# COHOMOLOGY AND DEFORMATIONS OF THE INFINITE DIMENSIONAL FILIFORM LIE ALGEBRA $\mathfrak{M}_2$

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Abstract. Denote $\mathfrak{m}_2$ the infinite dimensional $\mathbb{N}$-graded Lie algebra defined by the basis $e_i$ for $i \geq 1$ and by relations $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$, $[e_2, e_j] = e_{j+2}$ for all $j \geq 3$. We compute in this article the bracket structure on $H^1(\mathfrak{m}_2; \mathfrak{m}_2)$, $H^2(\mathfrak{m}_2; \mathfrak{m}_2)$ and in relation to this, we establish that there are only finitely many true deformations of $\mathfrak{m}_2$ in each weight by constructing them explicitely. It turns out that in weight 0 one gets only trivial and one formal non-converging deformations.

## Introduction

Recall the classification of infinite dimensional $\mathbb{N}$-graded Lie algebras $\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i$ with one-dimensional homogeneous components $\mathfrak{g}_i$ and two generators over a field of characteristic zero. A. Fialowski showed in [1] that any Lie algebra of this type must be isomorphic to $\mathfrak{m}_0$, $\mathfrak{m}_2$ or $L_1$. We call these Lie algebras infinite dimensional filiform Lie algebras in analogy with the finite dimensional case where the name was coined by M. Vergne in [9]. Here $\mathfrak{m}_0$ is given by generators $e_i$, $i \geq 1$, and relations $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$, $\mathfrak{m}_2$ with the same generators by relations $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$, $[e_2, e_j] = e_{j+2}$ for all $j \geq 3$, and $L_1$ with the same generators is given by the relations $[e_i, e_j] = (j - i)e_{i+j}$ for all $i, j \geq 1$. $L_1$ appears as the positive part of the Witt algebra given by generators $e_i$ for $i \in \mathbb{Z}$ with the same relations $[e_i, e_j] = (j - i)e_{i+j}$ for all $i, j \in \mathbb{Z}$. The result was also obtained later by Shalev and Zelmanov in [SZ].

The cohomology with trivial coefficients of the Lie algebra $L_1$ was studied in [7], the adjoint cohomology in degrees 1, 2 and 3 has been computed in [2] and also all of its non equivalent deformations were given. For the Lie algebra $\mathfrak{m}_0$, the cohomology with trivial coefficients has been studied in [4], and the adjoint cohomology in degrees 1 and 2 in [5]. The adjoint cohomology in degrees 1 and 2 of $\mathfrak{m}_2$ is the object of the present article. The cohomology of $\mathfrak{m}_0$ and $\mathfrak{m}_2$ rose interest only recently, and the reason is probably that - as happens usually for solvable Lie algebras - the cohomology is huge and therefore meaningless. Our point of view is that there still remain interesting features.

Indeed, it is true that the first and second adjoint cohomology of $\mathfrak{m}_2$ are infinite dimensional, but they are much less impressive than the analoguous results for $\mathfrak{m}_0$. We believe that this comes from the much more restrictive

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bracket structure for $\mathfrak{m}_2$. Actually, the bracket structure is so rigid that there is no infinite dimensional filiform Lie algebra "between" $\mathfrak{m}_2$ and $L_1$.

The space $H^1(\mathfrak{m}_2;\mathfrak{m}_2)$ becomes already interesting when we split it up into homogeneous components $H^1_l(\mathfrak{m}_2;\mathfrak{m}_2)$ of weight $l \in \mathbb{Z}$, this latter space being finite dimensional for each $l \in \mathbb{Z}$. The bracket structure on $H^1(\mathfrak{m}_2;\mathfrak{m}_2)$ is studied in Section 2.

The space $H^2(\mathfrak{m}_2;\mathfrak{m}_2)$ is discussed in Section 3. This space is here also finite dimensional in each weight separately. Given a generator of $H^2(\mathfrak{m}_2;\mathfrak{m}_2)$, i.e. an infinitesimal deformation, corresponding to the linear term of a formal deformation, one can try to adjust higher order terms in order to have the Jacobi identity in the deformed Lie algebra up to order $k$. If the Jacobi identity is satisfied to all orders, we will call it a true (formal) deformation, see Fuchs' book [6] for details on cohomology and [Fia2] for deformations of Lie algebras.

In Section 3.2 we discuss Massey products, in Sections 3.3 – 3.5 we describe all true deformations in negative weights. Section 3.6 identifies the deformations in weight zero.

As obstructions to infinitesimal deformations given by classes in $H^2(\mathfrak{m}_2;\mathfrak{m}_2)$ are expressed by Massey powers of these classes in $H^3(\mathfrak{m}_2;\mathfrak{m}_2)$, it is the vanishing of these Massey squares, cubes etc which makes it possible to prolongate an infinitesimal deformation to all orders. For $\mathfrak{m}_2$ here, on the one hand the cocycle equations are so rigid that they select already few cochains to be cocycles, but on the other hand, there are enough cochains to compensate all Massey powers, leading to formal, non-converging deformations. The main result reads

**Theorem 1.** *The true deformations of $\mathfrak{m}_2$ are finitely generated in each weight. More precisely, the space of unobstructed cohomology classes is zero in weight $l \leq -5$, because there are no non-trivial cocycles. It is in degree $l \geq -4$ of dimension two (but with changing representatives).*

*The two infinitesimal deformations in weight $l = 0$ can be prolongated to all orders and give a trivial deformation and a formal non-converging deformation.*

As a rather astonishing consequence, $\mathfrak{m}_2$ does not deform to $L_1$.

We believe that the discussion of these examples of deformations are interesting as they go beyond the usual approach where the condition that $H^2(\mathfrak{g},\mathfrak{g})$ should be finite dimensional is the starting point for the examination of deformations, namely the existence of a miniversal deformation [3].

Another attractive point of our study is the fact that here for $\mathfrak{m}_2$ the Massey squares, cubes etc. involved can all be compensated and lead to an interesting obstruction calculus. Thus the second adjoint cohomology of $\mathfrak{m}_2$ may serve as an example on which to study explicitely obstruction theory.

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## 1. Preliminaries

Recall the $\mathbb{N}$-graded Lie algebra $\mathfrak{m}_2 = \bigoplus_{i\geq 1}(\mathfrak{m}_2)_i$; all graded components $(\mathfrak{m}_2)_i$ are 1-dimensional, and we choose a basis $e_i$ of each of them. The brackets then read: $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$, $[e_2, e_j] = e_{j+2}$ for all $j \geq 3$.

We will compute in later sections of this paper the Lie algebra cohomology spaces $H^1(\mathfrak{m}_2;\mathfrak{m}_2)$ and $H^2(\mathfrak{m}_2;\mathfrak{m}_2)$ of $\mathfrak{m}_2$ with coefficients in the adjoint representation. Since $\mathfrak{m}_2$ is $\mathbb{N}$-graded, the cochain, cocycle, coboundary and cohomology spaces are also graded. Thus it makes sense to restrict attention to the graded components of *weight* $l$ denoted $C^*_l(\mathfrak{m}_2;\mathfrak{m}_2)$, $Z^*_l(\mathfrak{m}_2;\mathfrak{m}_2)$, $B^*_l(\mathfrak{m}_2;\mathfrak{m}_2)$ and $H^*_l(\mathfrak{m}_2;\mathfrak{m}_2)$ of the spaces of all cochains $C^*(\mathfrak{m}_2;\mathfrak{m}_2)$, cocycles $Z^*(\mathfrak{m}_2;\mathfrak{m}_2)$, coboundaries $B^*(\mathfrak{m}_2;\mathfrak{m}_2)$ and cohomology classes $H^*(\mathfrak{m}_2;\mathfrak{m}_2)$.

The cohomology spaces $H^*(\mathfrak{m}_2;\mathfrak{m}_2)$ for $* = 1, 2$ are interesting from the following point of view: $H^*(\mathfrak{m}_2;\mathfrak{m}_2)$ carries a graded Lie bracket

$$[,]\ :\ H^p(\mathfrak{m}_2;\mathfrak{m}_2)\otimes H^q(\mathfrak{m}_2;\mathfrak{m}_2)\to H^{p+q-1}(\mathfrak{m}_2;\mathfrak{m}_2),$$

which restricts to a Lie bracket on $H^1(\mathfrak{m}_2;\mathfrak{m}_2)$ which is graded with respect to the weight $l$. We will compute this bracket in the next section.

The space $H^2(\mathfrak{m}_2;\mathfrak{m}_2)$ draws its importance from the interpretation of being the space of *infinitesimal deformations* of the Lie algebra $\mathfrak{m}_2$. Such an infinitesimal deformation $[\omega]\in H^2(\mathfrak{m}_2;\mathfrak{m}_2)$ is the term of degree one in the expansion of a deformed bracket with respect to the deformation parameter. The question whether the infinitesimal term given by $[\omega]$ can be prolongated to degree two or even to all higher powers can be answered by studying the *Massey powers* of $[\omega]$. Indeed, it is a necessary condition for $[\omega]$ to admit a prolongation to degree two that the *Massey square* $[\omega]^2\in H^3(\mathfrak{m}_2;\mathfrak{m}_2)$ is zero, i.e. if for all $i, j, k \geq 1$

$$\omega(\omega(e_i, e_j), e_k) + \text{cycl.}\ =\ d\alpha,$$

for some 2-cochain $\alpha\in C^2(\mathfrak{m}_2;\mathfrak{m}_2)$. In this sense, the Massey square is the first obstruction for $[\omega]$ to give a (formal) deformation. The next obstruction is then the *Massey cube*, defined using $\omega$ and $\alpha$ by

$$\omega(\alpha(e_i, e_j), e_k) + \alpha(\omega(e_i, e_j), e_k) + \text{cycl.}.$$

In case all obstructions vanish, $[\omega]$ gives rise to a *formal deformation*. The bracket defined by $[,]_t = [,] + t\omega + t^2\alpha + \dots$ satisfies then the Jacobi identity up to all orders. But it is not clear whether setting $t = r$ for some $r\in\mathbb{R}$ defines a Lie bracket $[,]_r$, i.e. it is not clear whether the formal deformation *converges*. If this is the case, we call it a *true deformation*. A deformation having only a finite number of non-zero terms is always a true deformation.

A homogeneous cocycle $\omega$ of weight $l\in\mathbb{Z}$ for the Lie algebras $\mathfrak{m}_0$ or $\mathfrak{m}_2$ is given by coefficients $a_{i,j}$ such that $\omega(e_i, e_j) = a_{i,j}e_{i+j+l}$. The most important cocycle equation for $\mathfrak{m}_0$ was (cf [5]) for $i, j \geq 2$:

$$a_{i+1,j} + a_{i,j+1}\ =\ a_{i,j}.$$

In [5], we defined some fundamental solutions to this equation which we named *families*. The 2-*family* has $a_{2,k} = 1$ for all $k \geq 3$ and $a_{i,j} = 0$ for all $i > 2$, up to antisymmetry. The 3-*family* has $a_{3,k} = 1$ for all $k \geq 4$ and $a_{i,j} = 0$ for all $i > 3$, up to antisymmetry. The $a_{2,k}$ coefficients are then easily seen to be non-zero starting from $a_{2,5}$, and they grow linearly in $k$. For explicit formulae for the $m$-family, we refer to [5].

## 2. The space $H^1(\mathfrak{m}_2;\mathfrak{m}_2)$

We will compute the space $H^1_l(\mathfrak{m}_2;\mathfrak{m}_2)$ of homogeneous cohomology classes of weight $l\in\mathbb{Z}$ for each fixed $l$. A 1-cochain $\omega\in C^1(\mathfrak{m}_2;\mathfrak{m}_2)$ is called *homogeneous of weight* $l\in\mathbb{Z}$ in case $\omega(e_i)\,=\,a_ie_{i+l}$ for each $i\geq 1$. The cocycle identity reads then for a homogeneous cochain

$$d\omega(e_i,e_j)\,=\,\omega([e_i,e_j])-[e_i,\omega(e_j)]+[e_j,\omega(e_i)]\,=\,0$$

for all $i,j\geq 1$. We get different sets of equations for $i=1$, $j\geq 2$, $i=2$, $j\geq 3$, and $i,j\geq 3$.

(a) If $i=1$, $j\geq 3$, $j+l\geq 2$:

$$0\,=\,a_{j+1}-a_j-a_1\delta_{l,0}-a_1\delta_{l,1},$$

if $j\geq 3$, $j+l=0,1$, we get $0\,=\,a_{j+1}$, but there is no equation for $j+l\leq -1$.

If $i=1$ and $j=2$, $l\geq 1$:

$$0\,=\,a_3-a_2+a_1(1-\delta_{l,1}),$$

$0\,=\,a_3-a_2-a_1$ if $j=2$ and $l=0$, $0\,=\,a_3$ if $j=2$ and $l=-1,-2$, and no equation if $j+1+l\leq 0$.

(b) If $i=2$, $j\geq 3$, $j+l\geq 3$:

$$0\,=\,a_{j+2}-a_j-a_2\delta_{l,0}-a_2\delta_{l+1,0},$$

for $j+l=2$, we get $0=a_{-l+4}-a_2\delta_{j,3}$, for $j+l=1$, we get $0=a_{-l+3}+a_{-l+1}$, for $j+l=0$, we get $0=a_{-l+2}$, for $j+l=-1$, we get $0=a_{-l+1}$, and there is no equation for $j+l\leq -2$.

(c) If $i,j\geq 3$:

$$0\,=\,\delta_{j+l,1}a_j+\delta_{j+l,2}a_j-\delta_{i+l,1}a_i-\delta_{i+l,2}a_i.$$

Now let us discuss 1-cocycles in weight $l=0$. For $i=1$ and $j\geq 2$, we get by equations (a)

$$0\,=\,a_{j+1}-a_j-a_1,$$

and for $i=2$ and $j\geq 3$ by equations (b)

$$0\,=\,a_{j+2}-a_j-a_2.$$

Call $a_1=:a$ and $a_2=:b$, then we get on the one hand $a_3-b=a$, $a_4-a_3=a$, $a_5-a_4=a$ and so on, and on the other hand $a_5-a_3=b$. Therefore $b=2a$. In conclusion, we get a one parameter family of cocycles in weight $l=0$.

Now let us discuss 1-cocycles in weight $l=1$. For $i=1$ and $j\geq 3$, we get by equations (a)

$$0\,=\,a_{j+1}-a_j-a_1,$$

while for $j=2$, we get $0=a_3-a_2$. For $i=2$ and $j\geq 3$ by equations (b)

$$0\,=\,a_{j+2}-a_j.$$

We conclude $a_2=a_3$, $a_3=a_5$, $a_1=0$, $a_3=a_4$, and all $a_i$ for $i\geq 2$ are then equal. This means that we have one free parameter.

Now let us discuss 1-cocycles in weight $l\geq 2$. For $i=1$ and $j\geq 3$, we get by equations (a)

$$0\,=\,a_{j+1}-a_j,$$

while for $j = 2$, we get $0 = a_3 - a_2 + a_1$. For $i = 2$ and $j \geq 3$ by equations (b)

$$0 \;=\; a_{j+2} - a_j.$$

We have $a_4 = a_3$ and so on, and $a_1$ and $a_2$ are thus two free parameters.

Now let us discuss 1-cocycles in weight $l = -1$. For $i = 1$ and $j \geq 3$, we get by equations (a)

$$0 \;=\; a_{j+1} - a_j,$$

while for $j = 2$, we get $0 = a_3$. For $i = 2$ and $j \geq 4$, we get by equations (b)

$$0 \;=\; a_{j+2} - a_j - a_2,$$

while for $j = 3$, we get $0 = a_5 - a_2$. We have therefore $a_3 = 0$, $a_4 = a_3$, $a_5 = a_4$, $0 = a_6 - a_4 - a_2$, etc. This gives $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$ and so on. Remark that $a_1$ does not exist, because $\omega(e_i) = a_i e_{i-1}$.

Now let us discuss 1-cocycles in weight $l = -2$. Remark that here $a_1$ and $a_2$ do not exist. The equations (a), i.e. $i = 1$, $j \geq 2$, read

$$0 \;=\; a_{j+1} - \left\{ \begin{array}{ccc} 0 & \text{if} & j = 2,3 \\ a_j & \text{if} & j \geq 4 \end{array} \right.$$

The equations (b), i.e. $i = 2$, $j \geq 3$, read

$$0 \;=\; a_{j+2} + \left\{ \begin{array}{ccc} a_3 & \text{if} & j = 3 \\ 0 & \text{if} & j = 4 \\ -a_j & \text{if} & j \geq 5 \end{array} \right.$$

We get thus $a_3 = 0$, $a_4 = 0$, $a_5 = a_4$, $a_6 = 0$, and so on. One concludes that all coefficients are zero.

Now let us discuss 1-cocycles in weight $l \leq -3$. Remark that here $a_1$, $a_2$, up to $a_{-l}$ do not exist. The equations (a), i.e. $i = 1$, $j \geq 2$, read

$$0 \;=\; a_{j+1} - \left\{ \begin{array}{ccc} 0 & \text{if} & j = -l, -l+1 \\ a_j & \text{if} & j \geq -l+2 \end{array} \right.$$

The equations (b), i.e. $i = 2$, $j \geq 3$, read

$$0 \;=\; a_{j+2} + \left\{ \begin{array}{ccc} 0 & \text{if} & j = -l-1, -l \\ a_j & \text{if} & j = -l+1 \\ 0 & \text{if} & j = -l+2 \\ -a_j & \text{if} & j \geq -l+3 \end{array} \right.$$

One concludes that all coefficients are zero.

Next come the coboundaries. It is clear that $dC^0_l(\mathfrak{m}_2; \mathfrak{m}_2) = 0$ for all weights $l \leq 0$, because coboundaries are brackets with elements. It is also clear that $dC^0_l(\mathfrak{m}_2; \mathfrak{m}_2)$ is one-dimensional and generated by $de_l = [e_l, -]$ for $l \geq 1$. Observe that $[e_1, -]$ is zero on $e_1$ and non-trivial on all other $e_i$, that $[e_2, -]$ is zero on $e_2$, equal to a constant $a$ on all $e_i$ with $i \geq 3$ and equal to $-a$ on $e_1$, while $[e_i, -]$ for $i \geq 3$ is non-zero on $e_1$ and $e_2$ and zero on all others.

One sees that $Z^1_1(\mathfrak{m}_2; \mathfrak{m}_2) = dC^0_1(\mathfrak{m}_2; \mathfrak{m}_2)$. We therefore conclude that

**Theorem 2.**

$$\dim H^1_l(\mathfrak{m}_2; \mathfrak{m}_2) \;=\; \left\{ \begin{array}{ccccc} 0 & \text{if} & l = 1 & \text{or} & l \leq -1 \\ 1 & \text{if} & l = 0 & \text{or} & l \geq 2 \end{array} \right.$$

In order to compute the bracket structure, we need explicit non-trivial cocycles. Observe that the (non zero) coboundary for $l \geq 3$ is given by $a_1 \neq 0$ and $a_2 = a_1$. The explicit non-trivial cocycles are therefore:

- $l = 0$: the coefficients are growing linearly $a := a_1$, $a_2 = 2a$, $a_3 = 3a$ etc.
- $l = 2$: $b := a_2 \neq 0$ and $a_j = b$ for all $j \geq 3$.
- $l \geq 3$: $a_1 =: -\frac{c_l}{2}$ and $a_2 = \frac{c_l}{2}$. Then $a_3 = c_l$, $a_4 = c_l$, etc.

We express the previous description by introducing generators:

- $l = 0$: $\omega(e_k) = ke_k$ for all $k \geq 1$ (we took $a = 1$).
- $l = 2$:
$$\alpha(e_k) = \left\{ \begin{array}{ccc} be_{k+2} & \text{if} & k \geq 2 \\ 0 & \text{if} & k = 1 \end{array} \right.$$
- $l \geq 3$:
$$\gamma_l(e_k) = \left\{ \begin{array}{ccc} c_l e_{k+l} & \text{if} & k \geq 3 \\ -\frac{c_l}{2} e_{l+1} & \text{if} & k = 1 \\ \frac{c_l}{2} e_{l+2} & \text{if} & k = 2 \end{array} \right.$$

It is well known that $H^*(\mathfrak{g}, \mathfrak{g})$ carries a graded Lie algebra structure for any Lie algebra $\mathfrak{g}$, and that $H^1(\mathfrak{g}, \mathfrak{g})$ forms a graded Lie subalgebra. Let us compute this bracket structure on our generators:

Given $a \in C^p(\mathfrak{g}, \mathfrak{g})$ and $b \in C^q(\mathfrak{g}, \mathfrak{g})$, define
$$ab(x_1, \ldots, x_{p+q-1}) \;=\; \sum_{\sigma \in \mathrm{Sh}_{p,q}} (-1)^{\mathrm{sgn}\,\sigma} a(b(x_{i_1}, \ldots, x_{i_q}), x_{j_1} \ldots, x_{j_{p-1}})$$
for $x_1, \ldots, x_{p+q-1} \in \mathfrak{g}$. The bracket is then defined by
$$[a, b] \;=\; ab - (-1)^{(p-1)(q-1)} ba.$$
It thus reads on $H^1(\mathfrak{g}, \mathfrak{g})$ simply
$$[a, b](x) \;=\; a(b(x)) - b(a(x)).$$

We compute
$$\begin{aligned} \omega(\alpha(e_k)) - \alpha(\omega(e_k)) &= \left\{ \begin{array}{ccc} 0 & \text{if} & k = 1 \\ \omega(be_{k+2}) & \text{if} & k \geq 2 \end{array} \right\} - \alpha(ke_k) \\ &= \left\{ \begin{array}{ccc} 0 & \text{if} & k = 1 \\ b(k+2)e_{k+2} - bke_{k+2} & \text{if} & k \geq 2 \end{array} \right. \\ &= \left\{ \begin{array}{ccc} 0 & \text{if} & k = 1 \\ 2be_{k+2} & \text{if} & k \geq 2 \end{array} \right. \;=\; 2\alpha(e_k). \end{aligned}$$

$$\begin{aligned} \omega(\gamma_l(e_k)) - \gamma_l(\omega(e_k)) &= \left\{ \begin{array}{ccc} \omega(c_l e_{k+l}) & \text{if} & k =\geq 3 \\ \omega(-\frac{c_l}{2} e_{l+1}) & \text{if} & k = 1 \\ \omega(\frac{c_l}{2} e_{l+2}) & \text{if} & k = 2 \end{array} \right\} - \left\{ \begin{array}{ccc} kc_l e_{k+l}) & \text{if} & k =\geq 3 \\ -k\frac{c_l}{2} e_{l+1} & \text{if} & k = 1 \\ k\frac{c_l}{2} e_{l+2} & \text{if} & k = 2 \end{array} \right\} \\ &= \left\{ \begin{array}{ccc} c_l(k+l)e_{k+l} - kc_l e_{k+l} & \text{if} & k =\geq 3 \\ \left(-\frac{c_l}{2}(l+1) + \frac{c_l}{2}\right) e_{l+1} & \text{if} & k = 1 \\ \left(\frac{c_l}{2}(l+2) - c_l\right) e_{l+2} & \text{if} & k = 2 \end{array} \right\} \;=\; l\gamma_l(e_k). \end{aligned}$$

$$\alpha(\gamma_l(e_k)) - \gamma_l(\alpha(e_k)) = \left\{\begin{array}{ccc} \alpha(c_l e_{k+l}) & \text{if} & k = \geq 3 \\ \alpha(-\frac{c_l}{2}e_{l+1}) & \text{if} & k = 1 \\ \alpha(\frac{c_l}{2}e_{l+2}) & \text{if} & k = 2 \end{array}\right\} - \left\{\begin{array}{ccc} \gamma_l(be_{k+2}) & \text{if} & k = \geq 3 \\ 0 & \text{if} & k = 1 \\ \gamma_l(be_4) & \text{if} & k = 2 \end{array}\right\}$$

$$= \left\{\begin{array}{ccc} bc_l e_{k+l+2} - bc_l e_{k+l+2} & \text{if} & k = \geq 3 \\ -\frac{c_l}{2}be_{l+1+2} - 0 & \text{if} & k = 1 \\ \frac{c_l}{2}be_{l+4} - bc_l e_{l+4} & \text{if} & k = 2 \end{array}\right\}$$

$$= \left\{\begin{array}{ccc} 0 & \text{if} & k = \geq 3 \\ -\frac{c_l}{2}be_{l+1+2} & \text{if} & k = 1 \\ -\frac{c_l}{2}be_{l+4} & \text{if} & k = 2 \end{array}\right\}$$

$$= \left\{\begin{array}{ccc} 0 & \text{if} & k = \geq 3 \\ -\frac{c_l}{2}be_{k+l+2} & \text{if} & k = 1, 2. \end{array}\right.$$

This last cocycle is a coboundary, more precisely,

$$\alpha(\gamma_l(e_k)) - \gamma_l(\alpha(e_k)) = \left(\frac{c_l}{2}b\right)[e_{l+2}, -].$$

We conclude

$$\alpha(\gamma_l(e_k)) - \gamma_l(\alpha(e_k)) = 0$$

as cohomology classes. One easily computes that $\gamma_l$ and $\gamma_m$ commute. Therefore the bracket structure on $H^1(\mathfrak{m}_2; \mathfrak{m}_2)$ is described as follows:

**Theorem 3.** *$H^1(\mathfrak{m}_2; \mathfrak{m}_2)$ is a graded Lie algebra, generated in positive degrees by $\omega$ (degree 0), $\alpha$ (degree 2) and $\gamma_l$ (degree $l \geq 3$) such that $\omega$ acts as a grading operator on the trivial Lie algebra generated by $\alpha$ and the $\gamma_l$ for $l \geq 3$.*

## 3. The space $H^2(\mathfrak{m}_2; \mathfrak{m}_2)$

### 3.1. Cocycle identities.

For a 2-cochain $\omega$, the cocycle identity reads

$$\omega([e_i, e_j], e_k) + \omega([e_j, e_k], e_i) + \omega([e_k, e_i], e_j)$$
$$-[e_i, \omega(e_j, e_k)] - [e_j, \omega(e_k, e_i)] - [e_k, \omega(e_i, e_j)] = 0.$$

In the sequel, we will suppose $\omega$ homogeneous of weight $l \in \mathbb{Z}$ with $\omega(e_i, e_j) = a_{i,j}e_{i+j+l}$ for all $i, j \geq 1$. From the cocycle identity, we get the following equations on the coefficients $a_{i,j}$:

(a) Setting $i = 1$ and $j, k \geq 3$, we get for $j + k + l \geq 2$

$$(a_{j+1,k} + a_{j,k+1})e_{j+k+l+1} = (a_{j,k} - a_{k,1}\delta_{k+l,0} - a_{k,1}\delta_{k+l,1} - a_{1,j}\delta_{j+l,0} - a_{1,j}\delta_{j+l,1})e_{j+k+l+1},$$

and for $j + k + l = 0, 1$ (while there is no equation for $j + k + l < 0$)

$$(a_{j+1,k} + a_{j,k+1})e_{j+k+l+1} = 0.$$

(b) Setting $i = 1$, $j = 2$, and $k \geq 3$, we get for $k + l \geq 2$,

$$(a_{3,k} + a_{k+2,1} + a_{2,k+1})e_{k+l+3} = (a_{2,k} + a_{k,1} - a_{1,2}\delta_{2+l,0} - a_{1,2}\delta_{2+l,1})e_{k+l+3},$$

while for $k + l = 0$, we get

$$(a_{3,k} + a_{k+2,1} + a_{2,k+1})e_3 = (a_{2,k} - a_{k,1} - a_{1,2}\delta_{2+l,0} - a_{1,2}\delta_{2+l,1})e_3,$$

and for $k + l = 1$, we get

$$(a_{3,k} + a_{k+2,1} + a_{2,k+1})e_4 = (a_{2,k} - a_{1,2}\delta_{2+l,0} - a_{1,2}\delta_{2+l,1})e_4,$$

and for $k+l=-1,-2$, we get

$$(a_{3,k}+a_{k+2,1}+a_{2,k+1})e_{k+l+3}=(-a_{1,2}\delta_{2+l,0}-a_{1,2}\delta_{2+l,1})e_{k+l+3}.$$

(c) If $i=2$, and $j,k\geq 3$, we get for $j+k+l\geq 3$

$$(a_{j+2,k}+a_{j,k+2})e_{j+k+l+2}=(a_{j,k}-a_{k,2}\delta_{k+l+1,0}-a_{k,2}\delta_{k+l,0}-a_{2,j}\delta_{j+l+1,0}-a_{2,j}\delta_{j+l,0})e_{j+k+l+2},$$

for $j+k+l=1$

$$(a_{j+2,k}+a_{j,k+2})e_3=(-a_{j,k}-a_{k,2}\delta_{k+l+1,0}-a_{k,2}\delta_{k+l,0}-a_{2,j}\delta_{j+l+1,0}-a_{2,j}\delta_{j+l,0})e_3,$$

for $j+k+l\leq -2$, there is no equation, and for $j+k+l=0,-1,2$, we have

$$(a_{j+2,k}+a_{j,k+2})e_{j+k+l+2}=(-a_{k,2}\delta_{k+l+1,0}-a_{k,2}\delta_{k+l,0}-a_{2,j}\delta_{j+l+1,0}-a_{2,j}\delta_{j+l,0})e_{j+k+l+2}.$$

(d) If $i,j,k\geq 3$, we get

$$0=(-a_{j,k}\delta_{j+k+l,1}-a_{j,k}\delta_{j+k+l,2}-a_{k,i}\delta_{i+k+l,1}-a_{k,i}\delta_{i+k+l,2}-a_{i,j}\delta_{i+j+l,1}-a_{i,j}\delta_{i+j+l,2})e_{i+j+k+l}.$$

In equation (d), at most two terms can be non-zero for a given $l$ as $i$, $j$ and $k$ must be pairwise distinct.

Let us now compute the 2-coboundaries: a cocycle $\omega$ is a coboundary in case there exists a 1-cochain $\alpha$ such that

$$\omega(e_i,e_j)\;=\;\alpha([e_i,e_j])-[e_i,\alpha(e_j)]+[e_j,\alpha(e_i)].$$

As $\omega$ is homogeneous of weight $l$, $\alpha$ will be, and we set $\alpha(e_i)=a_ie_{i+l}$ for all $i\geq 1$. Then the previous equation gives:

(e) Suppose $i=1$ and $j\geq 3$, then

$$a_{1,j}e_{j+l+1}\;=\;(a_{j+1}-a_j-a_1\delta_{l,0}-a_1\delta_{l+1,0})e_{j+l+1}.$$

This equation makes sense only if $j+l\geq 2$. For $j+l=0,1$, one obtains

$$a_{1,j}\;=\;a_{j+1}.$$

(f) Suppose $i=1$ and $j=2$, then for $l\geq 2$

$$a_{1,2}\;=\;a_3-a_2+a_1,$$

while for $l=-1,-2$, one gets $a_{1,2}\;=\;a_3$, for $l=0$, one gets $a_{1,2}\;=\;a_3-a_2-a_1$, and for $l=1$, one gets $a_{1,2}\;=\;a_3-a_2$.

(g) Suppose $i=2$ and $j\geq 3$, then for $j+l\geq 2$

$$a_{2,j}\;=\;a_{j+2}-a_j(1-\delta_{j+l,2})+a_j\delta_{j+l,1}-a_2(\delta_{l,0}+\delta_{l,-1}),$$

while for $j+l=0,-1$, one gets $a_{2,j}\;=\;a_{j+2}$, and for $j+l=1$, one gets

$$a_{2,j}\;=\;a_{j+2}+a_j.$$

(h) For $i,j\geq 3$ with $i+j+l\geq 1$, $i+l\geq 1$ and $j+l\geq 1$, the coboundary equation reads

$$a_{i,j}\;=\;a_j\,(\delta_{j+l,1}+\delta_{j+l,2})-a_i(\delta_{i+l,1}+\delta_{i+l,2}).$$

Now stably, i.e. for a fixed $l$ and $j,k>>0$, we have just the following system of equations:

($\alpha$) $a_{3,k}+a_{k+2,1}+a_{2,k+1}\;=\;a_{2,k}+a_{k,1}-a_{1,2}\delta_{l,-2}-a_{1,2}\delta_{l,-1}$
($\beta$) $a_{j+1,k}+a_{j,k+1}\;=\;a_{j,k}$
($\gamma$) $a_{j+2,k}+a_{j,k+2}\;=\;a_{j,k}$

Equation ($\alpha$) means that the 1- and 2-coefficients determine the 3-coefficients. Equation ($\beta$) implies that the differences of adjacent 3- (resp. 4-) coefficients determine the 4- (resp. 5-) coefficients. But equation ($\gamma$) implies that differences of next to

adjacent 3-coefficients determine the 5-coefficents directly. We get stably on the one hand

$$a_{5,k} \;=\; a_{4,k}-a_{4,k+1} \;=\; (a_{3,k}-a_{3,k+1})-(a_{3,k+1}-a_{3,k+2}) \;=\; a_{3,k}-2\,a_{3,k+1}+a_{3,k+2},$$

and on the other hand

$$a_{5,k} \;=\; a_{3,k} - a_{3,k+2},$$

thus we conclude that for $l$ big enough $a_{3,k+1} = a_{3,k+2}$. Even if $k >> 0$, we take $j = 3$ in order to get these equations, thus there are extra terms (coming from equations (c)) for $j = -l$ and $j = -l - 1$, i.e. in case $l = -3$ and $l = -4$. In all other weights, we will finally (i.e. for $k >> 0$) have the conclusion $a_{3,k+1} = a_{3,k+2}$.

But now when the 3-coefficients are stably equal, the 4-coefficients are stably 0, and so are all higher coefficients. This limits considerably the choice of possible cocycles, at least stably. For example, let us suppose $l \geq -2$. In this case, equations (e) and (f) show that we can add coboundaries in order to have all 1-coefficients equal to zero. It is clear from equations (e), (f), (g) and (h) that once the 1-coefficients are set to zero, the 2-coefficients and higher coefficients cannot be changed by addition of a coboundary, because this would change the 1-coefficients, too.

(a), (b) and (c) then show that we have the system of equations

($\alpha'$) $a_{3,k} + a_{2,k+1} \;=\; a_{2,k}$

($\beta$) $a_{j+1,k} + a_{j,k+1} \;=\; a_{j,k}$

($\gamma$) $a_{j+2,k} + a_{j,k+2} \;=\; a_{j,k}$

for all $j, k \geq 3$. The system tells us that cocycles must have all 3-coefficients equal, all higher coefficients zero. Observe that the equations which determined the solutions for $\mathfrak{m}_0$ are a subset of the equations which must be satisfied for $\mathfrak{m}_2$. We conclude that in weight $l \geq -2$, there are at most two non-trivial families of true deformations: the 2-family and the 3-family. Whether they give indeed rise to true deformations will be determined in later subsections by studying their Massey powers.

3.2. **Massey powers.** Observe that the Massey square does not involve the bracket of the Lie algebra, so we get for $\mathfrak{m}_2$ the same Massey square as for $\mathfrak{m}_0$. For example, the 2-family has zero Massey square (as a cochain) in all weights (but observe that the 2-family is not necessarily a cocycle in all weights). We will examine the 3-family in positive or zero weight in the following proposition.

An important point is that for $\mathfrak{m}_0$, we had restrictions on the true deformations coming from the nullity of the Massey squares and higher Massey powers. For $\mathfrak{m}_2$ here, we have more possibilities to compensate non-zero Massey powers, so there are less restrictions. Most of the restrictions for deformations of $\mathfrak{m}_2$ come already from the cocycle equations.

**Proposition 1.** *Let $\omega \in Z^2_l(\mathfrak{m}_2;\mathfrak{m}_2)$ be the homogeneous 2-cocycle of weight $l \geq 0$ given by the 3-family and representing an infinitesimal deformation* *of $\mathfrak{m}_2$. Then $\omega$ can be prolongated to a formal deformation of $\mathfrak{m}_2$, i.e. all Massey powers $[\omega]^n \in H^3(\mathfrak{m}_2;\mathfrak{m}_2)$ of $\omega$ are trivial.*

**Proof**. Recall that the homogeneous 2-cocycle $\omega$ of weight $l$ is given by coefficients $a_{i,j}$ such that $\omega(e_i,e_j) = a_{i,j}e_{i+j+l}$. $\omega$ represents the 3-family, thus $a_{i,j} \neq 0$ (up to antisymmetry) only for $i = 2$ and $j \geq 5$ and $i = 3$ and $j \geq 4$. The Massey square of $\omega$ reads

$$M_{ijk} \,=\, a_{i,j}a_{i+j+l,k} \,+\, a_{j,k}a_{j+k+l,i} \,+\, a_{k,i}a_{k+i+l,j}.$$

We will always suppose $i < j < k$, up to anti-symmetry. Using $a_{i,j} \neq 0$ (up to antisymmetry) only for $(i = 2, j \geq 5)$ and $(i = 3, j \geq 4)$, we obtain as only possibly non-zero Massey squares $M_{2jk}$, $j,k \geq 4$, and $M_{3jk}$, $j,k \geq 4$. The squares $M_{3jk}$, $j,k \geq 4$ are zero because of the restriction $l \geq 0$; indeed,

$$M_{3jk} \,=\, a_{3,j}a_{3+j+l,k} \,+\, a_{k,3}a_{k+3+l,j} \,=\, a_{3+j+l,k} \,+\, a_{k+3+l,j},$$

and $l \geq 0$, $j,k \geq 4$ imply that $a_{3+j+l,k} = a_{k+3+l,j} = 0$.

The squares $M_{2jk}$, $j,k \geq 4$ are zero for $j \geq 4$, because then

$$M_{2jk} \,=\, a_{2,j}a_{2+j+l,k} \,+\, a_{k,2}a_{k+2+l,j},$$

and once again, $l \geq 0$, $j,k \geq 4$ imply that $a_{2+j+l,k} = a_{k+2+l,j} = 0$.

Therefore, the only Massey squares we have to compensate are $M_{23k}$, $k \geq 4$. We then introduce a homogeneous 2-cochain $\alpha$ of weight $2l$ with $\alpha(e_i,e_j) = b_{i,j}e_{i+j+2l}$. We have for $l \geq -1$

$$d\alpha(e_2,e_j,e_k) \,=\, (b_{2+j,k} - b_{k+2,j} - b_{j,k})e_{j+k+2l+2},$$

meaning $d\alpha(e_2,e_3,e_k) \,=\, (b_{5,k} - b_{k+2,3} - b_{3,k})e_{k+2l+5}$. We may then compensate the Massey square by just the 3-column of $b$-coefficients. This ensures that at most the 2- and 3-columns for the $a$- and the $b$-coefficients are non-zero.

Now suppose by induction that we have already compensated all Massey powers up to some level in such a way that at most the 2- and 3-columns for the coefficients of the intervening cochains are non-zero. Then we go on to compute the next Massey power

$$N_{ijk} \,=\, \beta(\gamma(e_i,e_j),e_k) \,+\, \gamma(\beta(e_i,e_j),e_k) \,+\, \text{cycl.},$$

where "cycl." means cyclic permutations in $i,j,k$ and $\beta$ and $\gamma$ are some 2-cochains satisfying the above restrictions. The weights of the cochains $\beta$ and $\gamma$ are positive or zero. Thus by compensating one step further, we will reproduce cochains such that at most the 2- and 3-columns for the coefficients are non-zero. This ends the inductive step. □

Let us summerize what we said about true deformations in weight $l \geq 0$:

**Proposition 2.** *In weight $l \geq 0$, the only non-trivial cocycles are given by (linear combinations of) the 2- and the 3-family. The 2-family gives rise to a true deformations (its Massey square is zero as a cochain), while the 3-family gives rise to a formal deformation.*

We will be more specific about the convergence of this formal deformation and about the $\mathbb{N}$-graded Lie algebras to which $\mathfrak{m}_2$ deforms in weight 0 in a later subsection.

3.3. **Cocycles in weight $l \leq -5$.** Let us show in this section that there are no non-trivial 2-cocycles in weight $l \leq -5$. This is somewhat surprising; we interprete it as being the fact that the cocycle equations for $\mathfrak{m}_2$ are very restrictive.

First of all, equations (e) mean that we can compensate the coefficients $a_{1,j}$ for $j+l \geq 0$ by a suitable coboundary. Observe that $a_{1,j}$ does not make sense for $j+l \leq -1$ as $a_{1,j}$ is the coefficient in front of $e_{j+1+l}$, so it can be set to zero. Therefore we will suppose in the following that $a_{1,j}=0$ for all $j \geq 2$. Thus, by antisymmetry, all coefficients involving an index 1 are zero.

With this in mind, the cocycle equations (a) and (b) become more simple:

- $a_{3,k} + a_{2,k+1} \ = \ a_{2,k}$
- $a_{j+1,k} + a_{j,k+1} \ = \ a_{j,k}$

for $k \geq 3$, $k+l \geq 2$, resp. $j+k+l \geq 2$, $j,k \geq 3$.

Let us write down the cocycle equations of type (c) with $j=3$ (this is the case of interest for the reasoning which eliminates higher non-zero terms) and $k \geq 4$:

$$\begin{aligned} -l-4 \leq k \leq -l-3 : a_{5,k} &= -a_{3,k+2} \\ k=-l-2 : a_{5,k} &= -a_{3,k} - a_{3,k+2} \\ k=-l-1 : a_{5,k} &= -a_{3,k+2} - a_{k,2} \\ k=-l : a_{5,k} &= a_{3,k} - a_{3,k+2} + a_{2,k} \\ k \geq -l+1 : a_{5,k} &= a_{3,k} - a_{3,k+2} \end{aligned}$$

Thus, for $k \geq -l+1$, we have on the one hand $a_{5,k} = a_{3,k} - a_{3,k+2}$, and on the other hand (for $k \geq -l-1$)

$$a_{5,k} \ = \ a_{4,k} - a_{4,k+1} \ = \ (a_{3,k} - a_{3,k+1}) - (a_{3,k+1} - a_{3,k+2}) \ = \ a_{3,k} - 2\,a_{3,k+1} + a_{3,k+2},$$

and one deduces $a_{3,k+1} = a_{3,k+2}$ for all $k \geq -l+1$. We call this coefficient $x := a_{3,k+1} = a_{3,k+2}$ for all $k \geq -l+1$.

The equation $a_{5,-l} \ = \ a_{3,-l} - a_{3,-l+2} + a_{2,-l}$ and the equation $a_{5,-l} \ = \ a_{3,-l} - 2a_{3,-l+1} + a_{3,-l+2}$ imply that $2a_{3,-l+2} \ = \ 2a_{3,-l+1} + a_{2,-l}$, and therefore with $a := a_{2,-l}$, we get $x \ = \ a_{3,-l+1} + \frac{a}{2}$.

**Step 1:** Using these equations, we fill in the table of coefficients $a_{i,j}$ starting from high $k$ values:

| | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $-l$ | $a$ | | | |
| $-l+1$ | | $x-\frac{a}{2}$ | $-\frac{a}{2}$ | $-\frac{a}{2}$ |
| $-l+2$ | | $x$ | 0 | 0 |
| $-l+3$ | | $x$ | 0 | 0 |
| $-l+4$ | | $x$ | 0 | 0 |
| $-l+5$ | | $x$ | 0 | 0 |

The $-\frac{a}{2}$ will repeat itself to the right of the table, meaning $a_{4+r,-l+1} = -\frac{a}{2}$ for all $r$. But $a_{-l+1,-l+1} = 0$ by antisymmetry, thus $a=0$.

**Step 2:** When we call $a_{3,-l} =: y$, the new table looks like:

| | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $-l$ | 0 | $y$ | $y-x$ | $y-x$ |
| $-l+1$ | | $x$ | 0 | 0 |
| $-l+2$ | | $x$ | 0 | 0 |
| $-l+3$ | | $x$ | 0 | 0 |
| $-l+4$ | | $x$ | 0 | 0 |
| $-l+5$ | | $x$ | 0 | 0 |

Once again, continuing the line with $y-x$ to the right, when we hit the diagonal, we get $y=x$.

**Step 3:** When we call $a_{3,-l-1} =: a$, the new table looks like:

| | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $-l-1$ | $a$ | $a-x$ | $a-x$ | $a-x$ |
| $-l$ | 0 | $x$ | 0 | 0 |
| $-l+1$ | $-x$ | $x$ | 0 | 0 |
| $-l+2$ | $-2x$ | $x$ | 0 | 0 |
| $-l+3$ | $-3x$ | $x$ | 0 | 0 |
| $-l+4$ | $-4x$ | $x$ | 0 | 0 |
| $-l+5$ | $-5x$ | $x$ | 0 | 0 |

The same argument as before gives us here $x=a$.

**Step 4:** This time, call $a_{3,-l-2} =: y$, then we get by the equation $a_{5,-l-2} = a_{3,-l-2} - a_{3,-l}$ that $a_{5,-l-2} = -y-x$ and $a_{5,-l-2} = -x+y = a_{4,-l-2} = -a_{3,-l-1} + a_{3,-l-2}$. One concludes $y=0$.

**Step 5:** Now write the new table:

| | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $-l-2$ | $x$ | 0 | $-x$ | $-x$ |
| $-l-1$ | $x$ | $x$ | 0 | 0 |
| $-l$ | 0 | $x$ | 0 | 0 |
| $-l+1$ | $-x$ | $x$ | 0 | 0 |
| $-l+2$ | $-2x$ | $x$ | 0 | 0 |
| $-l+3$ | $-3x$ | $x$ | 0 | 0 |
| $-l+4$ | $-4x$ | $x$ | 0 | 0 |
| $-l+5$ | $-5x$ | $x$ | 0 | 0 |

Finally, hitting once again the diagonal shows that $x = 0$. In order to conclude that all coefficients must be zero, it suffices to show that $a_{4,-l-3} = 0$. This follows from the (a) equation (with $j = 3$, $k = -l - 3$): $a_{4,-l-3} = -a_{3,-l-2} = 0$. $a_{4,-l-3} = 0$ suffices, because $a_{i,j}$ can only be non-zero starting from $i + j + l \geq 0$, i.e. $i = 2$ and $j \geq -l - 1$, $i = 3$ and $j \geq -l - 2$, $i = 4$ and $j \geq -l - 2$ and so on.

We summarize in the following

**Proposition 3.** *There are no non-trivial* 2*-cocycles in weight* $l \leq -5$.

3.4. **True deformations in weights $l = -1$ and $l = -2$.** Again, by the same reasoning as before, all coefficients involving an index 1 can be set to zero (up to addition of coboundaries).

The (a) and (b) equations are like in the general case. The (c) equations are not yet modified (only for $l = -3$ and $l = -4$). There is no non trivial (d) equation yet.

We are thus still in the range of validity of the reasoning which shows that there are as only possibly non-trivial cocycles the 2- and the 3-family.

The 2-family is still a cocycle of Massey square zero (as a cochain). The only thing which may be different here is the proof that the 3-family gives still rise to a formal deformation.

The first steps are like in the proof of proposition 1: the only Massey squares we have to compensate are $M_{23k}$, $k \geq 4$. We then introduce a homogeneous 2-cochain $\alpha$ of weight $2l$ with $\alpha(e_i, e_j) = b_{i,j}e_{i+j+2l}$. We have for $l \geq -2$, $j, k \geq 3$, $j < k$:

$$d\alpha(e_2, e_3, e_k) \;=\; b_{5,k} - b_{k+2,3} - b_{3,k} + \delta_{k+2l,0}b_{k,2} + \delta_{5+2l,1}b_{2,3} + \delta_{5+2l,2}b_{2,3}.$$

As for the 3-family $b_{2,3} = 0$, this reads more simply:

$$d\alpha(e_2, e_3, e_k) \;=\; b_{5,k} - b_{k+2,3} - b_{3,k} + \delta_{k+2l,0}b_{k,2}.$$

We may choose to compensate once again just by the 3-column, i.e. we may set $b_{5,k} = b_{k,2} = 0$ for all $k$. This ensures that at most the 2- and 3-columns for the $a$- and the $b$-coefficients are non-zero.

The next Massey power is then the Massey cube:

$$\begin{aligned} N_{ijk} &= \alpha(\omega(e_i, e_j), e_k) + \omega(\alpha(e_i, e_j), e_k) + \text{cycl.} \\ &= a_{i,j}b_{i+j+l,k} + b_{i,j}a_{i+j+2l,k} + \text{cycl..} \end{aligned}$$

We see that the terms we have to compensate are once again of type $N_{23k}$ (up to antisymmetry). We will have more and more Massey powers to compensate. This can be achieved by a growing, but finite number of cochains at each level. On the other hand, this process will not stop. We therefore get:

**Proposition 4.** *In weight $l = -1, -2$, the only homogeneous* 2*-cocycles are the* 2*- and the* 3*-family. The* 2*-family is of Massey square zero (as a cochain), and gives thus rise to a true deformation. The* 3*-family has zero Massey powers, and gives rise to a formal deformation with non-zero contributions at each level.*

3.5. **True deformations in weights $l = -3$ and $l = -4$.** Let us write down the cocycle equations. The important equations are those of type (c). They read:

$$a_{j+2,k} + a_{j,k+2} \ = \ a_{j,k} - a_{k,2}\delta_{k+l+1,0} - a_{k,2}\delta_{k+l,0} - a_{2,j}\delta_{j+l+1,0} - a_{2,j}\delta_{j+l,0}.$$

In weight $l = -3$, this means for $j = 3$ and $k \geq 4$ that

$$a_{5,k} \ = \ a_{3,k} - a_{3,k+2} - a_{2,3}.$$

Compare this equation to

$$a_{5,k} \ = \ a_{3,k} - 2a_{3,k+1} + a_{3,k+2},$$

which follows as usually from the (a) equations. In conclusion, we get:

$$-2a_{3,k+2} \ = \ a_{2,3} - 2a_{3,k+1}.$$

This means once again that the differences of 3-coefficients are constant, and thus that the 4-coefficients are equal, while the 5-coefficients are zero. More precisely

$$2a_{4,k+1} \ = \ 2(a_{3,k+1} - a_{3,k+2}) \ = \ a_{2,3},$$

and therefore $a_{4,k+1} = \frac{a_{2,3}}{2}$. Either $a_{4,k+1} \neq 0$ and we get a family with non-zero coefficients in the first three columns, or $a_{4,k+1} = 0$, i.e. $a_{2,3} = 0$, and we get the 3-family.

Observe that the 2-family does not satisfy the cocycle identities in weight $l \leq -3$. Indeed, for $j, k \geq 3$

$$a_{j+2,k} + a_{j,k+2} \ = \ a_{j,k} - a_{k,2}\delta_{k+l+1,0} - a_{k,2}\delta_{k+l,0} - a_{2,j}\delta_{j+l+1,0} - a_{2,j}\delta_{j+l,0},$$

and for $k >> 0$, all terms are zero, but one of the form $a_{2,j}$. This is a contradiction.

It remains thus (a linear combination of) the 3- and the 4-family. The 3-family is of Massey square zero in weight $l = -3$ (see the $\mathfrak{m}_0$-case !).

Let us turn to weight $l = -4$. Once again we look at a 2-cocycle $\omega$ given by coefficients $a_{i,j}$ such that $a_{1,k} = 0$ for all $k \geq 2$, which we can achieve possibly by adding a coboundary, cf equations (e). We cannot exploit independently equations (f) and (g), because in these equations the same coefficients occur.

Let us write down low degree (a) equations:

$$a_{j+1,k} + a_{j,k+1} \ = \ a_{j,k},$$

for $j, k \geq 3$. We therefore have for example $a_{3,4} = a_{3,5}$. The (b) equations read

- $k = 3$: $a_{2,4} = 0$.
- $k = 4$: $a_{3,4} + a_{2,5} = a_{2,4} = 0$.
- $k = 5$: $a_{3,5} + a_{2,6} = a_{2,5}$.
- $k \geq 6$: $a_{3,k} + a_{2,k+1} = a_{2,k}$.

And the (c) equations, which are the most interesting, read for $j = 3$:

- $k = 4$: $a_{5,4} + a_{3,6} = a_{3,4} - a_{4,2} - a_{2,3} = a_{3,4} - a_{2,3}$.
- $k = 5$: $a_{3,7} = a_{3,5} - a_{2,3}$.
- $k \geq 6$: $a_{5,k} + a_{3,k+2} = a_{3,k} - a_{2,3}$.

The (d) equations are still void.

Let us now start a table with the coefficients $a_{i,j}$ which varify these equations. First of all, we call $a := a_{2,3}$, and $a_{3,4} =: b$. Then on the one hand $-a_{4,5} = b - a - a_{3,6}$, and on the other hand $a_{4,5} = b - a_{3,6}$. This gives $a_{3,6} = b - \frac{a}{2}$, and $a_{4,5} = \frac{a}{2}$.

Now let us perform the same trick as in the other cases: on the one hand, we have $a_{5,k} = a_{3,k} - a_{3,k+2} - a_{2,3}$, and on the other hand, we have $a_{5,k} = a_{3,k} - 2a_{3,k+1} + a_{3,k+2}$ by the (a) equations, for $k \geq 6$. We get thus $a_{3,k+1} - a_{3,k+2} = \frac{a_{2,3}}{2}$, i.e. the differences of the 3-coefficients, which determine the 4-coefficients, are constant, and therefore the 5-coefficients zero. We now display the table:

| | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 3 | $a$ | | | |
| 4 | $0$ | $b$ | | |
| 5 | $-b$ | $b$ | $\frac{a}{2}$ | |
| 6 | $-2b$ | $b-\frac{a}{2}$ | $\frac{a}{2}$ | $0$ |
| 7 | $-3b+\frac{a}{2}$ | $b-\frac{2a}{2}$ | $\frac{a}{2}$ | $0$ |
| 8 | $-4b+\frac{3a}{2}$ | $b-\frac{3a}{2}$ | $\frac{a}{2}$ | $0$ |
| 9 | $-5b+\frac{6a}{2}$ | $b-\frac{4a}{2}$ | $\frac{a}{2}$ | $0$ |
| 10 | $-6b+\frac{10a}{2}$ | $b-\frac{5a}{2}$ | $\frac{a}{2}$ | $0$ |

We see that a 2-parameter family is building up. The remaining question is whether the Massey powers are zero, i.e. whether the family gives rise to a true or formal deformation. We will consider the two cases $a = 0$ and $b = 0$ separately. For $b = 0$, we have (a multiple of) the 4-family (up to a non-zero coefficient $a_{2,3}$). One easily varifies that the additional non-zero coefficient $a_{2,3}$ does not change the Massey square zero character of the 4-family in weight $l = -4$ (cf the $\mathfrak{m}_0$-case). For $a = 0$, we have the 3-family which has non-zero Massey squares. We compute that $M_{234} = 0$, $M_{235} = 0$, but $M_{23k} \neq 0$ for $k \geq 6$, that $M_{245} \neq 0$, but $M_{24k} = 0$ for $k \geq 6$, that $M_{25k} \neq 0$ for $k \geq 6$, that $M_{26k} = 0$ for $k \geq 7$, that $M_{34k} \neq 0$ for $k \geq 5$ and finally that $M_{35k} = 0$ for $k \geq 6$. These are all ordered Massey squares which are possibly non-zero.

We have thus a finite family of non-zero Massey squares which can be compensated by a finite sum of coboundaries. These give then rise to a finite number of higher dimensional Massey powers, which can also be compensated in the usual way. All in all we get a formal deformation.

**Proposition 5.** *In weights $l = -3$ and $l = -4$, the 3-family and the 4-family (and their linear combinations) are the only 2-cocycles. In weight $l = -3$, the 3-family gives a true and the 4-family a formal deformations, whereas in weight $l = -4$, the 4-family gives a true and the 3-family a formal deformation.*

3.6. **Identification of the deformations in weight $l = 0$.** We have seen in one of the previous sections that there are exactly two non-trivial cocycles (up to linear combinations) in weight $l = 0$. One is given by the 2-family, and the other is given by the 3-family. We then examined Massey powers, and found that while the 2-family is of Massey square zero in weight $l = 0$, the 3-family has Massey squares at each step and gives finally rise to a formal deformation. Let us identify in this section the Lie algebras to which $\mathfrak{m}_2$ deforms.

First consider the 2-family. Write down the corresponding deformation $\mathfrak{m}_2^1(t)$ (up to antisymmetry):

$$[e_1, e_j]_t = e_{j+1} \quad \forall j \geq 2,$$
$$[e_2, e_j]_t = e_{j+2} + te_{j+2} \quad \forall j \geq 3.$$

We claim that this deformation $\mathfrak{m}_2^1(t)$ is *trivial*, i.e. it does not leave the isomorphism class of $\mathfrak{m}_2$. In other words, for any $t \neq -1$, the deformed Lie algebra is isomorphic to $\mathfrak{m}_2$.

Indeed, define $\tilde{e}_j := (1 + t)e_j$ for all $j \geq 2$, and $\tilde{e}_1 = e_1$. Then if the $e_i$ obey the $\mathfrak{m}_2$-relations, the $\tilde{e}_i$ obey the $\mathfrak{m}_2^1(t)$-relations for all $t \neq -1$.

Now consider the deformation given by the 3-family. The corresponding deformation $\mathfrak{m}_2^2(t)$ reads (up to antisymmetry):

$$[e_1, e_j]_t = e_{j+1} \quad \forall j \geq 2,$$
$$[e_2, e_j]_t = e_{j+2} + t(1 - (j - 4))e_{j+2} \quad \forall j \geq 4,$$
$$[e_2, e_j]_t = te_{j+3} \quad \forall j \geq 4.$$

We already saw that this deformation has Massey corrections in any power of $t$, so that it is a formal deformation. Let us show that it gives a non-converging deformation. Indeed, if it were converging, the limiting object would be an $\mathbb{N}$-graded Lie algebra with one-dimensional graded components, generated in degrees 1 and 2. But by the classification theorem (Theorem p. 2 in [1]), $\mathfrak{m}_2^2(t)$ must be isomorphic to $L_1$. This is obviously not the case, as $\mathfrak{m}_2^2(t)$ has a codimension 3 abelian ideal, whereas $L_1$ does not have any abelian ideal.

Therefore we arrive at the conclusion:

**Proposition 6.** *The deformations of $\mathfrak{m}_2$ in weight $l = 0$ described in the following way: one of them is a trivial deformation, i.e. isomorphic to $\mathfrak{m}_2$, and the other is a formal non-converging deformation. In particular, $\mathfrak{m}_2$ does not deform to any other $\mathbb{N}$-graded Lie algebra with one-dimensional graded components, generated in degrees* 1 *and* 2*. In particular, it does not deform to $L_1$.*

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