# Nonconjugate subgroups of integral orthogonal groups

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# NONCONJUGATE SUBGROUPS OF INTEGRAL ORTHOGONAL GROUPS

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## §0 : Introduction :

In this note we consider a question raised by C. Okonek in connection with his joint work with W. Ebeling on diffeomorphisms of algebraic surfaces.

Question : Let $<,> : L \times L \to \mathbf{Z}$ be a nondegenerate symmetric bilinear form on a finitely generated free abelian group L. Does there exist an infinite family of isomorphic finitely generated subgroups $(\Gamma_\sigma)_{\sigma\in\Sigma}$ of $\mathrm{Aut}_{\mathbf{Z}}(L, <,>)$ such that $\Gamma_\sigma$ is not conjugate to $\Gamma_\tau$ for $\sigma \neq \tau$ ?

We answer the question affirmatively when $(L, < , >)$ splits as an orthogonal direct sum

$$(L, <,>) \cong (L_1, <,>) \perp (L_2, <,>)$$

where $(L_1, <,>)$ has signature (2,1), and $\mathrm{Aut}_{\mathbf{Z}}(L_2, <,>)$ has a subgroup of finite index which maps epimorphically onto $\mathbf{Z}$ ; this implies, amongst other things, that $\mathrm{rk}_{\mathbf{Z}}(L_2) \geq 3$. In particular, this always happens when $(L_1, <,>)$ and $(L_2, <,>)$ each have signature (2,1). More generally, one may show that the result holds when $(L, < , >)$ splits as an orthogonal direct sum

$$(L, <,>) \cong (L_1, <,>) \perp (L_2, <,>) \perp \ldots\ldots \perp (L_k, <,>)$$

where each $(L_i, <,>)$ has signature (2,1). The methods are a variation on those of our earlier paper [2], although to show finite generation we do actually use the main theorem of [2].

This work was done whilst the author was on sabbatical at the Max-Planck-Institut fur Mathematik, Bonn. We wish to thank Professor Okonek for raising the question here considered. We especially wish to thank Professor Hirzebruch and the staff of the MPI for their hospitality.

## §1 : Arithmetic subgroups and integral quadratic forms :

If $\mathbf{G}$ is a linear algebraic group defined and semisimple over $\mathbf{Q}$ ( which we may take to be imbedded $\mathbf{G_Q} \subset \mathbf{GL_n(Q)}$ ), by an *arithmetic subgroup* of $\mathbf{G}$ , we mean a subgroup $\Gamma$ of $\mathbf{G_R}$ which is commensurable with $\mathbf{G_Z} = \mathbf{G_Q} \cap \mathbf{GL_n(Q)}$ . This is independent of the particular imbedding $\mathbf{G_Q} \subset \mathbf{GL_n(Q)}$ chosen. Moreover, for such a subgroup $\Gamma$, $\mathbf{G_R}/\mathbf{\Gamma}$ has finite volume.

Theorem 1.1 [2] : Let $\mathbf{G}$ be a linear algebraic group defined and *simple* over $\mathbf{Q}$ with the property that $\mathbf{G_R}$ is noncompact. If $\Gamma$ is an arithmetic subgroup of $\mathbf{G}$ then $\overline{\Gamma} = \mathbf{G_C}$.

This has the following consequence, where $[\Gamma, \Gamma]$ denotes the commutator subgroup of $\Gamma$ :

Corollary 1.2 : Let $\mathbf{G}$ be a linear algebraic group defined and semisimple over $\mathbf{Q}$ with the property that $\mathbf{G_{i,R}}$ is noncompact for each $\mathbf{Q}$ -simple factor $\mathbf{G_i}$ . If $\Gamma$ is an arithmetic subgroup of $\mathbf{G}$ then $\overline{[\Gamma,\Gamma]} = \mathbf{G_C}$.

Proof : First observe that $\mathbf{G}$ is isogenous with the product of its $\mathbf{Q}$ -simple factors $\mathbf{G_1} \times \ldots . \times \mathbf{G_n}$ so that $\Gamma$ contains with finite index a subgroup of the form $\Gamma_1 \times \ldots . \times \Gamma_n$ where $\Gamma_i$ is an arithmetic subgroup of $\mathbf{G_i}$. Hence $[\Gamma_1, \Gamma_1] \times \ldots . \times [\Gamma_n, \Gamma_n]$ is contained in $[\Gamma, \Gamma]$ so we need only consider the case where $\mathbf{G}$ is $\mathbf{Q}$ -simple.

By Borel's Density Theorem in the from of [2], $\overline{\Gamma} = \mathbf{G_C}$ , and since $\mathbf{G_C}$ is nonabelian, $\Gamma$ is also non-abelian; hence $\overline{[\Gamma,\Gamma]}$ is nontrivial. $\Gamma$ normalises $[\Gamma,\Gamma]$ , so that $\overline{\Gamma}$ normalises $\overline{[\Gamma,\Gamma]}$. However, by (1.1), $\overline{\Gamma} = \mathbf{G_C}$. , so that $\overline{[\Gamma,\Gamma]}$ is a *normal* complex algebraic subgroup of $\mathbf{G_C}$. moreover, since $\overline{[\Gamma,\Gamma]}$ is the Zariski closure of a subset of $\mathbf{G_Q}$ , $\overline{[\Gamma,\Gamma]}$ is defined over $\mathbf{Q}$ , by Weil's Rationality Criterion [6]. Since $\overline{[\Gamma,\Gamma]}$ is $\mathbf{Q}$ -simple, and $[\Gamma,\Gamma]$ is nontrivial, it follows that $\overline{[\Gamma,\Gamma]} = \mathbf{G_C}$. as claimed. □

Let $<,> : \mathrm{L} \times \mathrm{L} \to \mathbf{Z}$ be a nondegenerate symmetric integral bilinear form on a free abelian group L of rank n, say. (L , < , >) is said to be *isotropic ( over* $\mathbf{Z}$ *)* when there exists a nonzero element $\mathbf{x} \in \mathrm{L}$ such that $< \mathbf{x}, \mathbf{x} > = 0$; otherwise (L , < , >) is said to be *anisotropic.*

Put $\Gamma = \mathrm{Aut}_{\mathbf{Z}}(\mathrm{L}, <,>)$. The associated real form

$$<,>: \mathbf{L} \otimes \mathbf{R} \times \mathbf{L} \otimes \mathbf{R} \to \mathbf{R}$$

is diagonalisable as

$$\sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{n} x_i y_i$$

assigning to $(L, <,>)$ the signature $(p, q)$, $p + q = n$, and $\Gamma$ imbeds as a discrete subgroup of finite covolume in the group

$$\mathrm{Aut}_{\mathbf{R}}(L \otimes R, <,>) = O(p,q);$$

moreover, $\Gamma$ is cocompact precisely when $(L, <,>)$ is anisotropic. (When $<,>$ is indefinite, we note that, by a classical theorem of Meyer [4], this can only happen if $n \leq 4$).

When the signature of $(L, <,>)$ is (2,1), the symmetric space of $O(2,1)$ is the upper half-plane, so that $\Gamma$ is a Fuchsian group. When $(L, <,>)$ is isotropic, $\Gamma$ contains a non-abelian free subgroup of finite index, whereas, when $(L, <,>)$ is anisotropic, $\Gamma$ contains, as a subgroup of finite index, a Surface group $\Sigma_g^+$ ; that is, the fundamental group of an orientable surface, of genus $g \geq 2$, having a presentation of the following form ;

$$\Sigma_g^+ = < X_1, \ldots, X_g, Y_1, \ldots, Y_g, : \prod_{i=1}^{g} [X_i, Y_i] > .$$

We summarise these observations thus :

Proposition 1.3 : Let $\Gamma$ be the automorphism group of a nondegenerate integral quadratic form of signature (2,1) ; then $\Gamma$ is finitely generated and

(i) $\Gamma$ contains a Surface subgroup of finite index when $(L, <,>)$ is anisotropic ;

(ii) $\Gamma$ contains a nonabelian free subgroup of finite index when $(L, <,>)$ is isotropic.

Let H be a subgroup of a group G ; we denote by $N_G H$ the normaliser of H in G ; that is

$$N_G H = \{g \in G : gHg^{-1}\}$$

Let $\mathbf{G}$ be a linear algebraic group defined and semisimple over $\mathbf{C}$. For any subgroup H of $\mathbf{G}$, we denote by $\overline{H}$ the closure of H in the Zariski topology of $\mathbf{G}$. $\overline{H}$ is then an algebraic subgroup of $\mathbf{G}$. Let H be a subgroup of a group G ; we denote by $N_G$ H the normaliser of H in G ; that is

$$N_G H = \{g \in G : gHg^{-1}\}.$$

If $\mathbf{H}$ is an algebraic subgroup of $\mathbf{G}$ , then $N_{\mathbf{G}}H$ is also an algebraic subgroup of $\mathbf{G}$ .

The obvious isomorphism

$$\mathbf{C}^{n_1} \oplus \ldots \oplus \mathbf{C}^{n_m} \cong \mathbf{C}^{n_1+\ldots+n_m}$$

induces an injection

$$O(n_1, \mathbf{C}) \times \ldots . \times O(n_m, \mathbf{C}) \subset O(n_1 + \ldots + n_m, \mathbf{C}).$$

A straightforward matrix calculation in the Lie algebra shows that

Proposition 1.4 : $O(n_1, \mathbf{C}) \times \ldots . \times O(n_m, \mathbf{C})$ is a self-normalising subgroup of $O(n_1 + \ldots + n_m, \mathbf{C})$, provided that each $n_i \geq 2$.

Proposition 1.5 : Let L be finitely generated free abelian group, and

$$<,>: L \times L \to \mathbf{Z}$$

a nondegenerate symmetric bilinear form which splits as an orthogonal direct sum

$$(L, <,>) \cong (L_1, <,>) \perp (L_2, <,>) \perp \ldots . . \perp (L_m, <,>)$$

where $m \geq 2$ , and each $rk_{\mathbf{Z}}(L_i) \geq 2$. Let $\mathbf{G}$ (resp. $\mathbf{G}_i$ ) be the linear algebraic group whose group of k-rational points is $Aut_k(L \otimes k, <,>)$ (resp. $Aut_k(L_i \otimes k, <,>)$ , and let

$$\mathbf{H} = \mathbf{G}_1 \times \ldots . \times \mathbf{G}_m \subset \mathbf{G} ;$$

Then $Aut_{\mathbf{Z}}(L_1, <,>) \times \ldots \times Aut_{\mathbf{Z}}(L_m, <,>)$ is contained as a subgroup of finite index in $N_{\mathbf{G}}(\mathbf{H}) \cap Aut_{\mathbf{Z}}(L, <,>)$.

Proof : Put $\lambda_i = rk_{\mathbf{Z}}(L_i)$, and $\lambda = \sum \lambda_i$. $\mathbf{H}$ and $(N_G H)$ are both linear algebraic groups defined over $\mathbf{Q}$ , so that the groups of real points, $\mathbf{H}_{\mathbf{R}}$ and $(\mathbf{N_G H})_{\mathbf{R}}$ respectively, are Lie groups having only finitely many connected components. Observe that $\mathbf{G}_{\mathbf{C}}$ (respectively $\mathbf{G}_{i,\mathbf{C}}$) is isomorphic to $O(\lambda , \mathbf{C})$ (repectively $O(\lambda_i, \mathbf{C} )$. It follows from (1.4) that

$$\mathbf{H}_{\mathbf{C}} = (\mathbf{N_G H})_{\mathbf{C}}.$$

Since $\mathbf{H} \subset \mathbf{N_G H}$ it follows easily that the identity components of the corresponding real groups are therefore equal ; that is, $\mathbf{H}_{\mathbf{R},0} = (\mathbf{N_G H})_{\mathbf{R},0}$. Since

$\mathrm{Aut}_{\mathbf{Z}}(L_1, <,>) \times \ldots \times \mathrm{Aut}_{\mathbf{Z}}(L_m, <,>)$ and $N_{\mathbf{G}}(\mathbf{H}) \cap \mathrm{Aut}_{\mathbf{Z}}(L, <,>)$ are both arithmetic in $\mathbf{N_G H}$, and $\mathrm{Aut}_{\mathbf{Z}}(L_1, <,>) \times \ldots \times \mathrm{Aut}_{\mathbf{Z}}(L_m, <,>)$ is contained in $N_{\mathbf{G}}(\mathbf{H}) \cap \mathrm{Aut}_{\mathbf{Z}}(L, <,>)$, the conclusion now follows.

§2 : Normal subdirect products :

By a *product structure* on a group G we mean a sequence $\mathcal{G} = (G_r)_{1 \leq r \leq n}$ of normal subgroups of G such that G is the internal direct product

$$G = G_1 \circ \ldots \circ G_n \ ;$$

that is, each $g \in G$ can be expressed uniquely as a product $g = g_1 \ldots g_n$ ; with $g_i \in G_i$. For any group H let $H^{ab}$ denote the abelianisation

$$H^{ab} = H/[H, H].$$

Observe that to any *product structure*

$$\mathcal{G} = (G_r)_{1 \leq r \leq n}$$

we may associate its abelianisation

$$\mathcal{G}^{ab} = (G_r^{ab})_{1 \leq r \leq n}.$$

Moreover, a product structure $\mathcal{G} = (G_r)_{1 \leq r \leq n}$ gives rise to projection maps

$$\pi_i : G_1 \circ \ldots \circ G_n \to G_i.$$

A subgroup H of $G = G_1 \circ \ldots \circ G_n$ is said to be a *subdirect product* of G (or, more accurately, of $\mathcal{G} = (G_r)_{1 \leq r \leq n}$) when, for all i , $\pi_i(H) = G_i$. Let $\mathcal{S}(G_1, \ldots G_n)$ denote the set of all *normal subdirect products* of $G_1 \circ \ldots \circ G_n$ ; that is, subdirect products which are also normal subgroups. If $\phi : G_1 \circ \ldots \circ G_n \to G_1^{ab} \circ \ldots \circ G_n^{ab}$ denotes the abelianisation map, $\phi$ induces a mapping

$$\phi^{-1} : \mathcal{S}(G_1^{ab} \ldots G_n^{ab}) \to \mathcal{S}(G_1 \ldots G_n)$$

by means of $H \mapsto \phi^{-1}(H)$. In [3], we showed

<u>Theorem 2.1:</u> For any product structure $\mathcal{G} = (G_r)_{1 \leq r \leq n}$

$$\phi^{-1} : \mathcal{S}(G_1^{ab} \ldots G_n^{ab}) \to \mathcal{S}(G_1 \ldots G_n) \quad \text{is bijective.}$$

We shall also need the following result of [3], which is important in the sequel ;

<u>Theorem 2.2:</u> Let H be a normal subdirect product of $G_1 \circ \ldots \circ G_n$. Then H is finitely generated (as a group, *not merely as a normal subgroup* ) if and only if each $G_i$ is finitely generated.

The conclusion of Theorem 2.2 is false if the assumption of normality on H is dropped.

## §3 : A construction for abelian subdirect products :

In this section, we will consider product structures with two factors on abelian groups ; for this reason we will write our groups additively . Thus suppose that

$$A = A_1 \oplus A_2$$

is a product structure on the finitely generated abelian group A, and suppose, moreover, that

(i) $A_2/\operatorname{Tor}(A_2)$ has rank $r_2 \geq 1$, and

(ii) $A_1$ is *free abelian* of rank $r_1 \geq 2$ .

By an *oriented splitting* for $A_1$ we shall mean a triple S of the form $S = (M_S, N_S, \epsilon_S)$ where

$$A_1 = M_S \oplus N_S$$

in which $N_S$ is free of rank 1 , and $\epsilon_S \in N_S$ is a generator. We shall denote by $\mathcal{S}$ the set of all oriented splittings of $A_1$.

Now make choices, once and for all, of a specific splitting

$$A_2/\operatorname{Tor}(A_2) = N' \oplus P$$

where $N'$ is also free of rank 1, and a specific generator $\phi \in N'$. For each $S \in \mathcal{S}$, put

$$\Delta(S) = M_S \oplus < \epsilon_S + \phi > \oplus P$$

and let A(S) denote the preimage of $\Delta(S)$ in $A_1 \oplus A_2$ under the natural mapping

$$A_1 \oplus A_2 \rightarrow A_1 \oplus (A_2/\operatorname{Tor}(A_2)).$$

It is easy to see that each A(S) is a (necessarily normal) subdirect product of $A_1 \oplus A_2$.

The group $\operatorname{Aut}(A_1)$ acts transitively on the set $\mathcal{S}$ of oriented splittings of $A_1$. Moreover, $\operatorname{Aut}(A_1)$ acts on $A_1 \oplus A_2$ by extending its natural action on $A_1$ by the identity on $A_2$. Fix a "basepoint splitting" $T \in \mathcal{S}$ . Since $\mathcal{S}$ is obviously infinite, we obtain the following :

<u>Theorem 3.1:</u> There is a subset $\Theta \subset \mathrm{Aut}(A_1)$ such that

$$\theta(A(T)) \qquad (\theta \in \Theta)$$

is an infinite family of (normal) subdirect products of $A_1 \oplus A_2$ having the property that $\theta(A(T)$ is distinct from $\sigma$ $(A(T))$ for $\theta \neq \sigma$.

## §4 : Infinite families of non-conjugate isomorphic imbeddings :

Let $\Lambda_1$ be a nonabelian free group of finite rank $m \geq 2$, and let $\Lambda_2$ be a finitely generated group such that $\Lambda_2^{ab}$ is infinite. Put $A_i = \Lambda_i^{ab}$ for i = 1 , 2. Since $A_1 \cong \mathbf{Z}^m$, and $A_2$ maps epimorphically onto $\mathbf{Z}$ , we may apply Theorem (3.1) above to obtain the existence of a faithfully indexed infinite family of

$$\theta(A(T))$$

of normal subdirect products of $A_1 \oplus A_2$, where $\theta$ ranges over some subset $\Theta$ of $\mathrm{Aut}(A_1) \cong GL_m(\mathbf{Z}))$. Let

$$\phi : \Lambda_1 \times \Lambda_2 \rightarrow A_1 \times A_2$$

denote the abelianisation map. As we have seen, $\phi$ induces a mapping

$$\phi^{-1} : \mathcal{S}(\Lambda_1 , \Lambda_2) \rightarrow \mathcal{S}(A_1 , A_2)$$

by means of $\Lambda \mapsto \phi^{-1}(\Lambda)$. Put $\Gamma = \phi^{-1}(A(T)$ ; then $\Gamma$ is a normal subdirect product of $\Lambda_1 \times \Lambda_2$, and so is finitely generated by Theorem (2.2). Furthermore, the group $\mathrm{Aut}(\Lambda_1) \times \mathrm{Aut}(\Lambda_2)$ acts naturally on subgroups of $\Lambda_1 \times \Lambda_2$ , and the orbit of $\Gamma$ under this action consists entirely of *normal* subdirect products of $\Lambda_1 \times \Lambda_2$. In fact, we only need consider the orbit of $\Gamma$ under the action of the subgroup $\mathrm{Aut}(\Lambda_1)$ ($\cong \mathrm{Aut}(\Lambda_1) \times \{1\}$) of $\mathrm{Aut}(\Lambda_1) \times \mathrm{Aut}(\Lambda_2)$.

Since $\Lambda_1$ is free, by a theorem of Nielsen [5], every automorphism $\theta$ of $\Lambda_1^{ab} = A_1$ lifts (nonuniquely) to an automorphism $\hat{\theta}$ of $\Lambda_1 \cong \Lambda_1 \times \{1\}$. Put

$$\Gamma_\theta = \hat{\theta}(\Gamma)$$

where for each $\theta \in \Lambda_1^{ab} = A_1$, $\hat{\theta}$ is some chosen lifting for $\theta$. It is clear that each $\Gamma_\theta$ is isomorphic to $\Gamma$. We may summarise our progress as follows ;

Theorem 4.1: Let $\Lambda_1$ be a nonabelian free group of finite rank $\geq 2$, and let $\Lambda_2$ be a finitely generated group which maps epimorphically onto $\mathbf{Z}$ ; there is a subset $\Theta \subset \mathrm{Aut}(\mathrm{A}_1)$ parametrising an infinite family

$$\Gamma_\theta = \hat{\theta}((\mathrm{A}(\mathrm{T})) \qquad (\theta \in \Theta)$$

of mutually isomorphic finitely generated normal subdirect products of $\Lambda_1 \times \Lambda_2$. with the property that $\Gamma_\theta$ is distinct from $\Gamma_\sigma$ for $\theta \neq \sigma$.

The analogue of Theorem (4.1) in which $\Lambda_1$ is replaced by the fundamental group of a closed orientable surface is also true ; we proceed to outline the necessary variations.

Let $\Sigma_+^g$ denote the closed surface of genus g $\geq 2$, and let $\Sigma_g^+$ denote its fundamental group

$$\Sigma_{\mathrm{g}}^+ = \pi_1(\Sigma_+^{\mathrm{g}}).$$

$$\Sigma_{\mathrm{g}}^+ = < \mathrm{X}_1, \ldots, \mathrm{X}_{\mathrm{g}}, \mathrm{Y}_1, \ldots, \mathrm{Y}_{\mathrm{g}}, : \prod_{\mathrm{i}=1}^{\mathrm{g}} [\mathrm{X}_\mathrm{i}, \mathrm{Y}_\mathrm{i}] > .$$

We may identify the abelianisation $\Lambda_1(\Sigma_{\mathrm{g}}^+;\mathbf{Z})$ of $\Sigma_g^+$ with $\mathbf{Z}^{2g}$ ; then the intersection form on $\Sigma_+^g$) gives rise to a nondegenerate symplectic form

$$<,>: \mathbf{Z}^{2\mathrm{g}} \times \mathbf{Z}^{2\mathrm{g}} \rightarrow \mathbf{Z}.$$

With this identification, *symplectic automorphisms* of $\mathbf{Z}^{2g}$ , that is elements of $\mathrm{Sp}_{2\mathrm{g}}(\mathbf{Z})$, lift back to automorphisms of $\Sigma_g^+ = \pi_1(\Sigma_+^g)$, with transvections lifting back to Dehn twists.

Let $\{\epsilon_1, \ldots, \epsilon_g Z^{2g}, \phi_1, \ldots, \phi_g\}$ be the standard symplectic basis for the form $<,>: \mathbf{Z}^{2\mathrm{g}} \times \mathbf{Z}^{2\mathrm{g}} \rightarrow \mathbf{Z}$ ; that is,

$$< \epsilon_i, \epsilon_j > = < \phi_i, \phi_j > = 0 \quad ; \quad < \epsilon_i, \phi_j > = \delta_{ij}.$$

In constructing subdirect products in $\mathrm{A}_1 \oplus \mathrm{A}_2$ as in §3, where now $\mathrm{A}_1 = \mathrm{H}_1(\Sigma_{\mathrm{g}}^+;\mathbf{Z}) \cong \mathbf{Z}^{2\mathrm{g}}$ , we take our "basepoint splitting" T of $\mathrm{A}_1 \cong \mathbf{Z}^{2\mathrm{g}}$ to be of the form

$$\mathbf{Z}^{2\mathrm{g}} = \mathbf{M_T} \oplus \mathbf{N_T}$$

where $\mathrm{Span}_{\mathbf{Z}}\{\epsilon_1, \ldots, \epsilon_{\mathrm{g}}\} \subset \mathrm{M_T}$ , and $\mathrm{N_T} \subset \mathrm{Span}_{\mathbf{Z}}\{\phi_1, \ldots, \phi_{\mathrm{g}}\}$. There is an infinite subset of such splittings which we may parametrise by suitable elements of the group $Sp_{2g}(\mathbf{Z})$. With these modifications, we obtain the following analogue of Theorem (4.1).

<u>Theorem 4.2:</u> Let $\Lambda_1$ be a Surface group of genus $g \geq 2$, and let $\Lambda_2$ be a finitely generated group which maps epimorphically onto $\mathbf{Z}$ ; there is a subset $\Theta \subset$ of $Sp_{2g}(\mathbf{Z})$ parametrising an infinite family

$$\Gamma_\theta = \hat{\theta}((A(T)) \qquad (\theta \in \Theta)$$

of mutually isomorphic finitely generated normal subdirect products of $\Lambda_1 \times \Lambda_2$ with the property that $\Gamma_\theta$ is distinct from $\Gamma_\sigma$ for $\theta \neq \sigma$.

Since the families $(\Gamma_\theta)_{\theta\in\Theta}$ just constructed consist of *normal* subgroups of $\Lambda_1 \times \Lambda_2$, we see that :

<u>Proposition 4.3 :</u> The families $(\Gamma_\theta)_{\theta\in\Theta}$ constructed in Theorems (4.1) and (4.3), possess the property that no two elements are conjugate in $\Lambda_1 \times \Lambda_2$.

Now let $< , > : L \times L \to \mathbf{Z}$ be a nondegenerate symmetric bilinear form on a finitely generated free abelian group L, such that $(L, < , >)$ splits as an orthogonal direct sum

$$(L, <,>) \cong (L_1, <,>) \perp (L_2, <,>).$$

Then

$$\mathrm{Aut}_{\mathbf{Z}}(L_1, <,>) \times \mathrm{Aut}_{\mathbf{Z}}(L_2, <,>) \subset \mathrm{Aut}_{\mathbf{Z}}(L, <,>).$$

Let $\mathbf{G}$ (resp. $\mathbf{G}_i$ ) be the linear algebraic group whose group of k-rational points is $\mathrm{Aut}_k(L \otimes k, <,>)$ (resp. $\mathrm{Aut}_k(L_i \otimes k, <,>)$ , and let $\mathbf{H} = \mathbf{G}_1 \times \mathbf{G}_2$. $\mathrm{Aut}_{\mathbf{Z}}(L_i, <,>)$ is a finitely generated linear group, and so, by Selberg's Theorem [1], has a torsion free subgroup, $\Lambda_i$ say, of finite index. Suppose that $\Lambda_2$ maps epimorphically onto $\mathbf{Z}$, and that $(L_1, <,>)$ has signature (2,1). If $(L \ ,< , >)$ is isotropic, then $\Lambda_1$ is free, whilst if $(L \ ,< , >)$ is anisotropic, $\Lambda_1$ is a Surface group of genus $g \geq 2$. Either way, if $\Lambda_2$ maps epimorphically onto $\mathbf{Z}$ , we may apply the results of Theorems 5 and 6 to conclude that there is an infinite family of mutually isomorphic finitely generated subgroups $\Gamma_\theta$ $(\theta \in \Theta)$ of $\Lambda_1 \times \Lambda_2$. with the property that no $\Gamma_\theta$ is conjugate to any $\Gamma_\sigma$ for $\theta \neq \sigma$. The $\Gamma_\theta$ are still subgroups of $\mathbf{H}_{\mathbf{Z}}$ so that, since $\Lambda_1 \times \Lambda_2$ has finite index in $\mathbf{H}_{\mathbf{Z}}$ , each $\Gamma_\theta$ is conjugate, in $\mathbf{H}_{\mathbf{Z}}$ , to at most finitely many $\Gamma_\sigma$. In particular, we may choose an infinite subfamily $(\Gamma_\sigma)_{\sigma \in \Sigma}$ such that no two distinct elements are conjugate. Thus we have proved ;

Theorem 4.4 : Let $<,> : L \times L \rightarrow \mathbf{Z}$ be a nondegenerate symmetric bilinear form on a finitely generated free abelian group L, such that $(L, <,>)$ splits as an orthogonal direct sum

$$(L, <,>) \cong (L_1, <,>) \perp (L_2, <,>)$$

where $(L_1, <,>)$ has signature (2,1), and $\mathrm{Aut}_{\mathbf{Z}}(L_2, <,>)$ has a subgroup of finite index which maps epimorphically onto $\mathbf{Z}$ . Then there exists an infinite family of isomorphic finitely generated subgroups $(\Gamma_\sigma)_{\sigma\in\Sigma}$ of $\mathrm{Aut}_{\mathbf{Z}}(L, <,>)$ such that $\Gamma_\sigma$ is not conjugate, in $\mathrm{Aut}_{\mathbf{Z}}(L_1, <,>) \times \mathrm{Aut}_{\mathbf{Z}}(L_2, <,>)$ , to $\Gamma_\tau$ for $\sigma \neq \tau$ .

Subgroups $\Gamma_\sigma$ , $\Gamma_\tau$ from the family just constructed , although not conjugate in $\mathrm{Aut}_{\mathbf{Z}}(L_1, <,>) \times \mathrm{Aut}_{\mathbf{Z}}(L_2, <,>)$, may become conjugate in $\mathrm{Aut}_{\mathbf{Z}}(L, <,>)$ . We show, however, that for each $\tau \in \Sigma$, the set

$$\{\sigma \in \Sigma : \Gamma_\sigma \text{ is conjugate to } \Gamma_\tau \text{ in} \mathrm{Aut}_{\mathbf{Z}}(L, <,>)\}$$

is finite.

Theorem 4.5: Let $<,> : L \times L \rightarrow \mathbf{Z}$ be a nondegenerate symmetric bilinear form on a finitely generated free abelian group L, such that $(L, <,>)$ splits as an orthogonal direct sum

$$(L, <,>) \cong (L_1, <,>) \perp (L_2, <,>)$$

where $(L_1, <,>)$ has signature (2,1), and $\mathrm{Aut}_{\mathbf{Z}}(L_2, <,>)$ has a subgroup of finite index which maps epimorphically onto $\mathbf{Z}$ . Then there exists an infinite family of isomorphic finitely generated subgroups $(\Gamma_\omega)_{\omega\in\Omega}$ of $\mathrm{Aut}_{\mathbf{Z}}(L, <,>)$ such that $\Gamma_\omega$ is not conjugate, in $\mathrm{Aut}_{\mathbf{Z}}(L, <,>)$ , to $\Gamma_\mu$ for $\omega \neq \mu$ .

Proof : Let $\Gamma_\sigma$ , $\Gamma_\tau$ be subgroups from the family constructed in (4.4), and suppose that for some $g \in \mathrm{Aut}_{\mathbf{Z}}(L, <,>)\}$

$$g\,\Gamma_\sigma\, g^{-1} = \Gamma_\tau.$$

Since $\Gamma_\sigma$ , $\Gamma_\tau$ are normal subdirect products of $\Lambda_1 \times \Lambda_2$, then, by [3], $[\Lambda_1 , \Lambda_1] \times [\Lambda_2 , \Lambda_2]$ is contained in both $\Gamma_\sigma$ and $\Gamma_\tau$. Moreover, from (1.2), we see that

$$\overline{[\Lambda_i , \Lambda_i]} = \mathbf{G}_i$$

so that

$$\overline{[\Lambda_1 , \Lambda_1]} \times \overline{[\Lambda_2 , \Lambda_2]} = \mathbf{H}.$$

from which it follows that g $\in N_G(\mathbf{H}) \cap Aut_{\mathbf{Z}}(L, <,>)$. Let $\nu$ denote the index of $Aut_{\mathbf{Z}}(L_1, <,>) \times Aut_{\mathbf{Z}}(L_2, <,>)$ in $N_G(\mathbf{H}) \cap Aut_{\mathbf{Z}}(L, <,>)$. By (1.5), $\nu$ is finite, so that, for each $\tau \in \Sigma$, the set

$$C_\sigma = \{\sigma \in \Sigma : \Gamma_\sigma \text{ is conjugate to } \Gamma_\tau \text{ in } Aut_{\mathbf{Z}}(L, <,>)\}$$

is finite, with cardinality bounded by $\nu$. Let $\Omega$ be a subset of $\Sigma$ obtained by choosing exactly one element from each $C_\sigma$ ; then $\Omega$ is infinite, and the family $(\Gamma_\omega)_{\omega\in\Omega}$ consists of isomorphic finitely generated subgroups of $Aut_{\mathbf{Z}}(L, <,>)$, and has the desired property that $\Gamma_\omega$ is not conjugate, in $Aut_{\mathbf{Z}}(L, <,>)$, to $\Gamma_\mu$ for $\omega \neq \mu$. □

By means of a more careful analysis, using the methods of [3], one may show :

<u>Theorem 4.6:</u> Let $<,> : L \times L \to \mathbf{Z}$ be a nondegenerate symmetric bilinear form on a finitely generated free abelian group L, such that $(L, <,>)$ splits as an orthogonal direct sum

$$(L, <,>) \cong (L_1, <,>) \perp (L_2, <,>) \perp \ldots\ldots \perp (L_k, <,>)$$

where $k \geq 2$, and each $(L_i, <,>)$ has signature (2,1); then there exists an infinite family $(\Gamma_\omega)_{\omega\in\Omega}$ of isomorphic, nonconjugate finitely generated subgroups of $Aut_{\mathbf{Z}}(L, <,>)$.

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