HIGHER CONJUGATION COHOMOLOGY IN COMMUTATIVE HOPF ALGEBRAS

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Let A be a graded, commutative Hopf algebra. We study an action of the symmetric group Σ_n on the tensor product of $n-1$ copies of A; this action was introduced by the second author in [8] and is relevant to the study of commutativity conditions on ring spectra in stable homotopy theory [6]. We show that for a certain class of Hopf algebras the cohomology ring $H^*(\Sigma_n; A^{\otimes n-1})$ is independent of the coproduct provided n and $(n-2)!$ are invertible in the ground ring. Then, by choosing a sufficiently simple coproduct, we are able to deduce significant information about the Σ_n invariants of $A^{\otimes n-1}$, including dimensions and algebra structure.

1. Introduction and Main Results

Let A be a graded, connected, unital, counital, associative, coassociative Hopf algebra. In section 8 of [5] it was shown how A has a 'conjugation' or 'antipode' χ satisfying the equality

$$\mu \circ (1 \otimes \chi) \circ \Delta = \eta \circ \epsilon,$$

where μ and Δ are the product and coproduct and η and ϵ are the unit and counit. In particular, $\chi(1) = 1$ and, for x of positive degree,

$$\chi(x) = -x + \sum_i x'_i \chi(x''_i)$$

where $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_i x'_i \otimes x''_i$. If A is commutative then $\chi^2 = 1$ and so gives an action of Σ_2 on A.

The second author extended this in [8] by providing, for each $n > 2$, an action of the symmetric group Σ_n on $A^{\otimes n-1}$ when A is commutative. If σ_i denotes the transposition $i \leftrightarrow i+1$, then Σ_n is generated by $\sigma_1, \ldots, \sigma_{n-1}$, and the action on $A^{\otimes n-1}$ is given by:

$$\begin{aligned}
\sigma_1 &= [(\mu \otimes 1) \circ (\chi \otimes \Delta)] \otimes 1^{\otimes n-3} \\
\sigma_i &= 1^{\otimes i-2} \otimes [(1 \otimes (\mu \circ (\mu \otimes 1)) \otimes 1) \circ (\Delta \otimes \chi \otimes \Delta)] \otimes 1^{\otimes n-i-2} \text{ if } 1 < i < n-1 \\
\sigma_{n-1} &= 1^{\otimes n-3} \otimes [(1 \otimes \mu) \circ (\Delta \otimes \chi)]
\end{aligned}$$

Note that each σ_i acts multiplicatively, as does χ in the case $n = 2$.

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Thus if A is commutative, as we will henceforth assume, we have a multiplicative action of Σ_n on $A^{\otimes n-1}$ for each $n \geq 2$. While it may seem unusual to have Σ_n acting on an $n-1$-fold product, this action does arise quite naturally in stable homotopy theory as will be explained in Section 3. Motivated by this application, we wished to calculate the cohomology of Σ_n with coefficients in $A^{\otimes n-1}$. However, in [3] we saw how complicated this calculation could be when we attempted it for $n = 2$ and for A an object familiar to algebraic topologists: the mod 2 dual Steenrod algebra.

Since these Σ_n actions explicitly involve χ and Δ, and the former involves the latter, it would be desirable if we could make the coproduct Δ as simple as possible. The following theorem, which is our main result and is proved in Section 2, gives conditions under which we can do this without changing the cohomology ring that we wish to calculate.

Theorem 1. *Let A be a graded, connected, coassociative Hopf algebra over the ring R. Suppose that A is isomorphic, as an algebra, to a tensor product of associative Hopf algebras, each of which has just a single algebra generator. Let $\tilde{A}$ be this tensor product, considered as a Hopf algebra. If n and $(n-2)!$ are invertible in R then there is an isomorphism of algebras*

$$\tilde{A}^{\otimes n-1} \longrightarrow A^{\otimes n-1}$$

which commutes with the Σ_n action, thus inducing an isomorphism of cohomology rings

$$H^*(\Sigma_n; \tilde{A}^{\otimes n-1}) \approx H^*(\Sigma_n; A^{\otimes n-1}).$$

Note that by assuming that the underlying algebra of A is a tensor product of associative monogenic algebras, we automatically have that A is commutative (and associative).

The importance of this theorem rests on the fact that the coproduct in $\tilde{A}$ will generally be much simpler than that in A since all the generators of $\tilde{A}$ are primitive. Thus calculations in $\tilde{A}$ will be simpler than in A and in the rest of this section we consider results that would be hard, if not impossible, to obtain by looking at A directly.

In particular, if A is a graded, connected, commutative, biassociative Hopf algebra, of finite type and R is a perfect field then, by the Borel-Hopf theorem ([5] Theorem 7.11), A satisfies the hypotheses of Theorem 1. Our Theorem is more general than this in that we can work over a ring instead of a field, and we do not need A to be of finite type so, for example, $A = \mathbb{Z}/8[x_1, x_2, \dots]$, where $|x_i| = 2$ for all i, satisfies the hypotheses of Theorem 1, but not of the Borel-Hopf theorem.

The hypothesis that $n.(n-2)!$ be invertible is rather curious and we will discuss this further at the end of Section 2.

Of course, if we make the stronger assumption that $n!$ is invertible in R, then the cohomology ring is zero in positive degrees, but even in this case the theorem is of significance since it greatly simplifies the calculation of $H^0(\Sigma_n; A^{\otimes n-1}) = (A^{\otimes n-1})^{\Sigma_n} = \{x \in A^{\otimes n-1} \mid \sigma x = x \text{ for all } \sigma \in \Sigma_n\}$. In particular, if R is a field and A is a polynomial algebra, then the theorem puts this calculation into the realm of classical invariant theory. This is because Σ_n acts multiplicatively on $\tilde{A}^{\otimes n-1}$ and preserves the R-subspace $Q \subset \tilde{A}^{\otimes n-1}$ spanned by the generators. So we have Σ_n

acting on a vector space Q, and the action on $\tilde{A}^{\otimes n-1}$ is just the action on the polynomial algebra $R[Q]$ $(= \tilde{A}^{\otimes n-1})$ induced from this Σ_n action on Q.[1]

In fact one does not necessarily need A to be polynomial. For example, Molien's theorem can be re-worked as follows.

Theorem 2.

1) Suppose R is a field whose characteristic is either 0 or coprime to $n!$. Let A be generated by $x_1, x_2, \ldots$ of positive degrees $d_1, d_2, \ldots$ and heights $h_1, h_2, \ldots$ (chosen from $\mathbb{N} \cup \infty$). Then the Poincaré series for $(A^{\otimes n-1})^{\Sigma_n}$ is given by

$$p(t) = \frac{1}{n!} \sum_{g \in \Sigma_n} \prod_i \frac{\det(1 - g t^{d_i h_i})}{\det(1 - g t^{d_i})},$$

where, in the i-th factor on the right, the determinants are those of the given operator acting on the space $\langle x_i \otimes 1^{\otimes n-2}, \ldots, 1^{\otimes n-2} \otimes x_i \rangle$.

2) If R is a field whose characteristic divides $n!$, then this Poincaré series bounds that for $(A^{\otimes n-1})^{\Sigma_n}$ coefficient-wise.

Proof. The proof of the first statement is a simple modification of the standard one as found in, e.g. [2], [7].

The proof of the second statement begins with the remark in [2] that in the modular case, the series above gives the Poincaré series for the multiplicity of the trivial representation *as a composition factor* in $\tilde{A}^{\otimes n-1}$. Thus the Poincaré series given above bounds that for $(\tilde{A}^{\otimes n-1})^{\Sigma_n}$. To complete the proof we will show that the Σ_n-invariants in $A^{\otimes n-1}$ can be mapped injectively into those for $\tilde{A}^{\otimes n-1}$.

Given an element θ of $A^{\otimes n-1}$ we write $l(\theta)$ for the 'least decomposable part' of θ. To be precise, filter $A^{\otimes n-1}$ by powers of the augmentation ideal, so that F_0 is the positive degree part of $A^{\otimes n-1}$ and $F_k = F_0 \cdot F_{k-1}$ for all $k > 0$. We may then write θ as $\theta_0 + \theta_1 + \cdots$, where θ_i is the sum of those monomials lying in F_i but not in F_{i+1}, and $l(\theta)$ will be θ_k where k is the least integer such that $\theta_k \neq 0$. The key point is that if θ is Σ_n-invariant in $A^{\otimes n-1}$ then $l(\theta)$ is Σ_n-invariant in $\tilde{A}^{\otimes n-1}$. This follows from the fact that if x is an indecomposable in $A^{\otimes n-1}$, then $\sigma_i(x)$ in $\tilde{A}^{\otimes n-1}$ is precisely the least decomposable part of $\sigma_i(x)$ in $A^{\otimes n-1}$.

We claim that there is some choice of basis $\varphi_1, \varphi_2, \ldots$ for $(A^{\otimes n-1})^{\Sigma_n}$ such that $l(\varphi_1), l(\varphi_2), \ldots$ are linearly independent in $(\tilde{A}^{\otimes n-1})^{\Sigma_n}$, enabling us to define an injection $(A^{\otimes n-1})^{\Sigma_n} \hookrightarrow (\tilde{A}^{\otimes n-1})^{\Sigma_n}$ by $\varphi_i \mapsto l(\varphi_i)$. To prove this, let $\phi_1, \phi_2, \ldots$ be any basis for $(A^{\otimes n-1})^{\Sigma_n}$. We will inductively modify this basis until we have $l(\phi_1), l(\phi_2), \ldots$ linearly independent. Clearly $l(\phi_1)$ on its own is linearly independent, so now assume that $l(\phi_1), \ldots, l(\phi_{m-1})$ are linearly independent. If this set remains linearly independent when we add $l(\phi_m)$ to it then there is nothing to do and the inductive step is complete. If not, then there is a relation $a_1 l(\phi_1) + \cdots + a_m l(\phi_m) = 0$, with $a_m \neq 0$. We replace ϕ_m by $a_1\phi_1 + \cdots + a_m\phi_m$, noting that the set $\phi_1, \phi_2, \ldots, \phi_m, \ldots$ still forms a basis for $(A^{\otimes n-1})^{\Sigma_n}$. If $l(\phi_m)$ is now linearly independent of $l(\phi_1), \ldots, l(\phi_{m-1})$ then the inductive step is complete. If not then, again, there is a relation $b_1 l(\phi_1) + \cdots + b_m l(\phi_m) = 0$ and we

[1] Ian Leary points out that the Σ_n representation Q is a tensor product of copies of the well-known $n-1$-dimensional Σ_n representation V constructed as follows. Take an n-dimensional vector space with basis $e_1, \ldots, e_n$ and let Σ_n permute the basis. Then V is the subspace spanned by $e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n$. There is one copy of V in Q for each generator of A.

replace ϕ_m by the corresponding linear combination of ϕ_i's. We repeat this process until we obtain some ϕ_m such that $l(\phi_1), \ldots, l(\phi_m)$ are linearly independent. This must occur within a finite number of steps because at each stage $l(\phi_m)$ is replaced by something of strictly higher filtration and in each degree the filtration is finite. Moreover $l(\phi_m)$ cannot ever be zero for this would imply a relation among $\phi_1, \ldots, \phi_m$. □

By way of example, if $n = 2$ then the first part of Theorem 2 leads to the following explicit formulae.

Corollary 3. *If R is a field of characteristic different from 2 and A is generated by $x_1, x_2, \ldots$ of positive degrees $d_1, d_2, \ldots$ and heights $h_1, h_2, \ldots$, then the Poincaré series for A^{Σ_2} is*

$$p(t) = \frac{\prod_i (1 + t^{d_i})(1 - t^{d_i h_i}) + \prod_i (1 - t^{d_i})(1 + t^{d_i h_i})}{2 \prod_i (1 - t^{2d_i})}.$$

If A is polynomial this simplifies to

$$p(t) = \frac{\prod_i (1 + t^{d_i}) + \prod_i (1 - t^{d_i})}{2 \prod_i (1 - t^{2d_i})}.$$

In fact, in the case $n = 2$ Theorem 1 leads to the following 'model' for the conjugation invariants.

Theorem 4. *Suppose A is a graded, connected, coassociative Hopf algebra which, as an algebra, is a tensor product of associative monogenic Hopf algebras and suppose that 2 is invertible in the ground ring. Let A^{Σ_2} denote the subalgebra of conjugation invariants, and let A_E denote the subalgebra of A spanned by the monomials whose exponents sum to an even number. Then there is an isomorphism of algebras*

$$A^{\Sigma_2} \approx A_E.$$

This is a simple consequence of Theorem 1 and the fact that $\chi(x) = -x$ if x is primitive.

We note that if $p > 2$ then Theorem 4 along with Corollary 3 completely solves the 'conjugation invariants' problem for the mod p dual Steenrod algebra, in marked contrast to the partial solution [3] available when $p = 2$.

2. Proof of Theorem 1

We wish to construct an isomorphism of algebras $f : \tilde{A}^{\otimes n-1} \to A^{\otimes n-1}$ that commutes with the Σ_n-action. Thus there are three components to the proof : we must define a multiplicative map, show that it is a bijection, and show that it commutes with the group action. We will begin by briefly discussing the significant features of each of the three components. It will be useful, however, to first fix a set of generators for $A^{\otimes n-1}$ and $\tilde{A}^{\otimes n-1}$. These objects are identical as algebras so the same set of generators will suffice for both. Let J be a set of generators for A. For each $x \in J$ and for $1 \leq i \leq n-1$, define $x[i] \in A^{\otimes n-1}$ by

$$x[i] = 1^{\otimes i-1} \otimes x \otimes 1^{\otimes n-1-i}.$$

Then a set of generators for both $A^{\otimes n-1}$ and $\tilde{A}^{\otimes n-1}$ is given by

$$\{x[i] \mid x \in J \text{ and } 1 \leq i \leq n-1\}.$$

Now, to define a multiplicative map $f : \tilde{A}^{\otimes n-1} \to A^{\otimes n-1}$ we will construct a map on these generators and extend it multiplicatively. This will give a well-defined, multiplicative map provided that we show that f preserves all relations which hold among the generators of $\tilde{A}^{\otimes n-1}$. Since $\tilde{A}$ is a tensor product of monogenic algebras we know that the only relations in $\tilde{A}$ are of the form $x^h = 0$ for some generator x and positive integer h and it follows that the only relations in $\tilde{A}^{\otimes n-1}$ are of the form $x[i]^h = 0$. Moreover, h cannot be chosen arbitrarily, because $R[x]/(x^h)$ is assumed to be a *Hopf* algebra (see the discussion in [4] §1-3). To be precise, h must be such that $\binom{h}{j} = 0$ in R for $0 < j < h$. With such an h we have that, if $g_1, g_2, \dots$ are multiplicative, e.g. elements of Σ_n, then $y^h = 0$ implies $(r_1 g_1(y) + r_2 g_2(y) + \cdots)^h = 0$ for any $r_1, r_2, \dots$ in R. In light of this, we will set $f(x[i]) = a_i x[1]$, where each a_i is an element of the group algebra $R\Sigma_n$. Since the height of $x[1]$ is the same as that of $x[i]$, the above argument shows that f preserves all the relations.

To show that f is bijective, it suffices to show that the map induced on the modules of indecomposables

$$\frac{(\tilde{A}^{\otimes n-1})^+}{(\tilde{A}^{\otimes n-1})^+ \cdot (\tilde{A}^{\otimes n-1})^+} \longrightarrow \frac{(A^{\otimes n-1})^+}{(A^{\otimes n-1})^+ \cdot (A^{\otimes n-1})^+}$$

is a bijection. Here the notation $(\ )^+$ denotes the augmentation ideal, i.e. the positive degree part of the algebra. We will show, at the end of the proof, that this induced map is a scalar multiple of the identity : for a generator $x[i]$ we have

$$f(x[i]) \equiv \lambda x[i] \mod (A^{\otimes n-1})^+ \cdot (A^{\otimes n-1})^+$$

for some $\lambda \in R$, and that the hypotheses ensure that λ is invertible so that the induced map is a bijection.

Most of the work will go into showing that f commutes with the Σ_n action. To explain further let us examine exactly what conditions must hold for f to commute with the group action. To do this we need to understand how Σ_n acts on $\tilde{A}^{\otimes n-1}$ but because the generators x are primitive in $\tilde{A}$ it is easy to see what $\sigma_k x[i]$ is, working directly from the definition. For example, if $k = i+1$ then $\sigma_k x[i] = x[i] + x[i+1]$. From this we can see that for f to commute with the Σ_n action, the following relations must hold for all generators x:

$$\sigma_k f(x[i]) = \begin{cases} f(x[i]) & \text{if } i < k-1 \\ f(x[i]) + f(x[i+1]) & \text{if } i = k-1 \\ -f(x[i]) & \text{if } i = k \\ f(x[i-1]) + f(x[i]) & \text{if } i = k+1 \\ f(x[i]) & \text{if } i > k+1 \end{cases}$$

We will ensure these relations hold by making appropriate choices of the elements of the group algebra used in the definition of f. For example, if $i > 1$, we will ensure that $\sigma_i f(x[i]) = -f(x[i])$ by defining $a_i = (\sigma_i - 1)a_{i-1}$, i.e. $f(x[i]) = (\sigma_i - 1) f(x[i-1])$. Since $\sigma_i^2 = 1$, we have the desired result that $\sigma_i f(x[i]) = -f(x[i])$.

Having thus outlined the strategy, we now present the details.

For each generator x of $\tilde{A}$, let

$$f(x[1]) = (\sigma_1 - 1)T_n x[1],$$

where σ_1 is interpreted as χ if $n = 2$, and $T_n \in R\Sigma_n$ is defined as follows. Let G_n be the subgroup of Σ_n generated by $\sigma_2, \ldots, \sigma_{n-1}$. Then T_n is the sum of the elements of G_n. If $n = 2$ then we understand T_n as 1.

Then $\sigma_1 f(x[1]) = -f(x[1])$, which, for $n = 2$, is all that is needed to show that f commutes with the Σ_2 action. For $n \geq 3$, we also have that $\sigma_k f(x[1]) = f(x[1])$ for all $k > 2$, since, for such k, σ_k commutes with σ_1 and $\sigma_k T_n = T_n$; this latter fact is true because $\sigma_k \in G_n$ and so left multiplication by σ_k simply permutes the elements of G_n, leaving their sum, T_n, unchanged.

Now $\sigma_i f(x[i-1])$ should equal $f(x[i-1]) + f(x[i])$, so we must have

$$f(x[i]) = (\sigma_i - 1)f(x[i-1]) \text{ for } i > 1.$$

Thus, having defined $f(x[1])$ above, we now use this formula to define $f(x[i])$ inductively for all i and then extend multiplicatively to define f on the whole of $\tilde{A}^{\otimes n-1}$. It follows that $\sigma_i f(x[i]) = -f(x[i])$ for $i > 1$. Furthermore, if $1 \leq i < k-1$, we also have that $\sigma_k f(x[i]) = f(x[i])$ since σ_k commutes with σ_i when $i < k-1$ and $\sigma_k f(x[1]) = f(x[1])$ if $k > 2$.

Next we consider $\sigma_i f(x[i+1])$:

$$\begin{aligned}
&\sigma_i f(x[i+1]) - f(x[i]) - f(x[i+1]) \\
&= [(\sigma_i - 1)(\sigma_{i+1} - 1) - 1](\sigma_i - 1)(\sigma_{i-1} - 1)\cdots(\sigma_1 - 1)T_n x[1] \\
&= (\sigma_i\sigma_{i+1}\sigma_i - \sigma_i\sigma_{i+1} - \sigma_{i+1}\sigma_i + \sigma_i + \sigma_{i+1} - 1)(\sigma_{i-1} - 1)\cdots(\sigma_1 - 1)T_n x[1] \\
&= (\sigma_{i+1}\sigma_i - \sigma_i + 1)(\sigma_{i+1} - 1)(\sigma_{i-1} - 1)\cdots(\sigma_1 - 1)T_n x[1] \\
&= (\sigma_{i+1}\sigma_i - \sigma_i + 1)(\sigma_{i-1} - 1)\cdots(\sigma_1 - 1)(\sigma_{i+1} - 1)T_n x[1]
\end{aligned}$$

and this is zero because $\sigma_{i+1}T_n = T_n$ by the same argument as above (σ_{i+1} being in G_n).

Finally, if $k < i - 1$, then

$$\begin{aligned}
\sigma_k f(x[i]) &= \sigma_k(\sigma_i - 1)\cdots(\sigma_{k+2} - 1)f(x[k+1]) \\
&= (\sigma_i - 1)\cdots(\sigma_{k+2} - 1)\sigma_k f(x[k+1]) \\
&= (\sigma_i - 1)\cdots(\sigma_{k+2} - 1)(f(x[k]) + f(x[k+1])) \\
&= (\sigma_i - 1)\cdots(\sigma_{k+3} - 1)(f(x[k+2])) \\
&= (\sigma_i - 1)\cdots(\sigma_{k+4} - 1)(f(x[k+3])) \\
&\cdots \\
&= f(x[i]).
\end{aligned}$$

Thus we see that $\sigma_k f(x[i])$ is as required for all i, k, i.e. f commutes with the Σ_n action.

We now finish by showing that f is a bijection on the modules of indecomposables. Note that since $f(x[i]) = (\sigma_i - 1)f(x[i-1])$ for $i > 1$ and $(\sigma_i - 1)(x[i-1]) \equiv x[i]$

modulo decomposables, it suffices to prove that $f(x[1]) \equiv \lambda x[1]$ modulo decomposables, for some unit $\lambda \in R$. For the remainder of the proof, the symbol $\equiv$ will denote equivalence modulo decomposables. First we need to calculate $T_n(x[1])$:

$$T_n(x[1]) \equiv (n-2)! \sum_{i=1}^{n-1} (n-i)x[i].$$

This is proved by induction using the fact that

$$T_n = (1 + \sigma_{n-1} + \sigma_{n-2}\sigma_{n-1} + \cdots + \sigma_2 \cdots \sigma_{n-2}\sigma_{n-1})T_{n-1},$$

i.e. the summands of the left hand factor form a set of coset representatives for G_n over G_{n-1}. This is easily seen to be the case since, under the natural action of Σ_n on $\{1, 2, \ldots, n\}$, the permutations $\sigma_j \cdots \sigma_{n-1}$ each map n differently.

Now,

$$(\sigma_1 - 1)x[i] \equiv \begin{cases} -2x[1] & \text{if } i = 1 \\ x[1] & \text{if } i = 2 \\ 0 & \text{if } i > 2. \end{cases}$$

Hence

$$\begin{aligned} f(x[1]) &\equiv (\sigma_1 - 1)T_n x[1] \\ &\equiv (\sigma_1 - 1)(n-2)! \sum_{i=1}^{n-1} (n-i)x[i] \\ &\equiv (n-2)!(-2(n-1)x[1] + (n-2)x[1]) \\ &\equiv -n.(n-2)!x[1]. \end{aligned}$$

So, as long as $n.(n-2)!$ is invertible in the ground ring R, f gives a bijection on the indecomposables. This explains the provenance of the curious condition on R in Theorem 1 and completes the proof of that theorem. $\square$

It remains unclear to what extent the hypothesis that $n.(n-2)!$ be invertible is necessary. If $n = 2$ and $R = \mathbb{Z}/2$ then the conclusion of the theorem is definitely false (see [3] for an example). Furthermore, calculations with Magma have revealed a number of other examples where, if n is not invertible, the invariants, or the higher cohomology, differ when one changes the coproduct. However, we have not found any cases where n *is* invertible in R and where the conclusion of the theorem does not hold.

3. Why should Σ_n act on $A^{\otimes n-1}$?

Let E be a ring spectrum and let $A = \pi_*(E \wedge E)$, the set of 'co-operations' in the cohomology theory associated to E. (For example, if E is the $\mathbb{F}_p$ Eilenberg-Mac Lane spectrum then A is the mod p dual Steenrod algebra.) If E is sufficiently nice then A is a commutative Hopf algebra (more generally it is a Hopf *algebroid*). The conjugation map, χ, on A is precisely the map induced on $\pi_*(E \wedge E)$ by switching the factors in the smash product $E \wedge E$ ([1] Lecture 3). Analogously, we can take the homotopy of a smash product of n copies of E, $\pi_*(E \wedge \cdots \wedge E)$, and there is a natural action of Σ_n induced by permuting the factors. But $\pi_*(E^{\wedge n}) = E_*(E^{\wedge n-1})$, the cohomology of an $n-1$-fold product of copies of E and, for suitable E, this is isomorphic to $A^{\otimes n-1}$. Thus there is an action of Σ_n on $A^{\otimes n-1}$ and it is shown in [8] that this action satisfies the formulae given at the start of this paper.

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