i^*)$

should be replaced by the value which makes $x(\partial_i\Sigma) = c_i$, and the integration over $dx(e_i^*)$ dropped. The normalizing constant $N_t(\Theta)$ is given by

$$N_t(\Theta) = \begin{cases} \int_{G^{2g}\times\Theta} Q_{t|\Sigma|}(c_r \dots c_1 b_g^{-1} a_g^{-1} b_g a_g \dots b_1^{-1} a_1^{-1} b_1 a_1)\, da_1 \dots db_g d_{\Theta_1} c_1 \dots d_{\Theta_r} c_r \\ \text{if } \Sigma \text{ is oriented of genus } g; \\ \int_{G^k\times\Theta} Q_{t|\Sigma|}(c_r \dots c_1 a_k^2 \dots a_1^2)\, da_1 \dots da_k \dots d_{\Theta_1} c_1 \dots d_{\Theta_r} c_r \\ \text{if } \Sigma \text{ is unorientable and } \pi_1(\Sigma) \text{ is generated} \\ \text{by } A_1, \dots, A_k, C_1, \dots, C_r \text{ subject to} \\ C_r \dots C_1 A_k A_k \dots A_1 A_1 = I, \text{ the identity in } \pi_1(\Sigma). \end{cases}$$

Next we have the following result on the limit of μ_t^Θ, as $t \to 0$, for the case Σ is orientable.

Theorem 7.2. *Assume that Σ is oriented of genus > 1. Then there is a dense subset $\mathcal{D}$ of full measure in G^r such that for any conjugacy class Θ in G^r containing a point of $\mathcal{D}$, and for any continuous function f on G^n which is invariant under the conjugation action of G*

$$\begin{aligned} \lim_{t\to 0} \int f\big(h(\kappa_1;\omega), \dots, h(\kappa_n;\omega)\big)\, d\mu_t^\Theta(\omega) &= \\ &\frac{1}{\mathrm{vol}_{\overline{\Omega}_\Theta}\big(\mathcal{M}(\Theta)^0\big)} \int_{\mathcal{M}(\Theta)^0} f\big(h(\kappa_1;\omega), \dots, h(\kappa_n;\omega)\big)\, d\mathrm{vol}_{[\overline{\Omega}_\Theta]}(\omega) \end{aligned} \tag{7.4}$$

The proof is very similar to the case when $\partial\Sigma$ has only one component [17, Theorem 8.3].

8. The discrete Yang-Mills measure and its classical limit

Let $\mathcal{T}$ be any triangulation of the surface Σ. Let $\mathcal{E}_\mathcal{T} = \{e_1, \bar{e}_1, \dots, e_M, \bar{e}_M\}$ be the set of all oriented edges of $\mathcal{T}$ and $\{\Delta_1, \dots, \Delta_m\}$ be the set of positively oriented 2–simplices of Σ. We pick positive numbers $t, |\Delta_1|, \dots, |\Delta_n|$. Denote by $V_\mathcal{T}$ the set of all vertices of $\mathcal{T}$.

Let $\mathcal{A}_\mathcal{T}$ be the set of all maps

$$x : \mathcal{E}_\mathcal{T} \to G$$

such that $x(\bar{e}_j)x(e_j)^{-1}$ for each $e_j \in \mathcal{E}_\mathcal{T}$. Let $\mathcal{A}_\mathcal{T}(\Theta)$ be the set of those $x \in \mathcal{A}_\mathcal{T}$ for which $x(\partial_i\Sigma) \in \Theta_i$ for $i = 1, \dots, r$, where $\Theta = \Theta_1 \times \cdots \Theta_r$ is a conjugacy class in G^r.

Let $\nu_t^\mathcal{T}$ be the discrete Yang-Mills measure on $\mathcal{A}_\mathcal{T}(\Theta)$, specified

$$\int_{\mathcal{A}_\mathcal{T}(\Theta)} f\, d\nu_t^\mathcal{T} = \int f(x) \prod_{k=1}^n Q_{t|\Delta_k|}\big(x(\partial\Delta_k)\big) \prod_{i=1}^r \delta\big(x(\partial_i\Sigma)c_i^{-1}\big)\, dx d_\Theta c \tag{8.1}$$

Thus Theorem 7.1 can be restated as (with hypotheses as before)

$$\int f\big(h(\kappa_1;\omega),...,h(\kappa_n;\omega)\big)\,d\mu_t^{\Theta}(\omega)$$

$$= \frac{1}{\nu_t^{\mathcal{T}}(\mathcal{A}_{\mathcal{T}})}\int_{\mathcal{A}_{\mathcal{T}}} f\big(x(\kappa_1),...,x(\kappa_n)\big)\,d\nu_t^{\mathcal{T}}(x) \tag{8.2}$$

Let

$$\mathcal{A}^0_{\mathcal{T}}(\Theta) \stackrel{\text{def}}{=} \{x : x(\partial\Delta_j) = e, \text{ for } i = 1,...,n\} \tag{8.3}$$

and

$$\mathcal{G}_{\mathcal{T}} \stackrel{\text{def}}{=} \{\text{all maps } \phi : V_{\mathcal{T}} \to G\} \tag{8.4}$$

This is a group acting on $\mathcal{A}^0_{\mathcal{T}}(\Theta)$ by

$$(\phi, x) \mapsto x^{\phi} : x^{\phi}(e) = \phi(i)^{-1}x(e)\phi(f)$$

where e is the 1–simplex running from i to f.

Then

$$\mathcal{A}^0_{\mathcal{T}}(\Theta)/\mathcal{G}_{\mathcal{T}} \simeq \mathcal{M}(\Theta) \simeq \Pi^{-1}(e)/G$$

Replacing $\mathcal{T}$ by an isomorphic copy if necessary, there is a subdivision of $\mathcal{T}$ which contains loops $A_1,...,B_g,C_1,...,C_r$, all based at some vertex o, such that the map

$$\mathcal{A}_{\mathcal{T}} \to G^{2g+r} : x \mapsto \big(x(A_1),...,x(B_g),x(C_1),...,x(C_r)\big)$$

induces a bijection

$$\mathcal{A}^0_{\mathcal{T}}(\Theta)/\mathcal{G}_{\mathcal{T}} \simeq \Pi^{-1}(e)/G \simeq \mathcal{M}^0(\Theta) \tag{8.5}$$

where $\Pi : G^{2g} \times \Theta \to G$ is the usual map

$$\Pi(a_1,b_1,...,a_g,b_g,c_1,...,c_r) = c_r \dots c_1 b_g^{-1}a_g^{-1}b_g a_g \dots b_1^{-1}a_1^{-1}b_1 a_1$$

Assume that Θ is such that Π is a submersion $G^{2g} \times \Theta \to G$ and $\Pi^{-1}(e)$ has a dense open subset $\Pi^{-1}(e)^0$ on which the isotropy of the conjugation action of G is $Z(G)$.

Then (8.5) induces a symplectic structure, again denoted $\overline{\Omega}_{\Theta}$ on the corresponding dense open subset $\mathcal{A}^0_{\mathcal{T}}(\Theta)^0/\mathcal{G}_{\mathcal{T}}$ of $\mathcal{A}^0_{\mathcal{T}}(\Theta)/\mathcal{G}_{\mathcal{T}}$. The proof of Theorem 7.2 contains also the proof of

Theorem 8.1. *Assume that Θ is such that Π is a submersion $G^{2g} \times \Theta \to G$ and $\Pi^{-1}(e)$ has a dense open subset $\Pi^{-1}(e)^0$ on which the isotropy of the conjugation action of G is $Z(G)$. With notation and hypotheses as above, and f a $\mathcal{G}_{\mathcal{T}}$–invariant function $\mathcal{A}_{\mathcal{T}}(\Theta)$,*

$$\lim_{t\to 0}\int_{\mathcal{A}_{\mathcal{T}}(\Theta)} f\,d\nu_t^{\mathcal{T}} = \int_{\mathcal{A}^0_{\mathcal{T}}(\Theta)^0/\mathcal{G}_{\mathcal{T}}} \tilde{f}\,d\mathrm{vol}_{\overline{\Omega}_{\Theta}} \tag{8.6}$$

where $\tilde{f}$ is the function on $\mathcal{A}^0(\Theta)/\mathcal{G}_{\mathcal{T}}$ induced by f.

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