TOWARD THE SECOND MAIN THEOREM ON COMPLEMENTS: FROM LOCAL TO GLOBAL

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Abstract. We prove the boundedness of complements modulo two conjectures: Borisov-Alexeev conjecture and effective adjunction for fibre spaces. We discuss the last conjecture and prove it in two particular cases.

Contents

1. Introduction

This paper completes our previous work [PS01] modulo two conjectures 1.1 and 7.12. The first one relates to Alexeev-Borisov conjecture:

Conjecture 1.1. *Fix a real number $\varepsilon > 0$. Let $(X, B = \sum b_i B_i)$ be a d-dimensional log semi-Fano variety (see Definition 2.6) such that*

(i) *$K + B$ is ε-lt; and*
(ii) *X is FT that is (X, Θ) is a klt log Fano variety with respect to some boundary Θ.*

Then X is bounded in the moduli sense, i.e., it belongs to an algebraic family $\mathcal{X}(\varepsilon, d)$.

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This conjecture was proved in dimension 2 by V. Alexeev [Ale94] and in toric case by A. Borisov and L. Borisov [BB92] (see also [Nik90], [Bor96], [Bor01], [McK02]).

Remark 1.2. Condition (ii) of Conjecture 1.1 can be replaced with another one. For example, we can assume that

(ii)′ X is rationally connected.

We prefer (ii) because it is better for our purposes.

We recall that the log pair (X, B) is said to be *ε-log terminal* (or simply *ε-lt*) if $\operatorname{totaldiscr}(X, B) > -1 + \varepsilon$, see Definition 2.2 below.

Theorem 1.3 ([Ale94])**.** *Conjecture* 1.1 *holds in dimension two.*

The second conjecture concerns with Adjunction Formula and will be discussed in Section 7.

Our main result is the following.

Theorem 1.4. *Fix a finite subset $\mathfrak{R} \subset [0, 1] \cap \mathbb{Q}$ and a positive integer I. Let (X, B) be a klt log semi-Fano variety of dimension d such that X is FT and the multiplicities of B are contained in $\Phi(\mathfrak{R})$ (see* 3.2*). Assume the LMMP in dimension $\leq d$. Further, assume that Conjectures* 1.1 *and* 7.12 *hold in dimension $\leq d$. Then $K + B$ has bounded complements. More precisely, there is a constant $C = C(d, \mathfrak{R}, I)$ such that $K + B$ is nI-complemented for some $n \leq C$.*

Addendum 1.5. Our proof shows that we need not Conjecture 1.1 in all generality. We need it only for some special value $\varepsilon_0 := \varepsilon_0(d, \mathfrak{R}, I) > 0$.

In the case when $K + B$ is numerically trivial our result is stronger:

Theorem 1.6 (cf. [Bla95], [Ish00])**.** *Fix a finite subset $\mathfrak{R} \subset [0, 1] \cap \mathbb{Q}$. Let (X, B) be a* 0*-pair of dimension d such that X is FT and the multiplicities of B are contained in $\Phi(\mathfrak{R})$. Assume the LMMP in dimension $\leq d$. Further, assume that Conjectures* 1.1 *and* 7.12 *hold in dimension $\leq d$. Then there is an integer $n = n(d, \mathfrak{R})$ such that $n(K_X + B) \sim 0$.*

Corollary 1.7 (cf. [Sho00])**.** *Fix a finite subset $\mathfrak{R} \subset [0, 1] \cap \mathbb{Q}$. Let (X, B) be a klt log semi-del Pezzo surface such that X is FT and the multiplicities of B are contained in $\Phi(\mathfrak{R})$. Then $K + B$ has bounded complements.*

Corollary 1.8. *Fix a finite rational subset $\mathfrak{R} \subset [0, 1] \cap \mathbb{Q}$. Let (X, B) be a klt log semi-Fano threefold such that X is FT and the multiplicities of B are contained in $\Phi(\mathfrak{R})$. Assume that Conjecture* 1.1 *hold in dimension* 3*. Then $K + B$ has bounded complements.*

We give a sketch of the proof of our main results in Section 4. One can see that our constructions are in fact from global to global, i.e., the proof essentially uses reduction to lower-dimensional global pairs. However it is expected that an improvement of our method can use reduction to local questions in the same dimension (local to global techniques).

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2. Preliminaries

2.1. **Notation.** All varieties are assumed to be algebraic and defined over an algebraically closed field of characteristic zero. We use standard terminology and notation of the Log Minimal Model Program (LMMP) [KMM87], [Kol92], [Sho93]. For the definition of complements and their properties we refer to [Sho93], [Sho00], [Pro01] and [PS01]. Recall that a *log pair* (or a *log variety*) is a pair (X, D) consisting of a normal variety X and a *boundary* D, i.e., an $\mathbb{R}$-divisor $D = \sum d_i D_i$ with multiplicities $0 \le d_i \le 1$. As usual K_X denotes the canonical (Weil) divisor of a variety X. Sometimes we will write K instead of K_X if no confusion is likely. Everywhere below $a(E, X, D)$ denotes the discrepancy of E with respect to $K_X + D$. We omit D if it is zero: $a(E, X) := a(E, X, 0)$. Recall the standard notation:

$$\begin{aligned}
\operatorname{discr}(X, D) &= \inf_E\{a(E, X, D) \mid \operatorname{codim} \operatorname{Center}_X(E) \ge 2\},\\
\operatorname{totaldiscr}(X, D) &= \inf_E\{a(E, X, D) \mid \operatorname{codim} \operatorname{Center}_X(E) \ge 1\}.
\end{aligned}$$

Definition 2.2. A log pair (X, B) is said to be *ϵ-log terminal* (*ϵ-log canonical*) if $\operatorname{totaldiscr}(X, B) > -1 + \epsilon$ (resp., $\operatorname{totaldiscr}(X, B) \ge -1 + \epsilon$).

2.3. Usually we work with $\mathbb{R}$-divisors. An $\mathbb{R}$-divisor is an $\mathbb{R}$-linear combination of prime Weil divisors. An $\mathbb{R}$-linear combination $D = \sum \alpha_i L_i$, where the L_i are integral Cartier divisors is called an $\mathbb{R}$-Cartier divisor. The pull-back f^* of an $\mathbb{R}$-Cartier divisor $D = \sum \alpha_i L_i$ under a morphism $f\colon Y \to X$ is defined as $f^*D := \sum \alpha_i f^* L_i$. Two $\mathbb{R}$-divisors D and D' are said to be *$\mathbb{Q}$- (resp., $\mathbb{R}$-)linearly equivalent* if $D - D'$ is a $\mathbb{Q}$- (resp., $\mathbb{R}$-)linear combination of principal divisors. For a positive integer I, two $\mathbb{R}$-divisors D and D' are said to be *I-linearly equivalent* if $I(D - D')$ is an (integral) principal divisor. The $\mathbb{Q}$-linear (resp., $\mathbb{R}$-linear, I-linear) equivalence is denoted by $\sim_{\mathbb{Q}}$ (resp., $\sim_{\mathbb{R}}$, $\sim_I$). Let $\Phi \subset \mathbb{R}$ and let $D = \sum d_i D_i$ be an $\mathbb{R}$-divisor. We say that $D \in \Phi$ if $d_i \in \Phi$ for all i.

2.4. For any contraction $f\colon X \to Z$ and a divisor D on X we decompose D as $D = D^{\mathrm{h}} + D^{\mathrm{v}}$, where D^{h} and D^{v} are horizontal and vertical parts of D, respectively.

2.5. Let $f\colon X \to Z$ be a morphism of normal varieties. For any $\mathbb{R}$-divisor Δ on Z define its *divisorial pull-back* $f^\bullet\Delta$ as the closure of the usual pull-back $f^*\Delta$ over $Z \setminus V$, where V is a closed subset of codimension ≥ 2 such that $V \supset \operatorname{Sing} Z$ and f is equidimensional over $Z \setminus V$. Thus each component of $f^\bullet\Delta$ dominates a component of Δ. It is easy to see that the divisorial pull-back $f^\bullet\Delta$ does not depend on the choice of V. Note however that in general $f^\bullet$ does not coincide with the usual pull-back f^* of $\mathbb{R}$-Cartier divisors.

Definition 2.6. Let (X, B) be a log pair of global type (the latter means that X is projective). Then it is said to be

log Fano variety if $K + B$ is lc and $-(K + B)$ is ample;
weak log Fano (WLF) variety if $K + B$ is lc and $-(K + B)$ is nef and big;
log semi-Fano (ls-Fano) variety if $K + B$ is lc and $-(K + B)$ is nef;
0-*pair* if $K + B$ is lc and numerically trivial*.

In dimension two we usually use the word *del Pezzo* instead of *Fano*.

Lemma 2.7. *Let (X, Θ) be a d-dimensional klt $\mathbb{Q}$-factorial* 0*-pair and let D be any* (*not necessarily effective*) *$\mathbb{Q}$-divisor such that* $\operatorname{Supp} D \subset \operatorname{Supp} \Theta$*. Assume the LMMP in dimension d. Then the D-MMP works* (*see* [Kol92, 2.26])*.*

Proof. In this situation, the D-MMP is nothing but the $K + \Theta + \varepsilon D$-MMP for $0 < \epsilon \ll 1$. □

Lemma-Definition 2.8. *Let X be a normal projective variety. We say that X is* FT (Fano type) *if it satisfies the following equivalent conditions:*

(i) *there is a $\mathbb{Q}$-boundary Ξ such that (X, Ξ) is a klt log Fano;*
(ii) *there is a $\mathbb{Q}$-boundary Ξ such that (X, Ξ) is a klt weak log Fano;*
(iii) *there is a $\mathbb{Q}$-boundary Θ such that (X, Θ) is a klt* 0*-pair and the components of Θ generate $N^1(X)$;*
(iv) *for any divisor Υ there is a $\mathbb{Q}$-boundary Θ such that (X, Θ) is a klt* 0*-pair and* $\operatorname{Supp} \Upsilon \subset \operatorname{Supp} \Theta$.

Proof. Implications (i) $\Longrightarrow$ (iv), (iv) $\Longrightarrow$ (iii), (i) $\Longrightarrow$ (ii) are obvious and (ii) $\Longrightarrow$ (i) follows by Kodaira's lemma (see, e.g., [KMM87, Lemma 0-3-3]). We prove (iii) $\Longrightarrow$ (i). Let (X, Θ) be such as in (iii). Take an ample divisor H such that $\operatorname{Supp} H \subset \operatorname{Supp} \Theta$ and put $\Xi = \Theta - \varepsilon H$, for $0 < \varepsilon \ll 1$. Clearly, (X, Ξ) is a klt log Fano. □

Lemma 2.9. (i) *Let $f \colon X \to Z$ be a* (*not necessarily birational*) *contraction of normal varieties. If X is FT, then so is Z.*
(ii) *The FT property is preserved under birational divisorial contractions and flips.*
(iii) *Let (X, D) be an ls-Fano variety such that X is FT. Let $f \colon Y \to X$ be a birational extraction such that $a(E, X, D) < 0$ for every f-exceptional divisor E over X. Then Y is also FT.*

Proof. The birational case of (i) and (ii) easily follows from from 2.8 (iii). To prove (i) in the general case we apply Ambro's result [Amb05] which is a variant of Log Canonical Adjunction (cf. 7.12, [Fuj99]). Let $\Theta = \sum_i \theta_i \Theta_i$ be a $\mathbb{Q}$-boundary on X whose components generate $N^1(X)$ and such that (X, Θ) is a klt 0-pair. Let A be an ample divisor on Z. By our assumption

*Such a log pair can be called also a *log Calabi-Yau variety*. However the last notion usually assumes some additional conditions such as $\pi_1(X) = 0$ or $q(X) = 0$.

$f^*A \equiv \sum_i \delta_i\Theta_i$. Take $0 < \delta \ll 1$ and put $\Theta' := \sum_i(\theta_i - \delta\delta_i)\Theta_i$. Clearly, $K + \Theta' \equiv -\delta f^*A$ and (X, Θ') is a klt log semi-Fano variety. By the base point free theorem $K + \Theta' \sim_{\mathbb{Q}} -\delta f^*A$. Now by [Amb05, Theorem 0.2] there is a $\mathbb{Q}$-boundary Θ_Z such that (Z, Θ_Z) is klt and $K_Z + \Theta_Z \sim_{\mathbb{Q}} -\delta f^*A$. Hence (Z, Θ_Z) is a klt log Fano variety. This proves (i).

Now we prove (iii). Let Ξ be a boundary such that (X, Ξ) is a klt log Fano. Let D_Y and Ξ_Y be proper transforms of D and Ξ, respectively. Then (Y, D_Y) is an ls-Fano, (Y, Ξ_Y) is klt and $-(K_Y + \Xi_Y)$ is nef and big. However Ξ_Y is not necessarily a boundary. To improve the situation we put $\Xi' := (1-\varepsilon)D_Y + \varepsilon\Xi_Y$ for small positive ε. Then (Y, Ξ') is a klt weak log Fano. □

Corollary 2.10. *Let X be an FT variety. Assume the LMMP in dimension $\dim X$. Then the D-MMP works on X with respect to any $\mathbb{R}$-divisor D.*

2.11. We say that a class of log pairs $(\mathcal{X}/\mathcal{Z}, \mathcal{B})$ has *bounded complements* if there is a constant Const such that for any log pair $(X/Z, B)$ form this class the log divisor $K + B$ is n-complemented near the fibre over o for some $n \le \text{Const}$.

2.12. **Notation.** Let X be a normal d-dimensional variety and let $\mathcal{B} = \sum_{i=1}^r B_i$ be any reduced divisor on X. Recall that $Z_{d-1}(X)$ usually denotes the group of Weil divisors on X. Consider the vector space $\mathfrak{D}_{\mathcal{B}}$ of all $\mathbb{R}$-divisors supported in $\mathcal{B}$:

$$\mathfrak{D}_{\mathcal{B}} := \left\{D \in Z_{d-1}(X) \otimes \mathbb{R} \mid \operatorname{Supp} D \subset \mathcal{B}\right\} = \sum_{i=1}^r \mathbb{R} \cdot B_i.$$

As usual, define a norm in $\mathfrak{D}_{\mathcal{B}}$ by

$$\|B\| = \max(|b_1|, \ldots, |b_r|),$$

where $B = \sum_{i=1}^r b_i B_i \in \mathfrak{D}_{\mathcal{B}}$. For any $\mathbb{R}$-divisor $B = \sum_{i=1}^r b_i B_i$ put $\mathfrak{D}_B := \mathfrak{D}_{\operatorname{Supp} B}$.

3. Hyperstandard multiplicities

Recall that *standard multiplicities* $1 - 1/m$ naturally appear as multiplicities in the divisorial adjunction formula $(K_X + S)|_S = K_S + \operatorname{Diff}_S$ (see [Sho93, §3], [Kol92, Ch. 16]). Considering the adjunction formula for fibre spaces and adjunction for higher codimensional subvarieties one needs to introduce a bigger class of multiplicities.

Example 3.1. Let $f \colon X \to Z \ni P$ be a minimal two-dimensional elliptic fibration over an one-dimensional germ (X is smooth). We can write a natural formula $K_X = f^*(K_Z + D_{\text{div}})$, where $D_{\text{div}} = d_P P$ is an effective divisor (cf. 7.2 below). ¿From Kodaira's classification of singular fibres (see [Kod63]) we obtain the following values of d_P:

Type	mI_n	II	III	IV	I_b^*	II^*	III^*	IV^*
d_P	$1-\frac{1}{m}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{2}{3}$

Thus the multiplicities of D_{div} are not necessarily standard.

3.2. Fix a subset $\mathfrak{R} \subset \mathbb{R}_{\geq 0}$. Define

$$\Phi(\mathfrak{R}) := \left\{1 - \frac{r}{m} \quad \Big| \quad m \in \mathbb{Z}, \quad m > 0 \quad r \in \mathfrak{R}\right\} \bigcap \Big[0,\, 1\Big].$$

We say that an $\mathbb{R}$-boundary B has *hyperstandard multiplicities* with respect to $\mathfrak{R}$ if $B \in \Phi(\mathfrak{R})$. For example, if $\mathfrak{R} = \{0, 1\}$, then $\Phi(\mathfrak{R})$ is the set of *standard multiplicities*. The set $\mathfrak{R}$ is said to be *rational* if $\mathfrak{R} \subset \mathbb{Q}$. Usually we will assume that $\mathfrak{R}$ is rational and finite. Denote

$$I(\mathfrak{R}) := \begin{cases} \mathrm{lcm}\begin{pmatrix} \text{denominators} \\ \text{of } r \in \mathfrak{R} \setminus \{0\} \end{pmatrix} & \text{if } \mathfrak{R} \text{ is rational and finite,} \\ \infty & \text{otherwise.} \end{cases}$$

(3.2.1) Fix a positive integer I such that $I(\mathfrak{R}) \mid I$. Denote by $\mathcal{N}_d(\mathfrak{R}, I)$ the set of all $m \in \mathbb{Z}$, $m > 0$ such that there exists a log semi-Fano variety (X, D) of dimension $\leq d$ satisfying the following properties:

(i) X is FT and $D \in \Phi(\mathfrak{R})$;
(ii) either (X, D) is klt, or $K_X + D \equiv 0$;
(iii) $K_X + D$ is m-complementary, $I \mid m$, and m is minimal under these conditions.

Put

$$N_{d,I} = N_d(\mathfrak{R}, I) := \max \mathcal{N}_d(\mathfrak{R}, I), \quad \epsilon_{d,I} = \epsilon_d(\mathfrak{R}, I) := 1/(N_{d,I} + 2).$$

Usually we omit I if $I = I(\mathfrak{R})$. We expect that $\mathcal{N}_d(\mathfrak{R}, I)$ is bounded whenever $\mathfrak{R}$ is finite and rational, see Theorems 1.4 and 1.6. In particular, $N_{d,I} < \infty$ and $\epsilon_{d,I} > 0$. For $\epsilon \geq 0$, define also the set of *semi-hyperstandard* multiplicities

$$\Phi(\mathfrak{R}, \epsilon) := \Phi(\mathfrak{R}) \cup [1 - \epsilon,\, 1].$$

Fix a positive integer n and define the set $\mathcal{P}_n$ by

$$\alpha \in \mathcal{P}_n \quad \Longleftrightarrow \quad 0 \leq \alpha \leq 1 \quad \text{and} \quad \lfloor (n+1)\alpha \rfloor \geq n\alpha.$$

This set obviously satisfies the following property:

Lemma 3.3. *If $D \in \mathcal{P}_n$ and D^+ is an n-complement, then $D^+ \geq D$.*

Lemma 3.4 (cf. [Sho00, Lemma 2.7])**.** *If $\mathfrak{R} \subset [0,\, 1] \cap \mathbb{Q}$, $I(\mathfrak{R}) \mid n$, and $0 \leq \epsilon \leq 1/(n+1)$, then*

$$\mathcal{P}_n \supset \Phi(\mathfrak{R}, \epsilon).$$

Proof. Let $1 \geq \alpha \in \Phi(\mathfrak{R}, \epsilon)$. If $\alpha \geq 1-\epsilon$, then

$$(n+1)\alpha > n+1-\epsilon(n+1) \geq n.$$

Hence, $\lfloor (n+1)\alpha \rfloor \geq n \geq n\alpha$ and $\alpha \in \mathcal{P}_n$. Thus we may assume that $\alpha \in \Phi(\mathfrak{R})$. It is sufficient to show that

$$\left\lfloor (n+1)\left(1-\frac{r}{m}\right)\right\rfloor \geq n\left(1-\frac{r}{m}\right) \tag{3.4.1}$$

for all $r \in \mathfrak{R}$ and $m \in \mathbb{Z}$, $m > 0$. We may assume that $r > 0$. It is clear that (3.4.1) is equivalent to the following inequality

$$(n+1)\left(1-\frac{r}{m}\right) \geq k \geq n\left(1-\frac{r}{m}\right), \tag{3.4.2}$$

for some $k \in \mathbb{Z}$ (in fact, $k = \lfloor (n+1)\left(1-\frac{r}{m}\right)\rfloor$). By our conditions, $N := nr \in \mathbb{Z}$, $N > 0$. Thus (3.4.2) can be rewritten as follows

$$mn - N + m - r \geq mk \geq mn - N. \tag{3.4.3}$$

Since $m - r \geq m - 1$, inequality (3.4.3) has a solution in $k \in \mathbb{Z}$. This proves the statement. □

Corollary 3.5 ([Pro01, Prop. 4.3.2], [PS01, Prop. 6.1])**.** *Fix positive integers I, n and a set $\mathfrak{R}$ such that $n \in \mathcal{N}_d(\mathfrak{R}, I)$ and $I(\mathfrak{R}) \mid I$. Let $f\colon Y \to X$ be a birational contraction of d-dimensional varieties and let D be a subboundary on Y such that*

(i) *$K_Y + D$ is nef over X,*
(ii) *f_*D is a boundary with multiplicities in $\Phi(\mathfrak{R}, \epsilon_{d,I})$.*

Assume that $K_X + f_*D$ is n-complemented. Then so is $K_Y + D$.

Corollary 3.6 ([Pro01, Prop. 4.4.1], [PS01, Prop. 6.2])**.** *Fix positive integers I, n and a set $\mathfrak{R}$ such that $n \in \mathcal{N}_d(\mathfrak{R}, I)$ and $I(\mathfrak{R}) \mid I$. Let $(X/Z \ni o, D = S + B)$ be a log variety. Set $S := \lfloor D \rfloor$ and $B := \{D\}$. Assume that*

(i) *$K_X + D$ is plt;*
(ii) *$-(K_X + D)$ is nef and big over Z;*
(iii) *$S \neq 0$ near $f^{-1}(o)$;*
(iv) *$D \in \Phi(\mathfrak{R}, \epsilon_{d,I})$.*

Further, assume that near $f^{-1}(o) \cap S$ there exists an n-complement $K_S + \mathrm{Diff}_S(B)^+$ of $K_S + \mathrm{Diff}_S(B)$. Then near $f^{-1}(o)$ there exists an n-complement $K_X + S + B^+$ of $K_X + S + B$ such that $\mathrm{Diff}_S(B)^+ = \mathrm{Diff}_S(B^+)$.

Adjunction on divisors cf. [Sho93, Cor. 3.10, Lemma 4.2]**.** Fix a subset $\mathfrak{R} \subset \mathbb{R}_{\geq 0}$. Define also the new set

$$\overline{\mathfrak{R}} := \left\{ r_0 - m\sum_{i=1}^{s}(1-r_i) \;\middle|\; r_0, \ldots, r_s \in \mathfrak{R},\ m \in \mathbb{Z},\ m > 0 \right\} \cap \mathbb{R}_{\geq 0}.$$

It is easy to see that $\overline{\mathfrak{R}} \supset \mathfrak{R}$. For example, if $\mathfrak{R} = \{0, 1\}$, then $\overline{\mathfrak{R}} = \mathfrak{R}$.

Lemma 3.7. (i) *If $\mathfrak{R} \subset [0, 1]$, then $\overline{\mathfrak{R}} \subset [0, 1]$.*

(ii) *If $\mathfrak{R}$ is finite and rational, then so is $\overline{\mathfrak{R}}$.*
(iii) $I(\mathfrak{R}) = I(\overline{\mathfrak{R}})$.
(iv) *Let $\mathfrak{G} \subset \mathbb{Q}$ be an additive subgroup containing* 1 *and let $\mathfrak{R} = \mathfrak{R}_{\mathfrak{G}} := \mathfrak{G} \cap [0,1]$. Then $\overline{\mathfrak{R}} = \mathfrak{R}$.*
(v) *If the ascending chain condition (a.c.c.) holds for the set $\mathfrak{R}$ then it holds for $\overline{\mathfrak{R}}$.*

Proof. (i)-(iv) are obvious. We prove (v). Indeed, let

$$q^{(n)} = r_0^{(n)} - m^{(n)} \sum_{i=1}^{s^{(n)}} (1 - r_i^{(n)}) \in \overline{\mathfrak{R}}$$

be an infinite increasing sequence, where $r_i^{(n)} \in \mathfrak{R}$ and $m^{(n)} \in \mathbb{Z}_{>0}$. By passing to a subsequence, we may assume that $m^{(n)} \sum_{i=1}^{s^{(n)}} (1 - r_i^{(n)}) > 0$, in particular, $s^{(n)} > 0$ for all n. There is a constant $\varepsilon = \varepsilon(\mathfrak{R}) > 0$ such that $1 - r_i^{(n)} > \varepsilon$ whenever $r_i^{(n)} \neq 1$. Thus, $0 \leq q^{(n)} \leq r_0^{(n)} - m^{(n)} s^{(n)} \varepsilon$ and $m^{(n)} s^{(n)} \leq (r_0^{(n)} - q^{(n)})/\varepsilon$. Again by passing to a subsequence, we may assume that $m^{(n)}$ and $s^{(n)}$ are constants: $m^{(n)} = m$, $s^{(n)} = s$. Since the numbers $r_i^{(n)}$ satisfy a.c.c., the sequence

$$q^{(n)} = r_0^{(n)} + m \sum_{i=1}^{s} r_i^{(n)} - ms$$

is not increasing, a contradiction. $\square$

Proposition 3.8. *Let $\mathfrak{R} \subset [0,\, 1]$, $1 \in \mathfrak{R}$, $\epsilon \in [0,\, 1]$, and let $(X, S + B)$ be a plt log pair, where S is a prime divisor, $B \geq 0$, and $\lfloor B \rfloor = 0$. If $B \in \Phi(\mathfrak{R}, \epsilon)$, then $\operatorname{Diff}_S(B) \in \Phi(\overline{\mathfrak{R}}, \epsilon)$.*

Proof. Write $B = \sum b_i B_i$, where the B_i are prime divisors and $b_i \in \Phi(\mathfrak{R}, \epsilon)$. Let $V \subset S$ be a prime divisor. By [Sho93, Cor. 3.10] the multiplicity d of $\operatorname{Diff}_S(B)$ along V is computed using the following relation:

$$d = 1 - \frac{1}{n} + \frac{1}{n} \sum_{i=0}^{s} k_i b_i = 1 - \frac{\beta}{n},$$

where n, $k_i \in \mathbb{Z}_{\geq 0}$, and $\beta := 1 - \sum k_i b_i$. It is easy to see that $d \geq b_i$ whenever $k_i > 0$. If $b_i \geq 1 - \epsilon$, this implies $d \geq 1 - \epsilon$. Thus we may assume that $b_i \in \Phi(\mathfrak{R})$ whenever $k_i > 0$. Therefore,

$$\beta = 1 - \sum k_i \left(1 - \frac{r_i}{m_i}\right),$$

where $m_i \in \mathbb{Z}_{>0}$, $r_i \in \mathfrak{R}$. Since $(X, S + B)$ is plt, $d < 1$. Hence, $\beta > 0$. If $m_i = 1$ for all i, then

$$\beta = 1 - \sum k_i (1 - r_i) \in \overline{\mathfrak{R}}.$$

So, $d \in \Phi(\overline{\mathfrak{R}})$ in this case. Thus we may assume that $m_0 > 1$. Since $1 - \frac{r_i}{m_i} \geq 1 - \frac{1}{m_i}$, we have $m_1 = \cdots = m_s = 1$ and $k_0 = 1$. Thus,

$$\beta = \frac{r_0}{m_0} - \sum_{i=1}^{s} k_i(1 - r_i) = \frac{r_0 - m_0 \sum_{i=1}^{s} k_i(1 - r_i)}{m_0}$$

and $m_0\beta = r_0 - m_0 \sum_{i=1}^{s} k_i(1 - r_i) \in \overline{\mathfrak{R}}$. Hence, $d = 1 - \frac{m_0\beta}{m_0 n} \in \Phi(\overline{\mathfrak{R}})$. $\square$

Proposition 3.9. *Let $1 \in \mathfrak{R} \subset [0, 1]$ and let (X, B) be a klt log semi-Fano of dimension $\leq d$ such that X is FT. Take I so that $I(\mathfrak{R}) \mid I$. Assume the LMMP in dimension d. If $B \in \Phi(\mathfrak{R}, \epsilon_{d,I})$, then there is an n-complement of $K + B$ for $n \in \mathcal{N}_d(\mathfrak{R}, I)$.*

Proof. If $\epsilon_{d,I} = 0$, there is nothing to prove. So we assume that $\epsilon_{d,I} > 0$. If X is not $\mathbb{Q}$-factorial, we replace X with its small $\mathbb{Q}$-factorial modification. Write $B = \sum b_i B_i$. Consider the new boundary $D = \sum d_i B_i$, where

$$d_i = \begin{cases} b_i & \text{if } b_i < 1 - \epsilon_{d,I}, \\ 1 - \epsilon_{d,I} & \text{otherwise.} \end{cases}$$

Clearly, $D \in \Phi(\mathfrak{R})$. Since $D \leq B$, there is an effective $\mathbb{Q}$-divisor Λ such that $K + D + \Lambda$ is klt and numerically trivial. Run $-(K + D)$-MMP. Since all the birational transformations are $K + D + \Lambda$-crepant, they preserve the klt property of $(X, D + \Lambda)$ and (X, D). Each extremal ray is Λ-negative, and therefore is birational. At the end we get a model $(\bar{X}, \bar{D})$ which is log semi-Fano. Since $\bar{D} \in \Phi(\mathfrak{R})$ there is an n-complement $\bar{D}^+$ of $K_{\bar{X}} + \bar{D}$ for some $n \in \mathcal{N}_d(\mathfrak{R}, I)$. By Corollary 3.5 we can pull-back this complement to X and as above this gives us an n-complement of $K_X + B$. $\square$

4. General reduction

For inductive purposes, we prove Theorems 1.4 and 1.6 in more general form:

Theorem 4.1. *Fix a positive integer I and a finite rational subset $\mathfrak{R} \subset [0, 1]$ such that $1 \in \mathfrak{R}$. Let (X, B) be a log semi-Fano variety of dimension d such that X is FT and $B \in \Phi(\mathfrak{R}, \epsilon_{d,I})$, where $\epsilon_{d-1,I} := \epsilon_{d-1}(\overline{\mathfrak{R}}, I)$. Assume the LMMP in dimension d. Further, assume that Conjectures* 1.1 *and* 7.12 *hold in dimension d. If either (X, B) is klt or $K_X + B \equiv 0$, then $K_X + B$ is nI-complemented for some $n \leq C(d, \mathfrak{R}, I)$.*

4.2. **Setup.** Let $(X, B = \sum b_i B_i)$ be a klt log semi-Fano variety of dimension d such that X is FT and

$$B \in \Phi(\mathfrak{R}, \epsilon_{d-1,I}), \tag{4.2.1}$$

where $\epsilon_{d-1,I} := \epsilon_{d-1}(\overline{\mathfrak{R}}, I)$. By our inductive hypothesis $\epsilon_{d-1,I} > 0$ whenever $\mathfrak{R} \subset [0, 1]$ is finite and rational. Clearly by changing I we may assume that $I(\mathfrak{R}) \mid I$.

(4.2.2) Assume that the pair (X, B) is $\epsilon_{d-1,I}$-lt. Then the multiplicities of B are contained in the finite set $\Phi(\mathfrak{R}) \cap [0,\, 1-\epsilon_{d-1,I}]$. By Conjecture 1.1 the pair (X, B) is bounded. Hence $(X, \operatorname{Supp} B)$ belongs to an algebraic family and we may assume that the multiplicities of B are fixed. The condition that $K + B$ is n-complemented is equivalent to the following

$$\exists\, \overline{B} \in |-K - \lfloor (n+1)B \rfloor| \quad \text{such that} \quad \left(X, \frac{1}{n}\left(\lfloor (n+1)B\rfloor + \bar{B}\right)\right) \quad \text{is lc}$$

(see [Sho93, 5.1]). Obviously, the last condition is closed. By Noetherian induction $K + B$ has bounded complements.

¿From now on we assume that (X, B) is not $\epsilon_{d-1,I}$-lt.

4.3. We replace (X, B) with $\mathbb{Q}$-factorial blowup of all divisors E of discrepancy $a(E, X, B) \le -1 + \epsilon_{d-1,I}$, see [Kol92, 21.6.1]. Condition (4.2.1) is preserved. Note that our new X is again FT by Lemma 2.9. ¿From now on we assume that X is $\mathbb{Q}$-factorial and

$$\operatorname{discr}(X, B) > -1 + \epsilon_{d-1,I}. \tag{4.3.1}$$

(4.3.2) For some $n_0 \gg 0$, the divisor $n_0 B$ is integral and the linear system $|-n_0(K+B)|$ is base point free. Let $\overline{B} \in |-n_0(K+B)|$ be a general member. Put $\Theta := B + \frac{1}{n_0}\overline{B}$. By Bertini's theorem $\operatorname{discr}(X, \Theta) = \operatorname{discr}(X, B)$. Thus we have the following

- $\Theta \ge B$,
- (X, Θ) is klt,
- $\operatorname{discr}(X, \Theta) \ge 1 + \epsilon_{d-1,I}$,
- $K_X + \Theta$ is $\mathbb{Q}$-linearly trivial, and
- $\Theta - B$ is supported in a movable (possibly trivial) divisor.

Define a new boundary

$$D := \sum d_i B_i, \qquad \text{where} \quad d_i = \begin{cases} 1 & \text{if} \quad b_i \ge 1 - \epsilon_{d-1,I}, \\ b_i & \text{otherwise.} \end{cases} \tag{4.3.3}$$

Clearly, $D \in \Phi(\mathfrak{R})$ and by (4.3.1)

$$\lfloor D \rfloor \ne 0. \tag{4.3.4}$$

Lemma 4.4 (the simplest case of the global-to-local statement)**.** *Fix a finite set $\mathfrak{R} \subset [0,1] \cap \mathbb{Q}$ and $I \in \mathbb{Z}_{>0}$ such that $I(\mathfrak{R}) \mid I$. Let $(X \ni o, D)$ be the germ of a $\mathbb{Q}$-factorial klt d-dimensional singularity, where $D \in \Phi(\mathfrak{R}, \epsilon_{d-1,I})$. There is an n-complement of $K_X + D$ with $n \in \mathcal{N}_{d-1}(\overline{\mathfrak{R}}, I)$.*

Proof. Consider a plt blowup $f\colon \tilde{X} \to X$ of (X, D) (see [PS01, Prop. 3.6]). By definition the exceptional locus of f is an irreducible divisor E, $(\tilde{X}, \tilde{D}+E)$ is plt, and $-(K_{\tilde{X}} + \tilde{D} + E)$ is f-ample, where $\tilde{D}$ is the proper transform of D. We can take f so that $f(E) = o$, i.e., E is projective. By Adjunction $-(K_E + \operatorname{Diff}_E(\tilde{D}))$ is nef and $(E, \operatorname{Diff}_E(\tilde{D}))$ is klt. By Proposition 3.8 we have $\operatorname{Diff}_E(\tilde{D}) \in \Phi(\overline{\mathfrak{R}}, \epsilon_{d-1,I})$. Hence there is an n-complement of $K_E + \operatorname{Diff}_E(\tilde{D})$

with $n \in \mathcal{N}_{d-1}(\overline{\mathfrak{R}}, I)$, see Proposition 3.9. This complement can be extended to $\tilde{X}$ by Corollary 3.6. $\square$

Claim 4.5. *The pair* (X, D) *is lc.*

Proof. By Lemma 4.4 near each point $P \in X$ there is an n-complement $K+B^+$ of $K+B$ with $n \in \mathcal{N}_{d-1}(\overline{\mathfrak{R}}, I)$. By Lemma 3.4, we have $\mathcal{P}_n \supset \Phi(\mathfrak{R}, \epsilon)$. Hence, by Lemma 3.3, $B^+ \geq B$. On the other hand, nB^+ is integral and for any component of $D - B$, its multiplicity in B is $\geq \epsilon_{d-1,I} > 1/(n+1)$. Hence, $B^+ \geq D$ and so (X, D) is lc near P. $\square$

4.6. Run $-(K + D)$-MMP (anti-MMP), see Lemma 2.7. It is clear that property (4.2.1) is preserved on each step. All birational transformations are $(K + \Theta)$-crepant. Therefore $K + \Theta$ is klt on each step. Since $B \leq \Theta$, so is $K + B$. By Claim 4.5 the log canonical property of (X, D) is also preserved and X is FT on each step by 2.9.

Claim 4.6.1. *None of components of* $\lfloor D \rfloor$ *is contracted.*

Proof. Let $\varphi\colon X \to \bar{X}$ be a $K + D$-positive extremal contraction and let E be the corresponding exceptional divisor. Assume that $E \subset \lfloor D \rfloor$. Put $\bar{D} := \varphi_* D$. Since $K_X + D$ is φ-ample, we can write

$$K_X + D = \varphi^*(K_{\bar{X}} + \bar{D}) - \alpha E, \qquad \alpha > 0.$$

Since $(\bar{X}, \bar{D})$ is lc, we have

$$-1 \leq a(E, \bar{X}, \bar{D}) = a(E, X, D) - \alpha = -1 - \alpha < -1,$$

a contradiction. $\square$

Corollary 4.7. *Condition* (4.3.1) *holds on each step of our MMP.*

Proof. Note that all our birational transformations are $(K + \Theta)$-crepant. Hence by (4.3.2), it is sufficient to show that none of the components of Θ with multiplicity $\geq 1 - \epsilon_{d-1,I}$ is contracted. Assume that on some step we contract a component B_i of multiplicity $b_i \geq 1 - \epsilon_{d-1,I}$. Then by (4.3.3) B_i is a component of $\lfloor D \rfloor$. This contradicts Claim 4.6.1. $\square$

4.8. **Reduction.** After a number of divisorial contractions and flips

$$X \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_N = Y, \tag{4.8.1}$$

we get a $\mathbb{Q}$-factorial model Y such that either

(4.8.2) there is a non-birational $K_Y + D_Y$-positive contraction $\varphi\colon Y \to Z$ to a lower-dimensional variety Z, or

(4.8.3) $-(K_Y + D_Y)$ is nef.

Here $\square_Y$ denotes the proper transform of $\square$ on Y.

Claim 4.9. *In case* 4.8.2, *Z is a point, i.e., $\rho(Y) = 1$ and $-(K_Y + B_Y)$ is nef.*

Proof. Let $F = \varphi^{-1}(o)$ be a general fibre. Since $\rho(Y/Z) = 1$ and $-(K + B) \equiv \Theta - B \geq 0$, the restriction $-(K + B)|_F$ is nef. It is clear that $B|_F \in \Phi(\mathfrak{R}, \epsilon_{d-1,I})$. Assume that Z is of positive dimension. Then $\dim F < \dim X$. By our inductive hypothesis and Proposition 3.9 there is a bounded n-complement $K_F + B|_F^+$ of $K_F + B|_F$ for some $n \in \mathcal{N}_{d-1}(\mathfrak{R}, I) \subset \mathcal{N}_{d-1}(\overline{\mathfrak{R}}, I)$. By Lemmas 3.3 and 3.4, we have $B|_F^+ \geq D|_F \geq B|_F$. On the other hand, $(K_X + D)|_F$ is φ-ample, a contradiction. □

4.10. Therefore we have a $\mathbb{Q}$-factorial FT variety Y and two boundaries $B_Y = \sum b_i B_i$ and $D_Y = \sum d_i B_i$ such that $\operatorname{discr}(Y, B_Y) > -1 + \epsilon_{d-1,I}$, $B_Y \in \Phi(\mathfrak{R}, \epsilon_{d-1,I})$, $D_Y \in \Phi(\mathfrak{R})$, $D_Y \geq B_Y$, and $d_i > b_i$ if and only if $d_i = 1$ and $b_i \geq -1 + \epsilon_{d-1,I}$. Moreover, one of the following two cases:

(4.10.1) $\rho(Y) = 1$, $K_Y + D_Y$ is ample, and (Y, B_Y) is a klt log semi-Fano variety, or

(4.10.2) (Y, D_Y) is a log semi-Fano variety.

These two cases will be treated in sections 6 and 9, respectively.

4.11. **Outline of the proof.** Now we sketch the basic idea in the proof of boundedness in case 4.10.1. Recall that on each step of (4.8.1) we contract an extremal ray which is $(K+D)$-positive. By Corollary 3.5 we can pull-back n-complements with $n \in \mathcal{N}_d(\mathfrak{R}, I)$ of $K_Y + D_Y$ to our original X. However it can happen in case $\rho(Y) = 1$ that $K_Y + D_Y$ has no any complements. In this case we will show in Section 6 below that the multiplicities of B_Y are bounded from the above: $b_i < 1 - c$, where $c > 0$. By Claim 4.6.1 divisorial contractions in (4.8.1) do not contracts components of B with multiplicities $b_i \geq 1 - \epsilon_{d-1,I}$. Therefore the multiplicities of B also are bounded from the above. Combining this with $\operatorname{discr}(X, B) > -1 + \epsilon_{d-1,I}$ and Conjecture 1.1 we get that $(X, \operatorname{Supp} B)$ belong to an algebraic family. By Noetherian induction (cf. (4.2.2)) we may assume that $(X, \operatorname{Supp} B)$ is fixed. Finally, by Proposition 5.4 and Remark 5.5 we have that (X, B) has bounded complements.

Case (4.10.2) will be treated in Sect. 9. In fact in this case we study the contraction $f\colon Y \to Z$ given by $-(K + D)$. When Z is a lower-dimensional variety, f is a fibration onto varieties with trivial log canonical divisor. The existence of desired complements can be established inductively, by using an analog of Kodaira's canonical bundle formula (see Conjecture 7.12)

5. Approximation and complements

The following Lemma 5.2 shows that the existence of n-complements is an open condition in the space of all boundaries B with fixed $\operatorname{Supp} B$.

5.1. **Notation.** Let $\mathcal{B}$ be a finite set of prime divisors B_i. Recall that $\mathfrak{D}_{\mathcal{B}}$ denotes the $\mathbb{R}$-vector space all $\mathbb{R}$-Weil divisors B with $\operatorname{Supp} B = \sum_{B_i \in \mathcal{B}} B_i$. Let

$$\mathfrak{I}_{\mathcal{B}} := \left\{ \sum \beta_i B_i \in \mathfrak{D}_{\mathcal{B}} \mid 0 \leq \beta_i \leq 1, \ \forall i \right\}$$

be the unit cube in $\mathfrak{D}_{\mathcal{B}}$.

Lemma 5.2. *Let (X, B) be a d-dimensional log pair where B is an $\mathbb{R}$-boundary. Assume that $K + B$ is n-complemented. Then there is a constant $\varepsilon = \varepsilon(X, B, n) > 0$ such that $K + B'$ is also n-complemented for any $\mathbb{R}$-boundary $B' \in \mathfrak{D}_B$ with $\|B - B'\| < \varepsilon$.*

Proof. Let $B^+ = B^\sharp + \Lambda$ be an n-complement, where Λ and B have no common components and $B^\sharp \in \mathfrak{D}_B$. Write $B = \sum b_i B_i$, $B' = \sum b'_i B_i$, $B^\sharp = \sum b_i^+ B_i$. Take ε so that

$$0 < (n+1)\varepsilon < \min(1 - \{(n+1)b_i\} \mid 1 \le i \le r, \quad b_i < 1).$$

We claim that B^+ is also an n-complement of B' whenever $\|B - B'\| < \varepsilon$. If $b_i = 1$, then obviously $b_i^+ = 1$. So, it is sufficient to verify the inequalities $nb_i^+ \ge \lfloor (n+1)b'_i \rfloor$ whenever $b_i^+ < 1$ and $b_i < 1$. Indeed, in this case,

$$\lfloor (n+1)b'_i \rfloor \le \lfloor (n+1)b_i + (n+1)(b'_i - b_i) \rfloor = \lfloor (n+1)b_i \rfloor \le nb_i^+.$$

(because $(n+1)(b'_i - b_i) < (n+1)\varepsilon < \min(1 - \{(n+1)b_i\})$). This proves the assertion. $\square$

Corollary 5.3. *For any $D \in Z_{d-1}(X)$ the subset*

$$\mathfrak{U}_D^n := \{B \in \mathfrak{I}_D \mid K + B \textit{ is } n\textit{-complemented}\}$$

is open in $\mathfrak{I}_D$.

Proposition 5.4. *Fix a positive integer I. Let $(X, \mathcal{B} = \sum_{i=1}^r B_i)$ be a projective log pair such that the following condition holds:*

(5.4.1) (Effective base point freeness) *There is a positive integer N such that for any integral nef Weil $\mathbb{Q}$-Cartier divisor of the form $mK + \sum m_i B_i$ the linear system $|N(mK + \sum m_i B_i)|$ is base point free.*

Then for any boundary $B \in \mathfrak{I}_{\mathcal{B}}$ such that $K + B$ is lc and $-(K + B)$ is nef, there is an n-complement of $K + B$ for some $n \le \mathrm{Const}\,(X, \mathcal{B})$ and $I \mid n$.

Remark 5.5. Condition (5.4.1) is satisfied if X is FT and all the B_i are $\mathbb{Q}$-Cartier.

Indeed, in this case $\mathrm{Pic}(X) \simeq \mathbb{Z}^\rho$ (see e.g. [IP99, Prop. 2.1.2]). In the space $\mathrm{Pic}(X) \otimes \mathbb{R} \simeq \mathbb{R}^\rho$ we have a closed convex cone $\mathrm{NEF}(X)$, the cone of nef divisors. This cone is dual to the Mori cone $\overline{\mathrm{NE}}(X)$, so it is rational polyhedral and generated by a finite number of semiample Cartier divisors $M_1, \dots, M_s$. Take a positive integer N' so that all the linear systems $|N' M_i|$ are base point free, and $N'K, N'B_1, \dots, N'B_r$ are Cartier. Write

$$N'K \sim_{\mathbb{Q}} \sum_{i=1}^s \alpha_{i,0} M_i, \quad N'B_j \sim_{\mathbb{Q}} \sum_{i=1}^s \alpha_{i,j} M_i, \qquad \alpha_{i,j} \in \mathbb{Q}, \qquad \alpha_{i,j} \ge 0.$$

Let N'' be the common multiple of denominators of the $\alpha_{i,j}$. Then

$$N'^2 N'' \left(mK + \sum_{j=1}^{r} m_j B_j \right) \sim N'N''m \left(\sum_{i=1}^{s} \alpha_{i,0} M_i \right) +$$

$$\sum_{j=1}^{r} N'N''m_j \left(\sum_{i=1}^{s} \alpha_{i,j} M_i \right) = \sum_{i=1}^{s} \left(mN''\alpha_{i,0} + \sum_{j=1}^{r} N''m_j\alpha_{i,j} \right) N'M_i$$

The last (integral) divisor generates a base point free linear system, so we can take $N = N'^2 N''$.

In the proof of Proposition 5.4 we follow arguments of [Sho00, Example 1.11], see also [Sho93, 5.2].

Proof of Proposition 5.4. Define the set

$$\mathcal{M} = \mathcal{M}_{\mathcal{B}} := \{B \in \mathfrak{I}_{\bar{B}} \quad | \quad K + B \text{ is lc and } -(K+B) \text{ is nef}\} \tag{5.5.1}$$

Then $\mathcal{M}$ is a closed compact convex polyhedron in $\mathfrak{I}_{\mathcal{B}}$. It is sufficient to show the existence of some n-complement for any $B \in \mathcal{M}$. Indeed, then $\mathcal{M} \subset \bigcup_{n \in \mathbb{Z}_{>0}} \mathfrak{U}^n_{\mathcal{B}}$. By taking a finite subcovering $\mathcal{M} \subset \bigcup_{n \in \mathcal{S}} \mathfrak{U}^n_{\mathcal{B}}$ we get a finite number of such n.

Assume that there is a boundary $B^o = \sum_{i=1}^{r} b_i^o B_i \in \mathcal{M}$ which has no any complements. By [Cas57, Ch. 1, Th. VII] there is infinite many rational points $(m_1/q, \dots, m_r/q)$ such that

$$\max \left(\left| \frac{m_1}{q} - b_1^o \right|, \dots, \left| \frac{m_r}{q} - b_r^o \right| \right) < \frac{r}{(r+1)q^{1+1/r}} < \frac{1}{q^{1+1/r}}.$$

Denote $b_i := m_i/q$ and $B := \sum b_i B_i$. Thus, $\|B - B^o\| < 1/q^{1+1/r}$. Then our proposition is an easy consequence of the following

Claim 5.6. *For $q \gg 0$ one has*

(5.6.1) $\lfloor (qN+1) b_i^o \rfloor \le qNb_i$ *whenever* $b_i < 1$;

(5.6.2) $B \equiv B^o$ *and* $-(K+B)$ *is nef; and*

(5.6.3) $K + B$ *is lc.*

Indeed, by (5.4.1) the linear system $|-qN(K+B)|$ is base point free. Let $F \in |-qN(K+B)|$ be a general member. Then $K + B + \frac{1}{qN}F$ is an qN-complement of $K + B^o$, a contradiction. $\square$

Proof of Claim. By the construction

$$\lfloor (qN+1) b_i^o \rfloor = m_i N + \lfloor b_i^o + qN(b_i^o - b_i) \rfloor$$

Put $c := \max_{b_i^o < 1} \{b_1^o, \dots b_r^o\}$. Then for $b_i < 1$ we have $b_i^o < c < 1$ and for $q \gg 0$,

$$b_i^o + qN(b_i^o - b_i) < c + \frac{qN}{q^{1+1/r}} < 1.$$

This proves (5.6.1).

Further, let $L_1, \dots, L_r$ be a finite set of curves generating $N_1(X)$. We have

$$|L_j \cdot (B - B^o)| = \left| \sum_i \frac{m_i}{q}(L_j \cdot B_i) - \sum_i b_i^o (L_j \cdot B_i) \right|$$
$$< \frac{1}{q^{1+1/r}} \sum_i (L_j \cdot B_i).$$

If $q \gg 0$, then the right hand side is $\ll 1/q$ while the left hand side is from the discrete set $\pm L_j \cdot (K + B^o) + \frac{1}{qN}\mathbb{Z}$ (because qB is an integral divisor and by our assumption (5.4.1)). Hence the left hand side is zero and $B \equiv B^o$. This proves (5.6.2).

Finally, we have to show that $K + B$ is lc. Assume the converse. By (5.4.1) the divisor $qN\,(K + B)$ is Cartier. So there is a divisor E of the field $\mathcal{K}(X)$ such that $a(E, X, B) \le -1 - 1/qN$ and $a(E, X, B^o) \ge -1$. On the other hand, $a(E, X, \sum \beta_i B_i)$ is an affine linear function in β_i:

$$\frac{1}{qN} \le a(E, X, B^o) - a(E, X, B) = \sum c_i (b_i^o - b_i) < \frac{\mathrm{Const}}{q^{1+1/r}}$$

which is a contradiction. □

The following is the first induction step to prove Theorem 4.1.

Corollary 5.7 (One-dimensional case)**.** *Fix a finite set $\mathfrak{R} \subset [0, 1] \cap \mathbb{Q}$ and a positive integer I. Then the set $\mathcal{N}_1(\mathfrak{R}, I)$ is finite.*

Proof. Let (X, B) be an one-dimensional log pair satisfying conditions of (3.2.1). Since X is FT, $X \simeq \mathbb{P}^1$. Since $B \in \Phi(\mathfrak{R})$ and $\mathfrak{R}$ is finite, we can write $B = \sum_{i=1}^r b_i B_i$, where $b_i \ge \delta$ for some fixed $\delta > 0$. Thus we may assume that r is fixed and $B_1, \dots, B_r$ are fixed distinct points. Then by Proposition 5.4 we have a desired complements. □

Example 5.8. Let $X \simeq \mathbb{P}^1$. If $\mathfrak{R} = \{0, 1\}$, then $I(\mathfrak{R}) = 1$ and $\Phi(\mathfrak{R})$ is the set of standard multiplicities. In this case, it is easy to compute that $\mathcal{N}_1(\mathfrak{R}) = \{1, 2, 3, 4, 6\}$ [Sho93, 5.2]. Consider more complicated case when $\mathfrak{R} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, 1\}$. Then $I = 12$ and one can compute that

$$\mathcal{N}_1(\mathfrak{R}) = 12 \cdot \{1, 2, 3, 4, 5, 7, 8, 9, 11\}.$$

Indeed, assume that (X, D) has no any $12n$-complements for $n \in \{1, 2, 3, 4, 5, 7, 8, 9, 11\}$. Write $D = \sum_{i=1}^r d_i D_i$, where $D_i \ne D_j$ for $i \ne j$. It is clear that the statement about the existence of n-complement D^+ such that $D^+ \ge D$ is equivalent to the following inequality

$$\sum_i \lceil n d_i \rceil \le 2n. \tag{5.8.1}$$

Since $d_i = 1 - r_i/m_i$, where $r_i \in \mathfrak{R}$, $m_i \in \mathbb{Z}_{>0}$, we have $d_i \ge 1/6$ for all i. We claim that at least one denominator of d_i does not divide 24. Indeed, otherwise $24D$ is an integral divisor and $D^+ := D + \frac{1}{24}\sum_{j=1}^{k} D_j$ is an 24-complement, where $D_j \in X$ are general points and $k = 24(2 - \deg D)$. Thus we may assume that the denominator of d_1 does not divide 24. Since $d_1 = 1 - r_1/m_1$, where $r_1 \in \mathfrak{R}$, we have $m_1 \geq 3$ and the equality holds only if $r_1 = 2/3$ or $5/6$. In either case, $d_1 \geq 13/18$.

Recall that a log pair (X, D) of global type is said to be *exceptional* if at has at least one $\mathbb{Q}$-complement and any $\mathbb{Q}$-complement is klt. If (X, D) is not exceptional, we may increase d_1: $d_1 = 1$. Then as above $13/18 \leq d_2 \leq 5/6$, so $r = 3$ and $d_3 \leq 5/18$. Now there are only a few possibilities for d_2 and d_3:

d_2	$\frac{13}{18}$	$\frac{3}{4}$	$\frac{7}{9}$	$\frac{19}{24}$	$\frac{4}{5}$	$\frac{13}{16}$	$\frac{5}{6}$
d_3	$\leq \frac{5}{18}$	$\leq \frac{1}{4}$	$\leq \frac{2}{9}$	$\leq \frac{5}{24}$	$\leq \frac{1}{5}$	$\leq \frac{3}{16}$	$\leq \frac{1}{6}$

In all cases $K + D$ has a $12n$ complement for some $n \in \{1, 2, 3, 4, 5\}$. In the exceptional case, there is a finite number of possibilities for $(d_1, \ldots, d_r)$. However the computations are much longer. We omit them.

6. The main theorem: Case $\rho = 1$

6.1. Now we begin to consider the case $\rho(Y) = 1$. Let $(X^{(m)}, B^{(m)})$ be a sequence of klt log pairs such as in Theorem 4.1. Assume that complements of $K_{X^{(m)}} + B^{(m)}$ are unbounded. By Corollary 5.7 $\dim X^{(m)} \geq 2$. Apply general reduction from Section 4. We get a sequence of birational maps $X^{(m)} \dashrightarrow Y^{(m)}$. Assume that we are in Case 4.10.1. In particular, $\rho(Y^{(m)}) = 1$, $Y^{(m)}$ is FT, and $K_{Y^{(m)}} + D^{(m)}$ is ample for all m. Recall that by Corollary 4.7

$$\operatorname{discr}(Y^{(m)}) \geq \operatorname{discr}(Y^{(m)}, B^{(m)}) \geq \operatorname{discr}(X^{(m)}, B^{(m)}) > -1 + \epsilon,$$

where $\epsilon = \epsilon_{d-1}(\overline{\mathfrak{R}}, I) > 0$. Note that $-(K_{Y^{(m)}} + B^{(m)})$ is nef, so by Conjecture 1.1 the sequence of varieties $Y^{(m)}$ is bounded. By Noetherian induction (cf. (4.2.2)) we may assume that $Y^{(m)}$ is fixed: $Y^{(m)} = Y$.

Let $Y \hookrightarrow \mathbb{P}^N$ be an embedding and let H be a hyperplane section of Y. Note that the multiplicities of $B_Y^{(m)} = \sum b_i^{(m)} B_i^{(m)}$ are bounded from below: $b_i^{(m)} \geq \epsilon_0 > 0$, where $\epsilon_0 := \min \Phi(\mathfrak{R}) \setminus \{0\}$. Then, for each $B_i^{(m)}$,

$$\epsilon_0 H^{d-1} \cdot B_i^{(m)} \leq H^{d-1} \cdot B_Y^{(m)} \leq -H^{d-1} \cdot K_Y.$$

This shows that the degree of $B_i^{(m)}$ is bounded and $B_i^{(m)}$ belongs to an algebraic family. Therefore we may assume that $\operatorname{Supp} B_Y^{(m)}$ is also fixed: $B_i^{(m)} = B_i$.

6.2. Assume that the multiplicities of $B^{(m)}$ are bounded from 1: $b_i^{(m)} \leq 1 - c$, where $c > 0$. Then we argue as in 4.11. By Claim 4.6.1 divisorial contractions in (4.8.1) do not contract components of $B^{(m)}$ with multiplicities $b_i^{(m)} \geq 1 - \epsilon$. Therefore the multiplicities of $B^{(m)}$ on $X^{(m)}$ are also bounded from the above. Combining this with $\operatorname{discr}(X, B) > -1 + \epsilon$ and

Conjecture 1.1 we get that $(X, \operatorname{Supp} B)$ belong to an algebraic family. By Noetherian induction (cf. (4.2.2)) we may assume that $(X, \operatorname{Supp} B)$ is fixed. Finally, by Proposition 5.4 and Remark 5.5 we have that (X, B) has bounded complements.

¿From now on we consider the case when the multiplicities of $B^{(m)}$ on Y are not bounded from 1. By passing to a subsequence we may assume that $\lfloor \lim B^{(m)} \rfloor \neq 0$.

Proposition 6.3. *If $\lfloor \lim B^{(m)} \rfloor \neq 0$, then $K_Y + B^{(m)}$ is n-complemented for some $n \in \mathcal{N}_{d-1}(\overline{\mathfrak{R}}, I)$ and for $m \gg 0$.*

Proof. By our assumption $\lim b_i^{(m)} = 1$ for some i and we can put $i = 1$. Denote $B^\infty := \lim_{m\to\infty} B^{(m)}$. It is clear that $-(K+B^\infty)$ is nef and $\lfloor B^\infty \rfloor \neq 0$ (more precisely, $b_1^\infty = 1$). Furthermore, $\operatorname{discr}(Y, B^\infty) \geq -1+\varepsilon$. In particular, (Y, B^∞) is plt.

Claim 6.4. *We have $b_j^\infty \leq 1 - \varepsilon$ for all $1 < j \leq r$. Moreover, by passing to a subsequence we may assume the following:*

(i) *If $b_j^\infty = 1$, then $j = 1$ and $b_j^{(m)}$ is strictly increasing.*
(ii) *If $b_j^\infty < 1 - \epsilon$, then $b_j^{(m)} = b_j^\infty$ is a constant.*
(iii) *If $b_j^\infty = 1 - \epsilon$, then $b_j^{(m)}$ is either a constant or strictly decreasing.*

In particular, $B^\infty \in \Phi(\mathfrak{R})$ and B^∞ is a $\mathbb{Q}$-boundary.

Proof. Since $\rho(Y) = 1$ and Y is $\mathbb{Q}$-factorial, the intersection $B_1 \cap B_j$ is of codimension one. For a general hyperplane section $Y \cap H$ we again have inequality

$$\operatorname{discr}(Y \cap H, B^\infty \cap H) \geq -1 + \varepsilon.$$

Thus by Lemma 6.6 below, we have $b_1^\infty + b_j^\infty \leq 2 - \varepsilon$, i.e, $b_j^\infty \leq 1 - \varepsilon$ for all $1 < j \leq r$. The rest follows from the fact that $\Phi(\mathfrak{R}) \cap [0, 1-\varepsilon]$ is finite. □

Corollary 6.5. *$b_j^\infty = 1 - \epsilon$ for some j.*

Proof. Indeed, otherwise $D^{(m)} = B^\infty$ and $-(K_Y + D^{(m)})$ is nef, a contradiction. □

Put $B' := B^\infty - B_1$. By the last corollary $B' \neq 0$. By Proposition 3.8 we have $\operatorname{Diff}_{B_1}(B') \in \Phi(\overline{\mathfrak{R}})$. If $-(K_Y + B^\infty)$ is ample, then we can extend complements of $K_{B_1} + \operatorname{Diff}_{B_1}(B')$ from $B_1 = \lfloor B^\infty \rfloor$ by Corollary 3.6. By Lemma 5.2 $K_Y + B^{(m)}$ is n-complemented for some $n \in \mathcal{N}_{d-1}(\overline{\mathfrak{R}}, I)$ in this case.

Consider the case when $K_Y + B^\infty \equiv 0$. There is an n-complement $K_{B_1} + \operatorname{Diff}_{B_1}(B')^+$ of $K_{B_1} + \operatorname{Diff}_{B_1}(B')$ for some $n \in \mathcal{N}_{d-1}(\overline{\mathfrak{R}}, I)$. By Lemma 3.4 $B' \in \mathcal{P}_n$. Take sufficiently small positive δ and let j be such that $b_j^\infty = 1 - \epsilon$. We claim that $B' - \delta B_j \in \mathcal{P}_n$. Indeed, otherwise

$$(n+1)b_j^\infty = (n+1)(1-\epsilon) = n + 1 - (n+1)/(N_{d-1}(\overline{\mathfrak{R}}, I) + 2)$$

is an integer. This is impossible. Again by Corollary 3.6 n-complement $K_{B_1}+\mathrm{Diff}_{B_1}(B')^+$ of $K_{B_1}+\mathrm{Diff}_{B_1}((1-\delta)B')$ can be extended to Y. Clearly, this gives us also an n-complement of K_Y+B_1+B'. □

Lemma 6.6 (cf. [Sho00, Prop. 5.2], [Pro01, §9]). *Fix $N\in\mathbb{Z}$, $N\geq 7$ and put $\varepsilon:=1/N$. Let $(S\ni o,\Lambda=\lambda_1\Lambda_1+\lambda_2\Lambda_2)$ be a log surface germ such that $\lambda_1,\lambda_2\geq 1-\varepsilon$. Assume that* $\mathrm{discr}(S,\Lambda)\geq -1+\varepsilon$ *at o. Then one of the following holds:*

(i) *$(S\ni o)$ is Du Val of type A_n, $n\geq 1$ and $\lambda_1=\lambda_2=1-\varepsilon$;*
(ii) *$(S\ni o)$ is smooth (moreover, $(S,\Lambda_1+\Lambda_2)$ is log non-singular) and $\lambda_1+\lambda_2\leq 2-\varepsilon$.*

Proof. According to [PS01, Th. 3.1] there is an n-complement $K_S+\Lambda^+$ near o for some $n\in\{1,2,3,4,6\}$. Since $\lambda_1,\lambda_2\geq 1-\varepsilon$, we have $\Lambda^+=\Lambda_1+\Lambda_2$. In this situation there is an analytic isomorphism

$$(S,\Lambda,o)\simeq(\mathbb{C}^2,\{xy=0\},0)/\mathbb{Z}_m(1,q),$$

where m is a positive integer such that $\gcd(m,q)=1$. Assume that $m\geq 2$. Take q so that $1\leq q\leq m-1$ and consider the weighted blow up with weights $\frac{1}{m}(1,q)$. We get an exceptional divisor E with discrepancy

$$-1+\varepsilon\leq a(E,S,\Lambda)=-1+\frac{1+q}{m}-\frac{\lambda_1}{m}-\frac{q\lambda_2}{m}.$$

Thus,

$$\begin{aligned}0\leq 1+q-\lambda_1-q\lambda_2-\frac{m}{N}&\leq 1+q\\ &-(1+q)(1-\varepsilon)-\frac{m}{N}=\frac{1+q-m}{N}.\end{aligned}$$

This gives as $q=m-1$ and equalities $\lambda_1=\lambda_2=1-\varepsilon$, i.e. case (i). If the point $(S\ni o)$ is smooth, then the usual blow up gives us

$$-1+\varepsilon\leq a(E,S,\Lambda)=1-\lambda_1-\lambda_2,\qquad \lambda_1+\lambda_2\leq 2-\varepsilon.$$

□

7. Effective adjunction

In this section we discuss the adjunction conjecture for fibre spaces. This conjecture can be considered as a generalization of the classical Kodaira canonical bundele formula for canonical bundle, see [Kod63], [Fuj86], [Kaw97], [Kaw98], [Amb99], [Fuj99], [FM00], [Fuj03], [Amb04], [Amb05].

7.1. **The set-up.** Let $f\colon X\to Z$ be a surjective morphism of normal varieties and let $D=\sum d_iD_i$ be an $\mathbb{R}$-divisor on X such that (X,D) is lc near the generic fibre of f and $K+D$ is $\mathbb{R}$-Cartier over the generic point of any prime divisor $W\subset Z$. In particular, $d_i\leq 1$ whenever $f(D_i)=Z$. Let $d:=\dim X$ and $d':=\dim Z$.

7.2. **Construction.** For a prime divisor $W \subset Z$ define a real number c_W as the log canonical threshold over the generic point of W:
(7.2.1)
$$c_W := \sup \{c \mid (X, D + cf^{\bullet}W) \quad \text{is lc over the generic point of } W\}.$$
It is clear that $c_W \in \mathbb{Q}$ whenever D is a $\mathbb{Q}$-divisor. Put $d_W := 1 - c_W$. Then the $\mathbb{R}$-divisor
$$D_{\mathrm{div}} := \sum_W d_W W$$
is called the *divisorial part* of adjunction (or *discriminant* of f) for $K_X + D$. It is easy to see that D_{div} is a divisor, i.e., d_W is zero except for a finite number of prime divisors.

Remark 7.3. (i) Note that the definition of the discriminant D_{div} is a codimension one construction, so computing D_{div} we can systematically remove codimension two subvarieties in Z and pass to general hyperplane sections $f_H \colon X \cap f^{-1}(H) \to Z \cap H$.
(ii) Let $h \colon X' \to X$ be a birational contraction and let D' be the crepant pull-back of D:
$$K_{X'} + D' = h^*(K_X + D), \quad h_*D' = D.$$
Then $D'_{\mathrm{div}} = D_{\mathrm{div}}$, i.e., the discriminant D_{div} does not depend on the choice of crepant birational model of (X, D) over Z.

The following lemma is an immediate consequence of the definition.

Lemma 7.4. *Notation as in* 7.1.
(i) (effectivity, cf. [Sho93, 3.2]) *If D is boundary over the generic point of any prime divisor $W \subset Z$, then D_{div} effective.*
(ii) (semiadditivity, cf. [Sho93, 3.2]) *Let Δ be an $\mathbb{R}$-divisor on Z and let $D' := D + f^{\bullet}\Delta$. Then $D'_{\mathrm{div}} = D_{\mathrm{div}} + \Delta$.*
(iii) *(X, D) is klt (resp., lc) over the generic point of W if and only if $d_W < 1$ (resp., $d_W \le 1$).*
(iv) *If (X, D) is lc and D is an $\mathbb{R}$- (resp., $\mathbb{Q}$-)boundary, then D_{div} is an $\mathbb{R}$- (resp., $\mathbb{Q}$-)boundary.*

7.5. **Construction.** ¿From now on assume that f is a contraction, $K_X + D$ is $\mathbb{R}$-Cartier, and $K + D \sim_{\mathbb{R}} f^*L$ for some $\mathbb{R}$-Cartier divisor L on Z. Recall that the latter means that there are real numbers α_j and rational functions $\varphi_j \in \mathcal{K}(X)$ such that

(7.5.1) $$K + D - f^*L = \sum \alpha_j\,(\varphi_j).$$

Define the *moduli part* D_{mod} of $K_X + D$ by

(7.5.2) $$D_{\mathrm{mod}} := L - K_Z - D_{\mathrm{div}}.$$

Then we have

(7.5.3) $$K_X + D = f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}) + \sum \alpha_j\,(\varphi_j).$$

In particular,
$$K_X + D \sim_{\mathbb{R}} f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}).$$
Clearly, D_{mod} depends on the choice of representatives of K_X and K_Z, and also on the choice of α_j and φ_j in (7.5.1). Any change of K_X and K_Z and change of α_j and φ_j gives a new D_{mod} which differs from the original one modulo $\mathbb{R}$-linear equivalence.

If $K+D$ is $\mathbb{Q}$-Cartier, the definition of the moduli part is more explicit. By our assumption (7.5.1) there is an integer I_0 such that $I_0(K+D)$ is linearly trivial on the generic fibre. Then for some rational function $\psi \in \mathcal{K}(X)$, the divisor $M := I_0(K+D) + (\psi)$ is vertical (and $\mathbb{Q}$-linearly trivial over Z). Thus,
$$M - I_0 f^*L = (\psi) + \sum I_0 \alpha_j\, (\varphi_j), \qquad \alpha_j \in \mathbb{Q}.$$
Rewrite it in a more compact form: $M - I_0 f^*L = \alpha\, (\varphi)$, $\alpha \in \mathbb{Q}$, $\varphi \in \mathcal{K}(X)$. The function φ vanishes on the generic fibre, hence it is a pull-back of some function $\upsilon \in \mathcal{K}(Z)$. Replacing L with $L + \frac{\alpha}{I_0}\, (\upsilon)$ we get $M = I_0 f^*L$ and

$$K_X + D - f^*L = \frac{1}{I_0}(\psi), \qquad \psi \in \mathcal{K}(X). \tag{7.5.4}$$

In other words, $K + D \sim_{I_0} f^*L$. Here L is $\mathbb{Q}$-Cartier. Then again we define the moduli part D_{mod} of $K_X + D$ by (7.5.2), where L is taken to satisfy (7.5.4). In this case, D_{mod} is $\mathbb{Q}$-Cartier and we have

$$K_X + D = f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}) + \frac{1}{I_0}\, (\psi). \tag{7.5.5}$$

In particular,
$$K_X + D \sim_{I_0} f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}).$$
As above, D_{mod} depends on the choice of representatives of K_X and K_Z, and also on the choice of I_0 and ψ in (7.5.4). Note that I_0 depends only on f and the horizontal part of D. Once these are fixed, we usually will assume that I_0 is a constant. Then any change of K_X, K_Z, and ψ gives a new D_{mod} which differs from the original one modulo I_0-linear equivalence.

Remark 7.5.1. By Lemma 7.4 $(D + f^\bullet\Delta)_{\mathrm{mod}} = D_{\mathrm{mod}}$. Roughly speaking this means that "the moduli part depends only on the horizontal part of D".

Remark 7.6. Let $g\colon Z' \to Z$ be a birational contraction. Consider the following diagram

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{g} & Z \end{array} \tag{7.6.1}$$

where X' is a resolution of the dominant component of $X \times_Z Z'$. Let D' be the crepant pull-back of D that is $K_{X'} + D' = h^*(K_X + D)$ and $h_*D' = D$. By Remark 7.3 we have $g_*D'_{\mathrm{div}} = D_{\mathrm{div}}$. Therefore, the discriminant defines a b-divisor $\mathbf{D}_{\mathrm{div}}$.

For a suitable choice of K'_X, we can write

$$h^*(K_X + D) = K_{X'} + D',$$

Now we fix the choice of K, α_j and φ_j in (7.5.1) (resp. K and ψ in (7.5.4)) and induce them naturally to X'. Then D_{mod} and D'_{mod} are uniquely determined and $g_*D'_{\mathrm{mod}} = D_{\mathrm{mod}}$. This defines a b-divisor $\mathbf{D}_{\mathrm{mod}}$.

We can write

$$K_{Z'} + D'_{\mathrm{div}} + D' = g^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}) + E,$$

where E is g-exceptional. Since

$$h^*f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}) \equiv K_{X'} + D' \equiv f'^*(K_{Z'} + D'_{\mathrm{div}} + D'_{\mathrm{mod}}),$$

we have $E = 0$ (see [Sho93, 1.1]), i.e., g is $(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}})$-crepant:

$$K_{Z'} + D'_{\mathrm{div}} + D'_{\mathrm{mod}} = g^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}). \tag{7.6.2}$$

Let us consider several examples.

Example 7.7. Assume that the contraction f is birational. Then by the ramification formula [Sho93, §2], [Kol92, Prop. 20.3] and negativity lemma [Sho93, 1.1] we have $D_{\mathrm{div}} = f_*D$, $K + D = f^*(K_Z + D_{\mathrm{div}})$, and $D_{\mathrm{mod}} = 0$.

Example 7.8. Let $X = Z \times \mathbb{P}^1$ and let f be the natural projection to the first factor. Take very ample divisors $H_1, \dots, H_4$ on Z. Let C be a section and let D_i be a general member of the linear system $|f^*H_i + C|$. Put $D := \frac{1}{2}\sum D_i$. Then $K_X + D$ is $\mathbb{Q}$-linearly trivial over Z. By Bertini's theorem $D + f^*P$ is lc for any point $P \in Z$. Hence $D_{\mathrm{div}} = 0$. On the other hand,

$$K_X + D = f^*K_Z - 2C + \frac{1}{2}f^*\sum H_i + 2C = f^*\left(K_Z + \frac{1}{2}\sum H_i\right).$$

This gives us that $D_{\mathrm{mod}} \sim_{\mathbb{Q}} \frac{1}{2}\sum H_i$.

Example 7.9. Let X be a hyperelliptic surface. Recall that it is constructed as the quotient $X = (E \times C)/G$ of the product of two elliptic curves by a finite group G acting on E and C so that the action of G on E is fixed point free and the action on C has fixed points. Let

$$f\colon X = (E \times C)/G \to \mathbb{P}^1 = C/G$$

be the projection. It is clear that degenerate fibres of f can be only of type $m\mathrm{I}_0$. Using the classification of such possible actions (see, e.g., [BPVdV84, Ch. V, Sect. 5]) we obtain the following cases:

Type	singular fibres	D_{div}
a) $(2K_X \sim 0)$	$2\mathrm{I}_0$, $2\mathrm{I}_0$, $2\mathrm{I}_0$, $2\mathrm{I}_0$	$\frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3 + \frac{1}{2}P_4$
b) $(3K_X \sim 0)$	$3\mathrm{I}_0$, $3\mathrm{I}_0$, $3\mathrm{I}_0$	$\frac{2}{3}P_1 + \frac{2}{3}P_2 + \frac{2}{3}P_3$
c) $(4K_X \sim 0)$	$2\mathrm{I}_0$, $4\mathrm{I}_0$, $4\mathrm{I}_0$	$\frac{1}{2}P_1 + \frac{3}{4}P_2 + \frac{3}{4}P_3$
d) $(6K_X \sim 0)$	$2\mathrm{I}_0$, $3\mathrm{I}_0$, $6\mathrm{I}_0$	$\frac{1}{2}P_1 + \frac{2}{3}P_2 + \frac{5}{6}P_3$

In all cases the moduli part D_{mod} is trivial.

Assumption 7.10. Under notation of 7.1 assume additionally that D is a $\mathbb{Q}$-divisor and there is a $\mathbb{Q}$-divisor Θ on X such that $K_X + \Theta$ is $\mathbb{Q}$-linearly trivial over Z and $(F, (1-t)D|_F + t\Theta|_F)$ is a klt log pair for any $0 < t \le 1$, where F is the generic fibre of f. In particular, Θ and D are $\mathbb{Q}$-boundaries near the generic fibre. In this case, both D_{div} and D_{mod} are $\mathbb{Q}$-divisors.

The following result is very important.

Theorem 7.11 ([Amb04]). *In notation and assumptions of* 7.1 *and* 7.10 *b-divisors* $\mathbf{K} + \mathbf{D}_{\mathrm{div}}$ *and* $\mathbf{D}_{\mathrm{mod}}$ *are b-Cartier.*

Proof. Put $D_t := tD + (1-t)\Theta$. For $0 \le t < 1$, $K_X + D_t$ is klt and $\mathbb{Q}$-linearly trivial over Z. By [Amb04, Theorem 0.2] $\mathbf{K} + (\mathbf{D}_t)_{\mathrm{div}}$ and $(\mathbf{D}_t)_{\mathrm{mod}}$ are b-Cartier. Hence so are

$$\mathbf{D}_{\mathrm{mod}} = \frac{1}{t}(\mathbf{D}_t)_{\mathrm{mod}} - \frac{1-t}{t}(\mathbf{D}_0)_{\mathrm{mod}}$$

and

$$\mathbf{K} + \mathbf{D}_{\mathrm{div}} = \frac{1}{t}(\mathbf{K} + (\mathbf{D}_t)_{\mathrm{div}}) - \frac{1-t}{t}(\mathbf{K} + (\mathbf{D}_0)_{\mathrm{div}}).$$

□

According to Kawamata [Kaw98, Theorem 2] (see also [Amb04, Theorem 0.2 (ii)], [Fuj99]), for $D \ge 0$ and (X, D) is klt near the generic fibre, $\mathbf{D}_{\mathrm{mod}}$ is b-nef. We expect more.

Conjecture 7.12. *Let notation and assumptions be as in* 7.1 *and* 7.10. *We have*

(7.12.1) (*Log Canonical Adjunction*) $\mathbf{D}_{\mathrm{mod}}$ *is b-semiample.*

(7.12.2) (*Particular Case of Effective Log Abundance Conjecture*) *Let* X_η *be the generic fibre of* f. *Then* $I_0(K_{X_\eta} + D_\eta) \sim 0$, *where* I_0 *depends only on* $\dim X_\eta$ *and horizontal multiplicities of* D.

(7.12.3) (*Effective Adjunction*) $\mathbf{D}_{\mathrm{mod}}$ *is effectively b-semiample, that is, there exists a positive integer* I *depending only on the dimension of* X *and the horizontal multiplicities of* D (*a finite set of rational numbers*) *such that* $I\mathbf{D}_{\mathrm{mod}}$ *is very b-semiample, that is,* $I\mathbf{D}_{\mathrm{mod}} = \overline{M}$, *where* M *is a base point free divisor on some model* Z'/Z.

Note that by (7.5.5) we may assume that

$$K + D \sim_I f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}). \tag{7.12.4}$$

Remark 7.13. We expect that hypothesis in 7.12 can be weakened as follows.

(7.12.1) It is sufficient to assume that $K + D$ is lc near the generic fibre, the horizontal part D^{h} of D is a $\mathbb{R}$-boundary, $K + D$ is $\mathbb{R}$-Cartier, and $K + D \equiv f^*L$.

(7.12.2) D^{h} is a $\mathbb{Q}$-boundary and $K + D \equiv 0$ near the generic fibre.

(7.12.3) D^{h} is a $\mathbb{Q}$-boundary, $K+D$ is $\mathbb{R}$-Cartier, and $K+D \equiv f^*L$.

This however is not needed for the proof of the main theorem.

Remark 7.14. In the notation of (7.12.2) we have $K_{X_\eta}+D_\eta \sim_{\mathbb{Q}} 0$. Assume that

(i) X is FT, and
(ii) LMMP and conjectures 1.1 and 7.12 hold in dimemsions $\le \dim X - \dim Z$.

Then the pair (X_η, D_η) satisfies the assumptions of Theorem 1.6 with $\mathfrak{R}$ depending only on horizontal multiplicities of D. Hence $I_0(K_{X_\eta}+D_\eta) \sim 0$, where I_0 depends only on $\dim X_\eta$ and horizontal multiplicities of D. Thus (7.12.2) holds automatically under additional assumptions (i)-(ii).

Example 7.15 (Kodaira formula [Kod63], [Fuj86]**).** Let $f\colon X \to Z$ be a fibration satisfying 7.5 whose generic fibre is an elliptic curve. Then $D^{\mathrm{h}}=0$ and $I_0=1$. Thus we can write $K_X+D^{\mathrm{v}}=f^*L$. The j-invariant defines a rational map $J\colon Z \dashrightarrow \mathbb{C}$. By blowing up Z and X we may assume that both X and Z are smooth and J is a morphism: $J\colon Z \to \mathbb{P}^1$. Let P be a divisor of degree 1 on $\mathbb{P}^1$. Take a positive integer n such that $12n$ is divisible by the multiplicities of all the degenerate fibres of f. In this situation, there is a generalization of the classical Kodaira formula [Fuj86]:

$$12nK_X = f^*\left(12nK_Z + 12nD_{\mathrm{div}} + nJ^*P\right),$$

We can rewrite it as follows

$$K_X = f^*\left(K_Z + D_{\mathrm{div}} + \frac{1}{12}J^*P\right). \tag{7.15.1}$$

Here $D_{\mathrm{mod}} = \frac{1}{12}J^*P$ is semiample and the coefficients of D_{div} are taken from the table in Example 3.1.

Example 7.16. Fix a positive integer m. Let $(E,0)$ be an elliptic curve with fixed group low and let $e_m \in E$ be an m-torsion. Define the action of $\boldsymbol{\mu}_m := \left\{\sqrt[m]{1}\right\}$ on $E\times\mathbb{P}^1$ by

$$\varepsilon(e,z) = (e+e_m, \varepsilon z), \qquad e\in E,\ z\in\mathbb{P}^1,$$

where $\varepsilon \in \boldsymbol{\mu}_m$ is a primitive m-root. The quotient map

$$X := (E\times\mathbb{P}^1)/\boldsymbol{\mu}_m \longrightarrow \mathbb{P}^1/\boldsymbol{\mu}_m \simeq \mathbb{P}^1$$

is an elliptic fibration having exactly two fibres of types $m\mathrm{I}_0$ over points 0 and $\infty\in\mathbb{P}^1$. Using the Kodaira formula one can show that

$$K_X = f^*K_Z + (m-1)F_0 + (m-1)F_\infty,$$

where $F_0 := f^{-1}(0)_{\mathrm{red}}$ and $F_\infty := f^{-1}(\infty)_{\mathrm{red}}$. Hence, in (7.5.4) we have $I_0=1$ and

$$L = K_Z + \left(1-\frac{1}{m}\right)\cdot 0 + \left(1-\frac{1}{m}\right)\cdot\infty.$$

Clearly,

$$D_{\text{div}} = \left(1 - \frac{1}{m}\right) \cdot 0 + \left(1 - \frac{1}{m}\right) \cdot \infty.$$

Hence $D_{\text{mod}} = 0$, $I = I_0 = 1$, and $K_X = f^*(K_Z + D_{\text{div}})$.

Corollary 7.17 (cf. [Amb04, Th. 3.1])**.** *Let notation and assumptions be as in* 7.1 *and* 7.10 (*cf.* 7.13).

(i) *If $(Z, D_{\text{div}} + D_{\text{mod}})$ is lc and $\mathbf{D}_{\text{mod}}$ is effective, then (X, D) is lc.*

(ii) *Assume that* (7.12.1) *holds. If (X, D) is lc, then so is $(Z, D_{\text{div}} + D_{\text{mod}})$ for a suitable choice of D_{mod} in the class of $\mathbb{Q}$-linear equivalence. Moreover, if (X, D) is lc and any lc centre of (X, D) dominates Z, then $(Z, D_{\text{div}} + D_{\text{mod}})$ is klt.*

Proof. For a log resolution $g \colon Z' \to Z$, consider base change (7.6.1). Take g so that Z' is smooth, $\mathbf{K} + \mathbf{D}_{\text{div}} = \overline{K_{Z'} + D'_{\text{div}}}$, $\mathbf{D}_{\text{mod}} = \overline{D'_{\text{mod}}}$, and $\operatorname{Supp} D_{\text{div}}$ is a simple normal crossing divisor.

(i) Assume that (X, D) is not lc. Let F be a divisor of discrepancy $a(F, X, D) < -1$. Since (X, D) is lc near the generic fibre, the centre of F on Z is a proper subvariety. Moreover, by [Kol96, Ch. VI, Th. 1.3] we can take g so that the centre of F on Z' is a prime divisor, say W. Put $D_Z := D_{\text{div}} + D_{\text{mod}}$. By (7.6.2) we have

$$-1 \le a(W, Z, D_Z) = a(W, Z', D'_{\text{div}} + D'_{\text{mod}}) \le a(W, Z', D'_{\text{div}}).$$

Therefore (X', D') is lc over the generic point of W (see (7.2.1)). In particular, $a(F, X', D') = a(F, X, D) \ge -1$, a contradiction.

(ii) By (7.12.1) we can take g so that D'_{mod} is semiample. By (7.6.2) g is $(K + D_{\text{div}} + D_{\text{mod}})$-crepant. If (Z', D'_{div}) is lc (resp. klt), then replacing D'_{mod} with an effective general representative of the corresponding class of $\mathbb{Q}$-linear equivalence we obtain

$$\operatorname{discr}(Z, D_{\text{div}} + D_{\text{mod}}) = \operatorname{discr}(Z', D'_{\text{div}} + D'_{\text{mod}}) = \operatorname{discr}(Z', D'_{\text{div}}) \ge -1.$$

(resp. > -1). We can suppose also that $\lfloor D_{\text{mod}} \rfloor = 0$. Hence $(Z, D_{\text{div}} + D_{\text{mod}})$ is lc (resp. klt) in this case. Thus we assume that (Z', D'_{div}) is not lc (resp. not klt). Let E be a divisor over Z of discrepancy $a(E, Z', D'_{\text{div}}) \le -1$. Clearly, we may assume that $\operatorname{Center}_{Z'} E \not\subset \operatorname{Supp} D'_{\text{mod}}$. Then $a(E, Z', D'_{\text{div}} + D'_{\text{mod}}) = a(E, Z', D'_{\text{div}}) \le -1$. Replacing Z' with its blowup we may assume that E is a prime divisor on Z' (and again $\operatorname{Center}_{Z'} E \not\subset \operatorname{Supp} D'_{\text{mod}}$). Since (X', D') is lc and by (7.2.1), $c_E = 0$, $d_E = 1$, and $a(E, Z', D'_{\text{div}}) = -1$. Then (Z', D'_{div}) is lc. Furthermore, by (7.2.1) the pair $(X', D' + cf'^{\bullet} E)$ is not lc for any $c > 0$. This means that $f^{-1}(\operatorname{Center}_Z(E))$ contains an lc centre. $\square$

The following example shows that the condition $\mathbf{D}_{\text{mod}} \ge 0$ in (i) of Corollary 7.17 cannot be omitted.

Example 7.18. Let $f \colon X \to Z = \mathbb{C}^2$ be a standard conic bundle given by $x^2 + uy^2 + vz^2$ in $\mathbb{P}^2_{x,y,z} \times \mathbb{C}^2_{u,v}$. The linear system $|-nK_X|$ is base point free for $n \ge 1$. Let $H \in |-2K_X|$ be a general member. Now let $\Gamma(t) := \Gamma_1 + t\Gamma_2$, where $\Gamma_1 := \{u = 0\}$ and $\Gamma_2 := \{v = 0\}$. Put $D(t) := \frac{1}{2}H + f^*\Gamma(t)$. Then $2(K + D) = 2f^*\Gamma(t)$ and $D(t)_{\mathrm{div}} = \Gamma(t)$. Since $K_Z = 0$, we have $D_{\mathrm{mod}} = 0$.

For $t = 1$, the log divisor $K_Z + \Gamma(t)$ is lc but $K + D(t) = f^*(K_Z + \Gamma(t))$ is not. Indeed, in the chart $z \neq 0$ there is an isomorphism

$$(X, f^*\Gamma) \simeq (\mathbb{C}^3_{x,y,u}, \{u(x^2 + uy^2) = 0\}). \tag{7.18.1}$$

The explanation of this fact is that the b-divisor $\mathbf{D}_{\mathrm{mod}}$ is non-trivial. To show this we consider the following diagram [Sar80, §2]:

$$\begin{tikzcd} X' \arrow[r, dashed, "\chi"] \arrow[dd, "f'"] & \tilde{X} \arrow[dr, "h"] & \\ & & X \arrow[d, "f"] \\ Z' \arrow[rr, "g"] & & Z \end{tikzcd}$$

where h is the blowup the central fibre $f^{-1}(0)_{\mathrm{red}}$, χ is the simplest flop, g is the blowup of 0, and f' is again a standard conic bundle. Put $t = 1/2$ and let $\tilde{D}$ and D' be the crepant pull-backs of $D := D(t)$ on $\tilde{X}$ and X', respectively. The h-exceptional divisor F appears in $\tilde{D}$ with multiplicity $1/2$. Let F' be the proper transform of F on X'. Then $F' = f'^*E$, where E is the g-exceptional divisor. It is easy to see from (7.18.1) that the pair (X, D) is lc but not klt at the generic point of $f^{-1}(0)_{\mathrm{red}}$. So is $(\tilde{X}, \tilde{D})$ at the generic point of the flopping curve. This implies that (X', D') is lc but not klt over the generic point of E. Therefore, $D'_{\mathrm{div}} = E + \Gamma'$, where Γ' is the proper transform of Γ. On Z', we have $K_{Z'} = E$ and $K + D' = f'^*g^*\Gamma$, so $D'_{\mathrm{mod}} = g^*\Gamma - E - D'_{\mathrm{div}} = -\frac{1}{2}E$. Thus $D' \leq 0$ and $2D'_{\mathrm{mod}}$ is free.

8. Two important particular cases of Effective Adjunction

Using the result of [Kaw97] we prove the following.

Theorem 8.1. *Conjectures* 7.12 *hold if* $\dim X = \dim Z + 1$.

Remark 8.2. We expect that in this case on can take $I = 12q$, where q is a positive integer such that qD^{h} is an integral divisor.

Proof. We may assume that a general fibre of f is a rational curve (see Example 7.15). Thus the horizontal part D^{h} of D is non-trivial. First we reduce the problem to the case when all components of D^{h} are generically sections. Write $D = \sum d_i D_i$ and take

$$\delta := \min\{d_i \mid D_i \text{ is horizontal and } d_i > 0\}.$$

(we allow components with $d_i = 0$). Let D_i be a horizontal component and let $D_i \to \hat{Z} \xrightarrow{g} Z$ be the Stein factorization of the restriction $f|_{D_i}$. Let

$n_i := \deg g$. Let l be a general fibre of f. Since $d_i D_i \cdot l \le D \cdot l = -K \cdot l = 2$, we have

$$n_i = D_i \cdot l \le 2/d_i \le 2/\delta. \tag{8.2.1}$$

Assume that $n_i > 1$. Consider the base change

$$\begin{array}{ccc} \hat{X} & \xrightarrow{h} & X \\ {\scriptstyle \hat{f}}\downarrow & & {\scriptstyle f}\downarrow \\ \hat{Z} & \xrightarrow{g} & Z \end{array}$$

where $\hat{X}$ is the normalization of the dominant component of $X \times_Z \hat{Z}$. Define $\hat{D}$ on $\hat{X}$ by

$$K_{\hat{X}} + \hat{D} = h^*(K_X + D). \tag{8.2.2}$$

More precisely, $\hat{D} = \sum_{i,j} \hat{d}_{i,j} \hat{D}_{i,j}$, where $h(\hat{D}_{i,j}) = D_i$, $1 - \hat{d}_{i,j} = r_{i,j}(1 - d_i)$, and $r_{i,j}$ is the ramification index along $\hat{D}_{i,j}$. By construction, the ramification locus Ξ of h is $\hat{f}$-exceptional, that is $\hat{f}(\Xi) \neq \hat{Z}$. Therefore, $\hat{D}$ is a boundary near the generic fibre. Similarly, we define $\hat{\Theta}$ as the crepant pull-back of Θ from 7.10. Thus the pair $(\hat{X}, \hat{D})$ satisfies assumptions of 7.1 and 7.10. It follows from (8.2.2) that

$$K_{\hat{X}} + \hat{D} = \hat{f}^* g^* (K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}).$$

According to [Amb99, Th. 3.2] for the discriminant $\hat{D}_{\mathrm{div}}$ of $\hat{f}$ we have

$$K_{\hat{Z}} + \hat{D}_{\mathrm{div}} = g^*(K_Z + D_{\mathrm{div}}).$$

For a suitable choice of $\hat{D}_{\mathrm{mod}}$ in the class of n_i-linear equivalence, we can write

$$K_{\hat{Z}} + \hat{D}_{\mathrm{div}} + \hat{D}_{\mathrm{mod}} = g^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}).$$

Therefore, $\hat{D}_{\mathrm{mod}} = g^* D_{\mathrm{mod}}$. If $\hat{I}\hat{D}_{\mathrm{mod}}$ is free for some positive integer $\hat{I}$, then so is $n_i \hat{I} D_{\mathrm{mod}}$. Thus we have proved the following.

Claim 8.3. *Assume that Conjecture* 7.12 *holds for $\hat{f} \colon \hat{X} \to \hat{Z}$ with constant $\hat{I}$. Then this conjecture holds for $f \colon X \to Z$ with $I := n_i \hat{I}$.*

Note that the restriction $\hat{f}|_{h^{-1}(D_i)} \colon h^{-1}(D_i) \to \hat{Z}$ is generically finite of degree n_i. Moreover, $h^{-1}(D_i)$ has a component which is a section over the generic point. Applying Claim 8.3 several times and taking (8.2.1) into account we obtain the desired reduction to the case when all the horizontal D_i's are generically sections.

8.4. Further by making a birational base change and by blowing up X we can get the situation when

(i) Z and X are smooth,
(ii) the D_i's are regular disjointed sections,

(iii) the morphism f is smooth outside of a simple normal crossing divisor $\Xi \subset Z$,

(iv) $f^{-1}(\Xi) \cup \operatorname{Supp} D$ is also a simple normal crossing divisor.

Let n be the number of horizontal components of D. Note that we alow multiplicities $d_i = 0$ on this step.

Let $\mathcal{M}_n$ be the moduli space of n-pointed stable rational curves, let $f_n \colon \mathcal{U}_n \to \mathcal{M}_n$ be the corresponding universal family, and let $\mathcal{P}_1, \dots, \mathcal{P}_n$ be sections of f_n which correspond to the marked points (see [Knu83]). It is known that both $\mathcal{M}_n$ and $\mathcal{U}_n$ are smooth and projective. Take $d_i \in [0, 1]$ so that $\sum d_i = 2$ and put $\mathcal{D} := \sum d_i \mathcal{P}_i$. Then $K_{\mathcal{U}_n} + \mathcal{D}$ is trivial on the general fibre. However, $K_{\mathcal{U}_n} + \mathcal{D}$ is not numerically trivial over $\mathcal{M}_n$ moreover it is not nef over $\mathcal{M}_n$:

Theorem 8.5 (see [Kee92], [Kaw97]). (i) *There exist a smooth projective variety $\bar{\mathcal{U}}_n$, a $\mathbb{P}^1$-bundle $\bar{f}_n \colon \bar{\mathcal{U}}_n \to \mathcal{M}_n$, and a sequence of blowups (blowdowns) with smooth centres*

$$\sigma \colon \mathcal{U}_n = \mathcal{U}^1 \to \mathcal{U}^2 \to \cdots \to \mathcal{U}^{n-2} = \bar{\mathcal{U}}_n.$$

(ii) *For $\bar{\mathcal{D}} := \sigma_* \mathcal{D}$, the (discrepancy) divisor*

$$\mathcal{F} := K_{\mathcal{U}_n} + \mathcal{D} - \sigma^*(K_{\bar{\mathcal{U}}_n} + \bar{\mathcal{D}})$$

is effective and essentially exceptional *on $\mathcal{M}_n$.*

(iii) *There exists a semiample $\mathbb{Q}$-divisor $\mathcal{L}$ on $\mathcal{M}_n$ such that*

$$K_{\bar{\mathcal{U}}_n} + \bar{\mathcal{D}} = \bar{f}_n^*(K_{\mathcal{M}_n} + \mathcal{L}).$$

Therefore,

$$K_{\mathcal{U}_n} + \mathcal{D} - \mathcal{F} = f_n^*(K_{\mathcal{M}_n} + \mathcal{L}).$$

Recall that for any contraction $\varphi \colon Y \to Y'$, a divisor G on Y is said to be *essentially exceptional* over Y' if for any prime divisor P on Y', the support of the divisorial pull-back $\varphi^{\bullet} P$ is not contained in $\operatorname{Supp} G$.

Corollary 8.6. *In the above notation we have*

$$(\mathcal{D} - \mathcal{F})_{\mathrm{div}} = 0, \qquad (\mathcal{D} - \mathcal{F})_{\mathrm{mod}} = \mathcal{L}.$$

Proof. See Example 8.10 below. $\square$

Since the horizontal components of D are sections, $(X/Z, D^{\mathrm{h}})$ is generically an n-pointed stable curve [Knu83]. Hence we have the induced rational maps

$$\begin{array}{ccc} X & \overset{\beta}{\dashrightarrow} & \mathcal{U}_n \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_n} \\ Z & \overset{\phi}{\dashrightarrow} & \mathcal{M}_n \end{array}$$

so that $f_n \circ \beta = \phi \circ f$ and $\beta(D_i) \subset \mathcal{P}_i$. Let $\Xi \subset Z$ be a closed subset such that f is smooth over $Z \setminus \Xi$. Replacing X and Z with its birational models and D with its crepant pull-back we may assume additionally to 8.4 that β

and ϕ are regular morphisms. Put $\mathcal{D} = \sum d_i \mathcal{P}_i$, where $D^{\mathrm{h}} = \sum_{f(D_i)=Z} d_i D_i$. Consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\mu"] \arrow[rr, bend left, "\beta"] \arrow[ddr, "f"'] & \hat{X} \arrow[r, "\psi"] \arrow[dd, "\hat{f}"] & \mathcal{U}_n \arrow[dd, "f_n"] \arrow[dr, "\sigma"] & \\
 & & & \bar{\mathcal{U}_n} \arrow[dl, "\bar{f}_n"] \\
 & Z \arrow[r, "\phi"] & \mathcal{M}_n &
\end{tikzcd}$$

where $\hat{X} := Z \times_{\mathcal{M}_n} \mathcal{U}_n$.

8.7. Since the fibres of f_n are stable curves, near every point $u \in \mathcal{U}_n$ the morphism f_n is either smooth or in a suitable local coordinate system is given by

$$(u_1,\, u_2, \ldots, u_{n-2}) \longmapsto (u_1 u_2,\, u_2, \ldots, u_{n-2}).$$

Then easy local computations show that $\hat{X}$ is normal and has only canonical singularities [Kaw97]. Moreover, the pair $(\hat{X}, \hat{D}^{\mathrm{h}} = \psi^* \mathcal{D})$ is canonical because f_n is a smooth morphism near $\operatorname{Supp} \mathcal{D}$.

We have

$$K_X + D = f^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}).$$

Put $\hat{D} := \mu_* D$. Then $K_X + D = \mu^*(K_{\hat{X}} + \hat{D})$, so

$$\hat{D}_{\mathrm{div}} = D_{\mathrm{div}}, \qquad \hat{D}_{\mathrm{mod}} = D_{\mathrm{mod}},$$

$$K_{\hat{X}} + \hat{D} = \hat{f}^*(K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}).$$

8.8. Let $\varphi\colon Y \to Y'$ be any contraction, where $\dim Y' \geq 1$. We introduce $G^{\perp} = G - \varphi^{\bullet} G_{\neg}$, where $G_{\neg}$ is taken so that the vertical part $(G^{\perp})^{\mathrm{v}}$ of $G^{\perp}$ is essentially exceptional and $G_{\neg}$ is maximal with this property. In particular, $(G^{\perp})^{\mathrm{v}} \leq 0$ over an open subset $U' \subset Y'$ such that $\operatorname{codim}(Y' \setminus U') \geq 2$. Note that our construction of $G_{\neg}$ and $G^{\perp}$ is in codimension one over Y', i.e., to find $G_{\neg}$ and $G^{\perp}$ we may replace Y' with $Y' \setminus W$, where W is a closed subset of codimension ≥ 2.

Lemma 8.9. *Let $\varphi\colon Y \to Y'$ be a contraction and let G be a $\mathbb{R}$-divisor on Y. Assume that* $\dim Y' \geq 1$. *Assume that* $(Y/Y', G)$ *satisfies conditions* 7.1. *The following are equivalent:*

(i) $G^{\mathrm{v}} - \varphi^{\bullet} G_{\mathrm{div}}$ *is essentially exceptional,*
(ii) $G_{\mathrm{div}} = G_{\neg}$,
(iii) $(G^{\perp})_{\mathrm{div}} = 0$.

Proof. Implications (ii) $\iff$ (iii) $\implies$ (i) follows by definition of G_{div} and semiadditivity (Lemma 7.4). Let us prove (i) $\implies$ (ii). Assume that $G^{\text{v}} - \varphi^\bullet G_{\text{div}}$ is essentially exceptional. Then by definition $G_{\text{div}} \le G_\neg$. On the other hand, for any prime divisor $P \subset Y'$, the multiplicity of $G^\perp$ along some component of $\varphi^\bullet P$ is equal to 0. Hence the log canonical threshold of $(K + G^\perp, \varphi^\bullet P)$ over the generic point of P is ≤ 1. So by definition of the divisorial part and Lemma 7.4 we have $0 \le (G^\perp)_{\text{div}} = G_{\text{div}} - G_\neg$. □

Example 8.10. Clearly, for $f_n \colon \mathcal{U}_n \to \mathcal{M}_n$, the discrepancy divisor $\mathcal{F}$ is essentially exceptional. Hence, $(\mathcal{D} - \mathcal{F})_{\text{div}} \ge 0$. On the other hand, by construction every fibre of f_n is reduced. Hence, for every prime divisor $W \subset Z$, the divisorial pull-back $f_n^\bullet W$ is reduced and $\operatorname{Supp}(f_n^\bullet W + \mathcal{D})$ is a simple normal crossing divisor over the generic point of W. This implies that $c_W \ge 1$ and so $(\mathcal{D} - \mathcal{F})_{\text{div}} = 0$.

Proof of Theorem 8.1 (*continued*). According to Ambro's theorem 7.11 by blowing up Z we may assume that the b-divisor $\mathbf{D}_{\text{mod}}$ stabilizes on Z, i.e., $\mathbf{D}_{\text{mod}} = \overline{D_{\text{mod}}}$. Thus it is sufficient to show that $D_{\text{mod}} = \phi^*\mathcal{L} = \phi^*(\mathcal{D} - \mathcal{F})_{\text{mod}}$. In this situation D_{mod} is effectively semiample because $N\mathcal{L}$ is an integral base point free divisor for some N which depends only on n. Since and ϕ is a regular morphism, to show $D_{\text{mod}} = \phi^*\mathcal{L}$ we will freely replace Z with an open subset $U \subset Z$ such that $\operatorname{codim}(Z \setminus U) \ge 2$. Thus all the statements below are valid over codimension one over Z. In particular, we may assume that $D_{\text{mod}} = (D^\perp)_{\text{mod}}$. Replacing D with $D^\perp$ we may assume that $D_\neg = 0$ (we replace Z with U as above). Thus $D^{\text{v}} \le 0$ and D^{v} is essentially exceptional. In particular, $D_{\text{div}} \ge 0$.

On the other hand, by construction the fibres $(\hat{f}^*(z), \hat{D}^{\text{h}} = \psi^*\mathcal{D})$, $z \in Z$ are stable (reduced) curves. In particular, they are slc (semi log canonical [KSB88, §4], [Kol92, Ch. 12]). By the inversion of adjunction [Sho93, §3], [Kol92, Ch. 16-17] for every prime divisor $W \subset Z$ and generic hyperplane sections $H_1, \dots, H_{\dim Z-1}$ the pair $(\hat{X}, \hat{D}^{\text{h}} + \hat{f}^\bullet W + \hat{f}^\bullet H_1 + \cdots + \hat{f}^\bullet H_{\dim Z-1})$ is lc. Since $\hat{D}^{\text{v}} \le 0$, so is the pair $(\hat{X}, \hat{D} + \hat{f}^\bullet W)$. This implies that $c_W \ge 1$ and so $D_{\text{div}} = \hat{D}_{\text{div}} = 0$.

We claim that $\hat{D}^{\text{v}} = \mu_* D^{\text{v}}$ is essentially exceptional. Indeed, otherwise $\hat{D}^{\text{v}}$ is strictly negative over the generic point of some prime divisor $W \subset Z$, i.e., μ contracts all the components E_i of $f^\bullet W$ of multiplicity 0. By 8.7 the pair $(\hat{X}, \hat{D} + \epsilon \hat{f}^\bullet W)$ is canonical over the generic point of W for some small positive ϵ. On the other hand, for the discrepancy of E_i we have $a(E_i, \hat{X}, \hat{D} + \epsilon \hat{f}^\bullet W) = a(E_i, \hat{X}, D + \epsilon f^\bullet W) = -\epsilon$. The contradiction proves our claim.

For relative canonical divisors we have

$$K_{\hat{X}/Z} = \psi^* K_{\mathcal{U}_n/\mathcal{M}_n}$$

(see, e.g., [Har77, Ch. II, Prop. 8.10]). Taking $\hat{D}^{\mathrm{h}} = \psi^*\mathcal{D}$ into account we obtain

$$K_{\hat{X}/Z} + \hat{D}^{\mathrm{h}} - \psi^*\mathcal{F} = \psi^*(K_{\mathcal{U}_n/\mathcal{M}_n} + \mathcal{D} - \mathcal{F}) = \psi^* f_n^*\mathcal{L} = \hat{f}^*\phi^*\mathcal{L}.$$

Hence,

$$-\hat{D}^{\mathrm{v}} - \psi^*\mathcal{F} \sim_{\mathbb{R}} K_{\hat{X}} + \hat{D}^{\mathrm{h}} - \psi^*\mathcal{F} \sim_{\mathbb{R}} \hat{f}^*\phi^*\mathcal{L} + \hat{f}^*K_Z$$

over Z, i.e., $\hat{D}^{\mathrm{v}} + \psi^*\mathcal{F}$ is $\mathbb{R}$-linearly trivial over Z.

Since $\psi^*\mathcal{F}$ is also essentially exceptional over Z, by Lemma 8.11 below we have $\hat{D}^{\mathrm{v}} = -\psi^*\mathcal{F}$ and

$$\hat{f}^*D_{\mathrm{mod}} = \hat{f}^*(D_{\mathrm{mod}} + D_{\mathrm{div}}) = K_{\hat{X}/Z} + \hat{D} = \hat{f}^*\phi^*\mathcal{L}.$$

This gives us $D_{\mathrm{mod}} = \phi^*\mathcal{L} = \phi^*(\mathcal{D} - \mathcal{F})_{\mathrm{mod}}$. Therefore D_{mod} is effectively semiample. This proves Theorem 8.1. □

Lemma 8.11 (cf. [Pro03, Lemma 1.6]). *Let $\varphi\colon Y \to Y'$ be a contraction with $\dim Y' \geq 1$ and let A, B be essentially exceptional over Y' divisors on Y such that $A \equiv B$ and $A, B \leq 0$ (both conditions are over codimension one over Y'). Then $A = B$ over codimension one over Y'.*

Proof. The statement is well-known in the birational case (see [Sho93, §1.1]), so we assume that $\dim Y' < \dim Y$. As in [Pro03, Lemma 1.6], replacing Y' with its general hyperplane section $H' \subset Y'$ and Y with $\varphi^{-1}(H')$ we may assume that $\dim \varphi(\operatorname{Supp} A) = 0$ and $\dim \varphi(\operatorname{Supp} B) \geq 0$. The essential exceptionality of A and B is preserved.

We may also assume that Y' is a sufficiently small affine neighborhood of some fixed point $o \in Y'$ (and $\varphi(\operatorname{Supp} A) = o$). Further, all the conditions of lemma are preserved if we replace Y with its general hyperplane section H. If $\dim Y' > 1$, then we can reduce our situation to the case $\dim Y = \dim Y'$. Then the statement of the lemma follows by [Sho93, §1.1] and from the existence of the Stein factorization. Finally, consider the case $\dim Y' = 1$ (here we may assume that $\dim Y = 2$ and φ has connected fibres). By the Zariski lemma $A = B + a\varphi^* o$ for some $a \in \mathbb{Q}$. Since A and B are essentially exceptional and ≤ 0, $a = 0$. □

□

Example 8.12. Assume that all the components $D_1, \dots, D_r$ of D^{h} are sections. If $r = 3$, then since $\mathcal{M}_3$ is a point, we have $D_{\mathrm{mod}} = 0$. For $r = 4$ the situation is more complicated: $\mathcal{M}_4 \simeq \mathbb{P}^1$, $\mathcal{U}_4$ is a del Pezzo surface of degree 5, and $f_4\colon \mathcal{U}_4 \to \mathcal{M}_4 = \mathbb{P}^1$ is a conic bundle with three degenerate fibres. Each component of degenerate fibre meets exactly two components of $\mathcal{D}$. Hence $\bar{\mathcal{D}}$ is a normal crossing divisor. It is easy to see that σ contracts a component of a degenerate fibre which meets $\mathcal{D}_i$ and $\mathcal{D}_j$ with $d_i + d_j \leq 1$. Clearly, $\bar{\mathcal{U}}_4 \simeq \mathbb{F}_e$ is a rational ruled surface, $e \geq 0$. We can write $\bar{\mathcal{D}}_i \sim \Sigma + a_i F$, where Σ is the minimal section and F is a fibre of $\bar{\mathcal{U}}_4 = \mathbb{F}_e \to \mathbb{P}^1$. Up to permutation we may assume that $\bar{\mathcal{D}}_i \neq \Sigma$ for $i = 2, 3, 4$. Taking $\sum d_i = 2$ into account we get

$$K_{\bar{\mathcal{U}}_4} + \bar{\mathcal{D}} \sim -2\Sigma - (2+e)F + \sum d_i(\Sigma + a_i F) = \left(\sum d_i a_i - e\right) F + \bar{f}_n^* K_{\mathcal{M}_4}.$$

Therefore,

$$\deg \mathcal{L} = \sum d_i a_i - e \geq e \sum d_i - e d_1 - e \geq 0.$$

8.13. Now we consider the case when the base variety is a curve.

Proposition 8.14. *Assume Conjectures* 1.1 *and* 7.12 *in dimensions* $\leq d-1$ *and LMMP in dimension* $\leq d$. *If* X *is FT (and projective), then Conjecture* 7.12 *holds.*

Corollary 8.15. *Conjecture* 7.12 *holds true in the following cases:*

(i) $\dim X = \dim Z + 1$,

(ii) $\dim X = 3$ *and* X *is FT.*

Proof. Immediate by Theorem 8.1 and Proposition 8.14. □

The rest of this section is devoted to proof of Proposition 8.14. Thus from now on and through the end of this section we assume that the base variety Z is a curve. First we note that $Z \simeq \mathbb{P}^1$ because X is FT.

Lemma 8.16. *Fix a positive integer* N. *Let* $f\colon X \to Z \ni o$ *be a contraction to a curve germ and let* D *be an* $\mathbb{R}$*-boundary on* X. *Let* D^{h} *be the horizontal part of* D. *Assume that*

(i) $\dim X \leq d$ *and* X *is FT over* Z,

(ii) ND^{h} *is integral,*

(iii) $K_X + D$ *is lc and numerically trivial over* Z.

Assume LMMP in dimension $\leq d$. *Further assume that the statement of Theorem* 4.1 *holds in dimensions* $\leq d-1$. *Then there is an* n*-complement* $K + D^+$ *of* $K + D$ *near* $f^{-1}(o)$ *such that* $N \mid n$, $n \leq \mathrm{Const}(N, \dim X)$, *and* $a(E, X, D^+) = -1$ *for some divisor* E *with* $\mathrm{Center}_Z\, E = o$.

Proof. Take a finite set $\mathfrak{R} \subset [0,\, 1] \cap \mathbb{Q}$ and I so that $D^{\mathrm{h}} \in \mathfrak{R}$ and $N \mid I$. Replacing D with $D + \alpha f^* o$ we may assume that (X, D) is maximally lc. Next we replace (X, D) with its $\mathbb{Q}$-factorial dlt modification. Let F be a component of $f^{-1}(o)$ of multiplicity 1 in D. Run $-F$-MMP over Z. This does not preserve the dlt property of $K + D$. However the $\mathbb{Q}$-factoriality and lc property are preserved and the reduced component $F \subset \lfloor D \rfloor$ is not contracted. On each step, the contraction is birational. So at the end we get a model with irreducible central fibre: $f^{-1}(o)_{\mathrm{red}} = F$. Then $D \in \Phi(\mathfrak{R})$. Applying D^{h}-MMP over Z, we may assume that D^{h} is nef over Z. By Corollary 3.5 we can pull-back complements to our original X. Note that the f-vertical part of D coincides with F, so it is numerically trivial over Z. Since X is FT over Z, $-K_X$ is big over Z. Therefore $D \equiv D^{\mathrm{h}}$ is nef and big over Z. Now apply construction of [PS01, §3] to (X, D) over Z. There are two cases:

(I) (X, F) is plt,

(II) (X, F) is lc but not plt (recall that $F \leq D$).

Consider, for example, the second case (the first case is much easier and can be treated in a similar way). First we define an auxiliary boundary to localize a suitable divisor of discrepancy -1. By Kodaira's lemma, for some effective $D^{\mho}$, the divisor $D - D^{\mho}$ is ample. Put $D_{\epsilon,\alpha} := (1-\epsilon)D + \alpha D^{\mho}$. Then $K_X + D_{\epsilon,\alpha} \equiv -\epsilon D + \alpha D^{\mho}$. So $(X, D_{\epsilon,\alpha})$ is a klt log Fano over Z for $0 < \alpha \ll \epsilon \ll 1$. Take $\beta = \beta(\epsilon, \alpha)$ so that $(X, D_{\epsilon,\alpha} + \beta F)$ is maximally lc and put $G_{\epsilon,\alpha} := D_{\epsilon,\alpha} + \beta F$. Thus $(X, G_{\epsilon,\alpha})$ is a lc (but not klt) log Fano over Z.

Let $g\colon \widehat{X} \to X$ be an *inductive blowup* of $(X, G_{\epsilon,\alpha})$ [PS01, Proposition 3.6]. By definition $\widehat{X}$ is $\mathbb{Q}$-factorial $\rho(\widehat{X}/X) = 1$, the g-exceptional locus is a prime divisor E of discrepancy $a(E, X, G_{\epsilon,\alpha}) = -1$, $(\widehat{X}, E)$ is plt, and $-(K_{\widehat{X}} + E)$ is ample over X. Since $(X, G_{\epsilon,\alpha} - \gamma F)$ is klt for $\gamma > 0$, $\mathrm{Center}_Z(E) = o$. Note that by construction E is not exceptional on some log resolution of $(X, G_{\epsilon,\alpha})$. Hence we may assume that E and g do not depend on ϵ and α if $0 < \epsilon \ll 1$. In particular, $a(E, X, D) = -1$.

By (iii) of Lemma 2.9 $\widehat{X}$ is FT over Z. Let $\widehat{D}$ and $\widehat{G}_{\epsilon,\alpha}$ be proper transforms on $\widehat{X}$ of D and $G_{\epsilon,\alpha}$, respectively. Then

$$\begin{aligned} 0 \equiv \quad g^*(K_X + D) \quad &= K_{\widehat{X}} + \widehat{D} + E, \\ g^*(K_X + G_{\epsilon,\alpha}) \quad &= K_{\widehat{X}} + \widehat{G}_{\epsilon,\alpha} + E, \end{aligned}$$

where $-(K_X + G_{\epsilon,\alpha})$ is ample over Z. Run $-(K_{\widehat{X}} + E)$-MMP starting from $\widehat{X}$ over Z:

$$\begin{tikzcd} & \widehat{X} \arrow[dl, "g"'] \arrow[dr, dashed] & \\ X \arrow[dr, "f"] & & \overline{X} \arrow[dl, "\bar{f}"'] \\ & Z & \end{tikzcd}$$

Since $-(K_{\widehat{X}} + E) \equiv \widehat{D}$, we can contract only components of $\widehat{D}$. At the end we get a model $(\overline{X}, \overline{D} + \overline{E})$ such that $-(K_{\overline{X}} + \overline{E})$ is nef and big over Z, $K_{\overline{X}} + \overline{E} + \overline{D} \equiv 0$, and $(\overline{X}, \overline{E} + \overline{D})$ is lc.

We claim that the plt property of $K_{\widehat{X}} + E$ is preserved under this LMMP. Indeed, for $0 < t \ll 1$, the log divisor $K_{\widehat{X}} + (1-t)\widehat{G}_{\epsilon,\alpha} + E$ is a convex linear combination of log divisors $K_{\widehat{X}} + \widehat{G}_{\epsilon,\alpha} + E$ and $K_{\widehat{X}} + E$. The first divisor is anti-nef and is trivial only on one extremal ray R, the ray generated by fibres of g. The second one is strictly negative on R. Since $\widehat{X}$ is FT over Z, the Mori cone $\overline{\mathrm{NE}}(\widehat{X}/Z)$ is polyhedral. Therefore $K_{\widehat{X}} + (1-t)\widehat{G}_{\epsilon,\alpha} + E$ is anti-ample (and plt) for $0 < t \ll 1$. By the base point free theorem there is a boundary $M \geq (1-t)\widehat{G}_{\epsilon,\alpha} + E$ such that $(\widehat{X}, M)$ is a plt 0-pair. Since E is not contracted, this property is preserved under our LMMP. Hence $(\overline{X}, \overline{M})$ is plt and so is $(\overline{X}, \overline{E})$. This proves our claim. In particular, $\overline{E}$ is normal and FT.

Take $\delta := 1/m$, $m \in \mathbb{Z}$, $m \gg 0$. For any such δ the pair $(\overline{X}, (1-\delta)\overline{D} + \overline{E})$ is plt and $-(K_{\overline{X}} + (1-\delta)\overline{D} + \overline{E})$ is nef and big over Z. By our inductive hypothesis there is an n-complement $K_{\overline{E}} + \operatorname{Diff}_{\overline{E}}(\overline{D})^+$ of $K_{\overline{E}} + \operatorname{Diff}_{\overline{E}}(\overline{D})$ with $n \in \mathcal{N}_{d-1}(\overline{\mathfrak{R}}, I)$. Clearly, this is also an n-complement of $K_{\overline{E}} + \operatorname{Diff}_{\overline{E}}((1-\delta)\overline{D})$. Note that nD is integral. We claim that $(1-\delta)\overline{D} \in \mathcal{P}_n$. Indeed, the vertical multiplicities of $(1-\delta)\overline{D}$ are contained in $\Phi(\mathfrak{R})$. Let d_i be the multiplicity of a horizontal component of $\overline{D}$. Then $nd_i \in \mathbb{Z}$. If $d_i = 1$, then obviously $(1-\delta)d_i \in \mathcal{P}_n$. So we assume that $d_i < 1$. Then $\lfloor (n+1)d_i \rfloor = nd_i$ and $\lfloor (n+1)(d_i - \delta) \rfloor = nd_i \geq n(d_i - \delta)$ for $\delta \ll 1$. This proves our claim. Now the same arguments as in [PS01, §3] shows that $K_X + (1-\delta)D$ is n-complemented near $f^{-1}(o)$. Since $D \in \mathcal{P}_n$, there is an n-complement $K_X + D^+$ of $K_X + D$ near $f^{-1}(o)$ and moreover, $a(E, X, D^+) = -1$. □

Corollary 8.17. *The multiplicities of D_{mod} are contained in a finite set.*

Proof. Consider a local n-complement D^+ of $K + D$ near $f^{-1}(o)$. Then $n(K_Z + D^+_{\mathrm{div}} + D^+_{\mathrm{mod}})$ is integral at o. By construction, (X, D^+) has a centre of log canonical singularities contained in $f^{-1}(o)$. Hence $D^+_{\mathrm{div}} = 0$. By semiadditivity (see Lemma 7.4) we have $D^+_{\mathrm{mod}} = D_{\mathrm{mod}}$. Thus nD_{mod} is integral at o. □

Proof of Proposition 8.14. The statement of (7.12.1) follows by [Amb04] (cf. [Kaw98]). Indeed, for any $0 < t < 1$ we put $D_t := (1-t)D + t\Theta$, where Θ is such as in 7.10. Then by [Amb04, Th. 0.1] $(D_t)_{\mathrm{mod}}$ is semiample. Hence so is D_{mod}.

Assertion (7.12.2) follows by Theorem 1.6 (in lower dimension).

Finally for (7.12.3) we note that by Corollary 8.17 ID_{mod} is integral and base point free for a bounded I because $Z \simeq \mathbb{P}^1$. □

Remark 8.18. It is possible that Proposition 8.14 can be proven by using results of [FM00], [Fuj03]. In fact, in these papers the authors write down the canonical bundle formula (for arbitrary $\dim Z$) in the following form (we change notation a little):

$$b(K + D) = f^*(bK_Z + L^{log,ss}_{X/Z}) + \sum_P s^D_P f^* P + B^D.$$

Here $D_{\mathrm{div}} = \frac{1}{b}\sum_P s^D_P P$, $D_{\mathrm{mod}} = \frac{1}{b} L^{log,ss}_{X/Z}$, and $\operatorname{codim} f(B^D) \geq 2$, so the term B^D is zero in our situation. Under additional assumption that D is a boundary it is proved that the denominators of D_{mod} are bounded (and D_{mod} is semiample because it is nef on $Z = \mathbb{P}^1$), see [FM00, Theorem 4.5], [Fuj03, Theorem 5.11]. This should imply our Proposition 8.14. We however do not know how to avoid the effectivity condition of D.

9. The main theorem: Case $-(K+D)$ is nef

In this section we prove Theorem 4.1 in case (4.10.2). The idea of the proof is to consider the contraction $f\colon X \to Z$ given by $-(K+D)$ and use Effective Adjunction to pull-back complements from Z. In practice, there are several technical issues which do not alow us to weaken the last assumptions in Theorem 4.1. Roughly speaking the inductive step work if the following two conditions hold:

(i) $0 < \dim Z < \dim X$, and
(ii) the pair $(Z, D_{\mathrm{div}} + D_{\mathrm{mod}})$ satisfies assumptions of Theorem 4.1.

The main step of the proof is Proposition 9.7.

9.1. Let (X, D) be an lc log pair and let $f\colon X \to Z$ be a contraction such that X is FT and $K + D \sim_{\mathbb{Q}} f^*L$ for some L. Our proof uses induction by $d := \dim X$. So we assume that Theorem 4.1 holds true for all X of dimension $< d$.

Lemma 9.2. *In notation of* 9.1 *assume that* $\dim Z > 0$. *Fix a finite rational set* $\mathfrak{R} \subset [0,1]$ *and let* $D \in \Phi(\mathfrak{R})$. *Then the multiplicities of horizontal components of* D *are contained in* $\frac{1}{N}\mathbb{Z} \cap [0,1]$, *where* N *depends only on* $\dim X$ *and* $\mathfrak{R}$.

Proof. Let F be a general fibre. Then $(F, D|_F)$ is a 0-pair satisfying conditions of Theorem 4.1. By our inductive hypothesis $K_F + D|_F$ is n-complemented for some $n \in \mathcal{N}_{d-1}(\mathfrak{R}) = \mathcal{N}_{d-1}(\mathfrak{R}, I(\mathfrak{R}))$. For this complement $D|_F^+$, we have $D|_F^+ \ge D|_F$. Since $K_F + D|_F \equiv 0$, $D|_F^+ = D|_F$. In particular, $nD|_F$ is integral for some $n \in \mathcal{N}_{d-1}(\mathfrak{R})$. Thus we can put $N := \operatorname{lcm}(\mathcal{N}_{d-1}(\mathfrak{R}))$. $\square$

Now we verify that under certain assumptions and conjectures the hyperstand coefficients go to hyperstandard after adjunction.

For a subset $\mathfrak{R} \subset [0,1]$, denote

$$\mathfrak{R}(n) := \left(\overline{\mathfrak{R}} + \frac{1}{n}\mathbb{Z}\right) \cap [0,1], \qquad \mathfrak{R}' := \bigcup_{n \in \mathcal{N}_{d-1}(\overline{R}, I)} \mathfrak{R}(n) \subset [0,\, 1].$$

These sets are rational and finite whenever so is $\mathfrak{R}$.

Proposition 9.3. *In notation of* 9.1, *fix a finite rational set* $\mathfrak{R} \subset [0,1]$.

(i) *If* $D \in \Phi(\mathfrak{R})$, *then* $D_{\mathrm{div}} \in \Phi(\mathfrak{R}')$.
(ii) *If* $D \in \Phi(\mathfrak{R}, \epsilon_{d-1,I})$, *then* $D_{\mathrm{div}} \in \Phi(\mathfrak{R}', \epsilon_{d-1,I}) \subset \Phi(\mathfrak{R}', \epsilon_{d-2,I})$.

Proof. By taking general hyperplane sections we may assume that Z is a curve. Furthermore, we may assume that X is $\mathbb{Q}$-factorial. Fix a point $o \in Z$. Let d_o be the multiplicity of o in D_{div}. Then $d_o = 1 - c_o$, where c_o is computed by (7.2.1). It is sufficient to show that $d_o \in \Phi(\mathfrak{R}(n)) \cup [1 - \epsilon_{d-1,I},\, 1]$ for any point $o \in Z$ and some $n \in \mathcal{N}_{d-1}(\overline{\mathfrak{R}}, I)$. Clearly, we can consider X and Z as germs near $f^{-1}(o)$ and o, respectively. We also may assume that

$c_o > 0$, so $f^{-1}(o)$ does not contain any centres of log canonical singularities of (X, D). By Lemma 8.16 there is an n-complement $K_X + D^+$ of $K_X + D$ near $f^{-1}(o)$ with $n \in \mathcal{N}_{d-1}(\overline{\mathfrak{R}}, I)$ and moreover, $a(E, X, D^+) = -1$.

Now we show that $d_o \in \Phi(\mathfrak{R}(n)) \cup [1 - \epsilon_{d-1,I},\, 1]$. By Lemma 3.4 $D \in \mathcal{P}_n$. Hence, $D^+ \geq D$, i.e., $D^+ = D + D'$, where $D' \geq 0$. Let $F \subset f^{-1}(o)$ be a reduced irreducible component. Since $K_X + D$ is $\mathbb{R}$-linearly trivial over Z, D' is vertical and $D' = c_o f^* P$. Let d_F and μ be multiplicities of F in D and f^*o, respectively (μ is a positive integer). Since $(X, D + D')$ is lc and $n(D + D')$ is an integral divisor, the multiplicity of F in $D + D'$ has the form k/n, where $k \in \mathbb{Z}$, $1 \leq k \leq n$. Then $k/n = d_F + c_o\mu$ and

$$c_o = \frac{1}{\mu}\left(\frac{k}{n} - d_F\right), \qquad d_o = 1 - \frac{1}{\mu}\left(\frac{k}{n} - d_F\right).$$

Consider two cases.

a) $d_F \in \Phi(\mathfrak{R})$, so $d_F = 1 - r/m$ ($r \in \mathfrak{R}$, $m \in \mathbb{Z}$, $m > 0$). Then we can write

$$d_o = 1 - \frac{km + rn - nm}{nm\mu} = 1 - \frac{r'}{m\mu} < 1,$$

where

$$0 \leq r' = r + \frac{km}{n} - m = \frac{km + rn - nm}{n} \leq \frac{nm + rn - nm}{n} \leq 1.$$

Therefore, $d_o \in \Phi(\mathfrak{R}(n))$, where $0 \leq r' = r + \frac{km}{n} - m \leq 1$. This proves, in particular, (i).

b) $d_F > 1 - \epsilon_{d-1,I}$. In this case,

$$1 > d_o = 1 - \frac{1}{\mu}\left(\frac{k}{n} - d_F\right) > 1 - \frac{1}{\mu}\left(\frac{k}{n} - 1 + \epsilon_{d-1,I}\right) > 1 - \epsilon_{d-1,I}.$$

This finishes the proof of (ii). □

Proposition 9.4. *Fix a finite rational subset $\mathfrak{R} \subset [0,\, 1]$ and a positive integer I divisible by $I(\mathfrak{R})$. Let (X, D) be a log semi-Fano variety of dimension d such that X is $\mathbb{Q}$-factorial FT and $D \in \Phi(\mathfrak{R}, \epsilon_{d-1,I})$. Assume that there is a $(K + D)$-trivial contraction $f\colon X \to Z$ with $0 < \dim Z < d$. Fix the choice of I_0 and ψ in 7.5 so that $\mathbf{D}_{\mathrm{mod}}$ is effective. We take I so that I_0 divides I. Assume the LMMP in dimension d. Further, assume that Conjectures* 1.1 *and* 7.12 *hold in dimension d. If $K_Z + D_{\mathrm{div}} + D_{\mathrm{mod}}$ is Im-complemented, then so is $K_X + D$.*

Proof. Put $D_Z := D_{\mathrm{div}} + D_{\mathrm{mod}}$. Apply (i) of Conjecture 7.12 to (X, D). We obtain

$$K + D = f^*(K_Z + D_Z),$$

and (Z, D_Z) is lc, where $D_{\mathrm{div}} \in \Phi(\mathfrak{R}', \epsilon_{d-1,I})$. By (7.12.3) ID_{mod} is integral for some bounded I''. Thus replacing $\mathfrak{R}'$ with $\mathfrak{R}' \cup \{1/I'', 2/I'', \dots, (I'' - 1)/I''\}$ we may also assume that $D_{\mathrm{mod}} \in \Phi(\mathfrak{R}', \epsilon_{d-1,I})$. Then $D_Z \in$

$\Phi(\mathfrak{R}', \epsilon_{d-1,I})$. Furthermore, by Lemma 2.9 Z is FT and by the construction, $-(K_Z+D_Z)$ is nef. By our inductive hypothesis K_Z+D_Z has bounded complements.

Let $K_Z+D_Z^+$ be an n-complement of K_Z+D_Z such that $I \mid n$. Then $D_Z^+ \geq D_Z$ (see Lemmas 3.3 and 3.4). Put $H_Z := D_Z^+ - D_Z$ and $D^+ := D + f^*H_Z$. Write $D^+ = \sum d_i^+ D_i$. By the above, $d_i^+ \geq d_i$. We claim that $K + D^+$ is an n-complement of $K + D$. Indeed, since $K + D \sim_I f^*(K_Z + D_Z)$, we have

$$\begin{aligned} n(K + D^+) =& n(K + D + f^*H_Z) = \\ & (n/I)I(K + D) + (n/I)If^*H_Z \sim \\ & (n/I)f^*I(K_Z + D_Z) + (n/I)f^*IH_Z = \\ & (n/I)f^*I(K_Z + D_Z^+) = f^*n(K_Z + D_Z^+) \sim f^*0 = 0. \end{aligned}$$

Thus, $n(K + D^+) \sim 0$. Further, since nd_i^+ is a nonnegative integer and $d_i^+ \geq d_i$, the inequality

$$nd_i^+ = \lfloor (n+1)d_i^+ \rfloor \geq \lfloor (n+1)d_i \rfloor$$

holds for every i such that $0 \leq d_i < 1$. Finally, by Corollary 7.17 the log divisor $K + D^+ = f^*(K_Z + D_Z)$ is lc. This proves our proposition. □

9.5. **Proof of Theorem 1.6 (= Theorem 4.1 in the case $K + D \equiv 0$).** Let $(X, B = D)$ be a 0-pair such that X is FT and $B \in \Phi(\mathfrak{R}, \epsilon_{d-1,I})$, where $\epsilon_{d-1,I} := \epsilon_{d-1}(\overline{\mathfrak{R}}, I)$ and $I(\mathfrak{R}) \mid I$. We assume that (X, B) is not klt. As in 4.3 we replace X with $\mathbb{Q}$-factorial blowup of all divisors E of discrepancy $a(E, X) \leq -1 + \epsilon_{d-1,I}$ and B with its crepant pull-back. Then X is $\mathbb{Q}$-factorial and $\epsilon_{d-1,I}$-lt. Moreover, $B \in \Phi(\mathfrak{R}, \epsilon_{d-1,I})$ and our new X is again FT by Lemma 2.9. Now run K-MMP. Since $K + B \equiv 0$, K cannot be nef. At the end we get a K-negative extremal contraction $f \colon X \to Z$ to a lower-dimensional variety. If $\dim Z > 0$, then we get our assertion by Proposition 9.4. So we assume that Z is a point and $\rho(X) = 1$. Then the pair $(X, \operatorname{Supp} B)$ is bounded by Conjecture 1.1 and because the multiplicities of B are bounded from below (cf. 6.1). In this situation, $K + B$ has desired bounded complement by Proposition 5.4. This proves Theorem 1.6.

9.6. **Proof of Theorem 4.1 in Case** (4.10.2)**.** To finish our proof of the main theorem we have to consider the case when (X, B) is klt and general reduction from Section 4 leads to case (4.10.2). Recall that in this situation X is FT and $B \in \Phi(\mathfrak{R}, \epsilon_{d,I})$. By (4.3.2) there is a boundary $\Theta \geq B$ such that (X, Θ) is a klt 0-pair. For the boundary D defined by (4.3.3) we also have $D \in \Phi(\mathfrak{R})$ and $\lfloor D \rfloor \neq 0$ by (4.3.4). All these properties are preserved under the (anti)-LMMP in 4.6. By our assumption at the end we have case (4.10.2), i.e., $-(K + D)$ is nef (and semiample). Therefore it is sufficient to prove the following.

Proposition 9.7. *Fix a finite rational subset $\mathfrak{R} \subset [0, 1]$ and a positive integer I such that $I(\mathfrak{R}) \mid I$. Let $(X, D = \sum d_i D_i)$ is a d-dimensional log semi-Fano variety such that*

(i) $D \in \Phi(\mathfrak{R})$, (X, D) *is not klt and* X *is FT,*
(ii) *there is boundary* $B = \sum b_i D_i \le D$ *such that either* $b_i = d_i < 1 - \epsilon_{d-1,I}$ *or* $b_i \ge 1 - \epsilon_{d-1,I}$ *and* $d_i = 1$ *(in particular,* $B \in \Phi(\mathfrak{R},\ \epsilon_{d-1,I})$*),*
(iii) (X, Θ) *is a klt* 0*-pair for some* $\Theta \ge B$*.*

Then $K + D$ *has a bounded* n*-complement such that* $I \mid n$*.*

The idea of the proof is to reduce the problem to Proposition 9.4 by considering the contraction $f\colon X \to Z$ given by $-(K + D)$. But here two technical difficulties arise. First it may happen that the divisor $-(K + D)$ is big and then f is birational. In this case one can try to extend complements from $\lfloor D \rfloor$ but the pair (X, D) is not necessarily plt and the inductive step (Corollary 3.6) does not work. We have to make some perturbations and birational transformations. Second to apply inductive hypothesis to $(Z, D_{\mathrm{div}} + D_{\mathrm{mod}})$ we have to check if this pair satisfies conditions of Theorem 4.1. In particular, we have to check the klt property of $(Z, D_{\mathrm{div}} + D_{\mathrm{mod}})$. By Corollary 7.17 this holds if any (X, D) any lc centre of (X, D) dominates Z. Otherwise we again need some additional work.

Proof. Note that we may replace B with $B_t := tB + (1 - t)D$ for $0 < t < 1$. This preserves all our conditions (i)-(iii). Indeed, (i) and (ii) are obvious. For (iii), we note that $(X, D^{\diamondsuit})$ is a 0-pair for some $D^{\diamondsuit} \ge D$ (because $-(K + D)$ is semiample). Hence one can replace Θ with $\Theta_t := t\Theta + (1 - t)D^{\diamondsuit}$.

Let $\mu\colon (\tilde{X}, \tilde{D}) \to (X, D)$ be a dlt modification of (X, D). By definition, μ is a $K + D$-crepant birational extraction such that $\tilde{X}$ is $\mathbb{Q}$-factorial, the pair $(\tilde{X}, \tilde{D})$ is dlt, and each μ-exceptional divisor E has discrepancy $a(E, X, D) = -1$ (see, e.g., [Kol92, 21.6.1], [Pro01, 3.1.3]). In particular, $\tilde{D} \in \Phi(\mathfrak{R})$ and $\tilde{X}$ is FT by Lemma 2.9. Let $\tilde{B}$ be the crepant pull-back of B. One can take t so that the multiplicities in $\tilde{B}$ of μ-exceptional divisors are $\ge 1 - \epsilon_{d-1,I}$. Thus for the pair $(\tilde{X}, \tilde{D})$ conditions (i)-(iii) hold. Therefore, we may replace (X, D) with $(\tilde{X}, \tilde{D})$ (and B, Θ with their crepant pull-backs).

Let $f\colon X \to Z$ be the contraction given by $-(K + D)$. By Theorem 1.6 (see 9.5) we may assume that $\dim Z > 0$. We apply induction by $N := \dim X - \dim Z$.

First, consider the case $N = 0$. Then $-(K + D)$ is big. We will show that $K + D$ is n-complemented for some $n \in \mathcal{N}_{d-1}(\overline{\mathfrak{R}}, I)$.

Fix $n_0 \gg 0$, and let $\delta := 1/n_0$. Then $D_\delta := D - \delta \lfloor D \rfloor \in \Phi(\mathfrak{R})$. It is sufficient to show that $K_X + D_\delta$ is n-complemented for some $n \in \mathcal{N}_{d-1}(\overline{\mathfrak{R}}, I)$. We will apply a variant of [PS01, Th. 5.1] with hyperstandard multiplicities. To do this, we run $-(K + D_\delta)$-MMP over Z. Clearly, this equivalent $\lfloor D \rfloor$-MMP over Z. This process preserve the $\mathbb{Q}$-factoriality and lc (but not dlt) property of $K + D$. At the end we get a model (X', D') such that $-(K_{X'} + D'_\delta)$ is nef over Z. Since X' is FT, the Mori cone $\overline{\mathrm{NE}}(X')$ is rational polyhedral. Taking our condition $0 < \delta \ll 1$ into account we get that $-(K_{X'} + D'_\delta)$ is nef. Since

$$-(K_{X'} + D'_\delta) = -(K_{X'} + D') + \delta \lfloor D' \rfloor ,$$

where $\lfloor D' \rfloor$ is effective, $-(K_{X'} + D'_\delta)$ is big. Note that (X', D') is lc but not klt. By our assumptions,

$$D'_\delta = (1-\delta)D' + \delta B' \leq (1-\delta)D' + \delta\Theta'$$

and $(X', (1-\delta)D' + \delta\Theta')$ is klt. Therefore so is (X', D'_δ). Now we apply [PS01, Th. 5.1] with $\Phi = \Phi(\mathfrak{R})$. This says that we can extend complements from some (possibly exceptional) divisor. By Proposition 3.8 the multiplicities of the corresponding different are contained in $\overline{\mathfrak{R}}$. We obtain an n-complement of $K_{X'} + D'_\delta$ for some $n \in \mathcal{N}_{d-1}(\overline{\mathfrak{R}}, I)$. By Proposition 3.5 we can pull-back this complement to X (we use the inclusion $D'_\delta \in \Phi(\mathfrak{R}) \subset \mathcal{P}_n$).

Now assume that Proposition 9.7 holds for all $N' < N$. Run $\lfloor D \rfloor$-MMP over Z. After some flips and divisorial contractions we get a model on which $\lfloor D \rfloor$ is nef over Z. Since X is FT, the Mori cone $\overline{\mathrm{NE}}(X)$ is rational polyhedral. Hence $-(K + D - \epsilon \lfloor D \rfloor)$ is nef for $0 < \epsilon \ll 1$. Put $B_\epsilon := D - \epsilon \lfloor D \rfloor$. We can take $\epsilon = 1/n$, $n \gg 0$ and then $B_\epsilon \in \Phi(\mathfrak{R})$. On the other hand, $B_\epsilon \leq (1-\delta)D + \delta B$ for some $\delta > 0$. Therefore, (X, B_ϵ) is klt. Now let $f^\flat \colon X \to Z^\flat$ be the contraction given by $-(K + B_\epsilon)$. Since $-(K + B_\epsilon) = -(K + D) + \epsilon \lfloor D \rfloor$, there is decomposition $f \colon X \xrightarrow{f^\flat} Z^\flat \longrightarrow Z$.

If $\dim Z^\flat = 0$, then $Z^\flat = Z$ is a point, a contradiction. If $\dim Z^\flat < \dim X$, then by Corollary 7.17 $(Z^\flat, (B_\epsilon)_{\mathrm{div}} + (B_\epsilon)_{\mathrm{mod}})$ is a klt log semi-Fano variety. We can apply Proposition 9.4 to the contraction $X \to Z^\flat$ and obtain a bounded complement of $K + B_\epsilon$. Clearly, this will be a complement of $K + D$.

Therefore, we may assume that $-(K + B_\epsilon)$ is big, $f^\flat$ is birational, and $\lfloor D \rfloor$ is big over Z. Replace (X, D) with its dlt modification. Assume that the horizontal part $\lfloor D \rfloor^{\mathrm{h}}$ does not coincide with $\lfloor D \rfloor$. As above, run $\lfloor D \rfloor^{\mathrm{h}}$-MMP over Z. For $0 < \epsilon \ll 1$, the divisor $-(K + D - \epsilon \lfloor D \rfloor^{\mathrm{h}})$ will be nef. Moreover, it is big over Z. Therefore, $-(K + D - \epsilon \lfloor D \rfloor^{\mathrm{h}})$ defines a contraction $f' \colon X \to Z'$ with $\dim Z' > \dim Z$. By our inductive hypothesis there is a bounded complement.

It remains to consider the case when $\lfloor D \rfloor^{\mathrm{h}} = \lfloor D \rfloor$. Then any lc centre of (X, D) dominates Z. By Corollary 7.17 and Proposition 9.4 there is a bounded complement of $K + D$.

□

This finishes the proof of Theorem 4.1.

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