DEGENERATION OF K3 SURFACES, II

by

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Introduction. This paper is a continuation of [7]. For the basic definitions and notations, we refer to [7].

Let $\pi : X \longrightarrow \Delta = \{t \in \mathbb{C} \mid |t| < \varepsilon\}$ be a semi–stable degeneration of K3 surfaces. The problem we study in the present paper is to find a "good" modification $\pi' : X' \longrightarrow \Delta$ of $\pi : X \longrightarrow \Delta$ and to describe the singular fiber X'_0 of such $\pi' : X' \longrightarrow \Delta$.

Several years ago, Kulikov [3] and Persson–Pinkham [8] proved that $\pi : X \longrightarrow \Delta$ has a modification $\pi' : X' \longrightarrow \Delta$ such that π' is also semi–stable and the canonical bundle $K_{X'}$ of X' is trivial, provided that every component of the singular fiber X_0 is algebraic (cf. Theorem 1.1). This $\pi' : X' \longrightarrow \Delta$ is considered as a "good" modification of $\pi : X \longrightarrow \Delta$. Moreover Kulikov [3] described the singular fiber X'_0 of $\pi' : X' \longrightarrow \Delta$, and classified it into three types I, II, III (cf. Theorem 1.2). In [3], he also obtained a result about the monodromy for the degeneration of each type I, II, III (cf. Theorem 1.3).

We study a semi–stable degeneration $\pi : X \longrightarrow \Delta$ of K3 surfaces, in general, without assuming that every component of X_0 is algebraic. Persson–Pinkham asked in their paper [8] whether or not the above theorem due to Kulikov and Persson–Pinkham (i.e., Theorem 1.1) would hold in this general analytic case. We know that the answer is no. Namely, the previous paper [7] showed that there exists $\pi : X \longrightarrow \Delta$ of which no semi–stable modification has trivial canonical bundle, and moreover that such a degeneration $\pi : X \longrightarrow \Delta$ must contain a Hopf surface or a (CB)–surface (see [6] for the definition) in the singular fiber X_0 (cf. Theorem 1.4). This $\pi : X \longrightarrow \Delta$ has no "good" modification at all in the above sense. In [7], it is also proved that if a semi–stable degeneration of K3 surfaces has trivial canonical bundle and contains a non–algebraic surface in the singular fiber, then it is classified into three types, II′, III′, II′+III′ (cf. Theorem 1.5). § 1 reviews these results.

This paper mainly treats with a semi–stable degeneration $\pi : X \longrightarrow \Delta$ of K3 surfaces of which no semi–stable modification has trivial canonical bundle. As mentioned above, such a degeneration $\pi : X \longrightarrow \Delta$ contains a Hopf surface or a (CB)–surface in the singular fiber. In § 2 (resp. § 3), we shall construct examples of $\pi : X \longrightarrow \Delta$ which contains a Hopf surface (resp. a (CB)–surface) and corresponds to a degeneration of each type I, II or III in algebraic case; The type of this $\pi : X \longrightarrow \Delta$ will be named $\tilde{\mathrm{I}}_0$, $\widetilde{\mathrm{II}}_0$ or $\widetilde{\mathrm{III}}_0$ (resp. $\tilde{\mathrm{I}}_1$, $\widetilde{\mathrm{II}}_1$ or $\widetilde{\mathrm{III}}_1$).

§ 4 gives several conjectures about the structure of a semi–stable degeneration $\pi : X \longrightarrow \Delta$ of K3 surfaces. The final one of the conjectures says that any $\pi : X \longrightarrow \Delta$ would have a semi–stable modification of type

$$\mathrm{I},\ \mathrm{II},\ \mathrm{III},\ \mathrm{II}',\ \mathrm{III}',\ \mathrm{II}'+\mathrm{III}',\ \tilde{\mathrm{I}}_0,\ \widetilde{\mathrm{II}}_0\ \widetilde{\mathrm{III}}_0,\ \tilde{\mathrm{I}}_1,\ \widetilde{\mathrm{II}}_1 \text{ or } \widetilde{\mathrm{III}}_1.$$

(See Conjecture 4.8.)

Last we would like to make a remark about an application to the cassification theory of higher dimensional complex manifolds. Using a certain semi–stable degeneration of K3 surfaces of which no semi–stable modification has trivial canonical bundle, one can easily construct a compact complex manifold Y of dimension three such that the Kodaira dimension $\kappa(Y) = 0$ but no bimeromorphic "model" Y' of Y has trivial m–th pluricanonical bundle $K_{Y'}^{\otimes m} = 1$ for any positive integer m . This phenomenon occurs first for complex manifolds of dimension three. Namely, let Z be a compact complex manifold with $\kappa(Z) = 0$, in general; if Z is a surface, it is classically known that the minimal model Z' of Z has trivial m–th pluricanonical bundle $K_{Z'}^{\otimes m} = 1$ for $m = 12$. If Z is a projective 3–fold, it is conjectured that there would exist a "good minimal model" Z' of Z , whose m–th pluricanonical bundle $K_{Z'}^{\otimes m} = 1$ would be trivial for large enough m . Recently, this conjecture was partially proved by Miyaoka [4], where he used the above Theorem due to Kulikov and Persson–Pinkham.

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§ 1.

In this section, we shall review some of known results about degenerations of K3 surfaces.

Let $\pi : X \longrightarrow \Delta$ be a semi–stable degeneration of K3 surfaces. Kulikov [3] and Persson–Pinkham [8] studied $\pi : X \longrightarrow \Delta$ under the assumption that every component of the singular fiber X_0 is algebraic, and they obtained the following result, which is the first and most important breakthrough:

Theorem 1.1 ([3], [8]). Let $\pi : X \longrightarrow \Delta$ be as above. Then $\pi : X \longrightarrow \Delta$ has a modification $\pi' : X' \longrightarrow \Delta$ such that π' is also semi–stable and the canonical bundle $K_{X'}$ of X' is trivial.

Such a modification $\pi' : X' \longrightarrow \Delta$ is considered as a "good model" of the original degeneration, and Kulikov [3] described the singular fiber of the model. Namely,

Theorem 1.2 ([3]). Let $\pi : X \longrightarrow \Delta$ be a semi–stable degeneration of K3 surfaces such that $K_X = 1$ and every component of the singular fiber X_0 is algebraic. Then X_0 is one of the following:

I. X_0 is a K3 surface.

II. $X_0 = V_1 + \dots + V_N$, where V_1 and V_N are rational surfaces and $V_2, \dots, V_{N-1}$ are elliptic ruled surfaces. The double curves are all elliptic curves. The dual graph $\pi(X_0)$ is given as follows.

III. $X_0 = V_1 + \dots + V_N$, where all V_i's are rational surfaces. The double curves form a rational cycle on each surface V_i . The dual graph $\pi(X_0)$ is a triangulation of a 2–sphere S^2 .

Kulikov [3] also obtained a result about the monodromy for degenerations of each type I, II, III. First we recall the definition of the Picard–Lefschetz transformation. For a semi–stable degeneration $\pi : X \longrightarrow \Delta$ of surfaces, let

$$\pi_1(\Delta^*) \longrightarrow \mathrm{Aut}\,(H^2(X_t,\mathbb{Z})) \quad (t \neq 0)$$

be a monodromy representation, and let P be the image of a generator of $\pi_1(\Delta^*) = \mathbb{Z}$. P is called the Picard–Lefschetz transformation for $\pi : X \longrightarrow \Delta$. Put $N = \log P$. Using the Hodge theory, Kulikov proved

<u>Theorem</u> 1.3 ([3]). Let $\pi : X \longrightarrow \Delta$ be a projective semi–stable degeneration of K3 surfaces with $K_X = 1$. Then we have

(i) if π is of type I, then $N = 0$;

(ii) if π is of type II, then $N^2 = 0$, $N \neq 0$;

(iii) if π is of type III, then $N^3 = 0$, $N^2 \neq 0$.

Now we shall study a semi-stable degeneration of K3 surfaces without assuming that every component of the singular fiber is algebraic. Then Theorem 1.1 is generalized as follows:

Theorem 1.4. Let $\pi : X \longrightarrow \Delta$ be a semi-stable degeneration of K3 surfaces. Then there exists a semi-stable modification $\pi' : X' \longrightarrow \Delta$ of $\pi : X \longrightarrow \Delta$ such that π' satisfies one of the following:

(1) K'_X is trivial

(2) the singular fiber X'_0 contains, as a component, a Hopf surface on which the double curves are just one elliptic curve,

(3) the singular fiber X'_0 contains, as a component, a (CB)-surface (see Nishiguchi [6] for the definition of a (CB)-surface).

Furthermore, we have examples of all cases, and the case (1) is disjoint from both the cases (2) and (3).

This theorem easily follows from Theorem 1.1, the proof of Theorem 1.1 and Proposition 1.4 all in Nishiguchi [7]. Examples of the cases (2) and (3) are given in § 4 of [7]. For further examples, see also Nishiguchi [5], § 5 of [7] and §§ 2, 3, 4 of this paper.

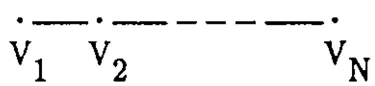
As in Theorem 1.2 for the algebraic case, one can classify a degeneration of the case (1), i.e., a semi-stable degeneration of K3 surfaces with trivial canonical bundle, even if it contains a non-algebraic surface in its singular fiber:

Theorem 1.5 ([7]). Let $\pi : X \longrightarrow \Delta$ be a semi–stable degeneration of K3 surfaces with $K_X = 1$. We assume that the singular fiber X_0 contains a non–algebraic surface. Then, after suitable Mod I and Mod II (see Kulikov [3] for their definition), X_0 becomes one of the following:

$$\text{I (see Theorem 1.2)}, \ \text{II}', \ \text{III}', \ \text{II}' + \text{III}'.$$

Here we define

II′. $X_0 = V_1 + \dots + V_N$, where V_1 and V_N are rational surfaces and $V_2, \dots, V_{N-1}$ are relatively minimal elliptic ruled surfaces or Hopf surfaces. The double curves are all elliptic curves. The dual graph $\pi(X_0)$ is as follows.



For the types III′ and II′ + III′, we shall not repeat their definition (see Theorem 2.1 in [7]), but we give the following

Conjecture–Definition 1.6. (i) In the singular fiber X_0 of a degeneration $\pi : X \longrightarrow \Delta$ of type III′, there exists just one hyperbolic Inoue surface, and all the other components are rational surfaces. Only such a degeneration will be called of type III′ in this paper. Namely one has

III′. $X_0 = V_1 + \dots + V_N + V_0$, where $V_i (1 \le i \le N)$ is a rational surface and V_0 is a hyperbolic Inoue surface. The double curves form a rational cycle on each rational surface $V_i (1 \le i \le N)$ and exactly two rational cycles on the hyperbolic Inoue surface V_0 . The dual graph $\pi(X_0)$ is a triangulation of a one–point union of two S^2's :

$$S^2 \underset{P}{\vee} S^2$$

where P corresponds to the component V_0 and points in S^2 other than P correspond to the components V_i's $(1 \leq i \leq N)$.

(ii) In the singular fiber X_0 of a degeneration $\pi : X \longrightarrow \Delta$ of type II$'$ + III$'$, there exist no hyperbolic Inoue surfaces but several parabolic Inoue surfaces. Only such a degeneration will be called of type II$'$ + III$'$. Namely one has

II$'$ + III$'$: $X_0 = V_1 + ... + V_N + V_0^{(1)} + ... + V_0^{(M)}$, where $V_i (1 \leq i \leq N)$ is a rational surface or a relatively minimal elliptic ruled surface, and $V_0^{(j)}$ $(1 \leq j \leq M)$ is a parabolic Inoue surface. The dual graph $\pi(X_0)$ is a triangulation of a one–point union of an S^2 and several line segments L_j's $(1 \leq j \leq M)$, where each L_j stems from a distinct point P_j on the 2–sphere S^2 ; The point P_j corresponds to the component $V_0^{(j)}$ $(j = 1,...,M)$, a point in S^2 other than P_i's corresponds to a rational surface, and a point in L_j other than P_j corresponds to a relatively minimal elliptic ruled surface unless it is an edge point of the triangulation of L_j which then corresponds to a rational surface. The double curves on each component consist of an elliptic curve and/or a rational cycle.

There are examples of degenerations of all types II$'$, III$'$, II$'$ + III$'$. See § 3 of Nishiguchi [7], and also Example 1.8 below.

We now have the following more general conjecture than the above 1.6, which would naturally give rise to the restriction of the number of hyperbolic Inoue surfaces in the singular fiber:

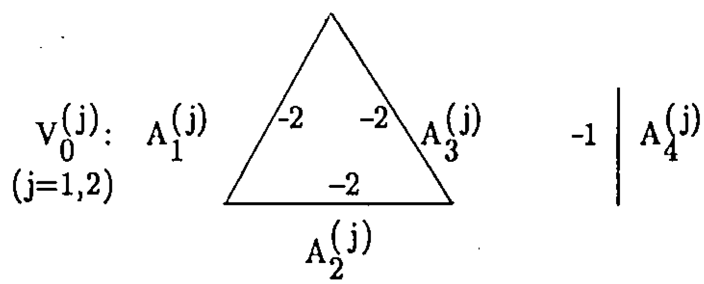
Conjecture 1.7. Let $\pi : X \longrightarrow \Delta$ be a semi–stable degeneration of K3 surfaces, $X_0 = \Sigma V_i$ the irreducible decomposition of the singular fiber X_0, and D_i the divisor of all double curves on V_i. Assume that one of D_i's has a negative definite intersection matrix. Then the others of D_i's do not have a negative definite or semi–definite intersection matrix.

The intersection matrix of curves on a hyperbolic Inoue surface or a (CB) – surface is negative definite, and moreover, that on any non–algebraic surface is negative definite or semi–definite. Therefore, Conjecture 1.7 says

Conjecture 1.7′. If the singular fiber X_0 of a semi–stable degeneration $\pi : X \longrightarrow \Delta$ of K3 surfaces contains a hyperbolic Inoue surface or a (CB) – surface, then the other components in X_0 are algebraic.

On the other hand, the intersection matrix of all curves on a parabolic Inoue surface is not definite, and one has an example of a degeneration of type II′ + III′ which contains two parabolic Inoue surfaces in the singular fiber. In fact, we have

Example 1.8. Let $V_0^{(j)}$ (j = 1,2) be a parabolic Inoue surface with the following configuration of curves:

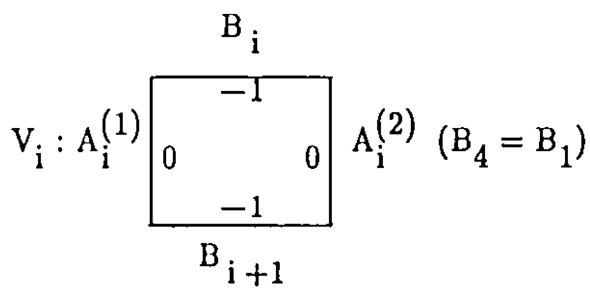


where $A_i^{(j)}$ $(i = 1,2,3)$ is a non–singular rational curve $\mathbb{P}^1$, $A_4^{(j)}$ is a non–singular elliptic curve, the canonical bundle $K_{V_0^{(j)}}$ is

$$K_{V_0^{(j)}} = -A_1^{(j)} - A_2^{(j)} - A_3^{(j)} - A_4^{(j)},$$

and the number near a curve in the picture denotes the self–intersection number of that curve (to keep this convention throughout this paper).

Let V_i $(i = 1,2,3)$ be a rational surface with the following configuration of $\mathbb{P}^1$'s :



where $K_{V_i} = -A_i^{(1)} - A_i^{(2)} - B_i - B_{i+1}$.

Let V_{3+j} $(j = 1,2)$ be a rational surface with the elliptic curve $A_4^{(j)}$ whose normal bundle is

$$N_{A_4^{(j)}/V_{3+j}} = \left[N_{A_4^{(j)}/V_0^{(j)}}\right]^*,$$

where $K_{V_{3+j}} = -A_4^{(j)}$.

Now we construct a two–dimensional variety X_0 with only normal crossings by gluing the surfaces V_i $(i = 1, \dots ,5)$ and $V_0^{(j)}$ $(j = 1,2)$ along the corresponding curves.

Then, as in Theorem 4.4 in Nishiguchi [7], X_0 can be made the singular fiber of a semi–stable degeneration $\pi : X \longrightarrow \Delta$ of K3 surfaces, by using the deformation theory due to Friedman [1]. It is clear that $K_X = 1$ and π is of type $II' + III'$.

Next we would like to consider the monodromy for a semi–stable degeneration of K3 surfaces with trivial canonical bundle when it is not necessarily projective. But nothing is known about this so far, and we only have

Conjecture 1.9. Let $\pi : X \longrightarrow \Delta$ be a semi–stable degeneration of K3 surfaces with $K_X = 1$. As in Theorem 1.3, let $N = \log P$ where P is the Picard–Lefschetz transformation for π. Then

(i) if π is of type I, then $N = 0$;

(ii) if π is of type II or II', then $N^2 = 0$, $N \neq 0$;

(iii) if π is of type III, III' or $II' + III'$ then $N^3 = 0$, $N^2 \neq 0$.

A main purpose of this paper is to study a semi–stable degeneration of K3 surfaces which has no semi–stable modification with trivial canonical bundle, i.e., which belongs to the case (2) and (3) in Theorem 1.4. The following two sections give examples of the cases (2) and (3), which look like degenerations of types I, II, III in Theorem 1.2.

§ 2.

In this section, we construct examples of semi–stable degenerations of K3 surfaces in the case (2) of Theorem 1.4. Those degenerations contain a Hopf surface in the singular fiber and have no semi–stable modification with trivial canonical bundle. The first example of the following was obtained in [7], but the others are new.

Example 2.1. Let V_0 be an elliptic Hopf surface H_α. By definition, H_α is $\mathbb{C}^2 - \{0\}/\langle g \rangle$, where g is the automorphism of $\mathbb{C}^2 - \{0\}$ in the form:

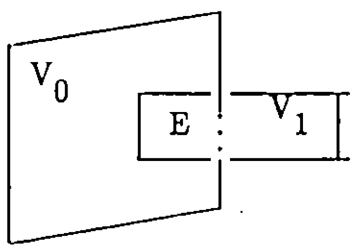
$$g : (z_1, z_2) \mapsto (\alpha z_1, \alpha z_2) \; ; \; \alpha \in \mathbb{C} \, , \; 0 < |\alpha| < 1 \, .$$

Let E be a fiber of H_α. Note that $K_{V_0} = -2E$ and all fibres are isomorphic to the elliptic curve $E = \mathbb{C} - \{0\}/\langle \alpha \rangle$.

Let V_1 be an elliptic K3 surface with the elliptic curve E as a smooth fiber.

We construct a two dimensional variety X_0 with only normal crossings by gluing V_0 and V_1 along the corresponding curve E as follows.

X_0 :

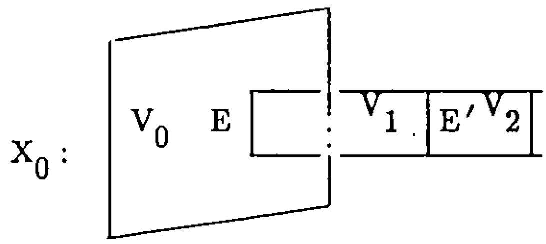


Then, by using the deformation theory due to Friedman [1], X_0 can be made the singular fiber of a semi–stable degeneration $\pi : X \longrightarrow \Delta$ of K3 surfaces. The canonical bundle K_X of X is written as $K_X = V_0$.

Example 2.2. Let V_0 and E be as in Example 2.1.

Let V_1 and V_2 be elliptic rational surfaces with an elliptic curve E' as smooth fibers on them. Assume that V_1 has the elliptic curve E as a smooth fiber.

We construct a variety X_0 with only normal crossings by gluing V_0, V_1 and V_2 along the corresponding curves E and E' as follows.

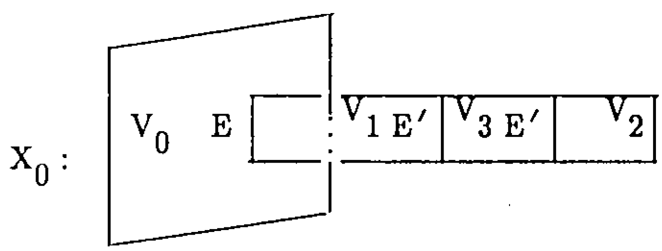


Then, as in Example 2.1, X_0 is the singular fiber of a semi-stable degeneration $\pi : X \longrightarrow \Delta$ of K3 surfaces with $K_X = V_0$.

Example 2.3. Let V_0, V_1, V_2, E and E' be as in Example 2.2. Put the elliptic curve $E' = \mathbb{C} - \{0\}/< \alpha' >$, $\alpha' \in \mathbb{C}$, $0 < |\alpha'| < 1$.

Let V_3 be an elliptic Hopf surface $H_{\alpha'}$. Recall that all fibers of $H_{\alpha'}$ are isomorphic to $\mathbb{C} - \{0\}/< \alpha' >$. By abuse of notation, distict two fibers are denoted by the same E'.

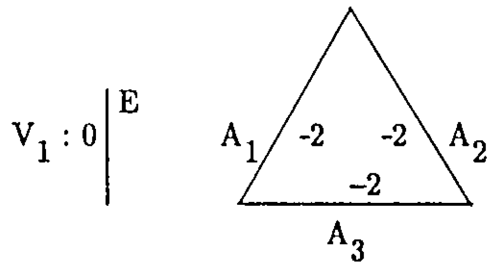
We construct a variety X_0 with only normal crossings by gluing V_0, V_1, V_2 and V_3 along the corresponding curves as follows.



Then X_0 is the singular fiber of a semi–stable degeneration $\pi : X \longrightarrow \Delta$ of K3 surfaces with $K_x = V_0$.

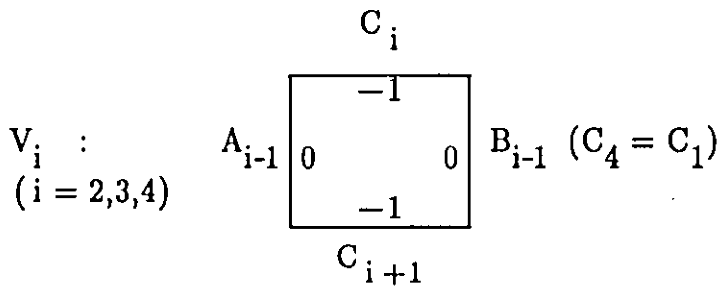
Example 2.4. Let V_0 and E be as in Example 2.1.

Let V_1 be a rational elliptic surface with the configuration of curves as follows.



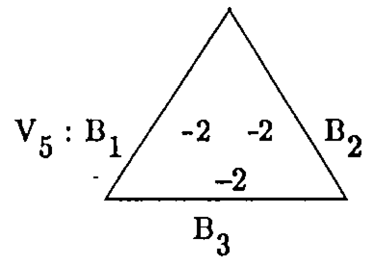
Here the elliptic curve E lies on V_1 as a smooth fiber; A_1 , A_2 and A_3 are $\mathbb{P}^1$'s and form a singular fiber of type I_3 ; and $K_{V_1} = - A_1 - A_2 - A_3$.

Let $V_i(i = 2,3,4)$ be a rational surface with the following configuration of $\mathbb{P}^1$'s:



where $K_{V_i} = - A_{i-1} - B_{i-1} - C_i - C_{i+1}$.

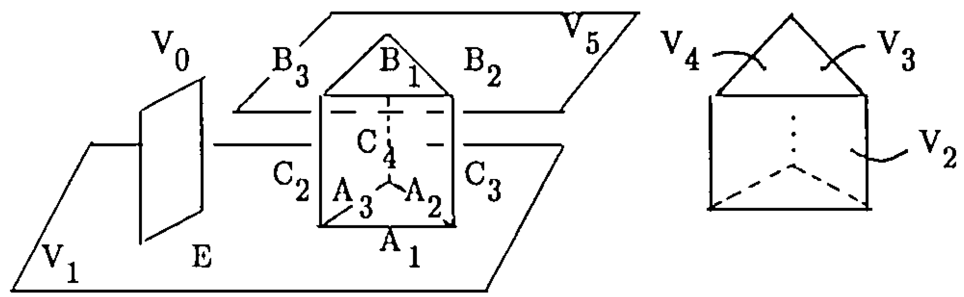
Let V_5 be a rational surface with the following configuration of $\mathbb{P}^1$'s :



where $K_{V_5} = -B_1 - B_2 - B_3$.

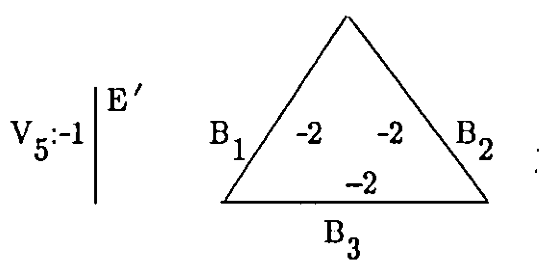
We construct a variety X_0 with only normal crossing by gluing $V_i (0 \le i \le 5)$ along the corresponding curves as follows.

X_0 :



Then X_0 is the singular fiber of a semi–stable degeneration $\pi : X \longrightarrow \Delta$ of K3 surfaces with $K_X = V_0$.

<u>Example</u> 2.5. Let $V_i (0 \le i \le 4)$, E , A_j , B_j and $C_j (1 \le j \le 3)$ be as in Example 2.4. As V_5 , the rational surface is replaced by a parabolic Inoue surface. Namely, let V_5 be a parabolic Inoue surface with the following configuration of curves:



where E' is an elliptic curve and $K_{V_5} = -E' - B_1 - B_2 - B_3$.

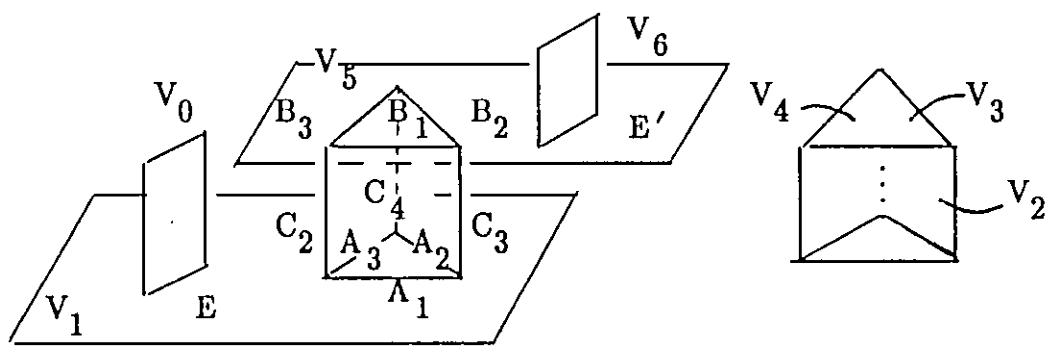
Let V_6 be a rational surface with the elliptic curve E' whose normal bundle is

$$N_{E'/V_6} = (N_{E'/V_5})^*,$$

where $K_{V_6} = -E'$.

We construct a variety X_0 with only normal crossings by gluing $V_i (0 \leq i \leq 6)$ along the corresponding curves as follows.

X_0 :

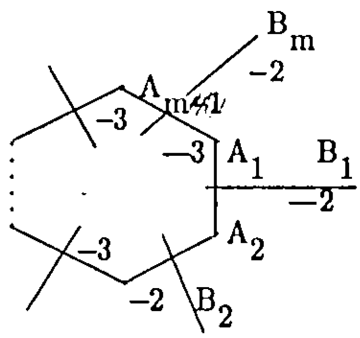


Then X_0 is the singular fiber of a semi–stable degeneration $\pi : X \longrightarrow \Delta$ of K3 surfaces with $K_X = V_0$.

§ 3.

In this section, we give three examples of semi–stable degenerations of K3 surfaces belonging to the case (3) of Theorem 1.4. Namely those degenerations contain (CB) – surfaces in the singular fibers. They seem to correspond to degenerations of types I, II and III in algebraic case (see Theorem 1.2). We remark that the first example was obtained in Nishiguchi [7], but the others are new.

Example 3.1. As in Example 4.3 in [7], let S be a (CB) – surface with the following configuration of $\mathbb{P}^1$'s :

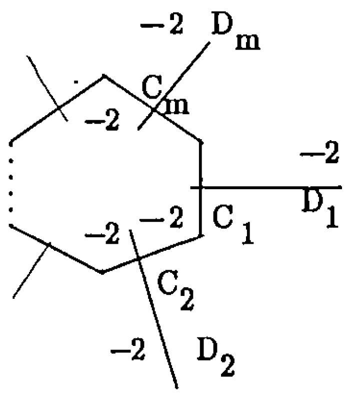


where the canonical bundle K_s of S is written as

$$K_S = -2(A_1 + \dots + A_m) - (B_1 + \dots + B_m) .$$

In fact, such a surface S can be obtained as a deformation of the blowing–up of the Hopf surface in Example 2.1, or equivalently constructed as a surface containing a global spherical shell (GSS for short). For more detail of the construction of S , see Kato [2].

Let Z_I be a K3 suraces with the configuration of $\mathbb{P}^1$'s as follows.



Then we have

Theorem 3.2. The (CB) – surface S and the K3 surface Z_I can be simultaneously made components of the singular fiber of a semi–stable degeneration of K3 surfaces.

This is nothing but Theorem 4.4 in [7].

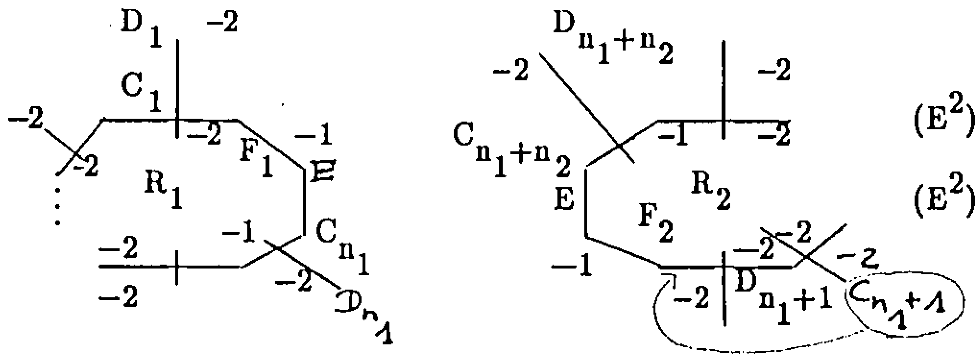
Remark 3.3. As mentioned also in Remark after Theorem 4.4 in [7], the Picard number of a K3 surface is less than 21, hence we have $m \leq 10$. For $m \leq 9$, there exists such a K3 surface Z_I , but for $m = 10$, the existence is not known.

Next we construct a semi–stable degeneration of K3 surfaces which contains a (CB) – surface and "a variety of type II" (see Theorem 1.2) in the singular fiber.

Example 3.4. Let S , A_i and $B_i (1 \leq i \leq m)$ be the (CB) – surface and the curves on it as in Example 3.1.

Let Z_{II} be a two–dimensional variety $Z_{II} = R_1 + R_2$ with only normal crossings; R_1 and R_2 are rational surfaces joining along a non–singular elliptic curve

E , and they have the following configuration of curves:



$(E^2)_{R_1} = -\ell_1$

$(E^2)_{R_2} = -\ell_2$

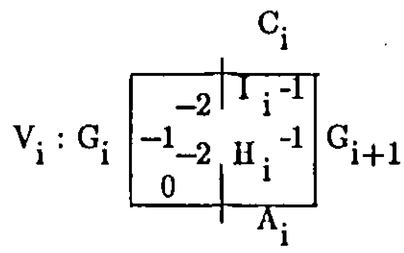
where C_i , $D_i (1 \le i \le n_1 + n_2)$ and $F_j (j = 1,2)$ are $\mathbb{P}^1$'s ; F_1 and $C_{n_1+n_2}$ sprout from the same point on E in Z_{II} ; So do C_{n_1} and F_2 ; The canonical bundles of R_1 and R_2 are $K_{R_1} = -E$, $K_{R_2} = -E$.

Then we have

Theorem 3.5. The (CB) – surface S and the variety Z_{II} with $n_1 + n_2 = m$ and $\ell_1 + \ell_2 = 2$ can be simultaneously sit in the singular fiber of a semi-stable degeneration of K3 surfaces.

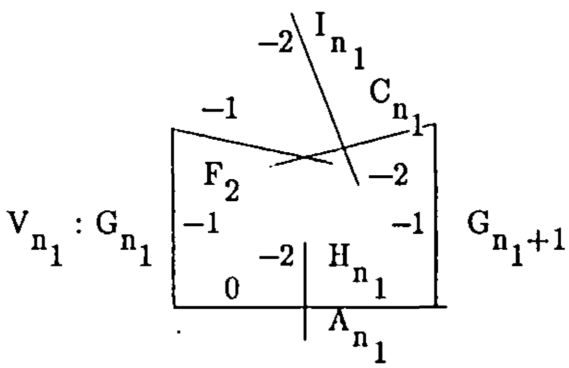
Proof. We construct the degeneration in the similar way to § 2 and [7, §§ 4, 5]. So we only describe each component which is in the singular fiber.

Let $V_i (1 \le i \le n_1 - 1\ ,\ n_1 + 1 \le i \le n_1 + n_2 - 1)$ be a rational surface with the following configuration of $\mathbb{P}^1$'s :



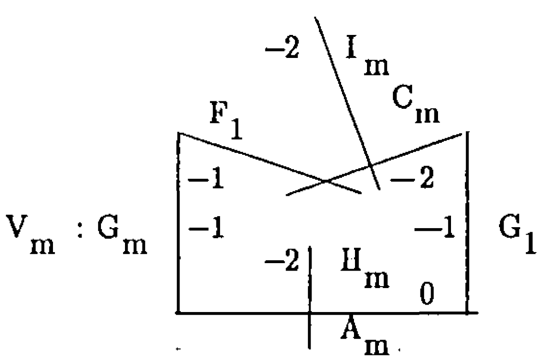
where $K_{V_i} = -G_i - G_{i+1} - 2C_i - I_i$.

Let V_{n_1} be a rational surface with the following configuration of $\mathbb{P}^1$'s :



where $K_{V_{n_1}} = -G_{n_1} - G_{n_1+1} - 2F_2 - 2C_{n_1} - I_{n_1}$.

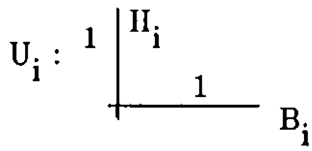
Similarly, let $V_{n_1+n_2} = V_m$ be a rational surface with the following configuration of $\mathbb{P}^1$'s :



where $K_{V_m} = -G_m - G_1 - 2F_1 - 2C_m - I_m$.

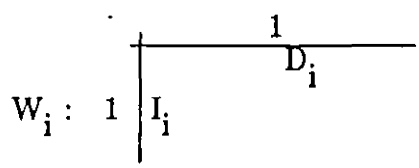
These $V_i (1 \leq i \leq n_1 + n_2 = m)$ are obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$. It is easy to find the blowing–ups, and left to the reader.

Let $U_i (1 \leq i \leq m)$ be a projective plane $\mathbb{P}^2$ with the following configuration of lines:



where $K_{U_i} = -B_i - 2H_i$.

Similarly, let $W_i (1 \leq i \leq m)$ be a $\mathbb{P}^2$ with the following configuration of lines:



where $K_{W_i} = -I_i - 2D_i$.

Now, gluing S, $R_j (j = 1,2)$, V_i, U_i and $W_i (1 \leq i \leq m)$ along the corresponding curves, we obtain a two–dimensional variety X_0 with only normal crossings. As in § 2, the deformation theory (cf. Friedman [1]) shows that X_0 is the singular fiber of a semi–stable degeneration $\pi : X \longrightarrow \Delta$ of K3 surfaces. Q.E.D.

Remark 3.6. (i) By the adjunction formula, the canonical bundle K_X of the above X is written as

$$K_X = 2S + \sum_{i=1}^{m} U_i + \sum_{i=1}^{m} V_i + \sum_{i=1}^{m} W_i$$

(ii) As in Remark 3.3, one has a restriction on $m = n_1 + n_2$, i.e., there does not exist Z_{II} with large n_1 and n_2 . In fact, we have $m = n_1 + n_2 \leq 8$. More precisely,

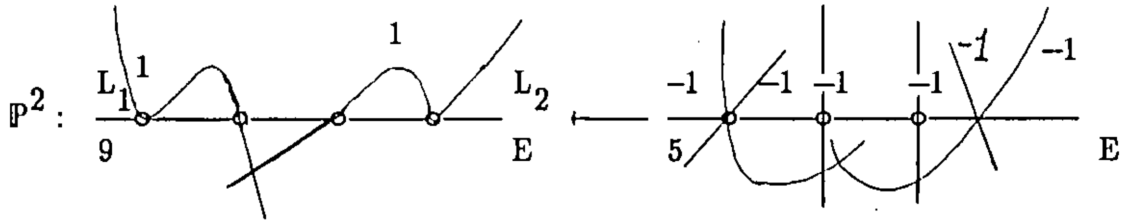
Proposition 3.7. Under the above notation, we must have

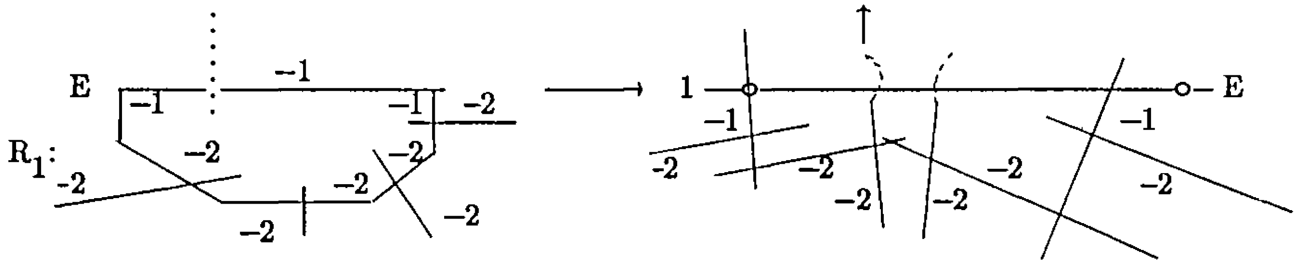
$$n_1 \leq 4 \quad \text{and} \quad n_2 \leq 4 .$$

Furthermore there exists such a variety Z_{II} with $\ell_1 + \ell_2 = 2$ for any $n_1 \leq 4$ and $n_2 \leq 4$.

Proof. The former easily follows from the Hodge index theorem. In order to prove the latter, it is enough to find R_1 with $\ell_1 = 1$ for any $n_1 \leq 4$. Here we only construct the rational surface R_1 with $\ell_1 = 1$ and $n_2 = 4$, because R_1 with $n_1 \leq 3$ is, as easily seen, obtained by suitable blowing–ups and blowing–downs of R_1 with $n_1 = 4$.

We start with $\mathbb{P}^2$, a non–singular cubic curve E and two lines L_1 and L_2 with the configuration as in the first picture below, and then proceed on the blowing–ups at the points indicated o in the following pictures.





(Here a dotted line means an exceptional curve of the first kind which does not appear explicitly in the configuration of curves on R_1 .) We used the same symbol E for the original cubic curve E and its proper transforms. Then note that the canonical bundle K of the surface at every stage satisfies $K = -E$.

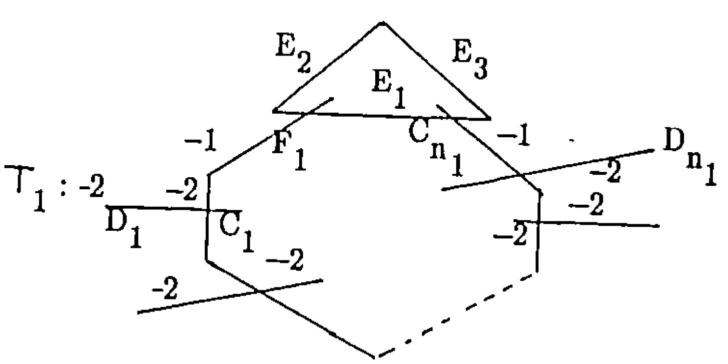
Thus we obtain the rational surface R_1 and therefore the variety $Z_{II} = R_1 + R_2$, which is desired.

Q.E.D.

Finally, we constructed a semi–stable degeneration of K3 surfaces which contains a (CB) – surface and a "variety of type III" (cf. Theorem 1.2) in the singular fiber.

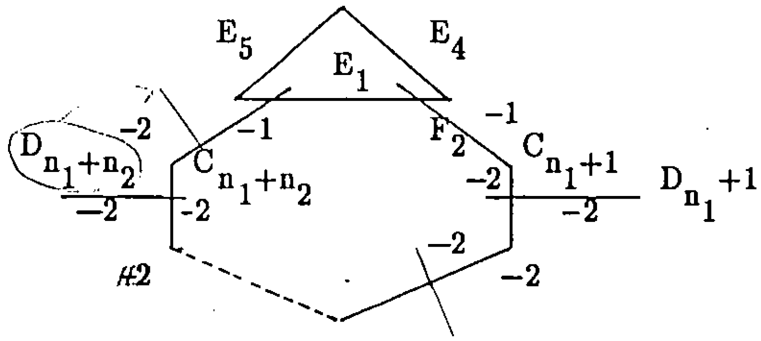
<u>Example</u> 3.8. Let S , A_i and $B_i (1 \leq i \leq m)$ be as in Example 3.1.

Let Z_{III} be a two–dimensional variety $Z_{III} = T_1 + T_2 + T_3 + T_4$ with only normal crossings; The T_j's $(1 \leq j \leq 4)$ are rational surfaces with the following configuration of $\mathbb{P}^1$'s :



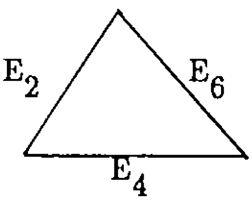
$$(E_1^2)_{T_1} = -\ell_1^{(1)}$$
$$(E_2^2)_{T_1} = -\ell_2^{(1)}$$
$$(E_3^2)_{T_1} = -\ell_3^{(1)}$$

T_2 :



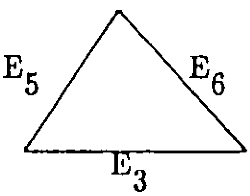
$$(E_1^2)_{T_2} = -\ell_1^{(2)}$$
$$(E_4^2)_{T_2} = -\ell_4^{(1)}$$
$$(E_5^2)_{T_2} = -\ell_5^{(1)}$$

T_3 :



$$(E_2^2)_{T_3} = -\ell_2^{(2)}$$
$$(E_4^2)_{T_3} = -\ell_4^{(2)}$$
$$(E_6^2)_{T_3} = -\ell_6^{(1)}$$

T_4 :



$$(E_3^2)_{T_4} = -\ell_3^{(2)}$$
$$(E_5^2)_{T_4} = -\ell_5^{(2)}$$
$$(E_6^2)_{T_4} = -\ell_6^{(2)}$$

and the T_j's $(1 \le j \le 4)$ meet one another transversally along the curves with the same name; F_1 and $C_{n_1+n_2}$ sprout from the same point on E_1 in Z_{III} ; So do F_2 and C_{n_1} ; The canonical bundles of T_i's are $K_{T_1} = -E_1 - E_2 - E_3$, $K_{T_2} = -E_1 - E_4 - E_5$, $K_{T_3} = -E_2 - E_4 - E_6$, $K_{T_4} = -E_3 - E_5 - E_6$.

Then we have

<u>Theorem</u> 3.9. The (CB) – surface S and the variety Z_{III} with $n_1 + n_2 = m$, $\ell_1^{(1)} + \ell_1^{(2)} = 4$ and $\ell_k^{(1)} + \ell_k^{(2)} = 2$ $(k = 2, \dots, 6)$ can be simultaneously sit in the singular fiber of a semi–stable degeneration of K3 surfaces.

Proof. As in the proof of Theorem 3.5, it is enough to describe each component which will be in the singular fiber. But components we need here are exactly the same as before. Namely, let V_i , U_i and $W_i (1 \leq i \leq m)$ be the rational surfaces in the proof of Theorem 3.5. Then, by gluing S , $T_j (1 \leq j \leq 4)$, V_i , U_i and $W_i (1 \leq i \leq m)$ along the corresponding curves, we construct a variety X_0 with only normal crossings. Due to the deformation theory, one obtains a semi–stable degeneration $\pi : X \longrightarrow \Delta$ of K3 surfaces whose singular fiber is isomorphic to the variety X_0 . Q.E.D.

Remark 3.10. (i) By the adjunction formula, the canonical bundle K_X of the above X is written as

$$K_X = 2S + \sum_{i=1}^{m} U_i + \sum_{i=1}^{m} V_i + \sum_{i=1}^{m} W_i .$$

(ii) As in Remarks 3.2 and 3.6 (ii), one also has a restriction on $m = n_1 + n_2$, i.e., there does not exist Z_{III} with large n_1 and n_2 . In fact, we have $m = n_1 + n_2 \leq 8$, again. More precisely,

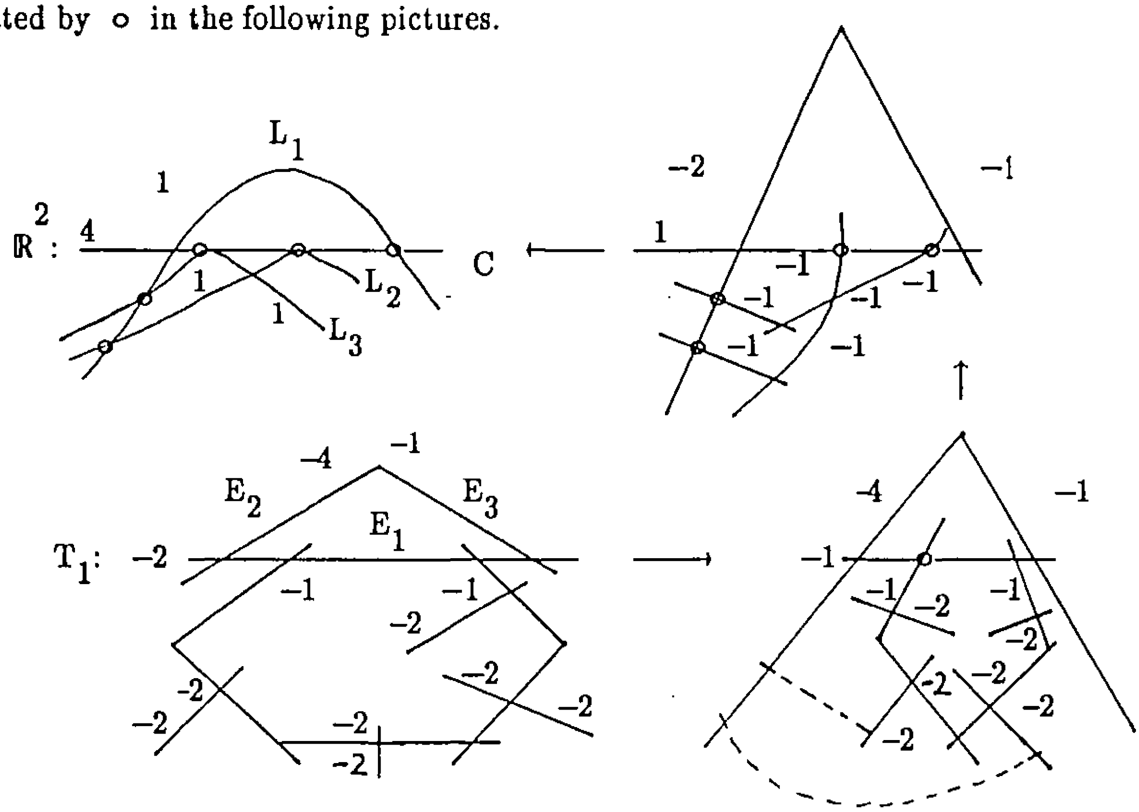
Proposition 3.11. Under the same notation, one must have

$$n_1 \leq 4 \quad \text{and} \quad n_2 \leq 4 .$$

Moreover, there exists such a variety Z_{III} with $\ell_1^{(1)} + \ell_1^{(2)} = 4$ and $\ell_k^{(1)} + \ell_k^{(2)} = 2\ (k = 2, \ldots, 6)$ for any $n_1 \leq 4$ and $n_2 \leq 4$.

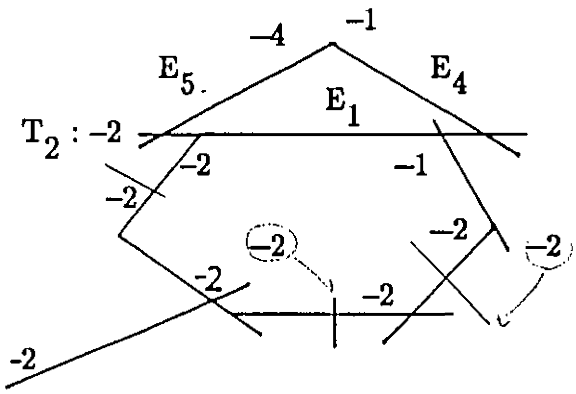
Proof. As in the proof of Proposition 3.7, the former part follows from the Hodge index theorem, and in order to prove the latter part, it is sufficient to give an example of Z_{III} for the maximal case $n_1 = n_2 = 4$:

To construct T_1 , we start with a projective plane $\mathbb{P}^2$, a non–singular conic curve C and three lines L_1 , L_2 , L_3 with the configuration as in the first picture below, where $K_{\mathbb{P}^2} = -C - L_1$, and then we proceed on the blowing–ups at the points indicated by o in the following pictures.



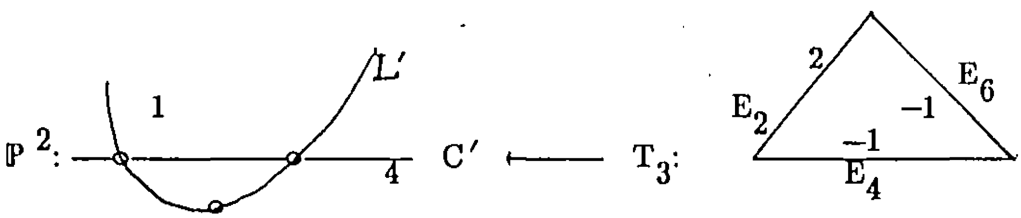
(For a dotted line, see the proof of Proposition 3.7.) Clearly the above T_1 has the canonical bundle $K_{T_1} = -E_1 - E_2 - E_3$ as desired. Thus one gets the rational surface T_1 .

Let T_2 be the copy of T_1 with the different names of curves as follows:



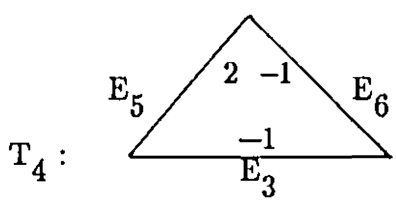
where $K_{T_2} = -E_1 - E_4 - E_5$.

Next we construct T_3 as follows: We take $\mathbb{P}^2$, a non–singular conic curve C' and a line L' , where $K_{\mathbb{P}^2} = -C' - L'$, and then blow up the $\mathbb{P}^2$ as follows.



Clearly $K_{T_3} = -E_2 - E_4 - E_6$, as desired. Thus the rational surface T_3 is obtained.

Let T_4 be the copy of T_3 with the different names of curves as follows:



where $K_{T_4} = -E_3 - E_5 - E_6$.

We put $Z_{III} = T_1 + T_2 + T_3 + T_4$, which is desired. Q.E.D.

§ 4.

In §§ 2 and 3, we obtained examples of non–algebraic degenerations of K3 surfaces, which seem to correspond to algebraic degenerations of types I, II, III. For such non–algebraic degenerations of K3 surfaces, we shall also define the types $\mathfrak{I}_0$, $\mathfrak{II}_0$, $\mathfrak{III}_0$ and the types $\mathfrak{I}_1$, $\mathfrak{II}_1$, $\mathfrak{III}_1$, where degenerations of the former three types contain a Hopf surface as examples in § 2 and ones of the latter three types contain a (CB)–surface as in § 3. To define those types, we must first make a preliminary

<u>Definition</u> 4.1. Let $\pi : X \longrightarrow \Delta$ be a semi–stable degneration of surfaces. Let $Y = V_1 + \dots + V_k$ be a set of some components in the singular fiber X_0 and $Z = V_{k+1} + \dots + V_\ell$ a set of other components than Y in X_0, i.e., $Y+Z = X_0$.

(1) Y is called a Hopf–set if

a) $Y \cap Z = E$, where E is a non–singular elliptic curve on V_k and V_{k+1},

b) Y has no triple point, and the double curves on each component consist of disjoint elliptic curves. The dual graph $\pi(Y)$ of Y is a tree,

c) Components which correspond to edge points in the tree $\pi(Y)$ and which are different from a component V_k are Hopf surfaces, and the other components are Hopf surfaces or relatively minimal elliptic ruled surfaces.

(2) Y is called a (CB)–set if

a) $Y \cap Z = D$, where D consists of a rational cycle and trees of $\mathbb{P}^1$'s sprouting from the rational cycle, (Note that D is not necessarily on a single component.)

b) Y contains just one (CB)–surface, and the other components of Y are rational surfaces,

c) The dual graph $\pi(Y)$ of Y is topologically contractible.

Now we can give the following

Definition 4.2. Let $\pi : X \longrightarrow \Delta$ be a semi–stable degeneration of K3 surfaces, and X_0 its singular fiber. Then we define types of $\pi : X \longrightarrow \Delta$ as follows:

I_0. $X_0 = Y + Z$, where Y consists of several (at least one) disjoint Hopf–sets and Z is a K3 surface or an elliptic surface with Kodaira dimension $\kappa = 1$ obtained from an elliptic K3 surface by a logarithmic transformation along a smooth fiber.

II_0. $X_0 = Y + Z$, where Y consists of several (at least one) disjoint Hopf–sets and Z is a two–dimensional variety with the configuration of normal crossing components which is same as that of the singular fiber in a degeneration of type II or II′ (see Theorems 1.2 or 1.5 respectively). We shall simply call such Z a variety of type II or II′.

III_0. $X_0 = Y + Z$, where Y consists of several (at least one) disjoint Hopf–sets and Z is a variety of type III or II′+III′ (see Theorems 1.2 or 1.5 respectively).

I_1. $X_0 = Y + Z$, where Y is a (CB)–set and Z is a K3 surface.

II_1. $X_0 = Y + Z$, where Y is a (CB)–set and Z is a variety of type II.

III_1. $X_0 = Y + Z$, where Y is a (CB)–set and Z is a variety of type III.

Remark 4.3. (i) An example of each type can be found as follows:

type I_0 with Z a K3 surface, in Example 2.1;

type I_0 with Z an elliptic surface of $\kappa = 1$, in Nishiguchi [5];

type II_0 with Z of type II, in Example 2.2;

type II_0 with Z of type II′, in Example 2.3;

type III_0 with Z of type III, in Example 2.4;

type III_0 with Z of type II′+III′, in Example 2.5;

type I_1, in Example 3.1;

type $\Upsilon\Upsilon_1$, in Example 3.4;

type $\Upsilon\Upsilon\Upsilon_1$, in Example 3.8 .

(ii) There exist degenerations of types Υ_0, $\Upsilon\Upsilon_0$, $\Upsilon\Upsilon\Upsilon_0$ with more than one Hopf–sets. One can easily construct such examples with minor change on Examples 2.1–2.5.

(iii) It is proposed that

(a) in the definitions of types Υ_1, $\Upsilon\Upsilon_1$ and $\Upsilon\Upsilon\Upsilon_1$, Y should not consist of more than one (CB)–sets;

(b) in the definition of type $\Upsilon\Upsilon\Upsilon_0$, Z should not be of type III′;

(c) in the definition of type $\Upsilon\Upsilon_1$, Z should not be of type II′;

(d) in the definition of type $\Upsilon\Upsilon\Upsilon_1$, Z should not by of type III′ or II′+III′.

These (a)–(d) would be relevant because of Conjecture 1.7′.

The previous paper [7] also gave several examples of semi–stable degenerations of K3 surfaces which contain, in the singular fiber, a (CB)–surface S belonging to certain series, especially every (CB)–surface containing a GSS with the second Betti number $b_2(S) \leq 5$. But those degenerations were all of type either Υ_1 or $\Upsilon\Upsilon_1$. Now we can construct degenerations of any type Υ_1, $\Upsilon\Upsilon_1$ or $\Upsilon\Upsilon\Upsilon_1$ for each of such (CB)–surfaces S above. The constructions are very similar to those in § 3 and [7, § 5], so left to the reader. In particular, one can prove the following theorem which is a generalization of Corollary 5.8 in [7]:

Theorem 4.4. Let S be a (CB)–surface containing a GSS with $b_2(S) \leq 5$. Then S can be made a component in the singular fiber of a semi–stable degeneration of K3 surfaces of any type Υ_1, $\Upsilon\Upsilon_1$ or $\Upsilon\Upsilon\Upsilon_1$.

We can now raise the following conjecture about the structure of a semi–stable degeneration of K3 surfaces in the cases (2) and (3) of Theorem 1.4. The conjecture would be a working hypothesis in the study of non–algebraic degenerations of K3 surfaces.

Conjecture 4.5. Let $\pi : X \longrightarrow \Delta$ be a semi–stable degeneration of K3 surfaces. Then $\pi : X \longrightarrow \Delta$ has no semi–stable modification with trivial canonical bundle if and only if $\pi : X \longrightarrow \Delta$ has a modification of type I_0, II_0, III_0, I_1, II_1 or III_1 .

Remark 4.6. "If" part in Conjecture 4.5 is easily verified by virtue of the proof of Theorem 1.1 in Nishiguchi [7].

Next we also have only a conjecture about the monodromy for a degeneration of

each type in Definition 4.2. The conjecture says that the analogue to Theorem 1.3 and Conjecture 1.9 would be true for a semi–stable degeneration of K3 surfaces which has no semi–stable modification with trivial canonical bundle.

Conjecture 4.7. Let $\pi : X \longrightarrow \Delta$ be a semi–stable degeneration of K3 surfaces, and N as in Theorem 1.3. Then

(i) if π is of type I_i $(i = 0,1)$, then $N = 0$;

(ii) if π is of type II_i $(i = 0,1)$, then $N^2 = 0, N \neq 0$;

(iii) if π is of type III_i $(i = 0,1)$, then $N^3 = 0$, $N^2 \neq 0$.

Finally, by summing up Conjectures 1.7′, 1.9, 4.5, 4.7 and Theorems 1.1–1.5, one obtains

<u>Conjecture</u> 4.8. Let $\pi : X \longrightarrow \Delta$ be a semi–stable degeneration of K3 surfaces. Then π has a modification of one of the following types:

I (cf. 1.2), Υ_0 (4.2), Υ_1 (4.2), ... C_I

II (1.2), II$'$(1.5), $\Upsilon\Upsilon_0$ (4.2), $\Upsilon\Upsilon_0$ (4.2), ... C_{II}

III (1.2), III$'$ (1.6), II$'$+III$'$ (1.6), $\Upsilon\Upsilon\Upsilon_0$ (4.2), $\Upsilon\Upsilon\Upsilon_1$ (4.2). .... C_{III}

(Here we divided the types I,....,$\Upsilon\Upsilon\Upsilon_1$ into three classes C_I, C_{II} and C_{III} as above.) Moreover, let $N = \log P$, where P is the Picard–Lefschetz transformation for π. Then we have

(i) if π is of a type in C_I, then $N = 0$;

(ii) if π is of a type in C_{II}, then $N^2 = 0, N \neq 0$;

(iii) if π is of a type in C_{III}, then $N^3 = 0$, $N^2 \neq 0$.

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