QUASICOHERENT SHEAVES IN COMMUTATIVE AND NONCOMMUTATIVE GEOMETRY.

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Intorduction

In this paper we try to survey a category of objects of noncommutative algebraic geometry. In the commutative case the notion of scheme is introduced in two steps. The first step is the definition of affine schemes, and the second one prescribes how to glue a scheme from affine parts. This way to proceed appeal to the topological properties of commuative schemes. It uses the functor $Spec$ from the category of commutative algebras to the category of topological spaces. On the other hand, any scheme can be considered as a presheaf of sets on the category of affine schemes. We will exploit this simple fact in our treatment of the noncomutative case.

Similar to commutative algebraic geometry, the category of *affine* noncommutative schemes is opposite to the category of algebras (we work over a basic ring k). Any object which can be regarded as a noncommutative scheme has to give rise to presheaf of sets on the category of affine objects. In this paper we consider all presheaves of sets on the categories of affine schemes both in the commutative and in noncomutative cases. To each presheaf X we associate a category $Qcoh_r(X)$ of (right) quasicoherent modules on it (2.6). Moreover we prove that any quasicoherent module is a sheaf in the canonical topology on X (3.5 and 4.9). In the commutative case we define a new topology on the category of affine schemes that is related to quasicoherent modules (3.12). The sheafification functor for this topology, (called here effective descent topology) preserves categories of quasicoherent modules (3.13).

Further, we find a "good" embeddding of the commutative case into the noncomutative one, using presheaves of groupoids on the category of noncommutative affine schemes (4.4). This embedding induces equivalence for categories of quasicoherent modules on any object X, when we consider it both in the commutative and in noncomutative worlds (4.7).

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1. Some categorical preliminaries. Sites.

1.1 We collect here some facts related to categories and sites. The main reference is [1] (see also [5]). Let $\mathcal{C}$ be a category. Denote by $\mathcal{C}^{\frown}$ the category of contravariant functors from $\mathcal{C}$ to the category of sets. An object of $\mathcal{C}^{\frown}$ is called a presheaf of sets. There is a natural functor $h : \mathcal{C} \longrightarrow \mathcal{C}^{\frown}$

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that sends an object $R \in \mathcal{C}$ to the functor $h_R(\cdot) := \mathrm{Hom}(\cdot, R)$. The presheaf h_R is called representable. Henceforce we will write R instead h_R . By Ioneda Lemma h is a full embedding, moreover for any presheaf $X \in C^{\frown}$ there is the natural isomorphism $\mathrm{Hom}_{\mathcal{C}^{\frown}}(R, X) = X(R)$.

For any object $X \in \mathcal{C}^{\frown}$ denote by $\mathcal{C}/X$ the category of pairs (R, ϕ) , where $R \in \mathcal{C}$ and $\phi \in X(R)$ with a canonical functor $j_X : \mathcal{C}/X \longrightarrow \mathcal{C}$ that sends (R, ϕ) to R . Note that any presheaf $X \in \mathcal{C}^{\frown}$ is a colimit in $\mathcal{C}^{\frown}$ of representable objects by the sistem $\mathcal{C}/X$.

1.2 Let $u : \mathcal{C} \to \mathcal{C}'$ be a functor. Denote by $\hat{u}^*$ a functor from $\mathcal{C}'^{\frown}$ to $\mathcal{C}^{\frown}$ that takes a presheaf $Y \in \mathcal{C}'^{\frown}$ to the composition Yu .

Since the category of sets has all small limits and colimits the functor $\hat{u}^*$ admits left and right adjoint functors $\hat{u}_!$ and $\hat{u}_*$ respectively (see [1], I, 5.1.). The functor $\hat{u}^*$ preserves all small limits and colimits. The functor $\hat{u}_*$ preserves all small limits. The functor $\hat{u}_!$ preserves all small colimits and the restriction of it on the subcategory $\mathcal{C} \subset \mathcal{C}^{\frown}$ coincides with u . Moreover for any presheaf $X \in \mathcal{C}^{\frown}$ we have

$$u_! X(S) \cong \varinjlim_{R \in C/X} \mathrm{Hom}(S, uR).$$

Furthermore, the functor $\hat{u}_!$ is fully faithful if and only if the functor u is fully faithful(see [1], I.5.6) (We recall that a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is called fully faithful if it induces an isomorphism $\mathrm{Hom}_{\mathcal{C}}(Y, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}'}(FY, FX)$ for any pair Y, X of objects from $\mathcal{C}$.)

1.3 A sieve in the category $\mathcal{C}$ is a full subcategory $\mathcal{D} \subset \mathcal{C}$ that satisfies the following condition: any object of $\mathcal{C}$ for which there exists a morphism from it to an object of $\mathcal{D}$ contains in $\mathcal{D}$. Let R be an object, a sieve in the category $\mathcal{C}/R$ will be called a sieve on R .

It is easy to see that a sieve on R is nothing more than a subpresheaves of R in the category $\mathcal{C}^{\frown}$. Henceforce, we will consider sieve on R as a presheaf. The notion of sieve is the main tool for the introduction of a topology on a category.

1.4 Definition. *A* Grothendieck topology $\mathcal{T}$ *on a category* $\mathcal{C}$ *is defined by giving for each object* $R \in \mathcal{C}$ *a set* $J(R)$ *of sieves on* R *which is called covering sieves. They have to satisfy the following conditions:*

T1) For any R *the maximal sieve* $\mathcal{C}/R$ *is in* $J(R)$ *.*
T2) If $T \in J(R)$ *and* $f : S \longrightarrow R$ *is a morphism of* $\mathcal{C}$ *, then the sieve* $f^*(T) := \left\{U \xrightarrow{\alpha} S \mid f\alpha \in T\right\}$ *is in* $J(S)$ *.*
T3) If $T \in J(R)$ *is a covering sieve and* U *is a sieve on* R *such that* $f^*(U) \in J(S)$ *for each* $S \xrightarrow{f} R$ *in* T *, then* $U \in J(R)$ *.*

1.5 Definition. *A* site $\Phi = (\mathcal{C}, \mathcal{T})$ *is a category* $\mathcal{C}$ *with a Grothendieck topology* $\mathcal{T}$ *.*

Let $\Phi = (\mathcal{C}, \mathcal{T})$ be a site, and R be an object of $\mathcal{C}$. A family of morphisms $F = (f_\alpha : R_\alpha \longrightarrow R),\ \alpha \in I$ is called covering if the sieve $T_F \hookrightarrow R$, which is generated by the family F , is a covering seive on R .

1.6 Definition. *Let* $\mathcal{C}$ *be a category with fiber products. A* pretopology *on* $\mathcal{C}$ *is defined by specifying for each object* R *of* $\mathcal{C}$ *, a set* $Cov(R)$ *of families of morphisms to* R *such that*

P1) If $(R_\alpha \to R)\ \alpha \in I$ *is in* $Cov(R)$ *and* $S \to R$ *is a morphism of* $\mathcal{C}$ *, then the family* $(R_\alpha \times S \to S),\ \alpha \in I$ *is in* $Cov(S)$ *.*

P2) If $(R_\alpha \to R)\ \alpha \in I$ *is in* $Cov(R)$ *and* $(R_{\beta_\alpha} \to R_\alpha)\ \beta_\alpha \in J_\alpha$ *is in* $Cov(R_\alpha)$ *for each* $\alpha \in I$ *, then the family* $(R_\gamma \to R)\ \gamma \in \coprod_{\alpha \in I} J_\alpha$ *is in* $Cov(R)$ *.*

P3) The family $R \xrightarrow{id_R} R$ *is in* $Cov(R)$ *.*

Any pretopology P on $\mathcal{C}$ generates a topology $\mathcal{T}$ such that a sieve is $\mathcal{T}$ -covering iff it contains some P -covering family.

1.7 A presheaf $X \in \mathcal{C}^\wedge$ is called sheaf (resp. separate presheaf) if for any object $R \in \mathcal{C}$ and for any covering sieve $S \in J(R)$ the canonical morphism

$$\mathrm{Hom}_{\mathcal{C}^\wedge}(R, X) \longrightarrow \mathrm{Hom}_{\mathcal{C}^\wedge}(S, X)$$

is a bijection (resp. an injection).

Let us define by $Shv(\mathcal{C}, \mathcal{T})$ the category of all sheaves on a site $\Phi = (\mathcal{C}, \mathcal{T})$. The category $Shv(\mathcal{C}, \mathcal{T})$ is a full subcategory of $\mathcal{C}^\wedge$.

The inclusion functor $i : Shv(\mathcal{C}, \mathcal{T}) \longrightarrow \mathcal{C}^\wedge$ has a left adjoint functor $\underline{a} : \mathcal{C}^\wedge \longrightarrow Shv(\mathcal{C}, \mathcal{T})$ that is known as the associated sheaf functor, or the sheafification functor. The functor $\underline{a}$ preserves all finite limits and colimits. For any presheaf X we can define a presheaf LX by the rule

$$LX(R) = \varinjlim_{T \in J(R)} \mathrm{Hom}_{\mathcal{C}^\wedge}(T, X) \tag{1}$$

This correspondence gives a functor from $\mathcal{C}^\wedge$ to itself which will be denoted by L . The composition of the functor L with itself is isomorphic to the functor $i\underline{a}$, i.e. we obtain an isomorphism $\underline{a}X \cong LLX$. (see [1], III)

1.8 Let $\Phi = (\mathcal{C}, \mathcal{T})$ and $\Phi' = (\mathcal{C}', \mathcal{T}')$ be two sites.

A functor $u : \mathcal{C} \longrightarrow \mathcal{C}'$ is called continue if for any sheaf $Y \in Shv(\mathcal{C}', \mathcal{T}')$ the presheaf $\hat{u}^*(Y)$ is a sheaf on site Φ . In this case $\hat{u}^*$ induces a functor from $Shv(\mathcal{C}', \mathcal{T}')$ to $Shv(\mathcal{C}, \mathcal{T})$, which will be denoted by u^* . The functor u^* has a left adjoint functor $u_! = \underline{a}'\hat{u}_! i$.

A functor $u : \mathcal{C} \longrightarrow \mathcal{C}'$ is called cocontinue if for any object $R \in \mathcal{C}$ and for any covering sieve $S \hookrightarrow u(R)$ the sieve on R , which is generated by all arrows $Z \longrightarrow R$ such that $u(Z) \longrightarrow u(R)$ factorized through S , is a covering sieve on R . Note that u is cocontinue iff the functor $\hat{u}_* : \mathcal{C}^\wedge \longrightarrow \mathcal{C}'^\wedge$ takes any sheaf on Φ to a sheaf on Φ' . Denote by $u_* : Shv(\mathcal{C}, \mathcal{T}) \longrightarrow Shv(\mathcal{C}', \mathcal{T}')$ the restriction of $\hat{u}_*$ on the categories of sheaves. It has a left adjoint functor $u^* = \underline{a}\hat{u}^* i'$ (see [1], III).

Let $u : \mathcal{C}' \longrightarrow \mathcal{C}$ be a functor and $\mathcal{T}$ is a topology on $\mathcal{C}$. The induced topology on $\mathcal{C}$ is the largest topology such that the functor u is continue.

1.9 Fix an object $X \in \mathcal{C}^\wedge$ and consider the category $\mathcal{C}/X$ with the canonical functor $j_X : \mathcal{C}/X \longrightarrow \mathcal{C}$. If $\mathcal{T}$ is a topology on $\mathcal{C}$, then denote by $\mathcal{T}_X$ the induced topology on $\mathcal{C}/X$ and by Φ_X denote the site $(\mathcal{C}/X, \mathcal{T}_X)$. It is proved in [1], III.5. that a seive S of an object $(Z \longrightarrow X)$ is covering in $\mathcal{C}/X$ iff the sieve $\widehat{j_{X!}}(R) \hookrightarrow Z$ is covering in $\mathcal{C}$. Therefore, the functor $j_X : \mathcal{C}/X \longrightarrow \mathcal{C}$ is continue and cocontinue. This gives us three adjoint functors:

$$\begin{array}{rlcl} j_{X*} : & Shv(\mathcal{C}/X, \mathcal{T}_X) & \longrightarrow & Shv(\mathcal{C}, \mathcal{T}) \\ j_X^* : & Shv(\mathcal{C}, \mathcal{T}) & \longrightarrow & Shv(\mathcal{C}/X, \mathcal{T}_X) \\ j_{X!} : & Shv(\mathcal{C}/X, \mathcal{T}_X) & \longrightarrow & Shv(\mathcal{C}, \mathcal{T}) \end{array} \tag{2}$$

The functor j_{X*} is called the direct image functor. The functor j_X^* is called the inverse image functor, or restriction functor to $\mathcal{C}/X$. The functor $j_{X!}$ is called extension by zero onto $\mathcal{C}$.

Let $m : Y \longrightarrow X$ be a morphism in $\mathcal{C}^\frown$. It induces a functor $j_m : \mathcal{C}/Y \longrightarrow \mathcal{C}/X$. As above, there are three functors

$$j_m^* : Shv(\mathcal{C}/X, \mathcal{T}_X) \longrightarrow Shv(\mathcal{C}/Y, \mathcal{T}_Y), \qquad j_{m*}, j_{m!} : Shv(\mathcal{C}/Y, \mathcal{T}_Y) \longrightarrow Shv(\mathcal{C}/X, \mathcal{T}_X)$$

such that j_{m*} and $j_{m!}$ are right and left adjoint to j_m^* respectively.

1.10 Proposition. [1] *The functor $j_{X!} : Shv(\mathcal{C}/X, \mathcal{T}_X) \longrightarrow Shv(\mathcal{C}, \mathcal{T})$ has the following decomposition*

$$Shv(\mathcal{C}/X, \mathcal{T}_X) \xrightarrow{e_X} Shv(\mathcal{C}, \mathcal{T})/\underline{a}X \longrightarrow Shv(\mathcal{C}, \mathcal{T}),$$

and the functor e_X is an equivalence.

Moreover, the canonical sheafification map $m : X \longrightarrow \underline{a}X$ for any presheaf $X \in \mathcal{C}^\frown$ induces the equivalences j_m^* and $j_{m*} = j_{m!}$ between the categories $Shv(\mathcal{C}/X, \mathcal{T}_X)$ and $Shv(\mathcal{C}/\underline{a}X, \mathcal{T}_{\underline{a}X})$.

1.11 A topology, for which every representable presheaf is a sheaf, is called subcanonical; the largest such topology is called canonical. The canonical topology will be denoted by can.

Let us describe covering families for the canonical topology.

We need to give the definition of equalizer. If $f_i : X \longrightarrow Y, i = 1, 2$ are two morphisms of a category $\mathcal{C}$, the an equalizer is a morphism $k : K \to X$ which is final in a full subcategory of $\mathcal{C}/X$ consisting of morphisms $k' : K' \to X$ with $f_1 k' = f_2 k'$. When it exists, the equalizer of two morphisms is unique up to isomorphism.

Suppose that all fiber products exist in a category $\mathcal{C}$. A family $(X_\alpha \to X)_{\alpha \in I}$ of maps in the category $\mathcal{C}$ is called a universal effectively epimorphic family if for all maps $Z \longrightarrow X$ and for all objects $Y \in \mathcal{C}$ the following diagram of sets

$$\mathrm{Hom}(Z, Y) \longrightarrow \prod_{\alpha \in I} \mathrm{Hom}(Z \times_X X_\alpha, Y) \rightrightarrows \prod_{\alpha, \beta \in I} \mathrm{Hom}(Z \times_X (X_\alpha \times_X X_\beta), Y)$$

is an equalizer.

It is clear that a family is covering for the canonical topology if and only if it is a universal effectively epimorphic family. If now $X \in \mathcal{C}$ is some object then a family $(W_\alpha \longrightarrow W)_{\alpha \in I}$ is covering for the object $(W \to X) \in \mathcal{C}/X$ in the canonical topology on $\mathcal{C}/X$ iff it is a universal effectively epimorphic family in $\mathcal{C}$.

1.12 Proposition. *Let X be a presheaf on the category $\mathcal{C}$. Then the topology on $\mathcal{C}/X$ induced by the canonical topology on $\mathcal{C}$ with respect to the functor $j_X : \mathcal{C}/X \longrightarrow \mathcal{C}$ coincides with the canonical topology on $\mathcal{C}/X$.*

It follows from Corollary 3.3. of [1] and from the description of covering families for the canonical topologies on $\mathcal{C}$ and $\mathcal{C}/X$.

2. Ringed sites and quasicoherent sheaves

2.1 A category $\mathcal{C}$ with a presheaf of rings is called a ringed category, i.e. we fix an object $\mathcal{A}$ of the category $\mathcal{C}^\frown$ with ring structure. A site $\Phi = (\mathcal{C}, \mathcal{T})$ with a sheaf of rings $\mathcal{A}$ is called a ringed site.

2.2 Let $\Phi = (\mathcal{C}, \mathcal{T})$ be a ringed site with a sheaf of rings $\mathcal{A}$. A presheaf of (right) $\mathcal{A}$ –modules is a presheaf $\mathcal{M}$ with a (right) module structure over ring object $\mathcal{A}$.

If $\mathcal{M}$ is a sheaf it will be called a sheaf of (right) $\mathcal{A}$ –modules.

Denote by $\mathrm{Mod}_r(\mathcal{A}, \mathcal{T})$ the category of sheaves of (right) $\mathcal{A}$ –modules on the site Φ . The category $\mathrm{Mod}_r(\mathcal{A}, \mathcal{T})$ is an abelian category that verifies the conditions AB5 and AB3* and possesses a family of generators (see [1], II).

2.3 Let X be an object of $\mathcal{C}^\frown$. The category $\mathcal{C}/X$ with the topology $\mathcal{T}_X$, induced by the functor $j_X : \mathcal{C}/X \longrightarrow \mathcal{C}$, forms the site $\Phi_X = (\mathcal{C}/X, \mathcal{T}_X)$. This site is ringed by the sheaf of rings $\mathcal{A}_X := j_X^* \mathcal{A}$. We will denote the category of sheaves of (right) $\mathcal{A}_X$ –modules by $\mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X)$ (or by $\mathrm{Mod}_r(X, \mathcal{T}_X)$ when the ring sheaf is fixed).

Let $\mathcal{M}$ be a sheaf of (right) $\mathcal{A}$ –modules. The sheaf $j_X^* \mathcal{M}$ has the structure of (right) $\mathcal{A}_X$ –module. Furhter, if $\mathcal{N}$ is a sheaf of (right) $\mathcal{A}_X$ –modules then $j_{X*}\mathcal{N}$ is a sheaf of (right) $j_{X*}\mathcal{A}_X$ –modules. Using the canonical morphism $\mathcal{A} \longrightarrow j_{X*}\mathcal{A}$, we obtain that $j_{X*}\mathcal{N}$ has the structure of (right) $\mathcal{A}$ –module. Thus there is a couple of adjoint functors:

$$\mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X) \quad \underset{j_{X*}}{\overset{j_X^*}{\rightleftarrows}} \quad \mathrm{Mod}_r(\mathcal{A}, \mathcal{T})$$

(for functors between categories of sheaves of modules we use the same notation as in 1.9 for functors between categories of sheaves of sets.)

The functor $j_X^* : \mathrm{Mod}_r(\mathcal{A}, \mathcal{T}) \longrightarrow \mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X)$ has a left adjoint functor $j_{X!}$ that is called the extension by zero. Note that, unlike the functors j_X^* and j_{X*} , the composition of the functor $j_{X!}$ for sheaves of modules with the forgetful functor to the category of sheaves of sets does not coincide with the functor $j_{X!}$ for sheaves of sets defined in 1.9 (see [1], IV.11.)

Let $m : Y \longrightarrow X$ be a morphism in $\mathcal{C}^\frown$. As above, the functor $j_m : \mathcal{C}/Y \longrightarrow \mathcal{C}/X$ induces three functors

$$j_m^* : \mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X) \longrightarrow \mathrm{Mod}_r(\mathcal{A}_Y, \mathcal{T}_Y), \qquad j_{m*}, j_{m!} : \mathrm{Mod}_r(\mathcal{A}_Y, \mathcal{T}_Y) \longrightarrow \mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X)$$

such that j_{m*} and $j_{m!}$ are right and left adjoint to j_m^* . Moreover the functors j_m^* and j_{m*} commute with the same functors between the categories of sheaves of sets defined in 1.9 with respect to forgetful functors.

2.4 Proposition. *Let $\Phi = (\mathcal{C}, \mathcal{T})$ be a site ringed by a sheaf $\mathcal{A}$. Suppose $m : X \longrightarrow \underline{a}X$ is the canonical sheafification map for a presheaf $X \in \mathcal{C}^\frown$. Then the functors j_m^* and j_{m*} between the categories $\mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X)$ and $\mathrm{Mod}_r(\mathcal{A}_{\underline{a}X}, \mathcal{T}_{\underline{a}X})$ are quasiinverse equivalences.*

It follows from Proposition 1.10.

2.5 Assume that a presheaf X is representable, i.e. $X \in \mathcal{C}$. Let M be a right $\mathcal{A}(X)$ –module. One can attach to it a presheaf of (right) $\mathcal{A}_X$ –modules $\widetilde{M}$ on the category $\mathcal{C}/X$. It is defined as

$$\widetilde{M}(S) := M \otimes_{\mathcal{A}(X)} \mathcal{A}(S).$$

for any object $S \to X$ of $\mathcal{C}/X$.

Fix an object $X \in \mathcal{C}^\frown$ and consider the category $\mathcal{C}/X$.

2.6 Definition. *A presheaf $\mathcal{F}$ of (right) $\mathcal{A}_X$ –modules on the category $\mathcal{C}/X$ will be called a (right)* quasicoherent module *on X if for any $m \in X(R)$ the presheaf $\widehat{j_m}^*(\mathcal{F})$ on $\mathcal{C}/R$ with the canonical structure of (right) $\mathcal{A}_R$ –module has a form $\widetilde{M}$, where M is a (right) $\mathcal{A}(R)$ –module.*

Morphisms between quasicoherent modules are , by definition, morphisms between them considered as presheaves of (right) $\mathcal{A}_X$ –modules. Denoted by $Qcoh_r(X)$ the category of (right) quasicoherent modules on X , which is a full subcategory of the category of presheaves of (right) $\mathcal{A}_X$ –modules.

Note that the definition of quasicoherent module on $X \in \mathcal{C}^\wedge$ does not depend from the topology on $\mathcal{C}$

2.7 Remark It follows immediately from the definition that each quasicoherent module on $R \in \mathcal{C}$ is isomorphic to $\widetilde{M}$ for some $\mathcal{A}(R)$ –module M , i.e the category $Qcoh_r(R)$ is equivalent to the category of (right) $\mathcal{A}(R)$ –modules. Hence a presheaf $\mathcal{F}$ of (right) $\mathcal{A}_X$ –modules is a quasicoherent module on $X \in \mathcal{C}^\wedge$ iff for each $(R \xrightarrow{s} X) \in \mathcal{C}/X$ the presheaf $\widehat{j_s}^*\mathcal{F}$ is a quasicoherent module on R .

2.8 The category $Qcoh_r(X)$ for any $X \in \mathcal{C}^\wedge$ has all cokernels and all direct sums. Hence it has all small colimits (see, for example, [4] I.5).

Let $m : Y \longrightarrow X$ be a morphism in $\mathcal{C}^\wedge$. The functor $\widehat{j_m}^* : (\mathcal{C}/X)^\wedge \longrightarrow (\mathcal{C}/Y)^\wedge$ induces the inverse image functor from $Qcoh_r(X)$ to $Qcoh_r(Y)$ which will be denoted by m^* . The functor m^* is right exact, this means that it preserves cokernels.

2.9 Assume now that the topology $\mathcal{T}$ on $\mathcal{C}$ satisfies the following condition: every quasicoherent module on any object $X \in \mathcal{C}^\wedge$ is a sheaf for the topology $\mathcal{T}_X$ on $\mathcal{C}/X$. In this case the category $Qcoh(X)$ is a full subcategory of $\mathrm{Mod}(\mathcal{A}_X, \mathcal{T}_X)$. Denote by φ_X the inclusion $Qcoh(X) \hookrightarrow \mathrm{Mod}(\mathcal{A}_X, \mathcal{T}_X)$.

2.10 Proposition. *Let $\Phi = (\mathcal{C}, \mathcal{T})$ be a site ringed by a sheaf $\mathcal{A}$. Suppose that all quasicoherent modules on any object X are sheaves for the topology $\mathcal{T}_X$ on $\mathcal{C}/X$. Let $m : X \longrightarrow \underline{a}X$ be the canonical sheafification map for a presheaf $X \in \mathcal{C}^\wedge$. Then the inverse image functor $m^* : Qcoh(\underline{a}X) \longrightarrow Qcoh(X)$ is fully faithful.*

Proof. There is the commutative diagram of functors

$$\begin{array}{ccc} Qcoh(X) & \xhookrightarrow{\varphi_X} & \mathrm{Mod}(\mathcal{O}_X, can_X) \\ {\scriptstyle m^*}\uparrow & & \uparrow{\scriptstyle j_m^*} \\ Qcoh(\underline{c}X) & \xhookrightarrow{\varphi_{\underline{c}X}} & \mathrm{Mod}(\mathcal{O}_{\underline{c}X}, can_{\underline{c}X}) \end{array}$$

where the horisontal arrows are full embeddings. By the Proposition 2.4, the functor j_m^* is an equivalence. Hence the functor m^* is fully faithful. □

2.11 Suppose that the inclusion functor $\varphi_X : Qcoh_r(X) \longrightarrow \mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X)$ has a right adjoint functor Q_X . The functor Q_X is called coherator.

Since φ_X is fully faithful the composition $Q_X\varphi_X$ is isomorphic to the identity functor on $Qcoh_r(X)$. If $\mathcal{F} \xrightarrow{f} \mathcal{G}$ is a morphism in $Qcoh_r(X)$ then it has a kernel $Ker(f)$ which can be defined as $Q_X Ker(\varphi_X(f))$.

It is easy to check that if coherator exists then the category $Qcoh_r(X)$ is abelian.

For a representable object $X \in \mathcal{C}$, the coherator Q_X exists and it is clearly the functor sending a sheaf $\mathcal{F}$ of $\mathcal{A}_X$ –modules to the quasicoherent module $\widetilde{\mathcal{F}(X)}$.

Let $m : Y \longrightarrow X$ be a morphism in $\mathcal{C}^\frown$. If there is a coherator Q_X then the functor m^* has a right adjoint functor m_* which is defined as the composition $Q_X j_{m*} \varphi_Y$.

2.12 To any presheaf $X \in \mathcal{C}^\frown$ we associate the category of (right) quasicoherent modules $Qcoh_r(X)$. This construction can be extended on a larger class of object by two ways.

First, let us consider an immersiom $i : Y \hookrightarrow X$. One can define a category of pair $Qcoh_r(X, Y)$ as a full subcategory of $Qcoh_r(X)$ consisting of such modules $\mathcal{F}$ for which $i^*(\mathcal{F}) \equiv 0$.

Any immersion $i : Y \hookrightarrow X$ induces the factor object X/Y . It is a pointed object and it is defined as a coproduct

$$\begin{array}{ccc} Y & \overset{i}{\hookrightarrow} & X \\ \downarrow & & \downarrow p \\ * & \hookrightarrow & X/Y \end{array}$$

We can compare the categories $Qcoh_r(X/Y, *)$ and $Qcoh_r(X, Y)$.

2.13 Proposition. *A natural functor* $p^* : Qcoh(X/Y, *) \longrightarrow Qcoh(X, Y)$ *is an equivalence.*

Proof. Let us construct a quasiinverse functor $p_* : Qcoh_r(X, Y) \longrightarrow Qcoh_r(X/Y, *)$. For any $R \in \mathcal{C}$ the set $X/Y(R) = X(R)/Y(R)$. Let $s \in X/Y(R)$ be a section. Eaither there is a unique pull back of s to the section $s' \in X(R)$ or it goes through $* \to X/Y$. If now $\mathcal{F}$ is a quasicoherent module on X such that $i^*(\mathcal{F}) \equiv 0$, then we put $p_*(\mathcal{F})(s) \cong \mathcal{F}(s')$ or put it by zero if it is pointed section. It is easy to see that this define a functor that is quasiinverse to p^* . □

2.14 Second, we can consider not only presheaves of sets but , say, presheaves of groupoids (We will not discuss more general case of presheaves of categories). By groupoid we mean a small category in which every arrow is invertible. All small groupoids generated the 2–category. If we identify all functors between groupoids which are equivalent then we obtain the 1–category. Denote this 1–category of all small groupoids by Grd . Any set can be considered as a groupoid, objects of which are elements of this set and morphisms are only identities. We can attach to each groupoid G two sets: the first $Ob(G)$ is the set of all objects of G , the second $\pi_0(G)$ is the seT of connected components of G . There are two natural morphisms $Ob(G) \longrightarrow G$ and $\pi_0 : G \longrightarrow \pi_0(G)$.

Let $\mathcal{C}^\frown_{gr}$ be the category of presheaves of groupoids on $\mathcal{C}$, i.e the category of all contravariant functors from $\mathcal{C}$ to the category of small groupoids Grd . It contains the category $\mathcal{C}^\frown$ as a full subcategory. Any presheaf $X \in \mathcal{C}^\frown_{gr}$ induces two presheves of sets $Ob(X)$ and $\pi_0(X)$ defined as

$$Ob(X)(R) = Ob(X(R)), \quad \pi_0(X)(R) = \pi_0(X(R)), \qquad R \in \mathcal{C}$$

There are natural morphisms $ob_X : Ob(X) \longrightarrow X$ and $\pi_{0X} : X \longrightarrow \pi_0(X)$ between the presheaves of groupoids. It is easy to see that these correspondences gives two functors Ob and π_0 from $\mathcal{C}^\frown_{gr}$ to $\mathcal{C}^\frown$.

The morphism $\pi_{0X} : X \longrightarrow \pi_0(X)$ splits for any $X \in \mathcal{C}^\frown_{gr}$.(Actually, choosing for each connected component some representative, we obtain a splitting). The functor $\pi_0 : \mathcal{C}^\frown_{gr} \longrightarrow \mathcal{C}^\frown$ is simultaneously right and left adjoint to the canonical immersion $\mathcal{C}^\frown \longrightarrow \mathcal{C}^\frown_{gr}$.

If $\Phi = (\mathcal{C}, \mathcal{T})$ is a site then it is defined the category of sheaves of groupoids $Shv_{gr}(\mathcal{C}, \mathcal{T})$.

We say that the presheaf $X \in \mathcal{C}^\frown_{gr}$ is a sheaf of groupoids if for any groupoid $G \in Grd$ the presheaf of sets:

$$R \mapsto \mathrm{Hom}_{Grd}(G, X(R))$$

is a sheaf.

2.15 Suppose now that site $\Phi = (\mathcal{C}, \mathcal{T})$ is ringed by a sheaf $\mathcal{A}$. For any object $X \in \mathcal{C}^{\frown}_{gr}$ let us define a category $\mathcal{C}/X$. The set of objects of $\mathcal{C}/X$ is the set of all $s \in Ob(X(S)),\ S \in \mathcal{C}$. A morphism from $s \in Ob(X(S))$ to $r \in Ob(X(R))$ is a pair (α, f), where $f : S \longrightarrow R$ in $\mathcal{C}$ and α is an isomorphism from $X(f)(s)$ to r in $X(R)$. There is a canonical functor $j_X : \mathcal{C}/X \longrightarrow \mathcal{C}$. This gives us the site $\Phi_X = (\mathcal{C}/X, \mathcal{T}_X)$ that is ringed by the sheaf $\mathcal{A}_X = j_X^* \mathcal{A}$. We can define the category of sheaves of $\mathcal{A}_X$ –modules $\mathrm{Mod}_r(\mathcal{A}_X, \mathcal{T}_X)$. Futhermore, repeating Definition 2.6, we define the category $Qcoh_r(X)$ of quasicoherent modules on X. The morphisms ob_X and π_{0X} induce the inverse image functors $ob_X^* : Qcoh(X) \longrightarrow Qcoh(Ob(X))$ and $\pi_{0X}^* : Qcoh(\pi_0(X)) \longrightarrow Qcoh(X)$ respectively.

3. Commutative case

3.1 Fix a commutative ring k. Denote by $Comalg/k$ the category of commutative algebras over k and by Aff/k denote the category of affine schemes that is opposite to $Comalg/k$. We note that the category of affine shemes has all fiber products and they correspond to the tensor products in the category $Comalg/k$. The canonical embedding of the category affine schemes to the category of all schemes Sch/k preserves all fiber products and finite disjoint unions. But it does not preserve all coproducts. For example, the infinite disjoint union of $\{Spec(R_\alpha)\}_{\alpha \in I}$ in Sch/k does not coincide, in general, with $Spec(\prod_{\alpha \in I} R_\alpha)$.

3.2 In this section by $\mathcal{C}$ we mean the category of all affine schemes Aff/k. The canonical contravariant functor $Aff/k \longrightarrow Comalg/k$ gives a presheaf of rings $\mathcal{O}$ on Aff/k with

$$\mathcal{O}(Spec(R)) := R$$

for any k –algebra R. This presheaf is represented by the ring scheme $Speck[t]$. Therefore it is a sheaf for any subcannonical topology on $\mathcal{C}$.

Further, the category $\mathcal{C}/X$ for any object $X \in \mathcal{C}^{\frown}$ is ringed by the presheaf $\mathcal{O}_X$ which is a sheaf for any subcanonical topology.

3.3 Denote by can the canonical topology on $\mathcal{C} = Aff/k$ and by $\underline{c} : \mathcal{C}^{\frown} \longrightarrow Shv(\mathcal{C}, can)$ denote the associative sheaf functor defined in 1.7, which is a left adjoint to the inclusion functor $i : Shv(\mathcal{C}, can) \longrightarrow \mathcal{C}^{\frown}$. The covering families for the canonical topology are the universal effectively epimorphic families defined in 1.11. In our case there is a simple but useful description of the universal effectively epimorphic families in terms of modules.

3.4 Proposition. *A family $(f_\alpha : Spec(R_\alpha) \to Spec(R))_{\alpha \in I}$ is universal effectively epimorphic in Aff/k if and only if for any R –module M the following diagram*

$$M \longrightarrow \prod_{\alpha \in I} M \otimes_R R_\alpha \rightrightarrows \prod_{\alpha, \beta \in I} M \otimes_R (R_\alpha \otimes R_\beta) \tag{3}$$

is an equalizer.

Proof. $\Rightarrow$ Take an R –module M and consider $S = R \oplus M$ as R –algebra with multiplicaition low given as $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_2 r_1)$. Denote by S_α an algebras $S \otimes_R R_\alpha$. We have the family $(g_\alpha : Spec(S_\alpha) \to Spec(S))_{\alpha\in I}$, where $g_\alpha = 1_S \otimes f_\alpha$. This family is effectively epimorphic. Therefore, considering morphisms to the scheme $Spec(k[t])$, we obtain that

$$S \longrightarrow \prod_{\alpha\in I} S\otimes_R R_\alpha \rightrightarrows \prod_{\alpha,\beta\in I} S\otimes_R (R_\alpha\otimes R_\beta) \qquad (4)$$

is an equalizer. And hence this is true for M as a direct summand.

$\Leftarrow$ Let $(f_\alpha : Spec(R_\alpha) \to Spec(R))_{\alpha\in I}$ be a family. We know that for any morphism $Spec(S) \longrightarrow Spec(R)$ the sequence (4) is exact. Further, for any algebra A the sequence of $\mathrm{Hom}'s$ from A to the sequence (4) in the category of commutative algebras is exact too. Hence $(f_\alpha)_{\alpha\in I}$ is an universal effectively epimorphic family. □

As above, we denote by $Qcoh(X)$ the category of quasicoherent modules on X for any presheaf $X \in \mathcal{C}^\wedge$ and by $\mathrm{Mod}(X, can_X)$ denote the category of sheaves of $\mathcal{O}_X$ –modules for the canonical topology on $\mathcal{C}/X$.

3.5 Corollary. *Let X be a presheaf on the category $\mathcal{C} = Aff/k$. Any quasicoherent module on X is a sheaf in the canonical topology on the category $\mathcal{C}/X$. Therefore the category $Qcoh(X)$ is a full subcategory of $\mathrm{Mod}(X, can_X)$.*

This corollary follows from the previous proposition and 1.12.

Denote by φ_X the inclusion $Qcoh(X) \hookrightarrow \mathrm{Mod}(X, can_X)$.

3.6 Remark Note that the category of all schemes Sch/k is embedding in $\mathcal{C}^\wedge = (Aff/k)^\wedge$ and for any scheme $X \in Sch/k$ a quasicoherent sheaf in our definition one-to-one corresponds to the quasicoherent sheaf in the usual definition. (Consider the small Zariski site X_{Zar} and the natural inclusion of the categories $X_{Zar} \longrightarrow Aff/X$. Then the inverse image functor takes a quasicoherent sheaf in our definition to a quasicoherent sheaf in the usual definition and this is one-to-one correspondence). Futhermore, if a scheme X is quasicompact and quasiseparated then there exists a coherator $Q_X : \mathrm{Mod}(X, can_X) \longrightarrow Qcoh(X)$. For the category of sheaves of modules on the small Zariski site it was proved in [2], II.3. Consequently, it is fulfilled in our situation too.

3.7 Corollary. *Let X be a presheaf on the category $\mathcal{C} = Aff/k$ and $m : X \longrightarrow \underline{c}X$ be a morphism of sheafification in the canonical topology. Then the inverse image functor $m^* : Qcoh(\underline{c}X) \longrightarrow Qcoh(X)$ is fully faithful.*

It immediately follows from Propositions 3.4 and 2.4.

3.8 Corollary. *Let $T \overset{i}{\hookrightarrow} Spec(R)$ be a sieve on an object $Spec(R) \in \mathcal{C}$. The sieve T is covering in the canonical topology if and only if the functor $i^* : Qcoh(Spec(R)) \longrightarrow Qcoh(T)$ is fully faithful*

Proof. We can assume that the sieve T is generated by the family $(f_\alpha : Spec(R_\alpha) \to Spec(R))_{\alpha\in I}$. It was said in 2.11 that functor i^* has a right adjoint functor i_* that takes a quasicoherent module $\mathcal{F}$ on T to $\widetilde{M}$ on $Spec(R)$, where the module M is an equalizer of

$$\prod_{\alpha\in I} \mathcal{F}(Spec(R_\alpha)) \rightrightarrows \prod_{\alpha,\beta\in I} \mathcal{F}(Spec(R_\alpha\otimes R_\beta)).$$

Assume that i^* is fully faithful. Then $i_* i^*(\widetilde{M}) \cong \widetilde{M}$ for any R –module M . Hence the sequence (3) is exact, and the sieve T is covering in the canonical topology.

Inverse statement direct follows from the previous corollary. □

3.9 On the category Aff/k there is a very important subcanonical topology. It is a well-known flat topology. The covering families for the flat topology are such universal effectively epimorphic families $(f_\alpha : Spec(R_\alpha) \to Spec(R))_{\alpha \in I}$ for which all morphisms f_α are flat, i.e. R_α are flat R –algebras for all α . The flat topology on the category Aff/k will be denoted by ft .

3.10 Now we introduce a new topology which is subcanonical but finer than the flat topology. We call it by the effective descent topology (edt).

Recall briefly the descent theory for modules over commutative ring. Let $f : R \longrightarrow S$ be a homomorphism of commutative rings. Let M be an S –module. There are two $S \otimes_R S$ –modules $M \otimes_R S$ and $S \otimes_R M$. Any $S \otimes_R S$ –homomorphism $\phi : M \otimes_R S \to S \otimes_R M$ defines three $S \otimes_R S \otimes_R S$ –homomorphisms

$$\begin{array}{rlcl} \phi_1 : & S \otimes_R M \otimes_R S & \longrightarrow & S \otimes_R S \otimes_R M \\ \phi_2 : & M \otimes_R S \otimes_R S & \longrightarrow & S \otimes_R S \otimes_R M \\ \phi_3 : & M \otimes_R S \otimes_R S & \longrightarrow & S \otimes_R M \otimes_R S \end{array}$$

by tensoring ϕ with 1_S in the first, second and third positions respectively. Moreover, in composition with the multiplication $\mu : S \otimes_R M \to M$, ϕ induces a homomorphism $\bar{\phi} : M \to M$ of S –modules:

$$\bar{\phi} : M \longrightarrow M \otimes_R S \xrightarrow{\phi} S \otimes_R M \xrightarrow{\mu} M$$

By a descent datum on an S –module M we mean an $S \otimes_R S$ –homomorphism $\phi : M \otimes_R S \to S \otimes_R M$ such that $\phi_2 = \phi_1 \phi_3$ and $\bar{\phi}$ is the identity on M .

If (M, ϕ) and (M', ϕ') are two descent data, then a morphism $g : (M, \phi) \to (M', \phi')$ of descent data is a S –homomorphism $g : M \to M'$ such that $(1_S \otimes g)\phi = \phi'(g \otimes 1_S)$.

Such way we obtain the category of descent data for any homomorphism of commutative rings $f : R \longrightarrow S$. Denote it by $Dd(f)$. There is a functor $d_f : \mathrm{Mod}(R) \longrightarrow Dd(f)$ that takes an R –module N to $(N \otimes_R S, \phi_N)$, where ϕ_N defined by the rule $\phi_N(n \otimes s_1 \otimes s_2) = s_1 \otimes n \otimes s_2$. (An R –morphism $h : N \longrightarrow N'$ goes to $h \otimes 1_S$).

A homomorphism $f : R \longrightarrow S$ of commutative rings is called an effective descent morphism if the functor $d_f : \mathrm{Mod}(R) \longrightarrow Dd(f)$ is an equivalence.

Denote by $(T_f \overset{i}{\hookrightarrow} Spec(R)) \in \mathcal{C}^\frown$ the sieve on $Spec(R)$ which consists of all maps $Spec(B) \longrightarrow Spec(R)$ which go through the map $f^{op} : Spec(S) \longrightarrow Spec(R)$. It is easy to see that the category $Dd(f)$ is nothing more than the category $Qcoh(T_f)$ of quasicoherent modules on T_f and the functor d_f coincides with the inverse image functor i^* .

Now we say that $F = (f_\alpha : Spec(R_\alpha) \to Spec(R))_{\alpha \in I}$ is an effective descent family if the functor $i^* : Qcoh(Spec(R)) \longrightarrow Qcoh(T_F)$ is an equivalence, where $T_F \overset{i}{\hookrightarrow} Spec(R) \in \mathcal{C}^\frown$ is the sieve generated by the family F .

3.11 Proposition. *The effective descent families satisfy conditions P1)–3) of Definition 1.6, i.e. they generate a pretopology on* $\mathcal{C} = Aff/k$.

Proof. Note that by Corollary 3.8 each effective descent family is an universal effectively epimorphic family. Consider some effective descent family $F = (Spec(R_\alpha) \to Spec(R))_{\alpha \in I}$. Let $Spec(S) \longrightarrow Spec(R)$ be a morphism and let $\bar{\mathcal{F}}$ be a quasicoherent module on the sieve $T_{\bar{F}}$ that is generated by the family $\bar{F} = (Spec(R_\alpha \otimes_R S) \to Spec(S))_{\alpha \in I}$. We can define a quasicoherent sheaf $\mathcal{F}$ on T_F by the rule $\mathcal{F}(\cdot) := \bar{\mathcal{F}}(\cdot \times_{Spec(R)} Spec(S))$ (this is the direct image of $\bar{\mathcal{F}}$ with respect to the canonical map from $T_{\bar{F}}$ to T_F . Since F is an effective descent family there is an R –module M such that $\mathcal{F} = i_F^*\widetilde{M}$. We see that the module M can be defined as an equalizer of

$$\prod_{\alpha\in I}\bar{\mathcal{F}}(Spec(S\otimes_R R_\alpha)) \rightrightarrows \prod_{\alpha,\beta\in I}\bar{\mathcal{F}}(Spec(S\otimes_R R_\alpha\otimes_R R_\beta))$$

Hence the R –module M has a structure of S –module and $\bar{\mathcal{F}} = i_{\bar{F}}^*\widetilde{M}$. Therefore, $\bar{F}$ is an effective descent family.

Let $F = (Spec(R_\alpha) \to Spec(R))_{\alpha\in I}$ be an effective descent family and, moreover, there fixed an effective descent family $G_\alpha = (Spec(R_{\beta_\alpha}) \to Spec(R_\alpha))_{\beta_\alpha\in J_\alpha}$ for each $\alpha\in I$. Let G be the family $(Spec(R_\gamma) \to Spec(R))_{\gamma\in\coprod\limits_{\alpha\in I} J_\alpha}$. We have the sieves $i_F : T_F \hookrightarrow Spec(R)$, $i_{G_\alpha} : T_{G_\alpha} \hookrightarrow Spec(R_\alpha)$ and the map of the sieves $p : T_G \hookrightarrow T_F$.

Let $f_\alpha : Spec(R_\alpha) \longrightarrow T_F$ and $g_\alpha : T_{G_\alpha} \to T_G$ be the canonical morphisms in the category $\mathcal{C}^\frown$. Consider the following diagram of functors.

$$\begin{array}{ccc}
\mathrm{Mod}(T_{G_\alpha}, can) & \xleftarrow{j^*_{g_\alpha}} & \mathrm{Mod}(T_G, can) \\
\wr\uparrow j^*_{i_{G_\alpha}} & & \wr\uparrow j^*_p \\
\mathrm{Mod}(Spec(R_\alpha), can) & \xleftarrow{j^*_{f_\alpha}} & \mathrm{Mod}(T_F, can)
\end{array}$$

The both vertical arrows are equivalence because all effective descent families are covering in the canonical topology.

Let $\mathcal{F}$ be a quasicoherent module on T_G . The sheaves of modules $\mathcal{F}_\alpha := j^*_{f_\alpha} j_{p*}(\mathcal{F})$ are quasicoherent for all α , because $j^*_{i_{G_\alpha}}(\mathcal{F}_\alpha) \cong j^*_{g_\alpha}(\mathcal{F})$ and G_α are effective descent families. Since the sieve T_F is generated by the family $(f_\alpha)_{\alpha\in I}$ the sheaf of modules $j_{p*}(\mathcal{F})$ is quasicoherent. By assumption, F is an effective descent family. Hence there is an R -module M such that $j_{p*}(\mathcal{F}) \cong i_F^*(\widetilde{M})$ and, consequently, $\mathcal{F} \cong i_G^*(\widetilde{M})$. □

3.12 The effective descent topology on the category $\mathcal{C} = Aff/K$ (edt) is a topology for which covering sieves are all sieves generated by the effective descent families. It follows from Corollary 3.8 that edt is subcanonical. Moreover it is finer than the flat topology. It was proved in a lot of places that a faithful flat homomorphism $f : R \longrightarrow S$ of commutative ring is an effective descent morphism (see, for example, [3], [6], [7]).

Denote by $\underline{e} : \mathcal{C}^\frown \longrightarrow Shv(\mathcal{C}, edt)$ the sheafification functor for the effective descent topology.

3.13 Theorem. *Let $X\in\mathcal{C}^\frown$ be a presheaf and $m : X \longrightarrow \underline{e}X$ be the sheafification morphism in the effective descent topology. Then the inverse image functor $m^* : Qcoh(\underline{e}X) \longrightarrow Qcoh(X)$ is an equivalence*

Proof. The functor $i\underline{e}$ is isomorphic to the composition $L\circ L$, where the functor L is defined by formula (1) in 1.7, i.e. the morphism m is the composition

$$X \xrightarrow{l_X} LX \xrightarrow{l_{LX}} \underline{e}X$$

Hence it is sufficient to prove that the functor $l_X^* : Qcoh(LX) \longrightarrow Qcoh(X)$ is an equivalence for all $X\in\mathcal{C}^\frown$.

By Corollary 3.5, the category $Qcoh(X)$ is a full subcategory of $\mathrm{Mod}(X, can_X)$. The functor l_X^* is induced by the functor $j^*_{l_X} : \mathrm{Mod}(LX, can_{LX}) \longrightarrow \mathrm{Mod}(X, can_X)$ which is an equivalence.

Hence, to prove the theorem we must show that for any coherent sheaf $\mathcal{F}$ on X the sheaf of $\mathcal{O}_{LX}$ –modules $\mathcal{F}' = j_{l_X *}\mathcal{F}$ is quasicoherent. By Remark 2.7 we need to check that for each morphism $s : Spec(R) \longrightarrow LX$ the sheaf $j_s^*\mathcal{F}'$ is quasicoherent. By definition of the functor L we have

$$LX(R) = \varinjlim_{T \in J(R)} \mathrm{Hom}_{\mathcal{C}^\frown}(T, X)$$

Therefore there exists a covering sieve $i : T \hookrightarrow Spec(R)$ such that the following diagram commutes

$$\begin{array}{ccc} T & \xrightarrow{s'} & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle l_X} \\ Spec(R) & \xrightarrow{s} & LX \end{array}$$

Since $j_{l_X}^*$ and $j_{l_X *}$ are quasi-inverse by Proposition 2.4 there is an isomorphism $j_i^* j_s^* \mathcal{F}' \cong j_{s'}^*\mathcal{F}$ of quasicoherent modules on T . As T is a covering sieve the functor j_i^* induces an equivalence $i^* : Qcoh(Spec(R)) \xrightarrow{\sim} Qcoh(T)$. Therefore, the sheaf $j_s^*\mathcal{F}'$ is quasicoherent on $Spec(R)$ for all $s : Spec(R) \longrightarrow LX$. Thus $\mathcal{F}' = j_{l_X *}\mathcal{F}$ is a quasicoherent module on LX . □

4. Noncomutative case

4.1 Fix a ring k . Denote by Alg/k the category of algebras over k . If $f_i : R \longrightarrow S_i,\ i = 1, 2$ be two homomorphisms in Alg/k then a coproduct $S_1 \coprod_R S_2$ always exists. We can take a presentation by generators and defining relations for both R –algebras S_1 and S_2 and write all these presentations together. The coproduct will be denoted by $S_1 \star_R S_2$.

Denote by $NAff/k$ the category that is opposite to Alg/k . We will call it by the category of noncommutative affine schemes. The object of $NAff/k$, which corresponds to an algebra R , will be denoted by $Sp(R)$. The category of noncommutative affine shemes has all fiber products and the product $Sp(S_1) \times_{Sp(R)} Sp(S_2)$ is isomorphic to $Sp(S_1 \star_R S_2)$

4.2 Let $N : Aff/k \longrightarrow NAff/k$ be the natural inclusion functor that takes $Spec(R)$ to $Sp(R)$. The functor N is fully faithful and has a right adjoint functor $Com : NAff/k \longrightarrow Aff/k$. The functor Com sends an object $Sp(R)$ to $Spec(R_c)$. Here R_c is the maximal commutative factor-algebra of R , i.e. $R_c = R/I$, where I is the ideal generated by all relations of the form $< ab - ba >$.

There is a functor $\widehat{N}_! : (Aff/k)^\frown \longrightarrow N(Aff/k)^\frown$ that is an extension of N onto the category of presheaves (see 1.2). By 1.2 the functor $\widehat{N}_!$ has the following description

$$\widehat{N}_!(X)(Sp(S)) = \varinjlim_{Spec(A) \to X} \mathrm{Hom}(Sp(S), Sp(A))$$

The functor $\widehat{N}_!$ is also fully faithful by 1.2 and it has a right adjoint functor $\widehat{N}^*$, restriction of which on $NAff/k$ coincides with the functor Com .

We mean the functor $\widehat{N}_!$ as a "course" embedding of the "commutative geometry" into the "noncommutative geometry". To get a "fine" embedding we need to consider presheaves of groupoids.

4.3 Now for each object $Sp(S) \in NAff/k$ we define a category I_S . The objects of I_S are couples $(Spec(A), m)$ with $Spec(A) \in Aff/k$ and $m : Sp(S) \longrightarrow Sp(A)$. A morphism from $(Spec(A), m)$ to $(Spec(A'), m')$ is a morphism $\xi : Spec(A) \to Spec(A')$ such that $m' = \xi m$.

Any morphism $f : Sp(S') \longrightarrow Sp(S)$ defines a functor $I_f : I_S \longrightarrow I_{S'}$. Moreover there is a functor $pr_S : I_S \longrightarrow Aff/k$ that takes an object $(Spec(A), m)$ to the $Spec(A)$. It is easy to see that the functor $\widehat{N}_!$ can be given by the following formula

$$\widehat{N}_!(X)(Sp(S)) = \varinjlim_{I_S} X(pr_S(\cdot)) \tag{5}$$

4.4 Consider the category I_S/X of objects pr_S –over X . It has as objects all triples $(Spec(A), m, a)$ with $m : Sp(S) \to Sp(A)$ and $a : Spec(A) \to X$ and as morphisms all morphisms in I_S that are compartible with maps to X . Denote by $Gr(I_S/X)$ the groupoid associated to the category I_S (i.e. we invert all arrows). Every morphism $g : Sp(S') \longrightarrow Sp(S)$ gives a functor $Gr(I_S/X) \longrightarrow Gr(I_{S'}/X)$. On the other hand, every $f : X' \longrightarrow X$ in $(Aff/k)\widehat{}$ induces a functor $Gr(I_S/X') \longrightarrow Gr(I_S/X)$.

Let us define for an object $X \in (Aff/k)\widehat{}$ a presheaf of groupoids $\overline{N}X \in N(Aff/k)\widehat{}_{gr}$ by rule

$$\overline{N}X(Sp(S)) := Gr(I_S/X) \tag{6}$$

This correspondence gives us a functor $\overline{N}$ from $(Aff/k)\widehat{}$ to the category of presheaves of groupoids $N(Aff/k)\widehat{}_{gr}$. It immediately follows from formula (5) that $\pi_0(\overline{N}X)$ is isomorphic to $\widetilde{N}_!X$. Hence we have the commutative diagram of functors

$$\begin{tikzcd}
(Aff/k)\widehat{} \arrow[r, "\overline{N}"] \arrow[dr, "\widehat{N}_!"'] & N(Aff/k)\widehat{}_{gr} \arrow[d, "\pi_0"] \\
 & N(Aff/k)\widehat{}
\end{tikzcd}$$

4.5 Proposition. *The functor* $\overline{N} : (Aff/k)\widehat{} \longrightarrow N(Aff/k)\widehat{}_{gr}$ *has the following properties:*

1) *there is an isomorphism* $\overline{N}(Spec(A)) \cong N(Spec(A)) = Sp(A)$ *, i.e. the restriction of the functor* $\overline{N}$ *on the category* Aff/k *is isomorphic to* N *;*
2) *there is an isomorphism* $\overline{N}X(Sp(A)) \cong X(Spec(A))$ *for any object* $Spec(A) \in Aff/k$ *;*
3) $\overline{N}$ *is fully faithfull.*

PROOF. The groupoid associated to a category with final or initial object is equivalent to the one-point set. This implies that $Gr(I_S/Spec(A))$ is equivalent to the set $\mathrm{Hom}_{NAff}(Sp(S), Sp(A))$ for any algebra S and commutative algebra A . Consequently, $\overline{N}(Spec(A)) \cong Sp(A)$ and 1) holds. By the same reason, the groupoid $Gr(I_A/X)$ is equivalent to the set $X(Spec(A)$ for any commutative algebra A . This proves the condition 2).

3) Since the functor $\widehat{N}_! = \pi_0\overline{N}$ is faithful by 1.2 the functor $\overline{N}$ is faithful too. Furhtermore, the composition $\widehat{N}^*\pi_0\overline{N}$ is equivalent to the identity functor on $(Aff/k)\widehat{}$.

Let $f : \overline{N}Y \longrightarrow \overline{N}X$ be a morphism. It is defined by the morphisms $f_S : Gr(I_S/Y) \longrightarrow Gr(I_S/X)$ for all $S \in Alg/k$. All these morphisms are compartible. Let $t = (Spec(A), m, a)$ be an element of $Gr(I_S/Y)$, where $m : Sp(S) \longrightarrow Sp(A)$ and $a \in Y(Spec(A))$. The morphism f_S takes this element to $t' = (Spec(A), m, a') \in Gr(I_S/X)$, where $a' \in X(Spec(A)$. Since f_S and f_A are compartible we have $a' = f_A(a)$. Hence if we know the morphism f_A then we can find the element a' . Thus every morphism f_S are unique defined by the set of morphisms f_A with commutative A . This implies that the functor $\overline{N}$ is full. $\square$.

4.6 Let X be an object of $(Aff/k)\hat{}$. We have two categories Aff/X and $NAff/\overline{N}X$, where the latter is defined in 2.15. There is a functor from Aff/X to $NAff/\overline{N}X$ that sends an element $a \in X(Spec(A))$ to the element $(Spec(A), id, a) \in Gr(I_{Spec(A)}/X)$ (see 4.4). Denote this functor by $\overline{n}_X$. This functor induces the inverse image functor $\overline{n}_X^* : Qcoh_r(\overline{N}X) \longrightarrow Qcoh(X)$.

The correspondence to a presheaf of sets $X \in (Aff/k)\hat{}$ the presheaf of groupoids $\overline{N}X \in N(Aff/k)\hat{}_{gr}$ is considered by us as a "fine" embeding of the "commutative case" in the "noncommutative" one, because the following statement holds

4.7 Theorem. *Let X be an object of $(Aff/k)\hat{}$. The functor $\overline{n}_X^* : Qcoh_r(\overline{N}X) \longrightarrow Qcoh(X)$, induced by the natural functor $\overline{n}_X : Aff/X \longrightarrow NAff/\overline{N}X$, is an equivalence.*

Proof. We construct a quasi-inverse functor for $\overline{n}_X$. For any object $X \in Aff/k$ we can define a presheaf of categories CX on $NAff/k$, i.e. contravariant functor fron $NAff/k$ to the category of all small categories by rule (compaire with (6)):

$$CX(Sp(S)) := I_S/X$$

There is a functor κ_X from $NAff/CX$ to Aff/X that sends an element $t = (Spec(A), m, a) \in I_S/X$ to the element $a \in X(Spec(A))$, where $m : Sp(S) \longrightarrow Spec(A)$.

Let $\mathcal{F}$ be a quasicoherent sheaf on X . We can define a presheaf $\mathcal{G}$ on the category $NAff/CX$ by rule

$$\mathcal{G}(t) = \mathcal{F}(a) \otimes_{(A,m)} S$$

(More formaly we can do it by the following way. The functor κ_X induces the functor $\hat{\kappa}_X^* : (Aff/X)\hat{} \longrightarrow (NAff/CX)\hat{}$ (see 1.2). Then the presheaf $\mathcal{G}$ can be defined as $\hat{\kappa}_X^*(\mathcal{F}) \otimes_{\hat{\kappa}_X^*(\mathcal{O}_X)} \mathcal{O}_{CX}$.)

Further, a morphism $\alpha : t \to t'$ in I_S/X is, by definition, a morphism $\alpha : Spec(A) \longrightarrow Spec(A')$ such that $\alpha m = m'$ and $a'\alpha = a$. Any such morphism gives us an isomorphism between $\mathcal{G}(t)$ and $\mathcal{G}(t')$, because

$$\mathcal{G}(t) = \mathcal{F}(a) \otimes_{(A,m)} S \cong \mathcal{F}(a') \otimes_{(A',\alpha)} A \otimes_{(A,m)} S \cong \mathcal{F}(a') \otimes_{(A',m')} S = \mathcal{G}(t')$$

Therefore, this presheaf $\mathcal{G}$ defines a presheaf $\mathcal{H}$ on $\overline{N}X$ which is, evidently, quasicoherent. We define a functor $\overline{n}_{*X}$ from $Qcoh(X)$ to $Qcoh_r(\overline{N}X)$ as such functor that takes object $\mathcal{F}$ to $\mathcal{H}$. It is easy to see that the functor $\overline{n}_{*X}$ is quasi-inverse of $\overline{n}_X^*$. $\square$

4.8 Futher, we will often write $\mathcal{C}$ instead of $NAff/k$. Let $\Phi = (\mathcal{C}, can)$ be a site ringed by the sheaf $\mathcal{O}$. Let X be an object of $\mathcal{C}\hat{}$ and $j_X : \mathcal{C}/X \longrightarrow \mathcal{C}$ is the canonical functor. Denote by can_X the canonical topology on $\mathcal{C}/X$. It is induced by the canonical topology on $\mathcal{C}$ by 1.12. We have a ringed site $\Phi_X = (\mathcal{C}/X, can_X)$ with the ring sheaf $\mathcal{O}_X$. In the commutative case we know that any quasicoherent module on X is a sheaf for the canonical topology on $\mathcal{C}/X$ (see 3.5). Moreover, it easy follows from the definition of the canonical topology. The same statement in the noncommutative case is also true, but the proof is not so direct.

4.9 Proposition. *Let X be a presheaf on the category $\mathcal{C} = NAff/k$. Then each quasicoherent module $\mathcal{M}$ on X is a sheaf for the canonical topology on $NAff/X$.*

PROOF. Let A be a k -algebra and let $(f_\alpha : A \longrightarrow B_\alpha)$, $\alpha \in I$ be a family of morphisms of algebras such that the opposite family of maps f_α^{op} in $NAff/k$ is covering in the canonical topology. To prove the proposition we have to show that for any (right) A -module M a homomorphism ϵ_M in the diagram

$$M \xrightarrow{\epsilon_M} \prod_{\alpha\in I} M \otimes_A B_\alpha \rightrightarrows \prod_{\alpha,\beta\in I} M \otimes_A (B_\alpha \star_A B_\beta)$$

is an equalizer.

Denote by N a A –bimodule $A \otimes_{\mathbb{Z}} M$. Define a ring S as $A \oplus N$ with multiplication low $(a_1, n_1)(a_2, n_2) = (a_1a_2, a_1n_2 + n_1a_2)$. By Lemma 4.10 the canonical map $\phi : M \longrightarrow N$ is an embedding. Denote by i the embedding N to S and by ψ_α the canonical maps $M \otimes_A B_\alpha \longrightarrow S \star_A B_\alpha$ for any $\alpha \in I$. Let ψ be the product of ψ_α by all $\alpha \in I$.

Consider the following commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{\epsilon_M} & \prod_{\alpha\in I} M \otimes_A B_\alpha & \rightrightarrows & \prod_{\alpha,\beta\in I} M \otimes_A (B_\alpha \star_A B_\beta) \\ {\scriptstyle i\phi}\big\downarrow & & \big\downarrow{\scriptstyle \psi} & & \big\downarrow \\ S & \xrightarrow{e_S} & \prod_{\alpha\in I} S \star_A B_\alpha & \rightrightarrows & \prod_{\alpha,\beta\in I} (S \star_A B_\alpha) \star_S (S \star_A B_\beta) \end{array} \qquad (7)$$

Since the family (f_α^{op}) is covering the map e_S is an equalizer. The maps ϕ and e_S are embeddings, hence the map ϵ_M is an embedding too. Thus we proved that any quasicoherent module is a separated sheaf.

Assume now that the following conditions hold

1. the maps ψ_α are embeddings for any α ,
2. intersection of $e_S(S)$ and $\psi(\prod_\alpha M \otimes_A B_\alpha)$ coincides with $e_S i\phi(M)$.

In this case we obtain that ϵ_M is an equalizer.

Thus, to finish the proof we need to show that our assumptions hold.

1) Denote an A -bimodule B_α/A by $\overline{B}_\alpha$. Consider a graded ring R_α for any $\alpha \in I$, defined as

$$R_\alpha = B_\alpha \oplus (B_\alpha \otimes_A N \otimes_A B_\alpha) \oplus (B_\alpha \otimes_A N \otimes_A \overline{B}_\alpha \otimes_A N \otimes_A B_\alpha) \oplus \cdots$$

with composition low

$$\begin{gathered}(b_0 \otimes n_1 \otimes \overline{b}_1 \otimes \cdots \otimes \overline{b}_{i-1} \otimes n_i \otimes b_i)(b'_i \otimes n_{i+1} \otimes \overline{b}_{i+1} \otimes \cdots \otimes \overline{b}_{i+j-1} \otimes n_{i+j} \otimes b_{i+j}) = \\ (b_0 \otimes n_1 \otimes \overline{b}_1 \otimes \cdots \otimes \overline{b}_{i-1} \otimes n_i \otimes \overline{b_i b'_i} \otimes n_{i+1} \otimes \overline{b}_{i+1} \otimes \cdots \otimes \overline{b}_{i+j-1} \otimes n_{i+j} \otimes b_{i+j})\end{gathered}$$

There are the ring homomorphisms $B_\alpha \longrightarrow R_\alpha$ and $S \longrightarrow R_\alpha$. Therefore , there is the canonical ring homomorphism $S \star_A B_\alpha \longrightarrow R_\alpha$. By Lemma 4.10 the map $M \otimes_A B_\alpha \longrightarrow B_\alpha \otimes_A N \otimes_A B_\alpha \cong B_\alpha \otimes_{\mathbb{Z}} M \otimes_A B_\alpha$ is an embedding. Hence the map $M \otimes_A B_\alpha \longrightarrow R_\alpha$ is an embedding too. It is the composition of ψ_α with the canonical map $S \star_A B_\alpha \longrightarrow R_\alpha$. Thus, we obtain that ψ_α is an embedding for any $\alpha \in I$.

2) The map i is the embedding of N into S . To check the second condition it is sufficient to show that intersection of $e_S i(N)$ and $\psi(\prod_{\alpha\in I} M \otimes_A B_\alpha)$ coincides with $e_S i\phi(M)$. Note that the maps $e_S i$ and ψ go through the canonical map $\prod_{\alpha\in I} B_\alpha \otimes_A N \otimes_A B_\alpha \longrightarrow \prod_{\alpha\in I} S \otimes_A B_\alpha$ On the other hand, there is a composition map

$$\prod_{\alpha\in I} S \otimes_A B_\alpha \longrightarrow R_\alpha \longrightarrow \prod_{\alpha\in I} B_\alpha \otimes_A N \otimes_A B_\alpha$$

which splits previous one. Hence $\prod_{\alpha\in I} B_\alpha \otimes_A N \otimes_A B_\alpha$ is a direct summand of $\prod_{\alpha\in I} S \otimes_A B_\alpha$.

Denote by ρ and θ the maps $N \longrightarrow \prod_{\alpha\in I} B_\alpha \otimes_A N \otimes_A B_\alpha$ and $\prod_{\alpha\in I} M \otimes_A B_\alpha \longrightarrow \prod_{\alpha\in I} B_\alpha \otimes_A N \otimes_A B_\alpha$ respectively. There is the commutative diagram

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M & \xrightarrow{\epsilon_M} & \prod\limits_{\alpha\in I} M \otimes_A B_\alpha \\ & & {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\theta} \\ 0 & \longrightarrow & N = A \otimes_{\mathbb{Z}} M & \xrightarrow{\rho} & \prod\limits_{\alpha\in I} B_\alpha \otimes_{\mathbb{Z}} M \otimes_A B_\alpha \end{array}$$

By Lemma 4.10 the embeddings ϕ and θ split as maps of abelian groups. Moreover, it is easy to see that these splits are commute. Therefore, the intersection of $\rho(N)$ and $\theta(\prod_{\alpha\in I} M \otimes_A B_\alpha)$ coincides with $\rho\phi(M)$. This proves condition two and the proposition. $\square$

4.10 Lemma. *Let R be a ring and M be a right R –module. Then the canonical map $M \longrightarrow R \otimes_{\mathbb{Z}} M$ is an embedding of rights R –modules and it splits as a morphism of abelian groups.*

Proof. It is clear that the composition of maps of abelian groups

$$M \longrightarrow R \otimes_{\mathbb{Z}} M \xrightarrow{\sim} M \otimes_{\mathbb{Z}} R \longrightarrow M$$

is the identity. (Here the last map is the action of the ring R on the right module M .) Hence $M \longrightarrow R \otimes_{\mathbb{Z}} M$ is an embedding of abelian groups and, consequently, it is an embeddidg of right R -modules. $\square$

4.11 Corollary. *Let X be a presheaf on the category $\mathcal{C} = NAff/k$. The category $Qcoh_r(X)$ is a full subcategory of the category $\mathrm{Mod}_r(X, can_X)$. If $m : X \longrightarrow \underline{c}X$ is the morphism of sheafification in the canonical topology, then the inverse image functor $m^* : Qcoh_r(\underline{c}X) \longrightarrow Qcoh_r(X)$ is fully faithful.*

This corollary follows from Propositions 4.9 and 2.4.

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