Arithmetical differential operators on nearly holomorphic Siegel modular forms

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§0. Introduction

Let p be a prime number (we often assume $p \geq 5$).

The purpose of this paper is to describe the action of certain arithmetical differential operators on algebraically defined nearly holomorphic Siegel modular forms (in the sense of Shimura, see [Shi2000]) over an $\mathcal{O}$-algebra A where $\mathcal{O}$ is the ring of integers in a finite extension K of $\mathbb{Q}_p$. Often we simply assume that $A = \mathbb{C}_p$.

We view such modular forms as certain formal expansions. The important property of these arithmetical differential operators is their commutation with Hecke operators (under an appropriate normalization, see Theorem 2.8). Their action produces natural families of distributions on a profinite group $Y = \varprojlim Y_i$ with values in these A-modules of nearly holomorphic Siegel modular forms inside a formal q-expansion ring like $A[\![q^B]\!][R_{i,j}]$ where B is an additive semi-group, $q^B = \{q^\xi | \xi \in B\}$ the corresponding formally written multiplicative semi-group (for example $B = B_m = \{\xi = {}^t\xi \in M_m(\mathbb{Q}) | \xi \geq 0, \xi \text{ half-integral}\}$ is the semi-group, important for the theory of Siegel modular forms), and the nearly holomorphic parameters $(R_{i,j}) = R$ correspond to the matrix $R = (4\pi \mathrm{Im}(z))^{-1}$ in the Siegel modular case.

We discuss some applications of this theory to construction of certain new p-adic families of modular forms (families of Siegel-Eisenstein series, families of theta-series wth spherical polynomials...). Main sources of this theory are:

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1) Serre's theory of p-adic modular forms as certain formal q-expansions (J.-P. Serre, Formes modulaires et fonctions zêta p-adiques, LNM 350 (1973) 191-268) [Se73].

2) Shimura's theory of arithmeticity for nearly holomorphic forms (Shimura, G., Arithmeticity in the theory of automorphic forms. AMS, 2000, [Sh2000]).

3) Hida's theory of p-adic modular forms and p-adic Hecke algebras (H. Hida, Elementary theory of L-functions and Eisenstein series, Cambridge University Press, 1993 [Hi93]).

4) Construction of p-adic Siegel-Eisenstein series by the author, appeared in Israel Journal of Mathematics in 2000, [PaSE].

Nerarly holomorphic Siegel modular forms provide additional arithmetical parameters. Moreover they admit an algebraic interpretation in terms of sections of certain de Rham cohomology sheaves on Siegel modular varieties (see [Ha2]), and a typical example of the arithmetical differential operators is given by the Gauss- Manin connection. On the other hand, differential operators preserving rational structures naturally come from the action of universal envelopping algebras of corresponding Lie algebras on automorphic forms.

Arithmetical nearly holomorphic Siegel modular forms where studied by G.Shimura [Shi2000] and they admit two different descriptions:

• as formal power series $g = \sum_{\xi\in B_m} a(\xi, R_{i,j})q^\xi \in \bar{\mathbb{Q}}[\![q^{B_m}]\!][R_{i,j}]$ such that for all $R = (R_{i,j})_{i,j=1,\cdots,m} = (4\pi\mathrm{Im}(z))^{-1} = (2\pi i(\bar z - z))^{-1}$ and $z \in \mathbb{H}_m$ in the Siegel upper half plane of degree m the series converges to a $\mathcal{C}^\infty$-Siegel modular form of a given weight k and character ψ;

• as certain $\mathcal{C}^\infty$-Siegel modular forms g taking values in $\bar{\mathbb{Q}}$ at all CM-points (up to a factor independnt of a concrete form) and satisfying certain reciprocity laws at these points.

The first description motivates our choice of arithmetical variables: q^ξ, $R = (R_{i,j}) = (4\pi\mathrm{Im}(z))^{-1} = 2\pi i(\bar z - z)^{-1}$. We describe the action of $\delta_k^{(r)}$ in terms of these arithmetical variables.

Consider the differential operator Δ_m (the *Maass differential operator*) of degree m, acting on complex C^∞-functions on $\mathbb{H}_m$, defined by the equality:

$$\Delta_m = \det(\tilde\partial_{ij}), \qquad \tilde\partial_{ij} = 2^{-1}(1+\delta_{ij})\partial/\partial_{ij}$$

For any $f \in \mathcal{C}^\infty(\mathbb{H}_m, \mathbb{C})$, one defines also the operators

$$M_k f(z) = \det(z-\bar{z})^{\varkappa-k}\Delta\left[\det(z-\bar{z})^{k-\varkappa+1}f(z)\right],$$

$$\delta_k f(z) = \left(-\frac{1}{4\pi}\right)^m \det(y)^{-1} M_k f(z).$$

Due to H. Maass [Maa71], for an integer k and a Dirichlet character ψ modulo N, the differential operator δ_k acts on $\mathcal{C}^\infty$-Siegel modular forms, $\delta_k : \tilde{\mathcal{M}}_k^m(N,\psi) \to \tilde{\mathcal{M}}_{k+2}^m(N,\psi)$. One defines then the Shimura differential operator as the composition

$$\delta_k^{(r)} = \delta_{k+2r-2} \circ \cdots \circ \delta_k : \tilde{\mathcal{M}}_k^m(N,\psi) \to \tilde{\mathcal{M}}_{k+2r}^m(N,\psi).$$

In order to describe explicitely the action of δ_k on Fourier expansions consider, for an integer $r \geq 0$ and for a complex number β, the polynomial

$$R_m(z;r,\beta) = (-1)^{mr} e^{\mathrm{tr}(z)} \det(z)^{r+\beta} \Delta_m^r \left[e^{-\mathrm{tr}(z)} \det(z)^{-\beta}\right].$$

According to its definition the degree of the polynomial $R_m(z;r,\beta)$ is equal to mr and the term of the highest degree coincides with $\det(z)^r$. We have also that for $\beta \in \mathbb{Q}$ the polynomial $R_m(z;r,\beta)$ has rational coefficients.

Notice also that for $m=1$ one has

$$R_1(z,r,\beta) = \sum_{t=0}^{r} \binom{r}{t} z^{r-t} \beta(\beta+1)\cdots(\beta+t-1).$$

M.Courtieu has established in his PhD thesis the following more explicit expression for the function $R_m(z;r,\beta)$ for arbitrary m (see [Cour], Chapter 3 and Theorem 2.6):

$$R_m(z;r,\beta) = \sum_{t=0}^{r} \binom{r}{t} \det(z)^{r-t} \sum_{|L| \leq mt-t} R_L(\beta) \lambda_{l_1}(z) \cdots \lambda_{l_t}(z),$$

where L runs over all the multi-indices $0 \leq l_1 \leq \cdots \leq l_t \leq m$, such that $|L| = l_1 + \cdots + l_t \leq mt - t$, the coefficients $R_L(\beta) \in \mathbb{Z}[1/2][\beta]$ are polynomials in β of degree $(mt - |L|)$ and with coefficients in the ring $\mathbb{Z}[1/2]$. The polynomials $\lambda_i(z)$ are defined by

$$\det(t1_m + z) = \sum_{i=0}^{m} \lambda_i(z) t^{m-i}$$

($\lambda_i(z)$ is the sum of all diagonal minors of size $i \times i$ of the matrix z).

Let $f \in \mathcal{M}_k^m(N,\psi)$ be a Siegel modular form whose Fourier expansion is given by $f(z) = \sum_{\xi \in B_m} c(\xi) e_m(\xi z)$.

Let us consider the natural representation $(0 \le r \le m)$

$$\begin{array}{cccc} \rho_r & : & \mathrm{GL}_m(\mathbb{C}) & \longrightarrow & \mathrm{GL}(\Lambda^r \mathbb{C}^m) \\ & & z & \longmapsto & \rho_r(z) \end{array}$$

of the group $\mathrm{GL}_m(\mathbb{C})$ on the vector space $\Lambda^r\mathbb{C}^m$ with respect to the basis

$$\{e_{i_1} \wedge \cdots \wedge e_{i_r} \ : \ i_1 < \cdots < i_r\}.$$

Thus $\rho_r(z)$ is a matrix of size $\binom{m}{r} \times \binom{m}{r}$ composed of the subdeterminants of z of degree r. Put

$$\rho_r^\star(z) = \det(z)\rho_{m-r}({}^tz)^{-1} \quad (r = 0, 1, \cdots, m),$$

in other words, $\rho^\star_{m-r}(z)$ is the matrix representing the action of z on $\Lambda^{m-r}\mathbb{C}^m$ with respect to the basis dual to the above basis of $\Lambda^r\mathbb{C}^m$. Then the representations ρ_r and $\rho_r^\star$ turn out to be polynomial representations so that for each $z \in \mathrm{M}_m(\mathbb{C})$ the linear operators $\rho_r(z)$, $\rho_r^\star(z)$ are well defined.

We prove (Theorem 3.2) then that the nearly holomorphic Siegel modular form $\delta_k^{(r)} f(z)$ is given by the following formal power series expansion

$$\begin{aligned} \delta_k^{(r)} f(z) = &\sum_{\xi \in B_m} c(\xi) \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \\ &\sum_{|L| \le mt-t} R_L(\kappa - k - r) \mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho^\star_{l_1}(\xi)\right) \cdots \mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho^\star_{l_t}(\xi)\right)) q^\xi \\ &= \sum_{\xi \in B_m} c(\xi) \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \sum_{|L| \le mt-t} R_L(\kappa - k - r) Q_L(R,\xi) q^\xi \\ &= \sum_{\xi \in B_m} c(\xi) Q(R,\xi) q^\xi \in \mathbb{Q}(f)[\![q^{B_m}]\!][R_{i,j}]_{i,j=1,\cdots,m}, \end{aligned}$$

where $\mathbb{Q}(f)$ is the subfield of $\mathbb{C}$ generated by the Fourier coefficients of f, L runs over all the multi-indices $0 \le l_1 \le \cdots \le l_t \le m$ as above,

$$\begin{aligned} Q(R,\xi) &= Q(R,\xi;k,r) = \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \sum_{|L| \le mt-t} R_L(\kappa - k - r) Q_L(R,\xi)), \\ Q_L(R,\xi) &= \mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho^\star_{l_1}(\xi)\right) \cdots \mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho^\star_{l_t}(\xi)\right)). \end{aligned}$$

For $r = 1$ this gives:

$$\delta_k f(z) = \sum_{\xi \in B_m} c(\xi) \sum_{l=0}^{m} (-1)^{m-l} c_{m-l}(k+1-\varkappa) \mathrm{tr}\left({}^t\rho_{m-l}(R) \cdot \rho_l^\star(\xi)\right) q^\xi,$$

where

$$c_{m-l}(k+1-\varkappa) = \prod_{j=0}^{m-l-1} (k+1-\varkappa+j/2) = \frac{\Gamma_{m-l}(k+1/2)}{\Gamma_{m-l}(k-1)/2)} \quad (l \le m-1),$$

and in the elliptic modular case $m = 1$, $\varkappa = 1$, the only possibilities are $t = 0, 1$ when $c_0(k) = 1$, $c_1(k) = k$, and one obtains again the classical formula

$$\delta_k(\sum_{n\ge 0} a_n q^n) = \sum_{n \ge 1} n a_n q^n - \frac{k}{4\pi y} \sum_{n \ge 0} a_n q^n = \sum_{n\ge 1} n a_n q^n - kR \sum_{n \ge 0} a_n q^n$$

in which

$$R = \frac{1}{4\pi y}, \quad \sum_{n \ge 1} n a_n q^n = \frac{1}{2\pi i} \frac{\partial}{\partial z} \left(\sum_{n \ge 0} a_n q^n \right) = q \frac{d}{dq} \left(\sum_{n\ge 0} a_n q^n \right).$$

Note that the operators δ_k act on nearly holomorphic forms and they are very different from operators studied in [Ch-Eh] and [Eh-Ib] although the polynomials which come up in their action on Fourier expansions have similar properties.

Applications to constructions of p-adic families of nearly holomophic Siegel modular forms are given in Sections 4-6 (Theorems 4.4, 5.4, and 6.2). In order to define p-adic families of nearly holomorphic Siegel modular forms one uses the notion of the p-adic integration with values in spaces of (formal) nearly holomorphic Siegel modular forms.

For any profinite group $Y = \varprojlim Y_i$ the completed group ring $\mathcal{O}[\![Y]\!] = \varprojlim \mathcal{O}[Y_i]$ is a compact ring which generalizes the Iwasawa algebra $\Lambda_\mathcal{O}$, and it may be viewed as the $\mathcal{O}$-algebra of all $\mathcal{O}$-valued distributions on Y (Section 4). The natural multiplication in $\mathcal{O}[\![Y]\!]$ corresponds to the *convolution* of distributions. In Section 5 we construct p-adic families of nearly holomophic Siegel-Eisenstein series obtained by applying $\delta_k^{(r)}$ to a holomorphic Siegel-Eisenstein series. In Section 6 we construct p-adic families of coming from theta series $\delta_{k_{m,P}}^{(r)}(\theta_{A,P}) = \delta_{k_{m,P}}^{(r)}\left(\sum_{x \in \mathbb{Z}^{mn}} P(x)) q^{A[x]}\right)$ where $A \in M_{m\times m}(\mathbb{Q})$ is a positive definite half-integral symmetric matrix which determines a quadratic form $Q_A(x) = {}^t x A x \overset{\mathrm{def}}{=} A[x]$ for $x \in \mathbb{Z}_p^{mn}$ and P run over homogeneous A-spherical polynomials which take integral values on the lattice $X = \mathbb{Z}^{mn}$ (we view P as a $\mathbb{Z}_p$-valued continuous function on $\mathbb{Z}_p^{mn}$).

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§2. Algebraical differential operators of Maass and Shimura.

§3. Arithmetical variables of nearly holomorphic Siegel modular forms and diffrerntial operators.

§4. Arithmetical families of nearly holomorphic Siegel modular forms.

§5. Families of distributions attached to Siegel-Eisenstein series.

§6. Families of distributions attached to theta functions with spherical polynomials.

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§1. Nearly holomorphic Siegel modular forms

1.1. We use an algebraic version of nearly holomorphic Siegel modular forms over an $\mathcal{O}$-algebra A where $\mathcal{O}$ is the ring of integers in a finite extension K of $\mathbb{Q}_p$. We fix embeddings $i_\infty\colon \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $i_p\colon \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ ($\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$ is the Tate field). We assume that A is a closed subring of $\mathbb{C}_p$ containing $i_p(\mathbb{Q}_p)$.

Recall some notations and definitions concerning Siegel modular forms.

Let A be a commutative ring with identity, then $\mathrm{M}_{r,s}(A)$ denote the set of all $(r \times s)$-matrices with coefficients in A. For $z \in \mathrm{M}_r(\mathbb{C})$ put $e_r(z) = e(\mathrm{tr}(z))$ with $e(u) = \exp(2i\pi u)$ for $u \in \mathbb{C}$. We denote by ${}^t z \in \mathrm{M}_{r,s}(A)$ the matrix, which is transpose to $z \in \mathrm{M}_{r,s}(A)$, and write $\xi[\eta]$ for ${}^t\eta\xi\eta$. For an invertible matrix ξ we put $\xi^\star = {}^t\xi^{-1}$. If ξ is a hermitian matrix then we write $\xi \geq 0$ or $\xi > 0$ according as ξ is non negative or positive definite.

Let $\mathbb{H}_m$ denote the Siegel upper half plane on the degree m,

$$\mathbb{H}_m = \{z \in \mathrm{M}_m(\mathbb{C}) \quad | \quad {}^t z = z = x + iy, \quad y > 0\},$$

so that $\mathbb{H}_m$ is a complex analytic variety whose dimension is denoted by $\langle m \rangle = m(m+1)/2$.

Let the symbol A_m denote the lattice of all half integral symetric matrices in the vector space $V = V_m = \{y \in \mathrm{M}_m(\mathbb{R}) \ | \ {}^t y = y\}$. This lattice is dual to the latice $L = \mathrm{M}_m(\mathbb{Z}) \cap V$ with respect to the pairing given by $(u, v) \longmapsto e_m(uv)$. Let $G = \mathrm{GSp}_{2m}$ be the algebraic subgroup of GL_{2m} defined by

$$G_A = \{\gamma \in \mathrm{GL}_{2m}(A) \quad | \quad {}^t\gamma J_m \gamma = \nu(\gamma) J_m, \nu(\gamma) \in A^\times\},$$

for any commutative ring A, where

$$J_{2m} = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix}.$$

The elements of G_A are characterized by the conditions

$$b^t a - a^t b = d^t c - c^t d = 0_m, \quad d^t a - c^t b = 1_m.$$

The multiplier ν defines a homomorphism $\nu : G_A \to A^\times$ so that $\nu(\gamma)^{2m} = \det(\gamma)^2$ and $\ker(\nu)$ is denoted by $\mathrm{Sp}_m(A)$. We also put

$$G_\infty = G_\mathbb{R}, G_\infty^+ = \{\gamma \in G_\infty \mid \nu(\gamma) > 0\}, G_\mathbb{Q}^+ = G_\infty^+ \cap G_\mathbb{Q}.$$

The group G_∞^+ acts transitively on the upper half plane $\mathbb{H}_m$ by the rule

$$z \longmapsto \gamma(z) = (az+b)(cz+d)^{-1} \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty^+, \quad z \in \mathbb{H}_m\right)$$

so that the scalar matrices acts trivially, and $\mathbb{H}_m$ can be identified with a homogeneous space of the group $\mathrm{Sp}_m(\mathbb{R})$. Let K_m denote the stabilizer of the point $i1_m \in \mathbb{H}_m$ in the group $\mathrm{Sp}_m(\mathbb{R})$,

$$K_m = \{\gamma \in \mathrm{Sp}_m(\mathbb{R}) \mid \gamma(i1_m) = i1_m\},$$

then there is a bijection $\mathrm{Sp}_m(\mathbb{R})/K_m \simeq \mathbb{H}_m$ and $K_m = \mathrm{Sp}_m(\mathbb{R}) \cap \mathrm{SO}_{2m}$. The group K_m is a maximal compact subgroup of the Lie group $\mathrm{Sp}_m(\mathbb{R})$ and it can be identified with the group $\mathrm{U}(m)$ of all unitary $m \times m$-matrices via the map $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longmapsto a + ib$.

For a function $f : \mathbb{H}_m \longrightarrow \mathbb{C}$ of the form

$$f = \sum_{\xi \in A_m} c(\xi) e_m(\xi z)$$

and for an integer k we use the notation of Petersson for the action of $\gamma \in \mathrm{GSp}_{2m}$ on f:

$$(f|_k\gamma)(z) = \det(\gamma)^{k-\kappa}\det(cz+d)^{-k} f(\gamma(z)) \quad (\text{where} \quad \kappa = (m+1)/2),$$

we put also for $N \in \mathbb{N}$:

$$\begin{aligned}
f|U(N)(z) &= \sum_{\xi \in A_m} c(N\xi) e_m(\xi z) \\
&= \sum_{{}^t u = u \in \mathrm{M}_m(\mathbb{Z}) \bmod N} f|_k \begin{pmatrix} 1_m & u \\ 0_m & N1_m \end{pmatrix}, \\
f|V(N)(z) &= \sum_{\xi \in A_m} c(\xi) e_m(N\xi z) \quad = \quad f(Nz) \\
&= N^{m(\kappa-k)} f|_k \begin{pmatrix} N1_m & 0_m \\ 0_m & 1_m \end{pmatrix}, \\
f|W(N)(z) &= \det(\sqrt{N}z)^{-k} f(-(Nz)^{-1}) \\
&= N^{m(\kappa-k/2)} f|_k W(N), \quad W(N) = \begin{pmatrix} 0_m & -1_m \\ N1_m & 0_m \end{pmatrix}.
\end{aligned}$$

There are the following relations for these operators:

$$(f|W(N))|W(N) = (-1)^{mk} f, \; f|W(N'N) = N^{mk/2} f|W(N')V(N).$$

We adopt also the notations

$$f^\rho = \sum_{\xi \in A_m} \overline{c(\xi)} e_m(\xi z), \quad Y = \{y \in \mathrm{M}_m(\mathbb{R}) \quad | \quad {}^t y = y > 0\},$$

$$B_m = \{h \in A_m \mid h \geq 0\}, \quad C_m = \{h \in A_m \mid h > 0\}.$$

The bloc notation $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}_{2m}(\mathbb{Z})$ defines the congruence subgroup

$$\Gamma_0^m(N) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2m}(\mathbb{Z}) | c \equiv 0 \bmod N \right\}$$

of the Siegel modular group $\Gamma^m = \mathrm{Sp}_{2m}(\mathbb{Z})$, and its normal subgroup

$$\Gamma_1^m(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2m}(\mathbb{Z}) \middle| \det(d) \equiv 1 \bmod N, c \equiv 0 \bmod N \right\} \subset \Gamma_0^m(N).$$

A classical Siegel modular form $f \in \mathcal{M}_k(N, \psi)$ of weight k and character ψ for $\Gamma_0^m(N)$ is a holomorphic function $f \colon \mathbb{H}_m \to \mathbb{C}$ such that for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^m(N)$ one has

$$f(\gamma(z)) = \psi(\det d) \det(cz + d)^k f(z)$$

(for $m > 1$ the regularity at ∞ is automatically satisfied by Koecher).

The Fourier expansion of such f uses the semi-group B_m:

$$f(z) = \sum_{\xi \in B_m} a(\xi) q^\xi \in \mathbb{C}[\![q^{B_m}]\!]$$

where the symbols

$$q^\xi = \exp(2\pi i \mathrm{tr}(\xi z)) = \prod_{i=1}^m q_{ii}^{\xi_{ii}} \prod_{i<j} q_{ij}^{2\xi_{ij}} \subset \mathbb{C}[\![q_{11}, \ldots, q_{mm}]\!][q_{ij}, q_{ij}^{-1}]_{i,j=1,\cdots,m}$$

are used (with $q_{ij} = \exp(2\pi(\sqrt{-1} z_{i,j})))$; they form a multiplicative semi-group so that one may consider f as a formal q-expansion and one can introduce Siegel modular forms over A as certain elements of $A[\![q^{B_m}]\!]$. Let $\widetilde{\mathcal{M}}_k(N, \psi)$ denote the complex vector space of $\mathcal{C}^\infty$-Siegel modular forms of weight k and character ψ for $\Gamma_0^m(N)$.

For a non negative integer r and variables q, $R = (R_{i,j})_{i,j=1,\ldots,m}$ consider the ring $A[\![q^{B_m}]\!][R_{i,j}]$ (over the complex numbers this notation corresponds to $q^\xi = \exp(2\pi i \mathrm{tr}(\xi z))$,

$R = (4\pi \mathrm{Im}(z))^{-1})$. Consider the following A-submodules

$$P_r(A) = \Big\{ g = \sum_{\xi \in B_m} a(\xi, R_{i,j}) q^\xi \in A[\![q^{B_m}]\!][R_{i,j}] \Big| \\ a(\xi, R_{i,j}) \in A[R_{i,j}], \deg_R a(\xi, R_{i,j}) \le r \Big\} \subset A[\![q^{B_m}]\!][R_{i,j}]$$

Consider also for any natural number a the A-modules

$$Q_{r,a} = \Big\{ g = \sum_{\xi \in B_m} a(\xi, R_{i,j}) q^{a^{-1}\xi} \Big\},$$

and put $Q_r = \bigcup_{a \ge 1} Q_{r,a}$.

Definition 1.2. Let $\mathcal{M}^m_{r,k}(N, \psi, A)$ denote the A-submodule of $A[\![q^{B_m}]\!][R_{i,j}]$ generated by all $F = \sum\limits_{\xi \in B_m} a(\xi, R) q^\xi$ such that all $a(\xi, R) \in A[R_{i,j}]_{i,j=1,\ldots,m}$ are polynomials of degree $\le r$ over $A \subset \mathbb{C}_p$ which belong to

$$i_p(\bar{\mathbb{Q}})[R_{i,j}]_{i,j=1,\ldots,m}$$

and

$$F = \sum_{\xi \in B_m} i_p^{-1}(a(\xi, R)) \Big|_{R=(4\pi Y)^{-1}} q^\xi \in \mathcal{M}^m_{r,k}(N), \psi, \bar{\mathbb{Q}}) \subset \mathcal{M}^m_{r,k}(N), \psi, \mathbb{C})$$

is a $\mathcal{C}^\infty$-complex function on $\mathbb{H}_m$ satisfying the following two conditions

$$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^m_0(N) \quad F|_k \gamma = \psi(\det(d)) F \tag{1.1}$$

$$\forall \gamma \in Sp_{2m}(\mathbb{Z}) \ F|_k \gamma \in Q_r \tag{1.2}$$

(a complex nearly holomorphic modular form of type r and weight k over $\bar{\mathbb{Q}}$ in the sense of Shimura, the second condition is necessary only for $m = 1$).

Put

$$\mathcal{M}^m_{r,k}(A) = \bigcup_{v \ge 1} \mathcal{M}^m_{r,k}(Np^v; A) \tag{1.3}$$

(an A-module of nearly holomorphic Siegel modular forms of type r and level Np^v ($v \ge 0$) with coefficients in A).

1.3. We may view a given cusp form

$$f = \sum_{\xi \in B_m} a(\xi; f) q^\xi \in A[\![q^{B_m}]\!],$$

as an element of $\mathcal{M}^m_{r,k}(A) \subset A[\![q^{B_m}]\!][R_{i,j}]$ in the formal q-expansion ring $A[\![q^{B_m}]\!][R_{i,j}])$. (over an $\mathcal{O}$-algebra A).

§2. Algebraic differential operators of Maass and Shimura

Now let us consider the differential operator Δ_m (the *Maass differential operator*) of degree m, acting on complex C^∞-functions on $\mathbb{H}_m$, defined by the equality:

$$\Delta_m = \det(\tilde{\partial}_{ij}), \qquad \tilde{\partial}_{ij} = 2^{-1}(1+\delta_{ij})\partial/\partial_{ij} \tag{2.1}$$

For any $f \in \mathcal{C}^\infty(\mathbb{H}_m, \mathbb{C})$, one defines also the operators

$$M_k f(z) = \det(z-\bar{z})^{\varkappa-k}\Delta\left[\det(z-\bar{z})^{k-\varkappa+1}f(z)\right],$$

$$\delta_k f(z) = \left(-\frac{1}{4\pi}\right)^m \det(y)^{-1} M_k f(z).$$

Due to H. Maass [Maa71], for an integer k and a Dirichlet character ψ modulo N, the differential operator δ_k acts on $\mathcal{C}^\infty$-Siegel modular forms, $\delta_k : \tilde{\mathcal{M}}_k^m(N,\psi) \to \tilde{\mathcal{M}}_{k+2}^m(N,\psi)$. One defines then the Shimura differential operator as the composition

$$\delta_k^{(r)} = \delta_{k+2r-2} \circ \cdots \circ \delta_k : \tilde{\mathcal{M}}_k^m(N,\psi) \to \tilde{\mathcal{M}}_{k+2r}^m(N,\psi).$$

In order to describe explicitely the action of δ_k on Fourier expansions consider, for an integer $r \geq 0$ and for a complex number β, the polynomial

$$R_m(z;r,\beta) = (-1)^{mr} e^{\mathrm{tr}(z)} \det(z)^{r+\beta} \Delta_m^r \left[e^{-\mathrm{tr}(z)}\det(z)^{-\beta}\right], \tag{2.2}$$

with $z \in V \otimes \mathbb{C}$, where the exponentiation is well defined by

$$\det(y)^\beta = \exp\left(\beta \log[\det(y)]\right),$$

for $\det(y) > 0$, $y \in Y \otimes \mathbb{C}$. According to definition (2.2) the degree of the polynomial $R_m(z;r,\beta)$ is equal to mr and the term of the highest degree coincides with $\det(z)^n$. We have also that for $\beta \in \mathbb{Q}$ the polynomial $R_m(z;r,\beta)$ has rational coefficients.

Notice also that for $m = 1$ one has

$$R_1(z,r,\beta) = \sum_{t=0}^{r} \binom{r}{t} \beta(\beta+1)\cdots(\beta+t-1) z^{r-t}.$$

M.Courtieu has established in his PhD thesis in 2000 the following more explicit expression for the function $R_m(z;r,\beta)$ for arbitrary m (see [Cour], Chapter 3)

$$R_m(z;r,\beta)=\sum_{t=0}^{r}\binom{r}{t}\det(z)^{r-t}\sum_{|L|\le mt-t}R_L(\beta)\lambda_{l_1}(z)\cdots\lambda_{l_t}(z),\tag{2.3}$$

where L runs over all the multi-indices $0\le l_1\le\cdots\le l_t\le m$, such that $|L|=l_1+\cdots+l_t\le mt-t$, the coefficients $R_L(\beta)\in\mathbb{Z}[1/2][\beta]$ are polynomials in β of degree $(mt-|L|)$ and with coefficients in the ring $\mathbb{Z}[1/2]$. The polynomials $\lambda_i(z)$ are defined by

$$\det(t1_m+z)=\sum_{i=0}^{m}\lambda_i(z)t^{m-i},\tag{2.4}$$

In other words $\lambda_i(z)$ is the sum of all diagonal minors of size $i\times i$ of the matrix z.

The polynomial $R_m(z;r,\beta)$.

Here we want to describe explicitely the polynomial $R_m(z;r,\beta)$ defined by (2.2). We show that this is a polynomial invariant for the action of conjugation on the space of symetric matrices $y\in V=\{\xi\in\mathrm{M}_m(\mathbb{R})\ \mid\ {}^t\xi=\xi\}$.

Let us consider the natural representation $(0\le r\le m)$

$$\begin{array}{cccc}\rho_r & : & \mathrm{GL}_m(\mathbb{C}) & \longrightarrow & \mathrm{GL}(\Lambda^r\mathbb{C}^m)\\ & & z & \longmapsto & \rho_r(z)\end{array}$$

of the group $\mathrm{GL}_m(\mathbb{C})$ on the vector space $\Lambda^r\mathbb{C}^m$ with respect to the basis

$$\{e_{i_1}\wedge\cdots\wedge e_{i_r}\ :\ i_1<\cdots<i_r\}.$$

Thus $\rho_r(z)$ is a matrix of size $\binom{m}{r}\times\binom{m}{r}$ composed of the subdeterminants of z of degree r. Put

$$\rho_r^\star(z)=\det(z)\rho_{m-r}({}^tz)^{-1}\quad(r=0,1,\cdots,m),$$

in other words, $\rho_{m-r}^\star(z)$ is the matrix representing the action of z on $\Lambda^{m-r}\mathbb{C}^m$ with respect to the basis dual to the above basis of $\Lambda^r\mathbb{C}^m$. Then the representations ρ_r and $\rho_r^\star$ turn out to be polynomial representations so that for each $z\in\mathrm{M}_m(\mathbb{C})$ the linear operators $\rho_r(z)$, $\rho_r^\star(z)$ are well defined. We consider the differential operators $\rho_r(\tilde\partial/\partial z)$ and $\rho_r^\star(\tilde\partial/\partial z)$ which associate to each $\mathbb{C}$-valued function on $\mathbb{H}_m$ a certain $\mathrm{M}_t(\mathbb{C})$ valued function on $\mathbb{H}_m$ with $t = \binom{m}{r}$. In particular, the Maass differential operator is given by

$$\Delta = \rho_m(\tilde{\partial}/\partial z) = \rho_m^\star(\tilde{\partial}/\partial z) = \det(2^{-1}(1+\delta_{ij})\tilde{\partial}/\partial z_{ij}).$$

We define the invariant polynomials $\lambda_r(z) = \mathrm{tr}(\rho_r(z))$. They are the coefficients of the characteristic polynomial of z

$$\det(t1_m - z) = \sum_{j=0}^{m} (-1)^j \lambda_j(z) t^{m-j}.$$

The following differentiation rules are valid (see [Sh83], lemma 9.1):

$$\Delta(fg) = \sum_{r=0}^{m} \mathrm{tr}\left({}^t\rho_r(\tilde{\partial}/\partial z)f \cdot \rho_{m-r}^\star(\tilde{\partial}/\partial z)g\right), \tag{2.5}$$

$$\rho_r(\tilde{\partial}/\partial z)\det(z)^\alpha = c_r(\alpha)\det(z)^{\alpha-1}\rho_{m-r}^\star(z), \tag{2.6}$$

$$\rho_r^\star(\tilde{\partial}/\partial z)\det(z)^\alpha = c_r(\alpha)\det(z)^{\alpha-1}\rho_{m-r}(z), \tag{2.7}$$

$$\rho_r(\tilde{\partial}/\partial z)e^{\mathrm{tr}(uz)} = \rho_r(u)e^{\mathrm{tr}(uz)}, \tag{2.8}$$

$$\Delta\left(e^{\mathrm{tr}(uz)}\det(z)^\alpha\right) = e^{\mathrm{tr}(uz)}\det(z)^{\alpha-1}\sum_{t=0}^{m} c_{m-t}(\alpha)\lambda_t(uz), \tag{2.9}$$

for $\alpha \in \mathbb{C}$ with

$$c_0(\alpha) = 1, \quad c_r(\alpha) = \prod_{j=0}^{r-1}(\alpha + j/2) = \frac{\Gamma_r(\alpha + (r+1)/2)}{\Gamma_r(\alpha + (r-1)/2)} \ \ (r \geq 1). \tag{2.10}$$

One can extend the formula (2.5) to the case of the action of Δ on n functions $f_1, \cdots, f_n$:

Lemma 2.1. *Let $f_1, \cdots, f_n : \mathbb{H}_m \to \mathbb{C}$ be n $\mathcal{C}^\infty$-functions, then we get*

$$\Delta(f_1 \cdots f_n) = \sum_{|T|=m} \sum_{K,L} (-1)^{\varepsilon(\sigma_{K,L})} \rho_{t_1}(\tilde{\partial}/\partial z)_{K_{t_1}}^{L_{t_1}}(f_1) \cdots \rho_{t_n}(\tilde{\partial}/\partial z)_{K_{t_n}}^{L_{t_n}}(f_n), \tag{2.11}$$

where T runs over the sets of multiplets $T = \{0 \leq t_1, \cdots, t_n \leq m\}$ which satisfy $|T| = t_1 + \cdots + t_n = m$. The second sum is extended over the sets of all multi-indices K and L consisting of strictely ordered subsets of multi-indices K_{t_j} and L_{t_j} such that $\mathrm{card}(K_{t_j}) =$

$\mathrm{card}(L_{t_j})$ *and* $\cup_{j=1}^n K_{t_j} = \cup_{j=1}^n L_{t_j} = \underline{m} = \{0,\cdots,m\}$ *are two partitions of* $\underline{m}$*. The permutation* $\sigma_{K,L}$ *of* $\underline{m}$ *is determined by the following rule: for any* $h \in \underline{m}$*, take* t_j *such that* $h = h_i \in K_{t_j}$ *where* i *the rank of* h *in* K_{t_j} *(with respect to the natural order), then* $\sigma_{K,L}(h_i) = h'_i$ *where* h'_i *is the element of rank* i *of* L_{t_j}*, and denote by* $\varepsilon(\sigma_{K,L})$ *the sign of the permutation* $\sigma_{K,L}$*.*

Proof of 2.1: One proves this lemma by induction on the number n of functions.

- If $n = 2$, this is the formula (2.5)
- We assume that (2.12) is true for $(n-1)$. By hypothesis we have

$$\Delta(f_1,\cdots,f_n) = \sum_{|T|=m}\sum_{K,L}(-1)^{\varepsilon(\sigma_{K,L})}\rho_{t_1}(\partial_z)^{L_{t_1}}_{K_{t_1}}(f_1)\cdots\rho_{t_{n-1}}(\tilde{\partial}_z)^{L_{t_{n-1}}}_{K_{t_{n-1}}}(f_{n-1}f_n). \quad (2.12)$$

But the operator $\rho_{t_{n-1}}(\tilde{\partial}_z)^{L_{t_{n-1}}}_{K_{t_{n-1}}}$ coincides with the operator $\Delta_{t_{n-1}}(z_1)$ (with respect to the variable $z_1 = (z_{ij})_{i\in K_{t_{n-1}}, j\in L_{t_{n-1}}}$). Thus formula (2.5) gives

$$\begin{aligned}\rho_{t_{n-1}}(\tilde{\partial}_z)^{L_{t_{n-1}}}_{K_{t_{n-1}}}&(f_{n-1}f_n)\\ &= \sum_{t'_{n-1}+t'_n=t_{n-1}}(-1)^{\varepsilon(\sigma'_{K,L})}\rho_{t'_{n-1}}(\tilde{\partial}_z)^{L_{t'_{n-1}}}_{K_{t'_{n-1}}}(f_{n-1})\cdots\rho_{t'_n}(\tilde{\partial}_z)^{L_{t'_n}}_{K_{t'_n}}(f_n),\end{aligned}$$

with $\mathrm{card}(K_{t'_{n-1}}) = \mathrm{card}(L_{t'_{n-1}}) = t'_{n-1}$, $\mathrm{card}(K_{t'_n}) = \mathrm{card}(L_{t'_n}) = t'_n$ and one has $K_{t'_{n-1}} \cup K_{t'_n} = K_{t_{n-1}}$, $L_{t'_{n-1}} \cup L_{t'_n} = L_{t_{n-1}}$ and (2.12) follows.

The following more delicate differentiation rule holds:

Lemma 2.2. *Let* $u = {}^t u$ *be a real symmetric* $m\times m$*-matrix,* α *a complex number,* r*,* j *two integers such that* $0 \le r \le j \le m$ *and* $z \in \mathbb{H}_m$ *then one has the following equality:*

$$\rho_r(\tilde{\partial})\left(\lambda_j(z)^\alpha\right) = c_r(\alpha)\lambda_j(z)^{\alpha-1}\mu^\star_{r,j-r}(z). \quad (2.13)$$

Here $\mu^\star_{r,j-r}(z)$ is a matrix of size $\binom{m}{r}\times\binom{m}{r}$ with the coefficients given by

$$\mu^\star_{r,j-r}(z)^L_K = \sum_{T\supset K,L}\rho^\star_{j-r}(z)^{\overline{T\backslash L}}_{\overline{T\backslash K}},$$

the sum is being taken over all the multi-indices $T : 1 \le t_1 < \cdots < t_j \le m$ containing the multiindices $K : 1 \le k_1 < \cdots < k_r \le m$ et $L : 1 \le l_1 < \cdots < l_r \le m$. The notation

$\bar{L}$ is used for the the complementary multi-index in $\underline{m} = \{0, \cdots, m\}$ of the multi-index L. Recall that the matrix coefficient $\rho^{\star}_{m-r}(z)^L_K$ is equal (up to a sign) to the determinant of the complementary matrix in z for the determinant $\rho_r(z)^L_K$ more precisely one has

$$\rho^{\star}_{m-r}(z)^L_K = (-1)^{\varepsilon(\sigma_{K,L})} \rho_{m-r}(z)^{\bar{L}}_{\bar{K}},$$

where $\sigma_{K,L}$ is the permutation of S_m given by the following rule: for every $k_i \in K$, $\sigma_{K,L}(k_i) = l_i$ and for every $\bar{k}_i \in \bar{K}$, $\sigma_{K,L}(\bar{k}_i) = \bar{l}_i$ with $\bar{l}_i \in \bar{L}$ and i denoting the rank of the index k_i (respectively l_i, $\bar{k}_i$, $\bar{l}_i$) with respect to the natural order in K (respectively L, $\bar{K}$, $\bar{L}$) (recall that $\varepsilon(\sigma_{K,L})$ denotes the signature of the permutation $\sigma_{K,L}$).

Proof of 2.2: is deduced from (2.8). Notice that $\lambda_j(z) = \sum_T \rho_j(z)^T_T$ is the sum of all diagonal minors of size j in z. If one applies $\rho_r(\tilde{\partial})^L_K$, one gets non zero terms only for those diagonal minors $\rho_j(z)^T_T$, when the multi-index T contains the multi-indices K et L. The corresponding minor is then the determinant of a submatrix z, so that the computation is similar to that in (2.8) but for a matrix of a lower size. This gives us then

$$\rho_r(\tilde{\partial})^L_K \left(\lambda_j(z)^{\alpha}\right) = c_r(\alpha)\lambda_j(z)^{\alpha-1} \sum_{T \supset K,L} \rho^{\star}_{j-r}(z)^{\overline{T\backslash L}}_{\overline{T\backslash K}},$$

and (2.13) follows.

Action of the Shimura differential operator on formal Fourier expansions

We now apply the above results on $R_m(z; r, \beta)$ in order to obtain an explicit dscripton of the action of the Shimura differential operator on formal Fourier expansions of nearly holomorphic Siegel modular forms. First of all we recall that the Shimura differential operator δ_k in the Siegel modular case is given as the twisted Maass operator Δ [Maa71].

Definition. *For any $f \in \mathcal{C}^{\infty}(\mathbb{H}_m, \mathbb{C})$, define the operators*

$$M_k f(z) = \det(z - \bar{z})^{\varkappa - k} \Delta \left[\det(z - \bar{z})^{k - \varkappa + 1} f(z)\right], \tag{2.14}$$

$$\delta_k f(z) = \left(-\frac{1}{4\pi}\right)^m \det(y)^{-1} M_k f(z). \tag{2.15}$$

Due to H. Maass [Maa71], for an integer k and a Dirichlet character ψ modulo N, the Shimura differential operator acts on $\mathcal{C}^{\infty}$-Siegel modular forms $\delta_k : \tilde{\mathcal{M}}^m_k(N, \psi) \to \tilde{\mathcal{M}}^m_{k+2}(N, \psi)$. One defines then $\delta^{(r)}_k = \delta_{k+2r-2} \circ \cdots \circ \delta_k$.

We show now that the Fourier expansion of $\delta_k^{(r)} f$, where $f = \sum_{B_m} c(\xi) e_m(\xi z) \in \mathcal{M}_k^m(N,\psi)$, is given by the formula

$$\delta_k^{(r)} f = \sum_{\xi \in B_m} c(\xi) \det(4\pi y)^{-r} R_m(4\pi \xi y; r, \beta) e_m(\xi z).$$

In order to describe the action of $\delta_k^{(r)} f$ one establishes first the following

Lemma 2.3. *Let α be a complex number, r a positive integer, u a real symmetric matrix and $z \in \mathbb{H}_m$ then one has the following relation*

$$\det(z)^{-r} \Delta \left(\det(z)^{r+1} \Delta^r \left(e^{\mathrm{tr}(uz)} \det(z)^{\alpha} \right) \right) = \Delta^{r+1} \left(e^{\mathrm{tr}(uz)} \det(z)^{\alpha+1} \right).$$

Proof of 2.3: Let us use the formula (2.9) giving $\Delta \left(e^{\mathrm{tr}(uz)} \det(z)^{\alpha} \right)$ in order to compare it with the action of the operator $\det(z)^{-r} \Delta \left(\det(z)^{r+1} \cdot \right)$ one has

$$\begin{aligned}
&\det(z)^{-r} \Delta \Big(e^{\mathrm{tr}(uz)} \det(z)^{\alpha+1} \cdot \\
&\cdot \sum_{l_1,\cdots,l_r=0}^{m} c_{m-l_1}(\alpha) \cdots c_{m-l_r}(\alpha - r + 1) P_{l_1 \cdots l_r}(u,z) \Big) = \\
&e^{\mathrm{tr}(uz)} \det(z)^{\alpha - r} \cdot \\
&\cdot \sum_{l_1,\cdots,l_{r+1}=0}^{m} c_{m-l_1}(\alpha) \cdots c_{m-l_r}(\alpha - r + 1) c_{m-l_{r+1}}(\alpha+1) P_{l_1 \cdots l_r l_{r+1}}(u,z).
\end{aligned}$$

The polynomials $P_{l_1 \cdots l_r}$ are defined recursively in such a way that

$$\begin{aligned}
&\Delta^r \left(e^{\mathrm{tr}(uz)} \det(z)^{\alpha} \right) = \\
&\quad e^{\mathrm{tr}(uz)} \det(z)^{\alpha - r} \sum_{l_1,\cdots,l_r=0}^{m} c_{m-l_1}(\alpha) \cdots c_{m-l_r}(\alpha - r + 1) P_{l_1 \cdots l_r}(u,z),
\end{aligned}$$

hence one has the following condition on $P_{l_1 \cdots l_r}(u,z)$:

$$P_{l_1 \cdots l_r}(u,z) = \sum_{i_1 + i_2 = m - l_r} \sum_{K,L} (-1)^{\varepsilon(\sigma_{K,L})} \rho_{i_1}(u)_{K_{i_1}}^{L_{i_1}} \rho^{\star}_{m-l_r}(z)_{K_{l_r}}^{L_{l_r}} \rho_{i_2}(\tilde{\partial})_{K_{i_2}}^{L_{i_2}} (P_{l_1 \cdots l_{r-1}}(u,z)),$$

(a polynomial in matrix variables u and z independent of α and symmetric in l_j). The independence of $P_{l_1 \cdots l_r}(u,z)$ on α results from the fact that the terms depending on α

factorise out using (2.6). The symmetry in l_j results from the theorem of Schwarz on the independence of the mixed partial derivatives of $\mathcal{C}^\infty$-functions on the order of differentiaion. Here the choice of $l_1, \cdots, l_r$ corresponds to another order of derivation applied to the factor $\det(z)^\star$. Replacing u by -1_m and α by $-\beta$, and using then the formula defining $R(z;r,\beta)$, one finds that this is indeed a polynomial as the factors $e^{\mathrm{tr}(z)}$ and $e^{-\mathrm{tr}(z)}$ cancel, as well as the factors $\det(z)^\beta$ and $\det(z)^{-\beta}$. One obtains then:

$$R(z;r,\beta) = \sum_{l_1,\cdots,l_r=0}^{m} c_{m-l_1}(-\beta)\cdots c_{m-l_r}(-\beta-r+1)P_{l_1\cdots l_r}(-1_m, z).$$

In particular the polynomials $P_{l_1\cdots l_r}$ are symmetric in l_j, and one can replace the index of summation $l'_1 = l_{r+1}$, $l'_2 = l_1$, $\cdots$, $l'_{r+1} = l_r$ so that one obtains the expression $\Delta^{r+1}\left(e^{\mathrm{tr}(uz)}\det(z)^{\alpha+1}\right)$. This gives exactly the announced property.

In order to compute the Fourier expansion of $\delta_k^{(r)} f$, one proceeds by induction on r. One computes the action of the Shimura differential operator on the test functions $e_m(\xi z)$, more precisely one shows that

$$\delta_k^{(r)}\left(e_m(\xi z)\right) = \det(4\pi y)^{-r} R_m(4\pi\xi y; r, \varkappa - k - r)e_m(\xi z).$$

In fact if one takes $f(z) = \sum_{\xi\in B_m} c(\xi)e_m(\xi z)$, one obtains then by linearity the Fourier expansion of $\delta_k^{(r)} f$, resulting

$$\delta_k^{(r)} f(z) = \sum_{\xi\in B_m} c(\xi)\det(4\pi y)^{-r} R_m(4\pi\xi y; r, \varkappa - k - r)e_m(\xi z),$$

It remains to prove the following

Lemma 2.4. *The action of the Shimura differential operator on the test functions $e_m(\xi z)$ (where $\xi \in B_m$ and $z \in \mathbb{H}_m$) is given by the formula*

$$\delta_k^{(r)}\left(e_m(\xi z)\right) = \det(4\pi y)^{-r} R_m(4\pi\xi y; r, \varkappa - k - r)e_m(\xi z). \tag{2.16}$$

Proof of 2.4: One proves this lemma by induction on r :

- If $r = 0$, it is obvious.

• Suppose next that it holds for some r then if one applies the Shimura operator δ_{k+2r} one obtains

$$\delta_k^{(r+1)}(e_m(\xi z)) =(-1)^m \det(4\pi y)^{-1}\det(z-\bar{z})^{\varkappa-k-2r}\cdot \\ \cdot \Delta\left[\det(z-\bar{z})^{k+2r+1-\varkappa}\det(4\pi y)^{-r}e_m(\xi z)\cdot \right. \\ \left. \cdot R_m(4\pi\xi y; r, \varkappa-k-r)\right].$$

Notice that $4\pi\xi y = -2i\pi\xi(z-\bar{z})$. On the other hand $\bar{z}$ is aniholomorphic hence it behaves as a constant under the action of Δ, and one can write, by putting $u=-2i\pi\xi$, $\beta=\varkappa-k-r$:

$$R_m(u(z-\bar{z}); r, \beta) = \\ (-1)^{mr} e^{\mathrm{tr}(u(z-\bar{z}))}\det(z-\bar{z})^{r+\beta}\Delta^r\left(e^{-\mathrm{tr}(u(z-\bar{z}))}\det(z-\bar{z})^{-\beta}\right).$$

One replaces this expression in the first formula pulling out the factor $e_m(\xi\bar{z}) = e^{\mathrm{tr}(-u\bar{z})}$ which is an antiholomorphic function. One obtains then

$$\delta_k^{(r+1)}(e_m(\xi z)) = \det(4\pi y)^{-(r+1)}e_m(\xi z)(-1)^{m(r+1)}e^{\mathrm{tr}(u(z-\bar{z}))}\det(z-\bar{z})^{\beta}\cdot \\ \cdot\Delta\left[\det(z-\bar{z})^{r+1}\Delta^r\left(e^{-\mathrm{tr}(u(z-\bar{z}))}\det(z-\bar{z})^{-\beta}\right)\right].$$

Let us now apply lemma 2.3 which gives

$$\delta_k^{(r+1)}(e_m(\xi z)) =\det(4\pi y)^{-(r+1)}e_m(\xi z)(-1)^{m(r+1)}e^{\mathrm{tr}(u(z-\bar{z}))}\cdot \\ \cdot\det(z-\bar{z})^{r+\beta}\Delta^{r+1}\left(e^{-\mathrm{tr}(u(z-\bar{z}))}\det(z-\bar{z})^{-\beta}\right) \\ =\det(4\pi y)^{-(r+1)}R_m(4\pi\xi y; r+1, \varkappa-k-(r+1))e_m(\xi z),$$

finishing the proof by induction.

Now we can write down the action of the Shimura operator on the Fourier expansions of Siegel modular forms in terms of the polynomial $R_m(z; r, \beta)$ as follows:

Proposition 2.5. *Let $f \in \mathcal{M}_k^m(N, \psi)$ be a Siegel modular form whose Fourier expansion is given by $f(z) = \sum_{\xi\in B_m} c(\xi)e_m(\xi z)$, then*

$$\delta_k^{(r)} f(z) = \sum_{\xi\in B_m} c(\xi)\det(4\pi y)^{-r}R_m(4\pi\xi y; r, \varkappa-k-r)e_m(\xi z).$$

Let us now use the explicite expressions (2.3) of M.Courtieu for the function $R_m(z; r, \beta)$ for arbitrary m in order to describe the action of $\delta_k^{(r)}$ on Fourier expansions.

Theorem 2.6. *a) Let $f \in \mathcal{M}_k^m(N,\psi)$ be a Siegel modular form whose Fourier expansion is given by $f(z) = \sum_{\xi \in B_m} c(\xi) e_m(\xi z)$, then*

$$\begin{aligned}\delta_k^{(r)} f(z) = &\sum_{\xi \in B_m} c(\xi) \det(4\pi y)^{-r} \sum_{t=0}^{r} \binom{r}{t} \det(2\pi i \xi(\bar z - z))^{r-t} \\ &\sum_{|L| \le mt-t} R_L(\varkappa - k - r) \lambda_{l_1}(2\pi i \xi(\bar z - z)) \cdots \lambda_{l_t}(2\pi i \xi(\bar z - z)) e_m(\xi z)\end{aligned} \tag{2.17}$$

where L runs over all the multi-indices $0 \le l_1 \le \cdots \le l_t \le m$, such that $|L| = l_1 + \cdots + l_t \le mt - t$, the coefficients $R_L(\beta) \in \mathbb{Z}[1/2][\beta]$ are polynomials in β of degree $(mt - |L|)$ and with coefficients in the ring $\mathbb{Z}[1/2]$ (recall that the polynomials $\lambda_i(z)$ are defined by $\det(t1_m + z) = \sum_{i=0}^{m} \lambda_i(z) t^{m-i}$.

b) In particular, if $r = 1$,

$$\begin{aligned}\delta_k f(z) = &(-1)^m \sum_{\xi \in B_m} c(\xi) \det(2\pi i(\bar z - z))^{-1} \\ &\sum_{t=0}^{m} c_{m-t}(k + 1 - \varkappa) \lambda_t(-2\pi i \xi(\bar z - z)) e_m(\xi z)\end{aligned} \tag{2.18}$$

where

$$c_0(\alpha) = 1, \quad c_r(\alpha) = \prod_{j=0}^{r-1} (\alpha + j/2) = \frac{\Gamma_r(\alpha + (r+1)/2)}{\Gamma_r(\alpha + (r-1)/2)}.$$

Proof of 2.6.: Let us write by (2.16)

$$\delta_k^{(r)} f(z) = \sum_{\xi \in B_m} c(\xi) \det(4\pi y)^{-r} R_m(4\pi \xi y; r, \varkappa - k - r) e_m(\xi z)$$

and substitute in it

$$R_m(z; r, \beta) = \sum_{t=0}^{r} \binom{r}{t} \det(z)^{r-t} \sum_{|L| \le mt-t} R_L(\beta) \lambda_{l_1}(z) \cdots \lambda_{l_t}(z),$$

with $4\pi \xi y = 2\pi i \xi(\bar z - z)$, $\beta = \varkappa - k - r$:

$$\begin{aligned}\delta_k^{(r)} f(z) = &\sum_{\xi \in B_m} c(\xi) \det(4\pi y)^{-r} \sum_{t=0}^{r} \binom{r}{t} \det(2\pi i \xi(\bar z - z))^{r-t} \\ &\sum_{|L| \le mt-t} R_L(\varkappa - k - r) \lambda_{l_1}(2\pi i \xi(\bar z - z)) \cdots \lambda_{l_t}(2\pi i \xi(\bar z - z)) e_m(\xi z)\end{aligned}$$

Note also that by (2.9)

$$\Delta\left(e^{\mathrm{tr}(uz)}\det(z)^{\alpha}\right)=e^{\mathrm{tr}(uz)}\det(z)^{\alpha-1}\sum_{t=0}^{m}c_{m-t}(\alpha)\lambda_t(uz),$$

for $\alpha\in\mathbb{C}$ with

$$c_0(\alpha)=1, c_r(\alpha)=\prod_{j=0}^{r-1}(\alpha+j/2)=\frac{\Gamma_r(\alpha+(r+1)/2)}{\Gamma_r(\alpha+(r-1)/2)}$$

and by definition (2.2)

$$R_m(z;n,\beta)=(-1)^{mn}e^{\mathrm{tr}(z)}\det(z)^{n+\beta}\Delta_m^n\left[e^{-\mathrm{tr}(z)}\det(z)^{-\beta}\right],$$

hence

$$R_m(z;1,\beta)=(-1)^{m}e^{\mathrm{tr}(z)}\det(z)^{1+\beta}\Delta_m\left[e^{-\mathrm{tr}(z)}\det(z)^{-\beta}\right]=$$

$$(-1)^{m}e^{\mathrm{tr}(z)}\det(z)^{1+\beta}e^{-\mathrm{tr}(z)}\det(z)^{-\beta-1}\sum_{t=0}^{m}c_{m-t}(-\beta)\lambda_t(-z)$$

giving

$$\delta_k f(z)=\sum_{\xi\in B_m}c(\xi)\det(4\pi y)^{-1}R_m(4\pi\xi y;1,\varkappa-k-r)e_m(\xi z)=$$

$$\sum_{\xi\in B_m}c(\xi)\det(2\pi i(\bar z-z))^{-1}(-1)^m e^{\mathrm{tr}(2\pi i\xi(\bar z-z))}\det(2\pi i\xi(\bar z-z))^{1+\beta}\cdot$$

$$\cdot\, e^{-\mathrm{tr}(2\pi i\xi(\bar z-z))}\det(2\pi i\xi(\bar z-z))^{-\beta-1}\sum_{t=0}^{m}c_{m-t}(k+1-\varkappa)\lambda_t(-2\pi i\xi(\bar z-z))e_m(\xi z)=$$

$$(-1)^m\sum_{\xi\in B_m}c(\xi)\det(2\pi i(\bar z-z))^{-1}\sum_{t=0}^{m}c_{m-t}(k+1-\varkappa)\lambda_t(-2\pi i\xi(\bar z-z))e_m(\xi z).$$

2.7. We conclude this section with a general result on commutation of the Shimura operator with Hecke operators (up to a normalization). Recall some basic fact about the action of Hecke algebras (see[And79]. Let q be a prime, $q\not| N$,

$$\Delta=\Delta_q^m(N)=$$
$$\left\{\gamma=\begin{pmatrix}a\ b\\ c\ d\end{pmatrix}\in G_{\mathbb{Q}}^{+}\cap\mathrm{GL}_{2m}(\mathbb{Z}[q^{-1}])\;\middle|\;\nu(\gamma)^{\pm}\in\mathbb{Z}[q^{-1}],\quad c\equiv 0_m\pmod N\right\}$$

be a subgroup in $G_{\mathbb{Q}}^{+}$ containing $\Gamma = \Gamma_0^m(N)$. The Hecke algebra

$$\mathcal{L} = \mathcal{L}_q^m(N) = \mathcal{D}_{\mathbb{Q}}(\Gamma, \Delta)$$

over $\mathbb{Q}$ is then defined as a $\mathbb{Q}$-linear space generated by the double cosets $(g) = (\Gamma g \Gamma)$, $g \in \Delta$ of the group S with respect to the subgroup Γ, for which multiplication is defined by the standard rule (see [An7], [Shi1] and in 1.7 below). We recall the description of the structure of $\mathcal{L} = \mathcal{L}_q^m(N)$, $(q \not| N)$: for each $j, 1 \le j \le m$ let us denote by w_j an automorphism of the algebra $\mathbb{Q}[x_0^{\pm 1}, x_1^{\pm 1}, \cdots, x_m^{\pm 1}]$ defined on its generators by the rule:

$$x_0 \mapsto x_0 x_j, \ x_j \mapsto x_j^{-1}, x_i \mapsto x_i \quad (1 \le i \le m, i \ne j).$$

Then the automorphisms w_j and the permutation group Σ_m of the variables x_i $(1 \le i \le m)$ generate together the Weyl group $W = W_m$, and there is the Satake isomorphism:

$$Sat : \mathcal{L} \xrightarrow{\sim} \mathbb{Q}[x_0^{\pm 1}, x_1^{\pm 1}, \cdots, x_m^{\pm 1}]^{W_m} \tag{2.19}$$

where W_m indicates the subalgebra of elements fixed by W_m. For any commutative $\mathbb{Q}$-algebra A the group W_m acts on the set $(A^{\times})^{m+1}$, therefore any homomorphism of $\mathbb{Q}$-algebras $\lambda : \mathcal{L} \to A$ can be identified with some element

$$(\alpha_0, \alpha_1, \cdots, \alpha_m) \in [(A^{\times})^{m+1}]^{W_m}. \tag{2.20}$$

Any double coset $(g) = (\Gamma g \Gamma)$ $(g \in \Delta = \Delta_q^m(N))$ can be represented is a disjoint union of left cosets:

$$(g) = \cup_{i=1}^{t(g)} \Gamma g_i,$$

therefore any element $X \in \mathcal{L}$ of the Hecke algebra $\mathcal{L}$ takes the form of a finite linear combination

$$X = \sum_{i=1}^{t(X)} v_i (\Gamma g_i),$$

with $v_i \in \mathbb{Q}, g_i \in \Delta$. In order to define the Hecke operators we use the notation by Petersson and Andrianov: for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta$

$$(f|_{k,\psi} g)(z) = (\det g)^{k-(m+1)/2} \psi(\det a) \det(cz+d)^{-k} f(g(z)) \tag{2.21}$$

$$(f|_{k,\psi}\gamma)(z) = f \quad \text{for all } \gamma \in \Gamma = \Gamma_0^m(N). \tag{2.22}$$

In this case for any

$$X = \sum_{i=1}^{t(X)} v_i(\Gamma g_i) \in \mathcal{L}$$

we have that the expression

$$f|X = \sum_{i=1}^{t(X)} v_i f|_{k,\psi}\, g_i, \tag{2.23}$$

is well defined and $f|X \in \mathcal{M}_k^m(N,\psi)$ so that the formula (2.23) gives a representation of the Hecke algebra $\mathcal{L} = \mathcal{L}_q^m(N)$ on the complex vector space $\mathcal{M}_k^m(N,\psi)$ $(q \not| N)$.

Action of the Hecke operators on Fourier expansions. Let

$$f(z) = \sum_{\xi \in B_m} c(\xi) e_m(\xi z) \in \mathcal{M}_m^k(N,\psi) \quad (\xi \in B_m)$$

and for a fixed prime q

$$X = \sum_{i=1}^{t(X)} v_i(\Gamma_0 g_i) \in \mathcal{L}_0 \quad (\Gamma_0 = \Gamma_0^m, \mathcal{L}_0 = \mathcal{L}_{0,q}^m)$$

be an arbitrary element of the extended Hecke algebra $\mathcal{L}_0$ with

$$g_i = \begin{pmatrix} q^{\nu_i} d_i^* & b_i \\ 0 & d_i \end{pmatrix} \in \Delta_{0,q}^m. \tag{2.24}$$

Let coefficients $(c|X)(\xi)$ be defined by the equality

$$(f|X)(z) = \sum_{\xi \in B_m} (c|X)(\xi) e_m(\xi z). \tag{2.25}$$

Then we have that

$$\begin{aligned}(c|X)(\xi) = \sum_{i=1}^{t(X)} & v_i q^{\nu_i(mk - l1m))} (\det d_i)^{-k} \times \\ & \times \psi(q^{m\nu_i}(\det d_i)^{-1}) e_m(q^{-\nu_i}\xi\, {}^t b_i d_i) c(q^{-\nu_i} d_i \xi {}^t d_i),\end{aligned} \tag{2.26}$$

(here we assume that $c(\xi) = 0$ for $\xi \notin B_m$ (see [An79], §3.2)).

As an example we consider the Frobenius elements (see [An7], §2.1):

$$\begin{aligned}\Pi_- = \Pi_-^m(q) &= (\Gamma_0 \begin{pmatrix} q1_m & 0 \\ 0 & 1_m \end{pmatrix} \Gamma_0), \\ \Pi_+ = \Pi_+^m(q) &= (\Gamma_0 \begin{pmatrix} 1_m & 0 \\ 0 & q1_m \end{pmatrix} \Gamma_0).\end{aligned} \tag{2.27}$$

Then for $f(z) = \sum_{\xi\in B_m} c(\xi)e_m(\xi z) \in \mathcal{M}_m^k(N,\psi)$ we get from (2.26)

$$(c|\Pi_-)(\xi) = \psi(q)^m q^{mk-l1m\rangle} c(q^{-1}\xi), \tag{2.28}$$

$$(c|\Pi_+)(\xi) = c(q\xi) \quad (\xi \in B_m \subset A_m) \tag{2.29}$$

Note that the operator $\Pi_+ = \Pi_+(q)$ is defined by the formula (2.29) also for $q|N$, when $\Pi_+(q)$ sends $\mathcal{M}_m^k(N,\psi)$ to itself.

Theorem 2.8. *Let*

$$f(z) = \sum_{\xi\in B_m} c(\xi)e_m(\xi z) \in \mathcal{M}_m^k(N,\psi) \quad (\xi \in B_m),$$

and let

$$\delta_k^{(r)} = \delta_{k+2r-2} \circ \cdots \circ \delta_k : \tilde{\mathfrak{M}}_k^m(N,\psi) \to \tilde{\mathfrak{M}}_{k+2r}^m(N,\psi)$$

be the Shimura differential operator defined by (2.15). Consider for a fixed prime q an arbitrary element

$$X = \sum_{i=1}^{t(X)} v_i(\Gamma_0 g_i) \in \mathcal{L}_0 \quad (\Gamma_0 = \Gamma_0^m, \mathcal{L}_0 = \mathcal{L}_{0,q}^m)$$

of the extended Hecke algebra $\mathcal{L}_0$ with

$$g_i = \begin{pmatrix} q^{\nu_i} d_i^* & b_i \\ 0 & d_i \end{pmatrix} \in \Delta_{0,q}^m,$$

and assume X homogeneous ($\nu_1 = \nu_2 = \cdots \nu_t = \nu$). Then

$$(\delta_k^{(r)} f)|_{k+2r} X = q^{\nu rm} \delta_k^{(r)}(f|_k X) \tag{2.30}$$

In particular

$$(\delta_k^{(r)} f)|_{k+2r}\Pi_+^m(q) = q^{rm}\delta_k^{(r)}(f|_k\Pi_+^m(q)), \quad (\delta_k^{(r)} f)|_{k+2r}\Pi_-^m(q) = q^{rm}\delta_k^{(r)}(f|_k\Pi_-^m(q)) \tag{2.31}$$

Proof of 2.8: We use the known property $\delta_k(f|_k\gamma) = (\delta_k f)|_{k+2}\gamma$ for all $\gamma \in \mathrm{Sp}_{2m}(\mathbb{R})$ and $k \in \mathbb{N}$. Taking into account $(\delta_k f)|_{k+2} g_i = \det(g_i)\delta_k(f|_k g_i)$, one immediately obtains (2.31) from the relation $\det g_i = q^{\nu_i rm}$ taking into account the homogeneousity of X.

§3. Arithmetical variables of nearly holomorphic Siegel modular forms and difererntial operators.

3.1. Arithmetical nearly holomorphic Siegel modular forms where studied by G.Shimura [Shi2000] and they admit two different descriptions:

• as formal power series $g = \sum\limits_{\xi \in B_m} a(\xi, R_{i,j}) q^{\xi} \in \bar{\mathbb{Q}}[\![q^{B_m}]\!][R_{i,j}]$ such that for all $R = (4\pi \mathrm{Im}(z))^{-1}$ and $z \in \mathbb{H}_m$ the series converges to a $\mathcal{C}^{\infty}$-Siegel modular form of a given weight k and character ψ.

• as certain $\mathcal{C}^{\infty}$-Siegel modular forms g taking values in $\bar{\mathbb{Q}}$ at all CM-points (up to a factor independnt of a concrete form) and satisfying certain reciprocity laws at these points.

The first description motivates our choice of arithmetical variables: q^{ξ}, $R = (R_{i,j}) = (4\pi \mathrm{Im}(z))^{-1} = 2\pi i(\bar{z} - z)^{-1}$. Our task in this Section is to expliciteltly describe the action of $\delta_k^{(r)}$ in terms of these arithmetical variables.

Theorem 3.2. *Let $f \in \mathcal{M}_k^m(N, \psi)$ be a Siegel modular form whose Fourier expansion is given by $f(z) = \sum_{\xi \in B_m} c(\xi) q^{\xi}$, then*

$$\begin{aligned}
\delta_k^{(r)} f(z) = &\sum_{\xi \in B_m} c(\xi) \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \\
&\sum_{|L| \le mt-t} R_L(\kappa - k - r) \mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho^{\star}_{l_1}(\xi)\right) \cdots \mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho^{\star}_{l_t}(\xi)\right) q^{\xi} \\
= &\sum_{\xi \in B_m} c(\xi) \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \sum_{|L| \le mt-t} R_L(\kappa - k - r) Q_L(R, \xi) q^{\xi} \\
= &\sum_{\xi \in B_m} c(\xi) Q(R, \xi) q^{\xi} \in \mathbb{Q}(f)[\![q^{B_m}]\!][R_{i,j}]_{i,j=1,\cdots,m},
\end{aligned}$$

where $\mathbb{Q}(f)$ is the subfield of $\mathbb{C}$ generated by the Fourier coefficients of f, L runs over all the multi-indices $0 \le l_1 \le \cdots \le l_t \le m$ as above,

$$\begin{aligned}
Q(R, \xi) &= Q(R, \xi; k, r) = \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \sum_{|L| \le mt-t} R_L(\kappa - k - r) Q_L(R, \xi)), \\
Q_L(R, \xi) &= \mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho^{\star}_{l_1}(\xi)\right) \cdots \mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho^{\star}_{l_t}(\xi)\right)).
\end{aligned}$$

If $r=1$ *one has*

$$\delta_k f(z) = \sum_{\xi\in B_m} c(\xi) \sum_{l=0}^{m} (-1)^{m-l} c_{m-l}(k+1-\varkappa) \mathrm{tr}\left({}^t\rho_{m-l}(R)\cdot\rho_l^\star(\xi)\right) q^\xi$$

where

$$c_{m-l}(k+1-\varkappa) = \prod_{j=0}^{m-l-1}(k+1-\varkappa+j/2) = \frac{\Gamma_{m-l}(k+1/2)}{\Gamma_{m-l}(k-1)/2)} \quad (l\le m-1).$$

Remark. In the elliptic modular case $m=1$, $\varkappa=1$, the only possibilities are $t=0,1$ when $c_0(k)=1$, $c_1(k)=k$, and one obtains again the classical formula

$$\delta_k(\sum_{n\ge 0} a_n q^n) = \sum_{n\ge 1} n a_n q^n - \frac{k}{4\pi y}\sum_{n\ge 0} a_n q^n = \sum_{n\ge 1} n a_n q^n - kR\sum_{n\ge 0} a_n q^n$$

in which

$$R = \frac{1}{4\pi y},\quad \sum_{n\ge 1} n a_n q^n = \frac{1}{2\pi i}\frac{\partial}{\partial z}\left(\sum_{n\ge 0} a_n q^n\right) = q\frac{d}{dq}\left(\sum_{n\ge 0} a_n q^n\right).$$

Proof of 3.2.: Let us change the variable by putting R equal $(4\pi y)^{-1} = -2\pi i z - \bar z)^{-1}$ and let us use the equality (2.9) with $u=-2\pi i\xi$ and the polynomial (2.2) with $n=r, \beta=\varkappa-k-r$

$$\begin{aligned}
\delta_k f(z) &= \left(-\frac{1}{4\pi}\right)^m \det(y)^{-1}\det(z-\bar z)^{\varkappa-k}\Delta\left[\det(z-\bar z)^{k-\varkappa+1} f(z)\right]\\
&= \det(2\pi i)^{-m}\det(z-\bar z)^{\varkappa-k-1}\Delta\left[\det(z-\bar z)^{k-\varkappa+1} f(z)\right].\\
\delta_k^{(r)}\left(e_m(\xi z)\right) &= \det(4\pi y)^{-r} e_m(\xi z) R(4\pi\xi y; r, \varkappa-k-r)\\
&= \det(2\pi i(\bar z - z))^{-r} e_m(\xi z) R(2\pi i\xi(\bar z - z); r, \varkappa-k-r)
\end{aligned}$$

We have by Theorem 2.6

$$\begin{aligned}
\delta_k^{(r)} f(z) = \sum_{\xi\in B_m} c(\xi)\det(4\pi y)^{-r}\sum_{t=0}^{r}\binom{r}{t}\det(2\pi i\xi(\bar z - z))^{r-t}\\
\sum_{|L|\le mt-t} R_L(\varkappa-k-r)\lambda_{l_1}(2\pi i\xi(\bar z - z))\cdots\lambda_{l_t}(2\pi i\xi(\bar z - z)) e_m(\xi z)
\end{aligned} \tag{3.1}$$

where L runs over all the multi-indices $0 \le l_1 \le \cdots \le l_t \le m$, such that $|L| = l_1 + \cdots + l_t \le mt - t$, the coefficients $R_L(\beta) \in \mathbb{Z}[1/2][\beta]$ are polynomials in β of degree $(mt - |L|)$ and with coefficients in the ring $\mathbb{Z}[1/2]$ (recall that the polynomials $\lambda_j(z)$ are defined by $\det(t1_m + z) = \sum_{j=0}^m \lambda_j(z) t^{m-j}$.

We now express all terms of type $\lambda_j(2\pi i\xi(\bar z - z))$ in terms of our arithmetical variable R: $2\pi i(\bar z - z) = R^{-1}$ and $\lambda_j(2\pi i \xi(\bar z - z)) = \lambda_j(\xi R^{-1}))$. Viewing symmetric matrices R and ξ as independent variables let us write the definition

$$\begin{aligned}
&\det(tR+\xi) = \det(R)\det(t1_m + \xi R^{-1}) = \det(R)\det(t1_m + \xi R^{-1}) \\
&= \sum_{j=0}^m \det(R)\lambda_j(\xi R^{-1}) t^{m-j} = t^m \det(\xi)\det(R\xi^{-1} + t^{-1}1_m) = \\
&t^m \det(\xi) \sum_{j=0}^m \lambda_j(R\xi^{-1}) t^{-m+j} = \sum_{j=0}^m \det(\xi)\lambda_j(R\xi^{-1}) t^j
\end{aligned} \tag{3.2}$$

Let us show that the coefficients of the polynomial $\det(tR+\xi)$ admit the following very explicite expression

$$\det(tR+\xi) = \sum_{r=0}^m t^r \operatorname{tr}\left({}^t\rho_r(R) \cdot \rho^\star_{m-r}(\xi))\right) \tag{3.3}$$

In order to prove (3.3) we use the differentiation rules (2.5) and (2.8)

$$\Delta(fg) = \sum_{r=0}^m \operatorname{tr}\left({}^t\rho_r(\tilde\partial/\partial z)f \cdot \rho^\star_{m-r}(\tilde\partial/\partial z)g\right),$$

$$\rho_r(\tilde\partial/\partial z)e^{\operatorname{tr}(uz)} = \rho_r(u)e^{\operatorname{tr}(uz)},$$

applied for the functions $f = e^{\operatorname{tr}(tRz)}$ and $g = e^{\operatorname{tr}(\xi z)}$:

$$\begin{aligned}
&\Delta(e^{\operatorname{tr}(tR+\xi)z}) = \Delta(e^{\operatorname{tr}(tRz)} \cdot e^{\operatorname{tr}(\xi z)}) = \sum_{r=0}^m \operatorname{tr}\left({}^t\rho_r(\tilde\partial/\partial z)e^{\operatorname{tr}(tRz)} \cdot \rho^\star_{m-r}(\tilde\partial/\partial z)e^{\operatorname{tr}(\xi z)}\right) = \\
&\sum_{r=0}^m \operatorname{tr}\left({}^t\rho_r(tRz)e^{\operatorname{tr}(tRz)} \cdot \rho^\star_{m-r}(\xi z)e^{\operatorname{tr}(\xi z)})\right) = \sum_{r=0}^m \operatorname{tr}\left({}^t\rho_r(tRz) \cdot \rho^\star_{m-r}(\xi z)\right) e^{\operatorname{tr}((tR+\xi)z)},
\end{aligned} \tag{3.4}$$

on the other hand

$$\begin{aligned}
&\Delta(e^{(\operatorname{tr}(tR+\xi)z)}) = \rho_m(\tilde\partial/\partial z)e^{\operatorname{tr}((tR+\xi)z)} = \\
&\rho_m((tR+\xi)z)e^{\operatorname{tr}((tR+\xi)z)} = \det((tR+\xi)z)e^{\operatorname{tr}((tR+\xi)z)}
\end{aligned} \tag{3.5}$$

and the substutution of $z = 1_m$ to (3.4) and (3.5) implies (3.3).

It follows from (3.3) and (3.2) that

$$\det(R)\lambda_j(\xi R^{-1}) = \mathrm{tr}\left({}^t\rho_{m-j}(R)\cdot\rho_j^\star(\xi)\right) \tag{3.6}$$

Let us substitute (3.6) to (3.1) writng $4\pi y = 2\pi i(\bar z - z) = R^{-1}$,

$$\lambda_j(2\pi i\xi(\bar z - z)) = \lambda_j(\xi R^{-1}) = \det(R)^{-1}\mathrm{tr}\left({}^t\rho_{m-j}(R)\cdot\rho_j^\star(\xi)\right)$$

$$\delta_k^{(r)} f(z) = \sum_{\xi\in B_m} c(\xi)\det(R^{-1})^{-r}\sum_{t=0}^{r}\binom{r}{t}\det(\xi R^{-1})^{r-t}$$
$$\sum_{|L|\le mt-t} R_L(\varkappa - k - r)\lambda_{l_1}(\xi R^{-1})\cdots\lambda_{l_t}(\xi R^{-1})e_m(\xi z)$$

$$\begin{aligned}
&= \sum_{\xi\in B_m} c(\xi)\sum_{t=0}^{r}\binom{r}{t}\det(\xi)^{r-t}\det(R)^t \\
&\qquad \sum_{|L|\le mt-t} R_L(\varkappa - k - r)\lambda_{l_1}(\xi R^{-1})\cdots\lambda_{l_t}(\xi R^{-1})e_m(\xi z) \\
&= \sum_{\xi\in B_m} c(\xi)\sum_{t=0}^{r}\binom{r}{t}\det(\xi)^{r-t} \\
&\qquad \sum_{|L|\le mt-t} R_L(\varkappa - k - r)\mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho_{l_1}^\star(\xi)\right)\cdots\mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho_{l_t}^\star(\xi)\right)e_m(\xi z)
\end{aligned} \tag{3.7}$$

where L runs over all the multi-indices $0 \le l_1 \le \cdots \le l_t \le m$, such that $|L| = l_1 + \cdots + l_t \le mt - t$, the coefficients $R_L(\beta) \in \mathbb{Z}[1/2][\beta]$ are polynomials in β of degree $(mt - |L|)$ and with coefficients in the ring $\mathbb{Z}[1/2]$ and the first statement of Theorem 3.2 follows.

In order to prove the second statement it suffices to specialize these formulae to the case of $r = 1$ using (2.18).

3.3. Note from (3.7) that the action of $\delta_k^{(r)}$ on the exponential $q^\xi = e_m(\xi z)$ written in variables $q^\xi, R = (R_{i,j})$ is given by

$$\begin{aligned}
\delta_k^{(r)}(q^\xi) &= \sum_{t=0}^{r}\binom{r}{t}\det(\xi)^{r-t} \\
&\qquad \sum_{|L|\le mt-t} R_L(\varkappa - k - r)\mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho_{l_1}^\star(\xi)\right)\cdots\mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho_{l_t}^\star(\xi)\right)q^\xi,
\end{aligned} \tag{3.8}$$

and one may view ξ also as an independent variable for which the notation

$$\partial_\xi = (\partial_{\xi,i,j}), \tilde{\partial}_\xi = (\tilde{\partial}_{\xi,i,j}), \tilde{\partial}_{\xi,i,j} = 2^{-1}(1+\delta_{i,j})\tilde{\partial}_{\xi,i,j} \tag{3.9}$$

will be used.

3.4. Our next task is to write down the formal action

$$\delta_k^{(r)}(R_{i_1,j_1}^{a_1}\cdots R_{i_n,j_n}^{a_n}q^\xi) \tag{3.10}$$

of $\delta_k^{(r)}$ using the differentiaion over the formal parameter ξ of the equality (3.8). Let us show first that

$$R_{i_1,j_1}^{a_1}\cdots R_{i_n,j_n}^{a_n}q^\xi = \det(R)^{a_1+\cdots+a_n}e^{\mathrm{tr}(2\pi i\xi(-\bar{z}))}\cdot D_\xi(e^{\mathrm{tr}(2\pi i\xi(\bar{z}-z))}) \tag{3.11}$$

for an appropriate differential polynomial on $\tilde{\partial}_{\xi,i,j}$ which we now specify. Recall that

$$(R_{i,j}) = (2\pi i(\bar{z}-z))^{-1} = \rho_1((2\pi i(\bar{z}-z))^{-1}) = \det(2\pi i(\bar{z}-z))^{-1}\cdot {}^t\rho_{m-1}^\star((2\pi i(\bar{z}-z)))$$

on the other hand

$${}^t\rho_{m-1}^\star((2\pi i(\bar{z}-z)))e^{\mathrm{tr}(2\pi i\xi(\bar{z}-z))} = {}^t\rho_{m-1}^\star((\tilde{\partial}_\xi))e^{\mathrm{tr}(2\pi i\xi(\bar{z}-z))}$$

hence

$$D_\xi = D_{\xi,i_1,j_1}^{a_1}\cdots D_{\xi,i_n,j_n}^{a_n} \tag{3.12}$$

where

$$(D_{\xi,i,j}) = {}^t\rho_{m-1}^\star((\tilde{\partial}_\xi)).$$

(we used the commutation of $\tilde{\partial}_{z,i,j}$ with $\tilde{\partial}_{\xi,i,j}$ with respect to the independent formal variables z and ξ).

Let us compute

$$\delta_k^{(r)}(R_{i_1,j_1}^{a_1}\cdots R_{i_n,j_n}^{a_n}q^\xi) = \delta_k^{(r)}\left(\det(R)^{a_1+\cdots+a_n}e^{\mathrm{tr}(2\pi i\xi(-\bar{z}))}\cdot D_\xi(e^{\mathrm{tr}(2\pi i\xi(\bar{z}-z))})\right)$$

using (3.11), (2.2), Lemma 2.4 and (3.8):

$$\delta_k^{(r)}(q^\xi) = \sum_{t=0}^{r}\binom{r}{t}\det(\xi)^{r-t}$$
$$\sum_{|L|\le mt-t} R_L(\varkappa-k-r)\mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho_{l_1}^\star(\xi)\right)\cdots\mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho_{l_t}^\star(\xi)\right)q^\xi,$$

$$
\begin{aligned}
&\delta_k^{(r)}(R_{i_1,j_1}^{a_1}\cdots R_{i_n,j_n}^{a_n}q^{\xi})\\
&\quad=\delta_k^{(r)}\left(\det(R)^{a_1+\cdots+a_n}e^{\mathrm{tr}(2\pi i\xi\bar z)}\cdot D_{\xi}(e^{\mathrm{tr}(2\pi i\xi(z-\bar z))})\right)\\
&\quad=e^{\mathrm{tr}(2\pi i(\xi\bar z))}\delta_k^{(r)}\left(\det(R)^{a_1+\cdots+a_n}\cdot D_{\xi}(e^{\mathrm{tr}(2\pi i\xi(z-\bar z))})\right)\\
&\quad=e^{\mathrm{tr}(2\pi i\xi\bar z))}D_{\xi}\left(\delta_k^{(r)}\left(\det(R)^{a_1+\cdots+a_n}\cdot e^{\mathrm{tr}(2\pi i\xi(z-\bar z))}\right)\right)
\end{aligned}
\tag{3.13}
$$

and our computation of (3.13) is reduced to computation of

$$
\delta_k^{(r)}\left(\det(R)^{a}\cdot e^{\mathrm{tr}(2\pi i\xi(z-\bar z))}\right)=(-2\pi i)^{-am}e^{\mathrm{tr}(-2\pi i(\xi\bar z))}\delta_k^{(r)}\left(\det(z-\bar z)^{-a}\cdot e^{\mathrm{tr}(2\pi i(\xi z))}\right)
$$

where $a=a_1+\cdots+a_n$.

Lemma 3.5. *Let $a\geq 0$ be a natural number then the action of the Shimura differential operator on the functions $\det(\bar z-z)^{-a}e_m(\xi z)$ (where $\xi\in B_m$ and $z\in\mathbb{H}_m$) is given by the formula*

$$
\begin{aligned}
&\delta_k^{(r)}\left(\det(z-\bar z)^{-a}e_m(\xi z)\right)\\
&\quad=(-2\pi i)^{-rm}\det(z-\bar z)^{-r-a}e_m(\xi z)R_m(2\pi i\xi(\bar z-z);r,\varkappa-k-r+a)\\
&\quad=(2\pi i)^{-rm}e^{\mathrm{tr}(2\pi i\xi(\bar z))}\det(z-\bar z)^{\varkappa-k-r}\Delta^r\left(e^{\mathrm{tr}(2\pi i\xi(z-\bar z))}\det(z-\bar z)^{-\varkappa+k+r-a}\right)
\end{aligned}
$$

where $\delta_k^{(r)}=\delta_{k+2r-2}\circ\cdots\circ\delta_k$,

$$
\delta_k f(z)=\left(-\frac{1}{4\pi}\right)^m\det(y)^{-1}M_kf(z)=(2\pi i)^{-m}\det(z-\bar z)^{\varkappa-k-1}\Delta\left[\det(z-\bar z)^{-\varkappa+k+1}f\right]
$$

$$
M_kf(z)=\det(z-\bar z)^{\varkappa-k}\Delta\left[\det(z-\bar z)^{k-\varkappa+1}f(z)\right],
$$

$$
R_m(z;r,\beta)=(-1)^{mr}e^{\mathrm{tr}(z)}\det(z)^{r+\beta}\Delta_m^r\left[e^{-\mathrm{tr}(z)}\det(z)^{-\beta}\right].
$$

Proof of 3.5: is very similar to that of 2.4. One proves this lemma by induction on r :

- If $r=0$, it is obviously true.
- Suppose next that it holds for some r then $\delta_k^{(r+1)}=\delta_{k+2r}\circ\delta_k^{(r)}$, and if one applies the Shimura operator δ_{k+2r} by induction one obtains

$$
\begin{aligned}
\delta_k^{(r+1)}\left(\det(z-\bar z)^{-a}e_m(\xi z)\right)&=(-2\pi i)^{-m}\det(z-\bar z)^{\varkappa-k-2r-1}\cdot\\
&\cdot\Delta\Big[\det(z-\bar z)^{k+2r+1-\varkappa}(2\pi i)^{-rm}\det(z-\bar z)^{-a-r}e_m(\xi z)\cdot\\
&\cdot R_m(2\pi i\xi(\bar z-z);r,\varkappa-k-r+a)\Big]
\end{aligned}
\tag{3.14}
$$

(recall that $4\pi\xi y = -2i\pi\xi(z-\bar{z})$). On the other hand $\bar{z}$ is aniholomorphic hence it behaves as a constant under the action of Δ, and it makes it possible to write, by putting $u = -2i\pi\xi$, $\beta = \varkappa - k - r + a$:

$$\begin{aligned} &R_m(u(z-\bar{z}); r, \beta) = \\ &\qquad (-1)^{mr} e^{\mathrm{tr}(u(z-\bar{z}))} \det(z-\bar{z})^{r+\beta} \Delta^r \left(e^{-\mathrm{tr}(u(z-\bar{z}))} \det(z-\bar{z})^{-\beta} \right) \end{aligned} \tag{3.15}$$

(compare with the proof of 2.4). One substituts (3.15) into (3.14) , then the factor $e_m(\xi\bar{z}) = e^{\mathrm{tr}(-u\bar{z})}$ which is an antiholomorphic function can be pulled out. One obtains then

$$\begin{aligned} &\delta_k^{(r+1)} \left(\det(z-\bar{z})^{-a} e_m(\xi z) \right) \\ &\quad = (-2\pi i)^{-m} \det(z-\bar{z})^{\varkappa-k-2r-1} \cdot \\ &\quad \cdot \Delta \Big[\det(z-\bar{z})^{k+2r+1-\varkappa} (2\pi i)^{-rm} \det(z-\bar{z})^{-a-r} e_m(\xi z) \cdot \\ &\quad \cdot (-1)^{mr} e^{\mathrm{tr}(u(z-\bar{z}))} \det(z-\bar{z})^{r+\beta} \Delta^r \left(e^{-\mathrm{tr}(u(z-\bar{z}))} \det(z-\bar{z})^{-\beta} \right) \Big] \\ &\quad = (2\pi i)^{-(r+1)m} (-1)^{m(r+1)} \det(z-\bar{z})^{\varkappa-k-2r-1} e_m(\xi\bar{z})) \cdot \\ &\quad \cdot \Delta \Big[\det(z-\bar{z})^{k+2r+1-\varkappa-a-r+r+\beta} \Delta^r \left(e^{-\mathrm{tr}(u(z-\bar{z}))} \det(z-\bar{z})^{-\beta} \right) \Big] \\ &\quad = (-2\pi i)^{-(r+1)m} \det(z-\bar{z})^{\varkappa-k-r-1} e_m(\xi\bar{z})) \cdot \\ &\quad \det(z-\bar{z})^{-r} \cdot \Delta \Big[\det(z-\bar{z})^{r+1} \Delta^r \left(e^{-\mathrm{tr}(u(z-\bar{z}))} \det(z-\bar{z})^{-\beta} \right) \Big] \end{aligned} \tag{3.16}$$

Let us now apply lemma 2.3 which gives

$$\begin{aligned} \det(z-\bar{z})^{-r} \cdot \Delta \Big[\det(z-\bar{z})^{r+1} \Delta^r \left(e^{-\mathrm{tr}(u(z-\bar{z}))} \det(z-\bar{z})^{-\beta+1} \right) \Big] \\ = \Delta^{r+1} \left(e^{-\mathrm{tr}(u(z-\bar{z}))} \det(z-\bar{z})^{-\beta+1} \right) \end{aligned}$$

and (3.16) transforms to

$$\begin{aligned} &\delta_k^{(r+1)} \left(\det(z-\bar{z})^{-a} e_m(\xi z) \right) \\ &= (-2\pi i)^{-(r+1)m} \det(z-\bar{z})^{\varkappa-k-r-1} e_m(\xi\bar{z}) \Delta^{r+1} \left(e^{-\mathrm{tr}(u(z-\bar{z}))} \det(z-\bar{z})^{-\beta+1} \right) \\ &= (-2\pi i)^{-(r+1)m} \det(z-\bar{z})^{\varkappa-k-(r+1)} e_m(\xi\bar{z}) \cdot \\ &\qquad \cdot \Delta^{r+1} \left(e^{-\mathrm{tr}(u(z-\bar{z}))} \det(z-\bar{z})^{-\varkappa+k+r+1-a} \right) \end{aligned} \tag{3.17}$$

finishing the proof by induction.

Corollary 3.6. *Let $a \geq 0$ be a natural number then the action of the Shimura differential operator on the functions $\det(2\pi i(\bar{z}-z))^{-a} e_m(\xi z)$ (where $\xi \in B_m$ and $z \in \mathbb{H}_m$) is given by the formula*

$$
\begin{aligned}
&\delta_k^{(r)} \left(\det(R)^a q^\xi\right) \\
&\quad = \det(R)^{r+a} R_m(\xi R^{-1}; r, \varkappa - k - r + a) q^\xi \\
&\quad = \det(R)^a q^\xi \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \\
&\quad \sum_{|L| \leq mt-t} R_L(\varkappa - k - r + a) \mathrm{tr}\left({}^t\rho_{m-l_1}(R) \cdot \rho^\star_{l_1}(\xi)\right) \cdots \mathrm{tr}\left({}^t\rho_{m-l_t}(R) \cdot \rho^\star_{l_t}(\xi)\right)
\end{aligned}
\tag{3.18}
$$

where $\delta_k^{(r)} = \delta_{k+2r-2} \circ \cdots \circ \delta_k$, and the coefficients $R_L(\beta) \in \mathbb{Z}[1/2][\beta]$ are polynomials in β of degree $(mt - |L|)$ and with coefficients in the ring $\mathbb{Z}[1/2]$.

Proof of 3.6.: Let us substitute into (2.3) $4\pi\xi y = 2\pi i \xi(\bar{z}-z) = \xi R^{-1}$ as z, $\beta = \varkappa - k - r + a$:

$$
R_m(z; r, \beta) = \sum_{t=0}^{r} \binom{r}{t} \det(z)^{r-t} \sum_{|L| \leq mt-t} R_L(\beta) \lambda_{l_1}(z) \cdots \lambda_{l_t}(z)
$$

giving

$$
\begin{aligned}
&R_m(\xi R^{-1}; r, \varkappa - k - r + a) \\
&= \sum_{t=0}^{r} \binom{r}{t} \det(\xi R^{-1})^{r-t} \sum_{|L| \leq mt-t} R_L(\varkappa - k - r + a) \lambda_{l_1}(\xi R^{-1}) \cdots \lambda_{l_t}(\xi R^{-1}) \\
&= \sum_{t=0}^{r} \binom{r}{t} \det(\xi)^{r-t} \det(R)^{-r} \\
&\quad \sum_{|L| \leq mt-t} R_L(\varkappa - k - r + a) \mathrm{tr}\left({}^t\rho_{m-l_1}(R) \cdot \rho^\star_{l_1}(\xi)\right) \cdots \mathrm{tr}\left({}^t\rho_{m-l_t}(R) \cdot \rho^\star_{l_t}(\xi)\right)
\end{aligned}
$$

using again (3.6): $\lambda_j(\xi R^{-1}) = \det(R)^{-1} \mathrm{tr}\left({}^t\rho_{m-j}(R) \cdot \rho^\star_j(\xi)\right)$, and (3.18) follows.

In a more general case let us apply Lemma 3.5 to (3.13):

$$\begin{aligned}
&\delta_k^{(r)}(R_{i_1,j_1}^{a_1}\cdots R_{i_n,j_n}^{a_n}q^{\xi})\\
&\quad = e^{\mathrm{tr}(2\pi i\xi\bar z))}D_\xi\left(\delta_k^{(r)}\left(\det(R)^{a_1+\cdots+a_n}\cdot e^{\mathrm{tr}(2\pi i\xi(z-\bar z))}\right)\right)\\
&\quad = e^{\mathrm{tr}(2\pi i\xi\bar z))}D_\xi\left((-2\pi i)^{-am}e^{\mathrm{tr}(-2\pi i(\xi\bar z))}\delta_k^{(r)}\left(\det(z-\bar z)^{-a}\cdot e^{\mathrm{tr}(2\pi i(\xi z))}\right)\right)\\
&\quad = (-2\pi i)^{-am}D_\xi\left(\delta_k^{(r)}\left(\det(z-\bar z)^{-a}\cdot e^{\mathrm{tr}(2\pi i(\xi z))}\right)\right)\\
&\quad = (-2\pi i)^{-am-rm}D_\xi\big(\det(z-\bar z)^{-r-a}e_m(\xi z)\\
&\qquad\qquad R_m(2\pi i\xi(\bar z-z);r,\varkappa-k-r+a)\big)
\end{aligned}\tag{3.19}$$

where $a=a_1+\cdots+a_n$. Let us use again the explicit expression (2.3) for the polynomial

$$R_m(z;r,\beta)=\sum_{t=0}^{r}\binom{r}{t}\det(z)^{r-t}\sum_{|L|\le mt-t}R_L(\beta)\lambda_{l_1}(z)\cdots\lambda_{l_t}(z),$$

where L runs over all the multi-indices $0\le l_1\le\cdots\le l_t\le m$, such that $|L|=l_1+\cdots+l_t\le mt-t$. Let us substitute (3.6) to (3.19) writng $4\pi y=2\pi i(\bar z-z)=R^{-1}$,

$$\lambda_j(2\pi i\xi(\bar z-z))=\lambda_j(\xi R^{-1})=\det(R)^{-1}\mathrm{tr}\left({}^t\rho_{m-j}(R)\cdot\rho_j^\star(\xi)\right):$$

$$\begin{aligned}
&\delta_k^{(r)}(R_{i_1,j_1}^{a_1}\cdots R_{i_n,j_n}^{a_n}q^{\xi})\\
&\quad = (-2\pi i)^{-am-rm}D_\xi\Big(\det(z-\bar z)^{-r-a}q^\xi R_m(2\pi i\xi(\bar z-z);r,\varkappa-k-r+a)\Big)\\
&\quad = (-2\pi i)^{-am-rm}D_\xi\Big(\det(z-\bar z)^{-r-a}q^\xi\sum_{t=0}^{r}\binom{r}{t}\det(2\pi i\xi(\bar z-z))^{r-t}\\
&\quad\sum_{|L|\le mt-t}R_L(\beta)\lambda_{l_1}(2\pi i\xi(\bar z-z))\cdots\lambda_{l_t}(2\pi i\xi(\bar z-z))\Big)\\
&\quad = D_\xi\Big(\det((2\pi i)(\bar z-z))^{-r-a}q^\xi\sum_{t=0}^{r}\binom{r}{t}\det(2\pi i\xi(\bar z-z))^{r-t}\\
&\quad\sum_{|L|\le mt-t}R_L(\beta)\lambda_{l_1}(2\pi i\xi(\bar z-z))\cdots\lambda_{l_t}(2\pi i\xi(\bar z-z))\Big)\\
&\quad = \det(R)^aD_\xi\Big(q^\xi\sum_{t=0}^{r}\binom{r}{t}\det(\xi)^{r-t}\\
&\quad\sum_{|L|\le mt-t}R_L(\beta)\mathrm{tr}\left({}^t\rho_{m-l_1}(R)\cdot\rho_{l_1}^\star(\xi)\right)\cdots\mathrm{tr}\left({}^t\rho_{m-l_t}(R)\cdot\rho_{l_t}^\star(\xi)\right)\Big)\Big)
\end{aligned}\tag{3.20}$$

§4. Arithmetical families of nearly holomorphic Siegel modular forms.

A preliminary definition of a p-adic family of Siegel modular forms is the following: $\{f_k\}$ is an infinite set of Siegel modular forms

$$f_k = \sum_{\xi \in B_m} a_k(\xi) q^\xi \in \bar{\mathbb{Q}}[\![q^{B_m}]\!]$$

(parametrized by the weight k), whose Fourier coefficients depend p-adic analytically on the weight k. If we fix embeddings $i_\infty\colon \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $i_p\colon \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ ($\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$ is the Tate field) then this amounts to analyticity of all maps $k \mapsto a_k(\xi)$ (where k is viewed as a p-adic parameter).

There are various approaches to construction of such families (at least for $m = 1$):

1) The technique of Hida and Wiles of Λ-adic modular forms where $\Lambda = \mathbb{Z}_p[\![T]\!]$ is the Iwasawa algebra: each element $A(T) = \sum\limits_n a_n T^n \in \Lambda$ converges on the unit disc $U = \{t \in \mathbb{C}_p \mid |t|_p < 1\}$ to a bounded $\mathbb{C}_p$-analytic function with a finite number of zeroes (in view of the p-adic Weierstrass preparation theorem, see [Ko]). We have therefore for each $k \geq 1$ a *specialization map*: $\Lambda \ni A(T) \mapsto A_k = A((1+p)^k - 1) \in \mathbb{Z}_p$; (a conceptual definition of this map uses the description of Λ in terms of p-adic distributions μ on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p$: one can attach to such a μ a unique power series $A_\mu(T) = \int_{\mathbb{Z}_p} (1+T)^x \mu(x) \in \Lambda$ and this correspondence is a bijection. Then the above specialization $A_\mu((1+p)^k - 1) = \int_{\mathbb{Z}_p} (1+p)^{kx} \mu(x)$ is the image at $A_\mu(T)$ of the $\mathbb{Z}_p$-ring homomorphism $\varkappa_k\colon \Lambda \to \mathbb{Z}_p$ given by $\varkappa_k(1+T) = (1+p)^k$). In order to introduce Λ-adic modular form consider the ring $\Lambda[\![q^{B_m}]\!]$ of all formal Siegel q-expansions with coefficients in Λ, and for each element $F = \sum\limits_{\xi \in B_m} A(\xi, T) q^\xi \in \Lambda[\![q^{B_m}]\!]$ and for each $k \geq 1$ we have a ring homomorphism $\varkappa_k\colon \Lambda[\![q^{B_m}]\!] \to \mathbb{Z}_p[\![q^{B_m}]\!]$, defined by

$$\varkappa_k(F) = \sum_{\xi \in B_m} A_k(\xi) q^\xi, \tag{4.1}$$

where

$$A_k(\xi = A(\xi, (1+p)^k - 1) \in \mathbb{Z}_p \ .$$

Let χ be a Dirichlet character modulo N and let us fix a finite extension $\mathcal{O}$ of $\mathbb{Z}_p$ containing the values of χ, and put $\Lambda_\mathcal{O} = \mathcal{O}[\![T]\!]$. Consider the ring

$$\Lambda_\mathcal{O}[\![q^{B_m}]\!][R_{i,j}]_{i,j=1,\cdots,m}$$

of all formal nearly holomorphic q-expansions (or (q,R)-expansions) with coefficients in $\Lambda_{\mathcal{O}}$.

Definition 4.1. An element $F = \sum_{\xi \in B_m} A(\xi, T)q^{\xi} \in \Lambda_{\mathcal{O}}[\![q^{B_m}]\!]$ is called a $\Lambda_{\mathcal{O}}$-adic Siegel modular form of level N and character χ if for all $k \gg 0$ the element

$$F_k = \sum_{\xi \in B_m} A(\xi, (1+p)^k - 1)q^n \in \mathcal{O}[\![q^{B_m}]\!]$$

represents the Fourier expansion of a complex nearly holomorphic Siegel modular form: $F_k \in \mathcal{M}_k^m(\Gamma_0(Np), \chi\omega^{-k})$, where $\omega\colon (\mathbb{Z}/p\mathbb{Z})^* \to \mu_{p-1} \subset \bar{\mathbb{Q}}^* \hookrightarrow \mathbb{C}^*$ is the Teichmüller character.

Let us denote by $\mathcal{M}^m(N, \chi, \Lambda_{\mathcal{O}})$ the $\Lambda_{\mathcal{O}}$-submodule of $\Lambda_{\mathcal{O}}[\![q^{B_m}]\!]$ generated by all such elements.

Note that such a family $F = \{F_k\}$ depends only on one parameter T, and in the above definition one has to use a variable Dirichlet character $\chi\omega^{-k}$ (which depends on k).

2) In order to define p-adic families of nearly holomorphic Siegel modular forms one uses the notion of the p-adic integration with values in spaces of (formal) nearly holomorphic Siegel modular forms.

Consider any profinite group $Y = \varprojlim Y_i$ (for example, $Y = \mathbb{Z}_p$, $Y = \Gamma = 1 + p\mathbb{Z}_p = \overline{\langle 1+p \rangle}$, $Y = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ etc.).

The completed group ring $\mathcal{O}[\![Y]\!] = \varprojlim \mathcal{O}[Y_i]$ is a compact ring which generalizes the Iwasawa algebra $\Lambda_{\mathcal{O}}$, and it may be viewed as the $\mathcal{O}$-algebra of all $\mathcal{O}$-valued distributions on Y; for the natural projection homomorphisms $\pi_i\colon Y \to Y_i$ put $U_{ij} = \pi_i^{-1}(y_j)$ $(y_j \in Y_i)$ then U_{ij} generate the profinite topology of Y and a distribution μ on Y corresponds to an element $f = \{f_i\} \in \mathcal{O}[\![Y]\!]$ where $f_i(y_j) = \mu(U_{ij})$. Recall that a distribution μ on Y with values in $\mathcal{O}$ is a finite-additive map on open-compact subsets $U \subset Y$:

$$\mu\colon \left\{ \begin{matrix} \text{open-compact sets} \\ U \subset Y \end{matrix} \right\} \longrightarrow \mathcal{O}$$

The natural multiplication in $\mathcal{O}[\![Y]\!]$ corresponds to the *convolution* of distributions:

$$(\mu_1 * \mu_2)(\pi_i^{-1}(y)) = \sum_{y = z\cdot w \in Y_i} \mu_1(\pi_i^{-1}(z)) \cdot \mu_2(\pi_i^{-1}(w))$$

Examples. a) $\Lambda_{\mathcal{O}} = \mathcal{O}[\![T]\!] \cong \mathcal{O}[\![\Gamma]\!]$, $1+T \mapsto 1+p \in \Gamma$.

b) $\Lambda_{\mathcal{O}} = \mathcal{O}[\![\mathbb{Z}_p^*]\!] \cong \Lambda_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}[\Delta]$ where $\mathbb{Z}_p^* = \Gamma \times \Delta$ (a direct product, where $\Delta = \mathbb{Z}_p^{*\mathrm{tors}}$).

For a continuous group homomorphism $\varkappa\colon Y \to \mathcal{O}^*$ we obtain an $\mathcal{O}$-algebra homomorphism $\varkappa\colon \mathcal{O}[\![Y]\!] \to \mathcal{O}$ which provides a group-theoretical interpretation of the above specialization maps: $\varkappa_k(x) = x^k$ for all $x \in \mathbb{Z}_p^*$.

In order to define more general p-adic families of nearly holomorphic Siegel modular forms, consider a family $\mathcal{P} = \{P\} \subset \mathcal{C}(Y, \mathcal{O})$ of continuous functions on Y with values in $\mathcal{O}$.

Definition 4.2. A p-adic family of nearly holomorphic Siegel modular forms parametrized by $\mathcal{P}$ is an element F of the ring $\mathcal{O}[\![Y]\!][\![q^{B_m}]\!][R_{i,j}] = \mathcal{D}istr\,(Y, \mathcal{O}[\![q^{B_m}]\!][R_{i,j}])$ such that if

$$F = \sum_{\xi \in B_m, \underline{n} \geq 0} \mu_{\xi,\underline{n}} R^{\underline{n}} q^{\xi} \in \mathcal{O}[\![Y]\!][\![q^{B_m}]\!][R_{i,j}], \tag{4.2}$$

then for almost all $P \in \mathcal{P}$ the integral

$$\int_Y P dF = \sum_{\xi \in B_m, \underline{n} \geq 0} \left(\int_Y P d\mu_{\xi,\underline{n}} \right) R^{\underline{n}} q^{\xi} \in \mathcal{O}[\![q^{B_m}]\!][R_{i,j}]$$

represents the Fourier expansion ((q, R)-expansion) of a complex nearly holomorphic Siegel modular form of weight $k = k(P)$, type $r = r(P)$, level $N = N(P)$ and character $\chi = \chi_P$. Here

$$R = (R_{i,j}) = (4\pi \mathrm{Im}(z))^{-1} = 2\pi i(\bar{z} - z)^{-1}, \underline{n} = (n_{ij})_{i,j=1,\cdots,m, i \geq j} \in \mathbb{N}^{m(m+1)/2},$$

$$R^{\underline{n}} = \prod_{i,j=1,\cdots,m, i\geq j} R_{i,j}^{n_{i,j}}.$$

Note that the parameter P could include all classical parameters: the weight k, the type r, and the character $\varepsilon \bmod p^{\alpha}$.

Let us use the notation $M_{\mathcal{P}}(\mathcal{O}[\![Y]\!])$ for the $\mathcal{O}[\![Y]\!]$-submodule of $\mathcal{O}[\![Y]\!][\![q^{B_m}]\!][R_{i,j}]$ generated by all such F.

As a version of this definition one can also consider, instead of an integral valued distribution $F \in \mathcal{D}istr\,(Y, \mathcal{O}[\![q^{B_m}]\!][R_{i,j}])$, growing measures (or "algebraic distributions") with values in $\mathcal{O}[\![q^{B_m}]\!][R_{i,j}]\otimes_{\mathcal{O}}\mathbb{C}_p$. Then such a construction provides us with more general families (for example, in the non-ordinary case, [CoMa]), see also [Colm 98].

One verifies that in the case of $Y = \Gamma$, $\Lambda_{\mathcal{O}} = \mathcal{O}[\![\Gamma]\!]$, and

$$\mathcal{P} = \left\{ P = P_{k,\varepsilon}\colon \longmapsto \varepsilon(u)u^k, \varepsilon \bmod p^\alpha, u \in \Gamma \right\} ,$$

this leads to the first definition (we assume that $\mathcal{O}$ contains the values of all Dirichlet characters ε).

An advantage of this alternative approach is a possibility to use the technique of construction of families F as distributions using the abstract Kummer congruences (which will be explained in more detail in this section).

Another advantage is a possibility to avoid the use of the variable Dirichlet character $\chi\omega^{-k}$ in the definition 4.1: if we replace Γ by $\mathbb{Z}_p^* = \Gamma \times \Delta$, $\Delta = \mu_{p-1} = (\mathbb{Z}_p^*)^{\mathrm{tors}}$ and put

$$F_k = \int_{\mathbb{Z}_p^*} \psi \left\langle x \right\rangle^k dF = \int_{\mathbb{Z}_p^*} \psi\omega^{-k}(x)x^k dF$$

then we may simply require that $\int_{\mathbb{Z}_p^*} \psi x^k dF \in M_k(Np, \chi\psi)$ for all characters $\psi\colon \mathbb{Z}_p^* \to \mathcal{O}^*$ of finite order, including $\psi\omega^{-k}$ (we use the decomposition $x^k = \omega^k(x)\cdot\langle x\rangle^k$, where $\omega(x) = \lim\limits_{n\to\infty} x^{p^n} \in \Delta$ and $\langle x\rangle = x/\omega(x) = (1+p)^s$, $s \in \mathbb{Z}_p$).

In order to recall the technique of *abstract Kummer congruences*, consider a complete p-adic ring $A = \varprojlim\limits_N A/p^N A$ and consider any family $\{\varphi_i\}_{i\in I} \subset \mathcal{C}(Y, \mathbb{Z}_p)$ which generate a dense $\mathbb{Q}_p$-linear subspace.

Theorem 4.1. *Let $\{a_i\} \subset A$ a collection of elements of A parametrized by the same set of indices I that we used for $\{\varphi_i\}$. Then the existence of a unique distribution $\mu \in \mathcal{D}istr\,(Y, A)$ with given integrals $a_i = \int\limits_Y \varphi_i d\mu$ is equivalent to the validity of the following congruences: for $\forall\, N \in \mathbb{N}$ and for each finite linear combination $\sum\limits_i b_i\varphi_i$, $b_i \in \mathbb{Z}_p$*

$$\left(\forall\, y \in Y \ \sum_i b_i\varphi_i(y) \equiv 0 \bmod p^N A \Longrightarrow \sum_i b_i a_i \equiv 0 \bmod p^N \right) \tag{4.1}$$

Proof: [Ka78] 1) If μ exists then $\int\limits_Y \sum b_i\varphi_i d\mu = \sum b_i a_i \equiv 0(p^N A)$ because $\sum b_i\varphi_i$ is divisible by p^N in the ring $\mathcal{C}(Y, A)$.

2) In order to construct μ out of $a_i \in A$, we use the equivalent definition of an A-distribution as an A-measure (a $\mathbb{Z}_p$-linear form on $\mathcal{C}(Y, \mathbb{Z}_p)$ with values in A): we need to construct such a linear form

$$\mu\colon \mathcal{C}(Y, \mathbb{Z}_p) \to A \quad \text{with the condition} \quad \forall i \in I \int\limits_Y \varphi_i d\mu = a_i \tag{4.2}$$

For any $\varphi \in \mathcal{C}(Y, \mathbb{Z}_p)$ let us choose an approximation $\sum b_i\varphi_i$. Then $\varphi = \lim\sum_i b_i\varphi_i \Rightarrow \int \varphi d\mu = \lim\sum_i b_i a_i \in A$. We see that the limit is well defined in view of the congruences (4.1).

The first example of p-adic families (given by Serre [Se73] in the elliptic modular case) is related to Eisenstein series. Let $k > m+1$, and let χ be a Dirichlet character modulo N. Consider in the Siegel modular case the series

$$E_k^{(m)}(z, \chi) = \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in P^{(m)} \cap \Gamma_0^{(m)}(N) \setminus \Gamma_0^{(m)}(N)} \chi(\det d)\det(cz+d)^{-k} \tag{4.3}$$

where

$$P^{(m)} = \left\{ \gamma = \begin{pmatrix} a & b \\ 0_m & d \end{pmatrix} \in \Gamma^{(m)} \right\},$$

so that for $m = N = 1$, $k \geq 4$ one has

$$E_k(z) = E_k^{(1)}(z) = \sum_{\gamma \in P\backslash\Gamma} 1|_k\gamma = \sum_{\substack{(c,d)=1, c>0 \\ c=0, d=1}} (cz+d)^{-k} = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \ ,$$

$$q = \exp(2\pi i z),\ \sigma_{k-1}(n) = \sum_{d|n} d^{k-1},\ \zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{\ell} (1-\ell^{-s})^{-1}\ (\mathrm{Re}(s) > 1)$$

$$\zeta(1-k) = -\frac{B_k}{k},\ \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = \frac{t}{e^t - 1}$$

$$E_4^{(1)}(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,\ E_6^{(1)} = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \ldots$$

Note also that (see [Nag2], p.408)

$$\begin{aligned} E_4^{(2)}(z) =& 1 + 240q_{11} + 240q_{22} + 2160q_{11}^2 + (240q_{12}^{-2} + 13440q_{12}^{-1} \\ & 30240 + 13440q_{12} + 240q_{12}^2)q_{11}q_{22} + 2160q_{22}^2 + \ldots \end{aligned}$$

$$E_6^{(2)}(z) = 1 - 504q_{11} - 504q_{22} - 16632q_{11}^2 + (-540q_{12}^{-2} + 44352q_{12}^{-1}$$
$$166320 + 44352q_{12} - 504q_{12}^2)q_{11}q_{22} - 16632q_{22}^2 + \dots.$$

Let us normalize, for $m = 1$, the Eisenstein series $E_k^{(1)}(z)$ by

$$G_k = G_k^{(1)} = \frac{\zeta(1-k)}{2} E_k^{(1)} = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

and put

$$G_k^*(z) = G_k(z) - p^{k-1} G_k(pz) \quad \text{then}$$
$$G_k^* = \frac{\zeta^*(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}^*(n) q^n,\ \sigma_{k-1}^*(n) = \sum_{\substack{d \mid n \\ (d,p)=1}} d^{k-1}, \quad \text{where}$$
$$\zeta^*(s) = \zeta(s)(1 - p^{-s}) = \sum_{\substack{n=1 \\ (p,n)=1}} n^{-s}$$ denotes the Riemenn zeta function with p-Euler factor removed.

Theorem 4.2. a) *Let* $k \equiv k' \bmod (p-1)p^{N-1}$ *then for all* $k \not\equiv 0(p-1)$ *we have that* $G_k^* \equiv G_{k'}^* \bmod p^N$ *in* $\mathbb{Q}[\![q]\!]$.

b) *Let* $k \equiv k' \bmod (p-1)p^{N-1}$ *then for any* $c \in \mathbb{Z}$, $(c,p) = 1$, $c > 1$ *we have that* $(1 - c^k)G_k^* \equiv (1 - c^{k'})G_{k'}^* \bmod p^N$ *(without restriction on* k*).*

c) $f_k = (1 - c^k)G_k^*$ *is a* p*-adic family on* $\mathbb{Z}_p^*$ *parametrized by the set* $\mathcal{P} = \{y \mapsto y^k, k \geq 4\}$ *of* p*-adic characters of* $\mathbb{Z}_p^*$.

Before giving a proof of Theorem 4.2, we explain how to avoid the use of a regularizing integer $c > 1$, $c \in \mathbb{Z}_p^*$ in 4.2. Consider $Y = \mathbb{Z}_p^*$, $\mathcal{O} \supset \mathbb{Z}_p$, $\mathcal{L}_\mathcal{O} = \operatorname{Quot} \Lambda_\mathcal{O}$, $\underline{\mathcal{L}}_\mathcal{O} = \operatorname{Quot} \underline{\Lambda}_\mathcal{O}$, the complete quotient ring of the completed group ring $\underline{\Lambda}_\mathcal{O} = \mathcal{O}[\![\mathbb{Z}_p^*]\!]$. Let us denote by $M(N, \chi, \underline{\Lambda}_\mathcal{O})$ the $\underline{\Lambda}_\mathcal{O}$-submodule of $\underline{\Lambda}_\mathcal{O}[\![q]\!]$ generated by all $F = \sum_{n \geq 0} \mu_n q^n \in \underline{\Lambda}_\mathcal{O}[\![q]\!]$ such that for all $k \gg 0$ $\int_{\mathbb{Z}_p^*} x^k dF = \sum_{n \geq 0} \left(\int_{\mathbb{Z}_p^*} x^k \mu_n \right) q^n \in \mathcal{O}[\![q]\!]$ is in $M_k(N_p, \chi, A)$ where A is a subring of $\bar{\mathbb{Q}}$ such that $i_p(A) \subset \mathcal{O}$.

Define $\mathcal{M}(N, \chi, \underline{\mathcal{L}}_\mathcal{O}) = \mathcal{M}(N, \chi, \underline{\Lambda}_\mathcal{O}) \otimes_{\underline{\Lambda}_\mathcal{O}} \underline{\mathcal{L}}_\mathcal{O}$ then the series f of Theorem 4.2 is an element of $\mathcal{M}(1, 1, \underline{\Lambda}_{\mathbb{Z}_p})$ depending on c, and $1 - c^k = \int_{\mathbb{Z}_p^*} x^k (\delta_1 - \delta_c)$ (δ_c denotes the Dirac distribution at $c \in \mathbb{Z}_p^*$), $\delta_c(U) = \begin{cases} 1, & \text{if } c \in U \\ 0, & \text{if } c \notin U \end{cases}$. The element $(\delta_1 - \delta_c)^{-1} \in \underline{\mathcal{L}}_{\mathbb{Z}_p}$ and

$G^* = f \otimes (\delta_1 - \delta_c)^{-1}$ is an element of $\mathcal{M}(1,1,\underline{\mathcal{L}}_{\mathbb{Z}_p})$ independent on the choice of c. It has the property $\varkappa_k(G^*) = G_k^*$ where $\varkappa_k$ is a unique extension of $\varkappa_k$: $\underline{\Lambda}_{\mathbb{Z}_p} \to \mathbb{Z}_p$ to $\underline{\mathcal{L}}_{\mathbb{Z}_p}$ (we apply $\varkappa_k$ coefficient-by-coefficient to $G^* \in \mathcal{M}(1,1,\underline{\mathcal{L}}_{\mathbb{Z}_p})$).

Proof of 4.2: The statements a) and b) follow from c). In order to show c) we put $f_k = \sum\limits_{n \geq 0} a_k(n) q^n$ then for $n > 0$ $a_k(n) = (1-c^k) \sum\limits_{\substack{d|n \\ (d,p)=1}} d^{k-1} = \int\limits_{\mathbb{Z}_p} x^k d\mu_{n,c}(x)$ where

$$\mu_{n,c} = (\delta_1 - \delta_c) * \sum_{\substack{d|n \\ (d,p)=1}} d^{-1} \cdot \delta_d, \quad \delta_c(U) = \begin{cases} 1, & \text{if } c \in U, \\ 0, & \text{if } c \notin U. \end{cases}$$

(note that for any $\mu^{(1)}, \mu^{(2)} \in \mathcal{D}istr\,(\mathbb{Z}_p^*, \mathcal{O})$, and for any character $P \in \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^*, \mathcal{O}^*)$ we have that $\int\limits_{\mathbb{Z}_p^*} Pd(\mu^{(1)} * \mu^{(2)}) = \int\limits_{\mathbb{Z}_p^*} Pd\mu^{(1)} \cdot \int\limits_{\mathbb{Z}_p^*} Pd\mu^{(2)}$.)

The case $n=0$: $a_k(0) = (1-c^k)\zeta^*(1-k)$ will be treated using the classical Kummer congruences given by the following

Theorem 4.3 (Kummer). *Put* $\zeta_{(p)}^{(c)}(-k) = (1-c^{k+1})(1-p^k)\zeta(-k)$, $k \geq 0$, *and let* $h(x) = \sum\limits_i \alpha_i x^i \in \mathbb{Z}[x]$ *such that* $h(a) \equiv 0 \bmod p^N$ *for all* $\alpha \in \mathbb{Z}_p^*$. *Then we have that* $\sum\limits_i \alpha_i \zeta_{(p)}^{(c)}(-i)$ $\equiv 0 \bmod p^N$.

Corollary 1 (p-adic continuity of $\zeta_{(p)}^{(c)}(-k)$ in a progression $\bmod\,(p-1)$). *If* $h(x) = x^k - x^{k'}$, $k \equiv k' \bmod (p-1)\,p^{N-1}$ *then*

$$\zeta_{(p)}^{(c)}(-k) \equiv \zeta_{(p)}^{(c)}(-k') \bmod p^N .$$

Proof of Corollary 1: We have that $h(a) \equiv 0 \bmod p^N$ by Euler's theorem: $a^{\varphi(p^N)} \equiv 1$ $(\mathrm{mod}\ p^N)$, $(a,p) = 1$.

Corollary 2 (Mazur). *There exists a unique measure* $\mu^{(c)}$ *on* $\mathbb{Z}_p^*$ *such that for all* $k \geq 1$ $\int\limits_{\mathbb{Z}_p^*} x^k d\mu^{(c)} = \zeta_{(p)}^{(c)}(1-k) = (1-c^k)(1-p^{k-1})\zeta(1-k)$.

Proof of Corollary 2: is implied by Theorem 4.1: the values

$$\int\limits_{\mathbb{Z}_p^*} x^{k+1} d\mu^{(c)} = \zeta_{(p)}^{(c)}(-k)$$

satisfy the abstract Kummer congruences in view of Theorem 4.3.

Let us show that Theorem 4.3 implies Theorem 4.2 c): putf

$$f=\frac{1}{2}\mu^{(c)}+\sum_{n\geq 0}\mu_{n,c}q^n\in\underline{\Lambda}[\![q]\!],\underline{\Lambda}=\mathcal{D}istr\left(\mathbb{Z}_p^*,\mathbb{Z}_p\right).$$

Then

$$\int_{\mathbb{Z}_p^*}x^k df=\frac{1}{2}\int_{\mathbb{Z}_p^*}x^k d\mu^{(c)}+\sum_{n\geq 0}\left(\int_{\mathbb{Z}_p^*}x^k d\mu_{n,c}\right)q^n=(1-c^k)G_k^*.$$

Proof of Theorem 4.3 [Ka78] : uses the sums

$$S_k(M)=\sum_{n=1}^{M-1}n^k=\frac{1}{k+1}[B_{k+1}(M)-B_{k+1}]$$

where

$$S_k^*(p^N)=\sum_{\substack{n=1\\p\nmid n}}^{p^N-1}n^k=S_k(p^N)-p^kS_k(p^{N-1}).$$

Let $f\in\mathcal{M}_k^m(N,\psi)$ be a Siegel modular form whose Fourier expansion is given by $f(z)=\sum_{\xi\in B_m}c(\xi)q^\xi$. As in §1, we fix embeddings $i_\infty\colon\bar{\mathbb{Q}}\hookrightarrow\mathbb{C}$, $i_p\colon\bar{\mathbb{Q}}\hookrightarrow\mathbb{C}_p$ and assume that $c(\xi)\in\bar{\mathbb{Q}}$ and $i_p(c(\xi))\in\mathcal{O}\subset\mathbb{C}_p$.

Let us construct p-adic families of nearly holomorphic Siegel modular forms starting from the action on a given form f of the arithmetical differential operator $\delta_k^{(r)}$:

$$\begin{aligned}\delta_k^{(r)}f(z)=&\sum_{\xi\in B_m}c(\xi)\sum_{t=0}^{r}\binom{r}{t}\det(\xi)^{r-t}\\&\sum_{|L|\leq mt-t}R_L(\varkappa-k-r)\mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho^\star_{l_1}(\xi)\right)\cdots\mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho^\star_{l_t}(\xi)\right))q^\xi\\&\in\mathbb{Q}(f)[\![q^{B_m}]\!][R_{i,j}]_{i,j=1,\cdots,m}\end{aligned}\tag{4.4}$$

where $\mathbb{Q}(f)$ is the subfield of $\mathbb{C}$ generated by the Fourier coefficients of f, L runs over all the multi-indices $0\leq l_1\leq\cdots\leq l_t\leq m$ as above. Let us define the family of polynomials

$$\begin{aligned}P_{r,k}(\xi,R)=&\sum_{t=0}^{r}\binom{r}{t}\det(\xi)^{r-t}\\&\sum_{|L|\leq mt-t}R_L(\varkappa-k-r)\mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho^\star_{l_1}(\xi)\right)\cdots\mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho^\star_{l_t}(\xi)\right))\end{aligned}\tag{4.5}$$

Theorem 4.4. *Put* $A_{m,p} = A_m \otimes_{\mathbb{Z}} \mathbb{Z}_p$ *then there exists a* p*-adic measure* μ_f *on the profinite group* $Y = A_{m,p}$ *with values in* $\mathcal{O}[\![q^{B_m}]\!][R_{i,j}]_{i,j=1,\cdots,m}$ *such that for each* $k > m+1$

$$\int\limits_{A_{m,p}} P_{r,k}(\xi, R)\mu_f(\xi) = \delta_k^{(r)} f \in \mathcal{O}[\![q^{B_m}]\!][R_{i,j}]_{i,j=1,\cdots,m} \ . \tag{4.6}$$

In this case $\mathcal{P} = \{P_{r,k}\}$, $k(P_{r,k}) = k + 2r, N(P_{r,k}) = N, r(P_{r,k}) = r$.

Proof of Theorem 4.4: Let $\delta_\xi \in \mathcal{D}istr\,(A_{m,p}, \mathcal{O}) = \mathcal{O}[\![A_{m,p}]\!]$ denote the Dirac distribution on $A_{m,p}$ concentrated in ξ. Then we define

$$\mu_f \in \mathcal{D}istr\,(A_{m,p}, \mathcal{O}[\![q^{B_m}]\!][R_{i,j}]_{i,j=1,\cdots,m})$$

as the following formal series

$$\mu_f = \sum_{\xi \in B_m} c(\xi)\delta_\xi q^\xi.$$

Let us compute the integral

$$\int\limits_{A_{m,p}} P_{r,k}(\xi, R)\mu_f(\xi) = \int\limits_{A_{m,p}} \sum_{\xi \in B_m} c(\xi)P_{r,k}(\eta, R)q^\xi \delta_\xi(\eta) = \sum_{\xi \in B_m} c(\xi)P_{r,k}(\xi, R))q^\xi. \tag{4.7}$$

If we take into account (4.4) and the definition (4.5) then the equality (4.6) follows from (4.7).

§5. Families of distributions attached to Siegel-Eisenstein series.

Recall that in the elliptic modular case $m=1$, for a Dirichlet character $\chi \bmod N$, and $k>2$, $\chi(-1)^k$, the following Eisenstein series is defined

$$G_k(z,\chi)=G_k^{(1)}(z,\chi)=\frac{L(1-k,\chi)}{2}+\sum_{n=1}^{\infty}\sigma_{k-1,\chi}(n)q^n\in \mathcal{M}_k(\Gamma_0(N),\chi,\mathbb{Q}(\chi)) \ .$$

If χ is primitive, then

$$G_k(z,\chi)=\frac{L(1-k,\chi)}{2}\sum_{\gamma\in P\setminus\Gamma_0(N)}\overline{\chi(d)}(cz+d)^{-k}, \gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(N) \ .$$

Let $N>1$ then the action of the involution $\tau_N=\left(\begin{smallmatrix} 0 & -1\\ N & 0\end{smallmatrix}\right)$ brings $G_k(z,\chi)$ to a similar series but without a constant term:

$$G_k'(z,\chi)=\sum_{n=1}^{\infty}\sigma_{k-1,\bar\chi}'(n)q^n\in\mathcal{M}_k(\Gamma_0(N),\bar\chi,\mathbb{Q}(\bar\chi)) \ ,$$

where

$$\sigma_{k-1,\bar\chi}'(n)=\sum_{d|n}\bar\chi(d)\left(\frac{n}{d}\right)^{k-1}=n^{k-1}\prod_{p|n}\Big(\sum_{i=1}^{\mathrm{ord}_p n}\chi(p)^{-i}p^{-i(k-1)}\Big)$$

is a finite Euler product.

In the Siegel modular case $\Gamma^m=Sp_{2m}(\mathbb{Z})\supset\Gamma_0^m(N)$, and there is an analogue

$$E_k(z,\chi)=\sum_{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in P^{(m)}\cap\Gamma_0^{(m)}(N)\setminus\Gamma_0^{(m)}(N)}\chi(\det d)\det(cz+d)^{-k}$$

of $G_k(z,\chi)$. However, for $N>1$, the Fourier expansion is known only for *the involuted series* $E_k|\tau_N$, where $\tau_N=\left(\begin{smallmatrix} 0_m & -1_m\\ N\cdot 1_m & 0_m\end{smallmatrix}\right)$ (for $N=1$ both series coincide). Here $z\in\mathbb{H}_m$ is in the Siegel upper half plane:

$$\mathbb{H}_m=\left\{z={}^tz\in \mathrm{M}_m(\mathbb{C})|\mathrm{Im}\, z>0\right\}, \quad \text{and} \quad P=\left\{\begin{pmatrix} a & b\\ 0 & c\end{pmatrix}\in Sp_{2m}(\mathbb{R})\right\}$$

is the Siegel parabolic subgroup.

Example. Involuted Siegel-Eisenstein series: let $k > m+1$, and let χ be a Dirichlet character modulo N. Let us define

$$E = E(k,\chi) = \sum_{\gamma \in P \diagdown \Gamma_0^m(N)} \chi(\det d)\det(cz+d)^{-k} \in \mathcal{M}_k(\Gamma_0(N),\bar{\chi}) \ , \tag{5.1}$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^m(N)$.

Let us also define

$$\begin{aligned} G^+(k,\chi) &= L^+(k,\chi)E(k,\chi)|\tau_N, \ \tau_N = \begin{pmatrix} 0_m & -1_m \\ N\cdot 1_m & 0_m \end{pmatrix} \\ E|\tau_N &= N^{km/2}(\det Nz)^{-k}E(-(Nz)^{-1}) \ , \\ L^+(k,\chi) &= \Gamma(k,m)L_N(k,\chi)\prod_{i=1}^{[m/2]} L_N(2k-2i,\chi^2) \ , \\ \Gamma(k,m) &= i^{mk}2^{-m(k+1)}\pi^{-mk}\Gamma_m(k), \Gamma_m(s) = \pi^{m(m-1)/4}\prod_{j=0}^{m-1}\Gamma(s-(j(2))) \ . \end{aligned} \tag{5.2}$$

and recall that $\varkappa = (m+1)/2$.

Theorem 5.1 (Siegel, Shimura [Sh83], P. Feit [Fe]). *If m is odd, $k > m+1$, $N > 1$, then*

$$G^+(k,\chi) = \sum_{0<\xi\in B_m} b^+(\xi,k,\chi)q^\xi, \ \ b^+(\xi,k,\chi) = \det \xi^{k-\varkappa}M(\xi,k,\chi) \ , \tag{5.3}$$

where $M(\xi,k,\chi) = \prod_{b|\det(2\xi)} M_q(\xi,\chi(\ell)\ell^{-k})$, $M_\ell(\xi,x) \in \mathbb{Z}[x]$ (a finite Euler product, e.g. for $m=1$ $M_\ell(n,x) = \sum_{i=0}^{\mathrm{ord}_\ell n} x^i$).

In order to construct a family, put $G_p^+(k,\chi) \overset{\det}{=} \sum_{\substack{\xi>0 \\ p\nmid\det(2\xi)}} b^+(\xi,k,\chi)q^\xi$.

Theorem 5.2. see [PaSE] *Let $\chi \bmod Np$ be a Dirichlet character, $\mathcal{O} \supset \mathbb{Z}_p[x]$ then there exists $E_{\lambda_\mathcal{O}}(\chi,i) \in \Lambda_\mathcal{O}[\![q^{B_m}]\!]$ such that for all $k \gg 0$, $k \equiv i \bmod (p-1)$*

$$E_{\Lambda_\mathcal{O}}(\chi,i)\big|_{T=(1+p)^k-1} = G_p^+(k,\chi^{-1}\omega^k) \tag{5.4}$$

$E_\Lambda(\chi)$ *is called the* Λ*-adic Siegel-Eisenstein series.*

Proof: For any $\delta \in \Delta = \mathbb{Z}_p^{*\mathrm{tors}}$ let us introduce the notation

$$G^+_{p,\delta}(k,\chi) = \sum_{\substack{\xi>0 \\ \det\xi\equiv\delta \bmod p}} b^+(\xi,k,\chi)q^\xi \tag{5.5}$$

then

$$\begin{aligned} G^+_p(k,\chi^{-1}\omega^k)q^\xi &= \sum_{\delta\in\Delta} \sum_{\substack{\xi>0 \\ \det\xi\equiv\delta \bmod p}} \det\xi^{k-((m+1)/2)} \cdot M(\xi,k,\chi^{-1}\omega^k)q^\xi \\ &= \sum_{\delta\in\Delta} \sum_{\substack{\xi>0 \\ \det\xi\equiv\delta (\bmod p)}} (\det\xi)^{-(m+1)/2} \\ &\qquad \cdot \langle\det\xi\rangle^k \prod_{\ell|\det(2\xi)} M_\ell(\xi,\chi^{-1}\omega^k(\ell)\ell^{-k})q^\xi \end{aligned} \tag{5.6}$$

Note that $\omega^k(\ell)\ell^{-k} = \langle\ell\rangle^{-k}$ hence the coefficient of q^ξ in (5.6) is a finite linear combination $\sum_j \alpha_j\beta_j^k$ where $\alpha_j \in \mathcal{O}$, $\beta_j = (1+p)^{\gamma_j} \in \Gamma$.

It follows that we can define

$$E_{\Lambda_\mathcal{O}}(\chi,i) = \sum_{\delta\in\Delta} \delta^i \sum_{\substack{\xi>0 \\ \det\xi\equiv\delta \bmod p}} \sum_j \alpha_j(1+T)^{\gamma_j} \cdot q^\xi \tag{5.7}$$

because if we substitute $T = (1+p)^k - 1$ to (5.7) we obtain

$$\sum_{\delta\in\Delta} \delta^i \sum_{\substack{\xi>0 \\ \det\xi\equiv\delta \bmod p}} \sum_j \alpha_j(1+p)^{\gamma_j k} \cdot q^\xi = G^+_p(k,\chi^{-1}\omega^k) \ .$$

Theorem 5.3. *Let* $A_m = \{\xi = {}^t\xi | \xi$ *half-integral*$\}$ *be the lattice in* $\mathrm{M}_m(\mathbb{Q})$ *dual to* $\{\eta \in \mathrm{M}_m(\mathbb{Z}) | {}^t\eta = \eta\}$ *with respect to the pairing* $(\xi,\eta) \mapsto tr(\xi\eta)$*. Put* $A_{m,p} = A_m \otimes_\mathbb{Z} \mathbb{Z}_p$ *then there exists a* p*-adic measure* $\mu_{\mathrm{Siegel-Eisenstein}} = \mu_{\mathrm{SE}}$ *on the profinite group* $Y = A_{m,p} \times \mathbb{Z}_p^*$ *with values in* $\mathcal{O}[\![q^{B_m}]\!]$ *such that for each* $k > m+1$ *and for each Dirichlet character* $\chi\colon \mathbb{Z}_p^* \to \mathcal{O}^*$

$$\int_{A_{m,p}\times\mathbb{Z}_p^*} \det\xi^{k-((m+1)/2)}\chi(x)x^k\mu_{\mathrm{SE}}(\xi,x) = G^+_p(k,\chi^{-1}) \ . \tag{5.8}$$

Proof: see [PaSE].

Remarks. a) For any k we can specialize $E_{\Lambda_{\mathcal{O}}}(\chi, i)$ to $T = (1+p)^k - 1$:

$$E_{\Lambda_{\mathcal{O}}}(\chi, i)\big|_{T=(1+p)^k-1} = \sum_{\delta\in\Delta} \delta^{i-k} \sum_{\substack{\xi>0 \\ \det\xi\equiv\delta}} (\det\xi)^{k-((m+1)/2)} M(\xi, k, \chi^{-1}\omega^k) q^\xi$$
$$= G_p^+(k, \chi^{-1}\omega^k)_{\omega^{i-k}} \ ,$$

where the subscript ω^{i-k} denotes the twist of $G_p^+(k, \chi^{-1}\omega^k) \in \mathcal{M}_k^m(\Gamma_0(N_p), \chi\omega^{-k})$ with the Dirichlet character ω^{i-k}.

b) Theorem 5.2 and Theorem 5.3 can be extended also to even m, but in this case the formula for $b^+(\xi, k, \chi)$ contains also the factor $L(k - (m/2), \chi\omega_\xi)$, where ω_ξ is a quadratic character associated to ξ,$\omega_\xi(d) = \left(\dfrac{(-1)^{m/2}\det(2\xi)}{d}\right)$. Note that we need to assume $\chi(-1) = (-1)^k$, and this additional factor is interpreted by the Kubota-Leopoldt theorem, see §1.

In order to obtain more general families of nearly holomorphic Eisenstein series one can apply to (5.8) the arithmetical differential operator $\delta_k^{(r)}$. Let us use again the family of polynomials (4.5).

Theorem 5.4. *Put $A_{m,p} = A_m \otimes_{\mathbb{Z}} \mathbb{Z}_p$ then there exists a p-adic measure $\mu_{\mathrm{SE}}^{(n.h)}$ on the profinite group $Y = A_{m,p} \times \mathbb{Z}_p^*$ with values in $\mathcal{O}[\![q^{B_m}]\!][R_{i,j}]_{i,j=1,\cdots,m}$ such that for each $k > m+1$ and for each Dirichlet character χ: $\mathbb{Z}_p^* \to \mathcal{O}^*$*

$$\int_{A_{m,p}\times\mathbb{Z}_p^*} \det\xi^{k-\varkappa}\chi(x)x^k P_{r,k}(\xi, R)d\mu_{\mathrm{SE}}^{(n.h)}(\xi, x) = \delta_k^{(r)}(G_p^+(k, \chi^{-1})) \ . \tag{5.9}$$

In this case

$$\mathcal{P} = \{Q_{r,k}(\xi, R, x) = \det\xi^{k-\varkappa}\chi(x)x^k P_{r,k}(\xi, R)\},$$

$k(Q_{r,k}) = k + 2r, N(Q_{r,k}) = N, r(Q_{r,k}) = r.$

Proof of Theorem 5.4: is very similar to that of Theorem 4.4, but it can be also easily deduced from the abstract Kummer congruences (Theorem 4.1) applied to the family $\mathcal{P}$.

§6. Families of distributions attached to theta functions with spherical polynomials.

In this section we describe certain p-adic families of Siegel modular forms whose parameter space is $Y = X_p = X \otimes \mathbb{Z}_p$ where $X = \mathbb{Z}^{mn}$ is a lattice. Let $A \in M_{m\times m}(\mathbb{Q})$ be a positive definite half-integral symmetric matrix which determines a quadratic form $Q_A(x) = {}^t x A x \overset{\text{def}}{=} A[x]$ for $x \in X$.

Theta-distribution attached to A is an element $\theta_A \in \mathcal{D}istr\,(Y, \mathcal{O}[\![B_n]\!])$ defined by

$$\theta_A(U) = \sum_{x \in U \cap X} q^{A[x]} \quad U \text{ a compact open subset of } X_p \tag{6.1}$$

In other words, $\theta_A(U)$ is a partial theta-series given by congruence conditions defining U, (see also [Hi85] where the elliptic modular case was treated).

Let $\Lambda_X = \mathcal{D}istr\,(X_p, \mathcal{O}) = \Lambda_{Y,\mathcal{O}}$ be the generalized Iwasawa algebra (which is isomorphic to $\mathcal{O}[\![T_{11}, \dots, T_{mn}]\!]$: for every $\mu \in \mathcal{D}istr\,(X_p, \mathcal{O})$ the corresponding power series is given by the Amice transform

$$A_\mu(T_{11}, \dots, T_{mn}) = \int\limits_{\mathbb{Z}_p^{mn}} (1 + T_{11})^{x_{11}} \cdots (1 + T_{mn})^{x_{mn}} \mu(x), \tag{6.2}$$

where $(1+T)^x = \sum\limits_{j \geq 0} \binom{x}{j} T^j$, $\displaystyle\binom{x}{j} = \frac{x(x-1)\cdots(x-j+1)}{j!}$.

One observes directly from definition (6.1) that

$$\theta_A = \sum_{x \in \mathbb{Z}^{mn}} \delta_x \cdot q^{A[x]} \in \Lambda_X[\![q^{B_n}]\!] \;. \tag{6.3}$$

Using the Amice transform, we then rewrite (6.3) in the form of a power series

$$\begin{gathered}\theta_A(T_{11}, \dots, T_{mn}; q) = \int\limits_{\mathbb{Z}_{[}^{mn}} (1 + T_{11})^{x_{11}} \cdots (1 + T_{mn})^{x_{mn}} d\theta_A(x) = \\ \sum_{k_{11}, \dots, k_{mn} = 0}^{\infty} T_{11}^{k_{11}} \cdots T_{mn}^{k_{mn}} \int\limits_{\mathbb{Z}_p^{mn}} \binom{x_{11}}{k_{11}} \cdots \binom{x_{mn}}{k_{mn}} d\theta_A(x) = \\ \sum_{\xi \in B_n} \sum_{k_{11}, \dots, k_{mn} = 0}^{\infty} \sum_{\substack{x \in \mathbb{Z}^{mn} \\ A[x] = \xi}} T_{11}^{k_{11}} \cdots T_{mn}^{k_{mn}} \binom{x_{11}}{k_{11}} \cdots \binom{x_{mn}}{k_{mn}} q^{\xi} = \\ \sum_{x \in \mathbb{Z}^{mn}} (1 + T_{11})^{x_{11}} \cdots (1 + T_{mn})^{x_{mn}} q^{A[x]}.\end{gathered} \tag{6.4}$$

Theorem 6.1. *Let us consider the family $\mathcal{P} = \{P\colon X \to \mathbb{Z}\}$ of homogeneous A-spherical polynomials which take integral values on the lattice $X = \mathbb{Z}^{mn}$. Then we may view P as a $\mathbb{Z}_p$-valued continuous function on $X_p = \mathbb{Z}_p^{mn}$ and for each such P*

$$\int_{X_p} P d\theta_A(x) = \theta_{A,P} = \sum_{x\in\mathbb{Z}} P(x) q^{A[x]} \tag{6.5}$$

is a Siegel modular form of degree n and of weight $\frac{m}{2} + \deg P$ (we assume that m is even).

Proof of Theorem 6.1: is very similar to that of Theorem 4.4, but it can be also easily deduced from the abstract Kummer congruences (Theorem 4.1) applied to the family $\mathcal{P} = \{P\}$.

In order to obtain more general families of nearly holomorphic Eisenstein series one can apply to (5.8) the arithmetical differential operator $\delta_k^{(r)}$. Let us use again the family of polynomials (4.5).

Theorem 6.2. *Let us consider the family $\mathcal{P} = \{P\colon X \to \mathbb{Z}\}$ of homogeneous A-spherical polynomials which take integral values on the lattice $X = \mathbb{Z}^{mn}$. Then we may view P as a $\mathbb{Z}_p$-valued continuous function on $X_p = \mathbb{Z}_p^{mn}$. There exists a p-adic measure $\theta_A^{(n.h)}$ on X_p with values in $\mathcal{O}[\![q^{B_n}]\!][R_{i,j}]_{i,j=1,\cdots,n}$ such that for each P*

$$\int_{X_p} P(x) \cdot P_{r,k_{m,P}}(A[x], R) d\theta_A^{(n.h)}(x) = \delta_{k_{m,P}}^{(r)}(\theta_{A,P}) \tag{6.6}$$

(where $k_{m,P} = \frac{m}{2} + \deg P$) is a Siegel modular form of degree n and of weight $k_{m,P} + 2r = \frac{m}{2} + \deg P + 2r$ (we assume that m is even). In this case

$$\mathcal{P} = \{H = P(x) P_{r,k_{m,P}}(A[x], R)\}.$$

$k(H) = k_{m,P} + 2r, N(H) = N, r(H) = r.$

Proof of Theorem 6.2: is deduced from the equality

$$\delta_{k_{m,P}}^{(r)}(\theta_{A,P}) = \sum_{x\in\mathbb{Z}^{mn}} P(x) P_{r,k_{m,P}}(A[x], R) q^{A[x]} \tag{6.7}$$

and very similar to that of Theorem 4.4. It can be also easily deduced from the abstract Kummer congruences (Theorem 4.1) applied to the family $\mathcal{P}$.

The modular forms (6.5) can be obtained as certain specialization of the power series (6.4), and we obtain a p-adic family of theta series with spherical polynomials.

Note that

$$\begin{aligned}\theta_{A,P} &= \sum_{x\in\mathbb{Z}^{mn}} P(x)(1+T_{11})^{x_{11}}\cdots(1+T_{mn})^{x_{mn}} q^{A[x]}\big|_{T_{11},\dots,T_{mn}=0} \\ &= \partial_P\theta_A(T_{11},\dots,T_{mn};q)\big|_{T_{11},\dots,T_{mn}=0}\end{aligned} \tag{6.6}$$

where

$$\partial_P = \left((1+T_{11})\frac{\partial}{\partial T_{11}},\dots,(1+T_{mn})\frac{\partial}{\partial T_{mn}}\right)$$

is a differential operator such that

$$\partial_P(1+T_{11})^{x_{11}}\cdots(1+T_{mn})^{x_{mn}} = P(x)(1+T_{11})^{x_{11}}\cdots(1+T_{mn})^{x_{mn}} \tag{6.7}$$

The map $F\mapsto \partial_P F\big|_{T_{11},\dots,T_{mn}=0}$ defines the following homomorphisms $\varkappa_P\colon \Lambda_X\to\mathbb{Z}_p$, and

$$\varkappa_P\colon \Lambda_X[\![q^{B_n}]\!]\to\mathbb{Z}_p[\![q^{B_n}]\!].$$

Let us consider the Laplace transform

$$L_\mu(T_{11}\dots,T_{mn}) = \sum_{k_{11},\dots,k_{mn}=0}^{\infty}\frac{T_{11}^{k_{11}}\cdots T_{mn}^{k_{mn}}}{k_{11}!\cdots k_{mn}!}\int\limits_{\mathbb{Z}_p^{mn}} x_{11}^{k_{11}}\cdots x_{mn}^{k_{mn}}\,d\mu(x)$$

of a p-adic measure $\mu\in\mathcal{D}istr\,(\mathbb{Z}_p^{mn},\mathcal{O})$ with values in a complete ring $\mathcal{O}$. Then

$$\begin{aligned}L_\mu(T_{11},\dots,T_{mn}) &= A_\mu(e^{T_{P11}}-1,\dots,e^{T_{m,n}}-1),\\ L_\mu(T_{11},\dots,T_{mn}) &= \int\limits_{\mathbb{Z}_p^{mn}} e^{T_{11}x_{11}+\dots+T_{mn}x_{mn}}\,d\mu(x)\end{aligned} \tag{6.8}$$

Note that the coefficients of the power series (6.8) may contain denominators $(k_{11}!\cdots k_{mn}!)$ so that

$$L_\mu(T_{11},\dots,T_{mn})\in K_{\mathcal{O}}[\![T_{11},\dots,T_{mn}]\!]$$

where $K_{\mathcal{O}} = \mathrm{Quot}\ \mathcal{O}$.

If

$$P = \sum_{k_{11},\ldots,k_{mn}} b_{k_{11},\ldots,k_{mn}} x_{11}^{k_{11}} \cdots x_{mn}^{k_{mn}}$$

is the spherical polynomial in (6.5), then the integral

$$\int\limits_{\mathbb{Z}_p^{mn}} P d\theta_A(x) = \sum_{k_{11},\ldots,k_{mn}} b_{k_{11},\ldots,k_{mn}} \int\limits_{\mathbb{Z}_p^{mn}} x_{11}^{k_{11}} \cdots x_{mn}^{k_{mn}} d\theta_A$$

is related to the Laplace transform by a more simple formula than (6.6), namely

$$\begin{aligned} L_{\theta_A}(T_{11},\ldots,T_{mn}) &= \sum_x e^{T_{11}x_{11}+\cdots+T_{mn}x_{mn}} q^{A[x]}, \\ \theta_{A,P} &= P\left(\frac{\partial}{\partial T_{11}},\ldots,\frac{\partial}{\partial T_{mn}}\right) L_\theta\big|_{T_{11}=\ldots=T_{mn}=0} \end{aligned} \tag{6.9}$$

A nice arithmetical example of a p-adic family of Hecke cusp eigenforms which are theta functions with spherical polynomials come from Hecke grössencharacters of an imaginary quadratic field $K = \mathbb{Q}\left(\sqrt{-D}\right)$.

Recall that such a χ of conductor (dividing) a given integral ideal $\mathfrak{m}$ of K is by definition a homomorphism

$$\{\text{prime-to-}\mathfrak{m} \text{ fractional ideals of } K\} \to \bar{\mathbb{Q}}^\times$$

such that for appropriate integers $n = n(\lambda)$

$$\chi((\alpha)) = \prod_\lambda \lambda(\alpha))^{n(\lambda)} = \sigma(\alpha)^{n(\sigma)}\overline{\sigma}(\alpha)^{n(\overline{\sigma})} \quad \text{if} \quad \alpha \equiv 1 \bmod^\times \mathfrak{m}$$

where λ: $K \hookrightarrow \bar{\mathbb{Q}}$, $\lambda = \sigma, \overline{\sigma}$, σ: $K \hookrightarrow \bar{\mathbb{Q}}$ is a fixed embedding, and $\overline{\sigma}$ is its composition with the nontrivial Galois automorphism of K.

In order to define $L(s,\chi)$ one uses a fixed complex embedding of $\bar{\mathbb{Q}}$. If $n(\sigma) > 0$ and $n(\overline{\sigma}) = 0$ then there exists an elliptic cups eigenform $f_\chi \in S_k(\Gamma_0(N(\mathfrak{m})),\psi)$ of weight $k = n(\sigma)+1$ with a Dirichlet character ψ (depending on χ) such that $L(s,\chi) = L(s,f_\chi)$ (the Mellin transform of f_χ). Moreover f_χ is a linear combination of binary theta-series with spherical polynomials defined by the values of the grössencharacters χ (see [Bl86]).

The function $\chi \mapsto f_\chi$ provides us with a p-adic family of Hecke cusp eigenforms (see [Hi85], [Ka78],).

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