ENUMERATIVE GEOMETRY AND INTERSECTION THEORY

An introduction to Fulton's and Macpherson's intersection theory

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ENUMERATIVE GEOMETRY AND INTERSECTION THEORY
(An introduction to FULTON's and MACPHERSON's intersectiion theory)

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Abstract

These notes deal with the following

Intersection construction: Let X be a smooth variety (over a field) of dimension n, U, $V \subseteq X$ subvarieties, $\dim U = k$, $\dim V = \ell$. Then there is a canonical intersection class

$$U \circ V \in A_{k+\ell-n}(U \cap V)$$

(where A_* denotes CHOWhomology) with any good functorial property one might wish. If each component of $U \cap V$ has dimension $k + \ell - n$, it is the cycle

$$U \circ V = \sum_{\lambda} i(Z_\lambda, U \circ V; X)\, Z_\lambda$$

in $A_{k+\ell-n}(U \cap V) = \bigoplus_\lambda \mathbb{Z} \cdot Z_\lambda$, where the Z_λ are the irreducible components of $U \cap V$ and $i(Z_\lambda, U \circ V; X)$ is

the intersection multiplicity of U and V along Z in X as defined by SAMUEL or SERRE.

This is done in the book

[F] : William FULTON, Intersection theory, Springer 1984

The construction of the class $U \circ V$ is in direct geometric terms; there is no need to move either U or V, in fact:

1) no assumption of quasi-projectivity (no need for a 'moving lemma', usually based on the homogeneity of projective space)

2) no need for an a priori theory of intersection multiplicities (these come out as a consequence of the construction)

3) intersection passes to rational equivalence and defines the intersection product

$$\circ \ : \ A_k(X) \otimes A_\ell(X) \longrightarrow A_{k+\ell-n}(X)$$

in the CHOWring $A_*(X)$.

Moreover, this construction is but a special case of a general intersection construction which works in the singular case as well (for a precise stament see 2.1 below).

These are the worked- out notes of a series of lectures I gave in Cologne in the Oberseminar HERRMANN - LAMOTKE in January - February 1984. The topic was to give an account on the basic intersection construction of [F] , together with an overview of motivating ideas and underlying philosophy, coming either from classical enumerative geometry or the papers [5] - [6] (which still needed the assumption on quasiprojectivity, though).

Those interested to go directly to FULTON's and MACPHERSON's construction could read section 2 of Part A and then go directly to Part B . Here, I have tried to give a complete, but as short as possible, presentation of the construction in [F] , where complete means that I present all main lines of thought while refering to [F] for the proofs of some minor statements when they don't add to clarity. Perhaps the only minor deviations from [F] are that I put a little more emphasis on the notion of rational equivalence as varying in a projective family, and on making explicit the role of the THOM isomorphism for vectorbundles in the intersection construction, namely, that its geometric meaning is exactly 'intersecting with the zero section'. In the Appendix to Part A , I give some more details on classical enumerative geometry, mainly based on [10] , and mention some recent results ([1], [2], [3], [7]) - the first real improvements since SCHUBERT's book [11] after more then 100 years.

I want to thank the Max - Planck - Institut für Mathematik for support.

Part A: Motivation

1. Classical intersection theory

1.1. From the beginning of the development of algebraic geometry in the last century the following topics were closely interrelated:

a) enumerative geometry

b) intersection multiplicities

c) rational equivalence of cycles

d) canonical classes (characteristic classes, or CHERN classes).

This comes about as follows. In enumerative geometry, one considers families of geometric figures and asks how many members of the family satisfy given conditions of incidence imposed on them (e.g. pass through given points, be tangent to given lines with given order of contact etc.). A natural restriction, of course, is that the figures and the conditions imposed should be defined by algebraic equations.

Example 1 a) The circles in the plane are parametrized by $\mathbb{P}^3$. The circles tangent to a given circle form a quadric hypersurface in $\mathbb{P}^3$.

b) The plane conics are parametrized by $\mathbb{P}^5$. Given a line ℓ and a conic C , the conics tangent to ℓ form a quadric hypersurface H_ℓ , and those tangent to C a hypersurface H_C of degree 6 (whose explicit equation can be calculated from classical elimination theory as the discriminant of the equations of C and a general conic, see [6]) .

In this way, the given family of figures is considered as as an algebraic variety X , the points of X correspon-ding to the individual members of the family, and the conditions imposed define positive cycles on X whose intersection corresponds to those members of the family which satisfy the given conditions.

An important rôle in the development of these ideas was played by the 'principle of consevation of number' (PCN, also called 'principle of continuity'), stated by PONCE-LET 1822; roughly, the idea behind it was that the number of points in the intersection of cycles should not vary under general change of the parameters on which the cycles depended, so that the solution to an enumerative problem could be obtained by moving the generic situation to a special one by specializing the parameters which one could possibly handle. Cycles in an allowable family of movement should be thought of as being equivalent (with respect to the enumerative problem).

Example 2 a) The problem 'How many lines meet 4 given lines in 3- space' would be solved as follows. Move the first two lines until they intersect, and similarly the second two. Then there are exactly two lines meeting the given four lines:

1st line = intersection line of the planes spanned by the two pairs of lines

2nd line = line connecting the intersection points of the two pairs

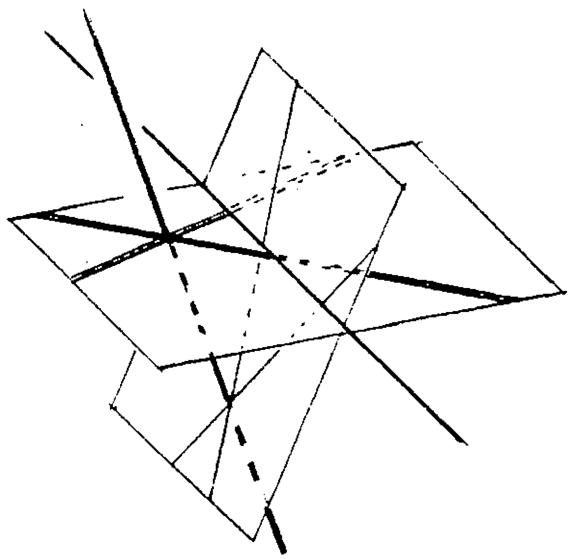


Fig. 1

So the answer is 2 in general (or ∞).

b) 'Given a curve C in the plane, whose defining equation has degree d , and given a family $\mathcal{F}$ of plane curves, how many members of $\mathcal{F}$ are tangent to C ?'

Call this number N . Move C until it decomposes into d lines in general position. Then for a member D of $\mathcal{F}$ being tangent to C means

1) D is tangent to one of the d lines which make up C

or

2) D passes through one of the d(d-1)/2 intersection points of the lines making up C .

Now, by PCN, the number ν of members of $\mathcal{F}$ touching a line in general position is independent of that line and hence a constant of $\mathcal{F}$, and the same is true for the number μ of members passing through a general point. Now the contribution of 1) to N is $d \cdot \nu$, while the contribution of 2) is $d(d-1) \cdot \mu$, each member of $\mathcal{F}$ passing through an intersection point being counted twice, since one thinks of it as a degenerate limit position of two members of $\mathcal{F}$ coming together when moving C to its final position C_{fin} :

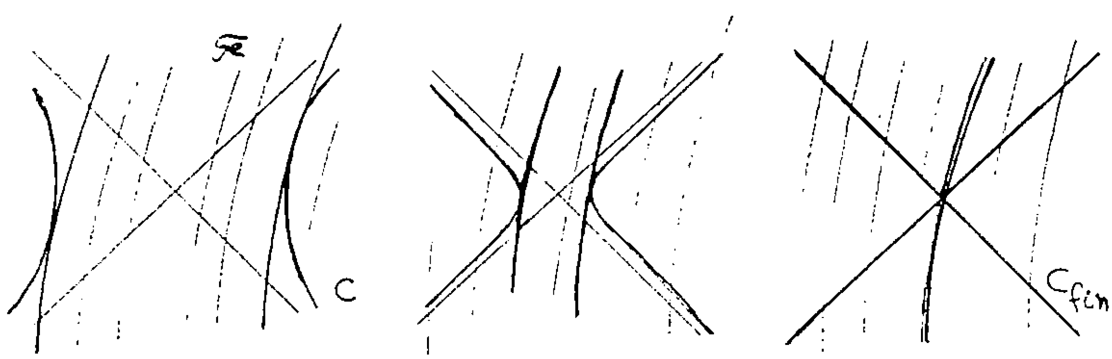


Fig. 2

Thus one arrives at the famous expression of CHASLES(1864)*):

(1) $$\boxed{N = \alpha\cdot\mu + \beta\cdot\nu} ,$$

where α , β depend only on the given condition and were called the <u>characteristics of the condition</u>, and μ , ν depend only on the family, called the <u>characteristics of the family</u>. The right hand side of (1) was called the <u>module of the condition</u>. In our special case, the condition is 'being tangent to C' , with $\alpha = d(d-1)$ and $\beta = d$.

Note the following cases:

(i) Choose $\mathcal{F}$ to be the linear system of lines through a general point. Then one sees easily $\mu = 1$, $\nu = 0$, and so

$$\alpha = \# \{\text{tangents to } C \text{ through a general point}\},$$

a number classically called the <u>class of C</u>, and denoted $d^{\vee}$, since it depends only on the degree d of C ; in fact, we now see $d^{\vee} = d(d-1)$, so that we have derived the classical expression for $d^{\vee}$.

Dually:

$$\beta = \# \{\text{points of } C \text{ on a general line}\},$$

so the number of these points is just the degree of the -

*) this type of classical argumentation has been made rigorous only recently, see e.g. [7] , and the appendix to Part A .

defining equation of C , and therefore classically was called the <u>degree of C</u> . (Incidentally, one should note that it may happen for this definition to make sense with respect to PCN one has to count the intersection points of C with a line with multiplicities; e.g. let C be defined by a degree 2 equation which becomes a square under specialization of the coefficients).

(ii) Choose C to be a conic; then $\alpha = \beta = 2$. Choose $\mathfrak{F}$ to be a linear system of conics (a 'pencil'); e.g. $\mathfrak{F}$ = family of circles touching a line ℓ at a point P :

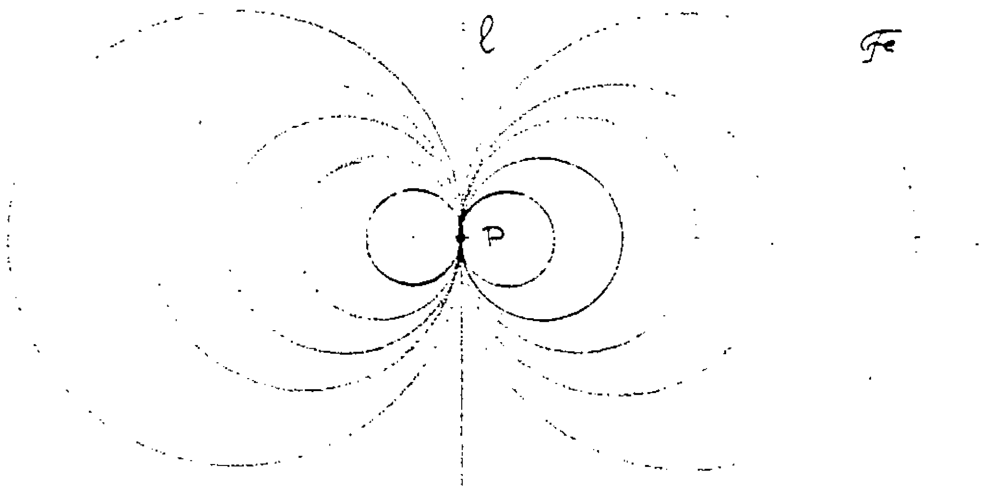


Fig. 3

Then $\mu = 2$, $\nu = 1$, and $N = 2\cdot 2 + 2\cdot 1 = 6$; hence, there are six conics common to the hypersurface H_C of conics touching C and the line $\mathfrak{F}$ in the $\mathbb{P}^5$ parametrizing the conics; in other words

$$\deg H_C = 6 \quad ,$$

a result confirmed by rigorous computation (see example 1, b)).Note how elegantly the geometric argument gives this number, compared to the brute force of computation.

From these simple examples it is clear that, for PCN to be valid, one has to assign multiplicities to the components of intersection of intersecting cycles; the assignment of these multiplicities being done 'with great virtuosity but little explanation' (KLEIMAN) by the 19th century geometers. The systematic procedure of attaching multiplicities turned out to be a difficult problem. Another complication for making the results derived with the help of PCN rigorous was caused by <u>excess intersections</u> (see 1.3 below), this meaning the codimension of intersection being too small in certain 'degenerate' situations, where the enumerative problem forces the corresponding cycles to be in special position.

Besides the desire to make PCN rigorous, a further impact for developing a theory of equivalence of cycles came from the challenge to generalize the theory of the <u>canonical divisor class</u> of a projective variety (i.e. the linear equivalence class of the divisor of a meromorphic differential form of highest degree), known from the beginning of algebraic geometry, to higher codimension; in other words, it came from the desire to define <u>canonical cycles</u> of any codimension, well - defined up to an equivalence generalizing linear equivalence of divisors, which should be intrinsic invariants of the variety encoding some important geometry*) (this problem was solved by TODD and EGER in 1937 for smooth projective varieties; these 'canonical classes' were shown, over $\mathbb{C}$, to give the CHERN classes of the cotangent bundle by NAKANO and SERRE in 1956).

Finally, after more than a century and the work of a great many mathematicians (CHASLES, SCHUBERT, SEVERI, V.D.WAERDEN, CHEVALLEY, WEIL, SAMUEL, CHOW,...) a definite form for intersection theory was developed in the 1958 CHEVALLEY seminar. To put it roughly, one proceeds as follows:

Let there be given a smooth variety X of dimension n .

(i) Given subvarieties U , V of dimension k , ℓ , and a proper component Z of $U \cap V$ (that is, $\dim Z = k + \ell - n$, define the intersection multiplicity $i(Z, U \circ V; X)$ of U and V along Z in X ;

(ii) define 'virtual varieties' (= cycles) and a notion of equivalence between them (<u>rational equivalence</u> = variation in a family parametrized by $\mathbb{P}^1$) ;

(iii) given any two cycles, move one of them in

*) enumerative problems and the theory of canonical (=characteristic)classes are inextricably interwoven. Historically, intinsic invariants were found by enumerative data(e.g.the ZEUTHEN-SEGRE-invariant) thus ultimately leading to the theory of characteristic classes, which nowadays in turn is used to solve enumerative problems. See [9] (the story is too long to be told here).

a system of equivalence as to intersect the other one properly (unfortunately, this forces X to be quasi- projective: CHOW's moving lemma), define the intersection cycle via (i) , and show the result is well- defined up to rational equivalence.

Then the equivalence classes of cycles form a graded ring $A_*(X) = \bigoplus_{k \in \mathbb{Z}} A_k(X)$ (graded by dimension), with the product

(2) $$\circ : A_k(X) \otimes A_\ell(X) \longrightarrow A_{k+\ell-n}(X)$$

given by intersection. This is the <u>CHOWring of X</u> .

In 1975, FULTON generalized this by defining, for possibly singular quasi- projective varieties, CHOW homology groups $A_k(X)$, $k \in \mathbb{Z}$, in a similar way (see [12]) . Furthermore, there are CHOW cohomology groups $A^k(X)$, $k \in \mathbb{Z}$, and there is an intersection pairing ('cap- product')

(3) $$\circ : A^{n-k}(X) \otimes A_\ell(X) \longrightarrow A_{k+\ell-n}(X)$$

generalizing (2) ; see again [12]

1.2. Refined intersections

<u>Refined intersection problem</u> ([5]) : Given subvarieties

U, V of the smooth variety X, $\dim X = n$, $\dim U = k$, $\dim V = \ell$, define, in a 'canonical way' a refined intersection class

$$U \circ V \in A_{k+\ell-n}(U \cap V)$$

with the two properties

1) U and V intersect properly $\Rightarrow$ $U \circ V$ is the intersection cycle of U and V in $A_{k+\ell-n}(U \cap V) = \bigoplus_{Z \text{ irr. comp of } U \cap V} \mathbb{Z} \cdot Z$ as given by the classical theory;

2) it maps to the intersection given by (2) under the mapping $A_*(U \cap V) \longrightarrow A_*(X)$ induced by the inclusion $U \cap V \hookrightarrow X$.

Remark 1 In Algebraic Topology, doing so is possible: If σ, τ are simplicial cycles on a triangulated manifold of (real) dimension k, ℓ, there is a canonical intersection class

$$\sigma \circ \tau \in H_{k+\ell-n}(|\sigma| \cap |\tau|; \mathbb{Z}) .$$

Over $\mathbb{C}$, there is a natural transformation $A_* \longrightarrow H_{ev}(\ ; \mathbb{Z})$, under which the constructions should correspond.

Remark 2 (SEVERI) There are situations, where a refined intersection class is needed to solve enumerative problems (see Example 1 revisited, b), below, and, for detailed exposition, [5] ,[6] , and [F] , 10.4):

<u>Example 1 revisited</u> a) Ask (APPOLONIUS) 'How many circles in the plane are tangent to three given circles in general position?'

The number of these is given by the intersection number $\deg(H_1 \circ H_2 \circ H_3)$ of the three quadrics H_i , $i = 1,2,3$, in $\mathbb{P}^3$ representing the families of circles tangent to just one general circle, and hence, by BÉZOUT, the number should be

$$2^3 = 8 \quad ,$$

which classically is known to be right (of course, such a simple answer is only true when allowing complex solutions).

b) Ask (STEINER 1848) 'How many non singular conics in the plane are tangent to five given conics in general position?'

Let C_i , $i = 1,\ldots,5$ be the five given conics. As we have seen, the conics tangent to a C_i form a hypersurface H_i in $\mathbb{P}^5$ of degree 6 ; so, arguing as in the APPOLONIUS problem, the asked- for number should be

$$6^5 = 7776 \quad \text{(STEINER-BISCHOFF 1859)} ,$$

which is wrong. Instead of analyzing this situation here (for the history I refer to the article of Steven KLEIMAN, 'Chasles's enumerative theory of conics. A historical introduction', in: Studies in algebraic geometry, Math. Ass. Am. Stud. Math. 20(1980), 117-138, and for the correct solution also to this article as well as to [F] and [6]) consider the following analogous problem where the same reasoning gives a number which plainly is absurd: ask the question 'How many non singular conics in the plane are tangent to five given lines?' The BÉZOUT argument of above then yields the number

$$2^5 = 32 ,$$

which obviously is wrong (just consider the dual problem; by elementary analytic geometry, there is a unique conic passing through five general points, so the number should be 1). The reason for this is that even if the five lines ℓ_i , $i = 1,\dots,5$, are in general position, the five corresponding hypersurfaces H_i in $\mathbb{P}^5$ whose points parametrize the conics tangent to the ℓ_i <u>never are</u>; in fact, they all contain the VERONESE surface $V \subseteq \mathbb{P}^5$ representing the double lines (V consists of the conics with the homogeneous equation $(a_0z_0+a_1z_1+a_2z_2)^2 = 0$, hence of those conics whose homogeneous coordinates in $\mathbb{P}^5$ are just the degree two monomials in a_0,a_1,a_2 , so

$V = \mathbb{P}^2$ embedded in $\mathbb{P}^5$ via the usual VERONESE embedding) as a scheme theoretic component. Hence, the number 32 has no enumerative significance, since it is the intersection number of the six hypersurfaces H_i, in special position, when moving them into general position, which destroys the geometric meaning of the intersection. Instead, one has to isolate the contribution of V to the intersection number, which turns out to be 31, leaving 1 as the true solution. The same analysis applies to the STEINER- BISCHOFF- problem; here, the VERONESE V contributes 4512 thus giving as the correct answer 7776 - 4512 = 3264. This analysis was done by FULTON and MACPHERSON in [6], based on their solution to the refined intersection problem (see also the chapters 9 and 10 of [F]). This solution to the refined intersection problem was based on the classical intersection theory and hence depended on the assumption of quasi- projectivity; but a closer analysis shows that it yields a new approach to defining *ab initio* intersections with the only prerequisite a theory for intersecting with *divisors*, which amounts to a theory of the first CHERN class of a line bundle. This point of view is worked out in the book [F] and will be described in Part B of these notes; for motivation of the construction, I describe in the following the solution of the refined intersection problem given in [5] and [6].

Before doing so, however, I would like to make some more remarks on classical enumerative geometry, especially on the impressive work of SCHUBERT and some modern approaches to making his spectacular results rigorous, as far as it is connected with the correctness of the number 3264 of solutions of the STEINER- BISCHOFF-problem ; for more details see the appendix.-

This number appeared in print for the first time in the work of CHASLES (1864) , but was rederived independently several times by others. CHASLES based his derivation on his famous expression $\alpha\mu+\beta\nu$; for an outline of his argument see also the appendix.
The first derivation
meeting our today standards of rigour was given only recently, namely by KLEIMAN in the 1980 paper cited above (it existed in preprint form in 1974).

In his famous book 'Kalkül der abzählenden Geometrie' SCHUBERT based his computations on a vast generalization of CHASLES's expression $\alpha\mu+\beta\nu$ to the case of the enumeration of varieties in a p - parameter - family ($p \in \mathbb{N}$) touching p given varieties in general position to obtain such spectacular results as

1) $\#$ $\{$quadrics in 3- space touching 9 given quadrics in general position$\}$ = 666,841,088 (confirmed 1982 by VAINSENCHER and DE CONCINI - PROCESI [17]).

2) # { twisted cubics in 3- space touching 12 given quadrics in general position } = 5,819,539,783,680 (as far as I know not confirmed up to the present date, but PIENE and others are working on this case)

In modern terms, SCHUBERT's general formula by which he arrives at these results is as follows: Let $V \subseteq \mathbb{P}^N$ be a projective variety (which we allow to be reducible, but it should be reduced) . Define the module m_V for the simple condition of 'being tangent to V ' to be the following formal expression in the indeterminates $\mu_0, \dots, \mu_{N-1}$:

$$m_V \quad := \quad \alpha_0(V)\mu_0 + \dots + \alpha_{N-1}(V)\mu_{N-1}$$

with $\alpha_i(V)$:= (weighted) number of tangent spaces to $V_{reg} \cap H^{(i)}$ meeting a given general (N-i-2)- plane and limits of such

= degree of the polar locus of V of dimension i (classically the i-th class of V),

where V_{reg} is the manifold of regular points of V and $H^{(i)}$ a general plane of codimension i . Let N be the number of varieties in a p - parameter family touching p given varieties $V_1, \dots, V_p$ in general position; then SCHUBERT's generalization of 1.1 (1) is the formula

(1) $$\boxed{N = \prod_{i=1}^{p} m_{V_i}} \quad ,$$

with the following prescription: Expand the right hand side into a sum of monomials $\mu_o^{j_o} \dots \mu_{N-1}^{j_{N-1}}$, and use the interpretation

(2) $\mu_o^{j_o} \dots \mu_{N-1}^{j_{N-1}}$:= number of varieties in the given family simultaneously touching j_o 0- planes, j_1 1- planes,...,j_{N-1} (N-1)-planes in general position.

The case $p = 1$, $N = 2$ and the varieties being conics (or more generally curves) gives back the famous CHASLES expression 1.1 (1) .

The numbers (2) are again called the characteristics of the given family , and solving the given enumerative problem basically consists of computing these characteristics, which is very difficult in general (this being the reason that 1) has been confirmed only recently and 2) is still open.) In his book, SCHUBERT fills page after page with tables for the determination of these numbers, which are, in modern terms, CHERNnumbers of parameterspaces or suitable blowups of these, and are worked out by SCHUBERT in many special cases by means of a subtle analysis of suitable degenerations into special configurations with impressive zeal. So the problem of enumeration of contacts has been broken up into two steps:

Step 1 : Reduction to a linear problem of determining the contacts with linear spaces; this reduction is given by the formula (1)

Step 2 : Determination of the characteristics (2) ; this is the difficult part and generally requires a good description for the parameter space of the family and the computation of its CHOWring.

For instance, in the case of the STEINER- BISCHOFF- problem, the correct parameterspace is not the space $\mathbb{P}^5$ of all conics in the plane, but $\mathbb{P}^5$ blown up along the VERONESE, which is the space of complete conics in the sense of STUDY (see the discussion in the appendix, based on KLEIMAN's article cited above); this is equivalent to the validity of (1) in this case.

For a modern and rigorous treatment of SCHUBERT's formula (1) see [7].

I now finally turn to the description of the solution to the refined intersection problem given in [5] , [6] , where classical intersection theory is assumed to be given. So let X be a smooth quasi- projective variety of dimension n , U , V $\subseteq$ X subvarieties of dimensions k , ℓ .

We want to define a canonical intersection class $U \circ V$ in $A_{k+\ell-n}(X)$.

For this, we may assume V smooth; for, if this case is settled, the general case follows by considering $U \times V$ and the diagonal Δ_X in $X \times X$ and observing $(U \times V) \cap \Delta_X \cong U \cap V$ via the diagonal map (this standard trick is known under the heading 'reduction to the diagonal').

Now, to find a possible candidate for $U \circ V$, use the following

<u>Heuristic principle</u>: Intersections should be 'preserved under reasonable deformations' ([6] , p. 12).

In fact, for any inclusion $V \overset{i}{\hookrightarrow} X$ of smooth varieties, there <u>is</u> a very reasonable deformation, which deforms i into the inclusion $V \hookrightarrow \nu^X_V$ of V as the zero section of its normal bundle ν^X_V (philosophically, the normal bundle should be thought of as the algebraic geometer's infinitesimal substitute for the topologist's tubular neighbourhood). If X is quasi- projective, this deformation may be visualized as follows. Choose a vectorbundle $E \xrightarrow{\pi} X$ and a section $s : X \longrightarrow E$ which defines V scheme- theoretically. Identifying X with the zero section, and thinking of the total space E as a 'box' containing X, we have the following picture:

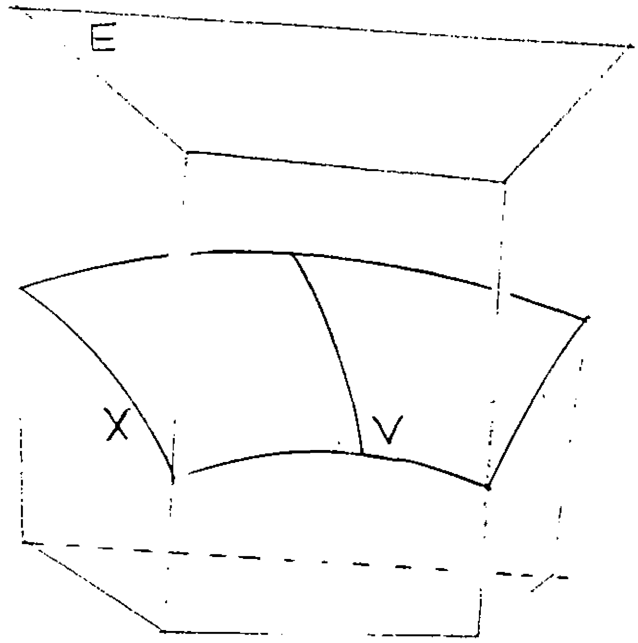


Fig. 4

We consider the section $s_t := \frac{1}{t} s$, $t \in \mathbb{A} - \{0\}$, put $i_t := V \hookrightarrow X_t$ to be the inclusion of V in $X_t := \operatorname{im} s_t \subseteq E$ and push X_t to infinity; in other words, we form

$$P := \overline{X \times \mathbb{A}^*} \quad \text{in} \quad E \times \mathbb{A} ,$$

where $X \times \mathbb{A}^*$ is embedded as a locally closed subscheme in $E \times \mathbb{A}$ via $(x,t) \mapsto (\frac{1}{t} s(x),t)$, and the bar denotes schematic closure. The projection $pr_2 : E \times \mathbb{A} \longrightarrow \mathbb{A}$ restricts to $p : P \longrightarrow \mathbb{A}$, and provides us with the commutative diagram

$$\begin{array}{ccc} V \times \mathbb{A} & \xhookrightarrow{j} & P \\ & {\scriptstyle pr_2} \searrow \quad \swarrow {\scriptstyle p} & \\ & \mathbb{A} & \end{array}$$

It then can be shown that

(i) p is a flat map,

(ii) Pulling (3) back to $t \in \mathbb{A}$ gives an embedding $j_t : V \hookrightarrow P_t$ with $P_t := p^{-1}(t)$. Then

a) For $t \neq 0$, this embedding is isomorphic to the embedding $i_t : V \hookrightarrow X_t$ (and hence to the embedding $i : V \hookrightarrow X$, since $i_t \cong i$ via $\pi | X_t$)

b) For $t = 0$, this embedding is isomorphic to the embedding of V as the zero section of the normal bundle ν_V^X of V in X .

The diagram (3) is referred to as 'deformation to the normal bundle' and is due to MACPHERSON. The following figure schematically sketches some stages of the deformation process:

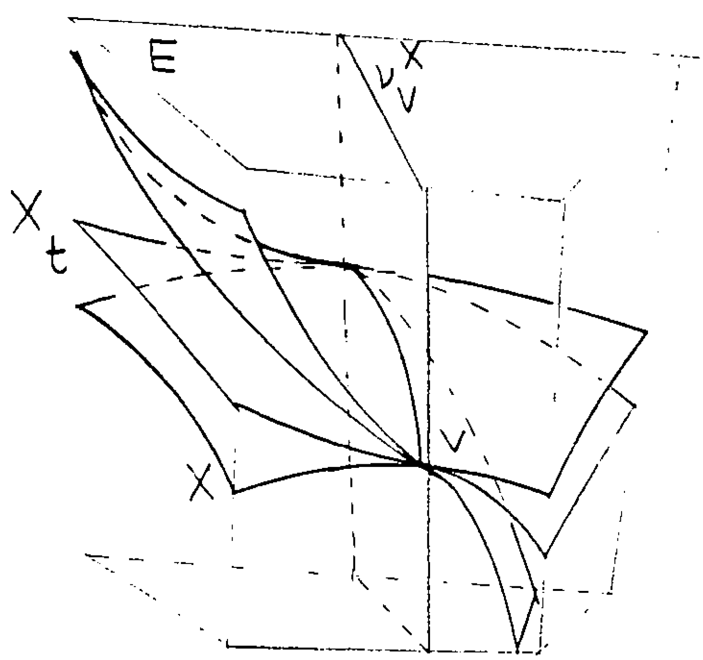


Fig. 5

Now suppose U is any subvariety of X. Then $s(U)$ defines $U \cap V$ in U scheme- theoretically, and the part of U outside $U \cap V$ gets pushed away to ∞ in the deformation for $t \to 0$, while $U \cap V$ remains fixed. In fact, one may show that under $t \to 0$ the inclusion $U \hookrightarrow X$ goes to the inclusion $C_{U \cap V} U \hookrightarrow \nu_V^X$, where $C_{U \cap V} U$ is the <u>normal cone of</u> $U \cap V$ <u>in</u> U.

To make this more explicit, recall the definition of the normal cone. For this, let U be any scheme and W a closed subscheme, defined by the coherent $\mathcal{O}_U$ - ideal $\mathcal{J}$ (all schemes will be algebraic schemes over a field k. Locally, choosing generators $f_1,\dots,f_d$ of $\mathcal{J}$ amounts to representing W locally as the fibre $f^{-1}(0)$ of the morphism $f : U \longrightarrow \mathbb{A}^d$ defined by the f_i's. Geometrically, one thinks of the differentials df_i as defining coordinates on the normal space of W in U, and since one wants to keep track of the relations between them, one thinks of the normal space $C_W U$ as being locally given by the relations between the differentials of the equations defining W in U, i.e. in the setting above to be locally the subscheme of $W \times \mathbb{A}^d$ defined by the relations between the df_i. This shows that $C_W U$ is a cone, this meaning that the equations defining it are homogeneous in the inderminates corresponding to the coordinates on $\mathbb{A}^d$, which accounts for the name 'normal cone'. A little thinking shows that this informal description amounts to the definition

$C_W U$:= spectrum of the graded $\mathcal{O}_W$ - algebra $\bigoplus_{k \geq 0} \mathcal{J}^k / \mathcal{J}^{k+1}$.

We now return to the situation U , $V \subsetneq X$ with V smooth. Now, given independent equations $f_1, \ldots, f_d$ for V in X (locally) , the $f_i | U$ define $U \cap V$ in U (but need not be independent). The epimorphism

$$\mathcal{O}_{U \cap V}[df_1, \ldots, df_d] \longrightarrow\!\!\!\!\rightarrow \bigoplus_{k \geq 0} \mathcal{J}^k / \mathcal{J}^{k+1}$$

given by $f_i \mapsto f_i | U$ then corresponds to the inclusion

(4) $$C_{U \cap V} U \hookrightarrow \nu_V^X | U \cap V$$

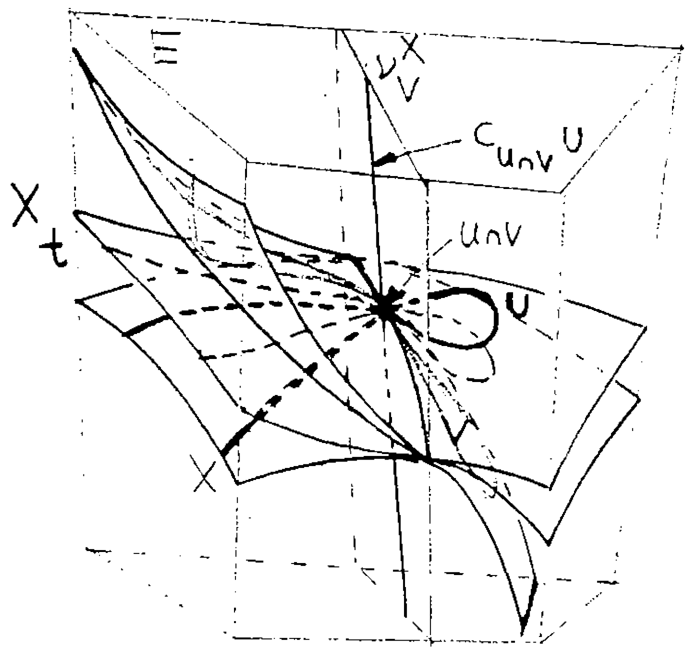


Fig. 6

<u>Remark</u> Note that it is important to regard $U \cap V$ as a

scheme; otherwise (4) would not always hold.

Example As an example, consider the following simple situation:

$$X := \mathbb{A}^2 \ , \quad U := \{xy = 0\} \ , \quad V := \{x-y = 0\}$$

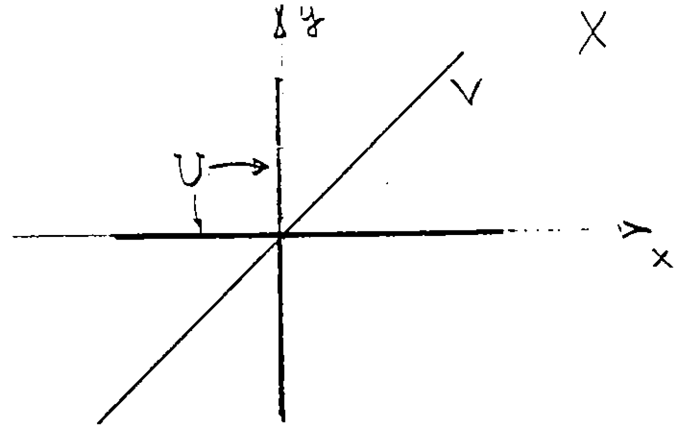


Fig. 7

Then $U \cap V$ has the equation $x^2 = 0$ in U ; set-theoretically, $(U \cap V)_{red} = \{0\}$, whereas $U \cap V$ has to be thought of as a 'fat point' of multiplicity two. One has

$$C_{(U \cap V)_{red}} U = U$$

$C_{U \cap V} U$ = a double line, namely spec($A[t]$), where A is the ARTIN ring $A = k[x]/(x^2)$

(The last line comes from the general fact that, if U = spec R , and the ideal $I \subseteq R$ defining the subscheme W of U is generated by the regular sequence $(f_1,\dots,f_d)$, $C_W U$ = spec $(R/I[t_1,\dots,t_d])$ for indeterminates $t_1,\dots,t_d$.)

These considerations are confirmed by the deformation to the normal bundle, which in this case can easily be worked out explicitely:

$E \longrightarrow X$ is the trivial bundle $\mathbb{A}^2 \times \mathbb{A} \longrightarrow \mathbb{A}^2$

$s : X \longrightarrow E$ is given by $(x,y) \longmapsto (x,y,x-y)$

If $E \times \mathbb{A} = \mathbb{A}^4$ has coordinates (x,y,w,t) , the embedding $X \times \mathbb{A}^* \hookrightarrow E \times \mathbb{A}$ is given by

$$\begin{aligned} x &= x \\ y &= y \\ w &= \tfrac{1}{t}(x-y) \\ t &= t \quad , \end{aligned}$$

and the total space P of the deformation is the hypersurface $x-y = wt$ in $\mathbb{A}^4$. For fixed t , $P_t = \{(x,y,w,t) \mid x-y = wt\}$, a plane. Choosing $u := x+y$ and w as coordinates on P_t , the image of U in P_t has the equation $u^2-w^2t^2 = 0$, and putting $t = 0$ gives the equation $u^2 = 0$ for $C_{U\cap V}U$ in $P_0 = \nu^X_V = X$.

The behaviour of the embeddings $U \hookrightarrow P_t$ for $t \to 0$ can then be visualized as follows:

For fixed t , P_t is a 2 - plane in $\mathbb{A}^3$ with coordinates (x,y,w) , so we imagine the various P_t as a family of 2 - planes in 3 -space. P_0 is the plane spanned by the line $x-y = 0$ in the (x,y) - plane and the w - axis. For $t \neq 0$, U is embedded as a pair of distinct lines passing through the origin of P_t and lying over U embedded in the (x,y) - plane, and for $t \to 0$, these two lines get pushed more and more away from the (x,y) - plane, until they both coalesce into the w - axis, which makes it plausible that $C_{U\cap V}U$ should be a <u>double</u> line.

An analogous example is: $X := \mathbb{A}^2$, $U := \{y^2(y+1)-x^2 = 0\}$, $V = \{y = 0\}$

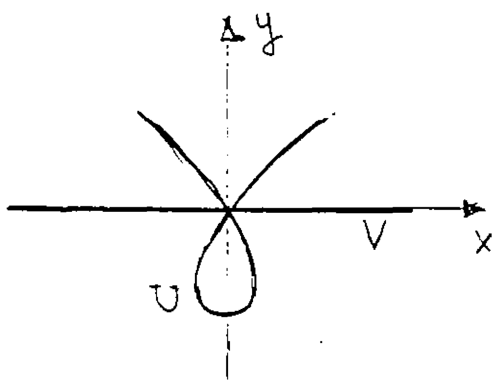


Fig. 8

We now return to the refined intersection problem. By the heuristic principle on p.21 , applied to the deformation into the normal bundle, we conclude: Intersecting V and U on X should be the same as intersecting V and $C_{U\cap V}U$ on ν^X_V .

We now suppose the total space ν^X_V to be quasi- projective. (In fact, for the general intersection construction this is no restriction at all, since, by the trick of reducing to the diagonal, to intersect general U and V means to intersect $U \times V$ and the diagonal Δ_X in $X \times X$, and if X is quasi- projective, so is $\nu^{X\times X}_X = TX$, the tangent bundle of X).

Now, by (4) , $C_{U\cap V}U \hookrightarrow \nu^X_V|U\cap V$. By quasi- projectivity, we can move the zero section of $\nu^X_V|U\cap V$ as to classically intersect with $C_{U\cap V}U$ on $\nu^X_V|U\cap V$, i. e. we can form

$$[U \hat{\cap} V] \circ [C_{U\cap V}U] \in A_{k+\ell-n}(\nu^X_V|U\cap V)$$

under FULTON's capproduct 1.1 (3) :

$$A^{n-\ell}(\nu^X_V|U\cap V) \otimes A_k(\nu^X_V|U\cap V) \longrightarrow A_{k+\ell-n}(\nu^X_V|U\cap V) .$$

And, by the projection $\nu^X_V|U\cap V \longrightarrow U\cap V$, we can push this class down to $A_{k+\ell-n}(U\cap V)$.

This class is the answer to the refined intersection problem.

Remark If $E \xrightarrow{\pi} X$ is a vectorbundle over a variety X, pushing down cycles by π does not immediately makes sense, since π is not proper. Since, however, the normal cone $C_{U\cap V}U$ has a canonical extension to the projective completion $\mathbb{P}(\nu \oplus \mathbb{1})$ of $\nu := \nu_V^X \mid U\cap V$, we can push down via the projection $p : \mathbb{P}(\nu \oplus \mathbb{1}) \longrightarrow U\cap V$, which is proper. We will incorporate this remark in the generalization of the refined intersection construction which we are going to describe now.

For this, we first make a

Digression on cones Let X be a variety, or more generally an algebraic scheme. Consider a positively graded $\mathcal{O}_X$ - algebra $\mathcal{S}$ and assume:

(5) $\mathcal{S}$ is locally of finite presentation, i.e. X can be covered by open sets U so that over each U there is an exact sequence

$$\mathcal{O}_U[T_1',\dots,T_l'] \to \mathcal{O}_U[T_1,\dots,T_k] \to \mathcal{S}|U \to 0$$

of graded $\mathcal{O}_U$ - algebras, where the T_i, T_j' have degree one.

Then the cone $\pi : C \longrightarrow X$ defined by $\mathcal{S}$ is defined to be

$\mathrm{spec}\,\mathcal{S} \longrightarrow X$. This means the following: Over U as in 2) we have a presentation

$$\mathcal{S}|U \cong \mathcal{O}_U[T_1,\dots,T_k]/(f_1,\dots,f_\ell)$$

for $f_1,\dots,f_\ell \in \mathcal{O}_X(U)[T_1,\dots,T_k]$. The f_j define the subscheme $C_U \subseteq U \times \mathbb{A}^k$, and define $\pi_U : C_U \longrightarrow U$ to be the restriction of the first projection. These local pieces π_U then glue into the global $\pi : C \longrightarrow X$.

Furthermore, there is defined the projective cone $p : \mathbb{P}(C) \longrightarrow X$ defined by $\mathcal{S}$ as $\mathrm{proj}(\mathcal{S}) \longrightarrow X$. Locally, p_U is defined by the f_j in $U \times \mathbb{P}^{k-1}$. The line bundle $\mathcal{O}(1)$ on $\mathbb{P}^{k-1}$ lifts to $U \times \mathbb{P}^{k-1}$ and restricts to $\mathbb{P}(C)_U$, and these locally defined line bundles glue into a global line bundle $\mathcal{O}(1) \longrightarrow \mathbb{P}(C)$, called the canonical line bundle.

Finally, let $\mathcal{S}$ be given and put a positive grading on $\mathcal{S}[t]$, t an indeterminate, by letting t having degree one. Then the projective cone defined by $\mathcal{S}[t]$ is called the projective completion of $p : C \longrightarrow X$ and denoted $\bar{p} : \mathbb{P}(C \oplus 1) \longrightarrow X$ or $\bar{p} : \bar{C} \longrightarrow X$.

As the most important example to us consider an ideal $\mathcal{J} \subseteq \mathcal{O}_X$ defining $Y \subseteq X$; then $\mathcal{S} := \bigoplus_{k \geq 0} \mathcal{J}^k/\mathcal{J}^{k+1}$ defines the normal cone as described on p. 24 .

We now return to the intersection construction. We generalize the refined intersection construction in the situation

$$\begin{array}{ccc} U \cap V & \hookrightarrow & U \\ \cap\downarrow & & \downarrow\cap \\ V & \hookrightarrow & X \end{array}$$

to the following construction(assume all schemes to be quasi-projective):

<u>Refined intersection construction</u> Let X be a variety of dimension n. Consider the cartesian square

(6)
$$\begin{array}{ccc} W & \overset{j}{\hookrightarrow} & U \\ \downarrow & & \downarrow f \\ V & \underset{i}{\hookrightarrow} & X \end{array}$$

where i is a regular embedding of codimension d (i.e. the ideal $\mathcal{J} \subseteq \mathcal{O}_X$ defining V is locally generated by a regular sequence of d elements) and $f : U \longrightarrow X$ is any morphism, $\dim U = k$. Let $\nu_V^X \longrightarrow V$ be the normal bundle of i, $\nu \xrightarrow{\pi} W$ the lift of ν_V^X to W, and $p : \mathbb{P}(\nu \oplus \mathbb{1}) \longrightarrow W$ the projective completion of π. Let $\overline{\pi} : \overline{C_W U} \longrightarrow W$ be the completed normal cone of j; then $C_W U \subseteq \mathbb{P}(\nu \oplus \mathbb{1})$ (see p. 31).

The zero section $W \subseteq V \subseteq \mathbb{P}(V \oplus \mathbb{1})$ defines a class in $A^d(\mathbb{P}(V \oplus \mathbb{1}))$ (see 12), and the refined intersection class $U \circ V$ is defined to be

(7) $$U \circ V \; := \; p_*([W] \circ [C_W U]) \in A_{k-d}(W)$$

under the cap product $\circ : A^d(P) \otimes A_k(P) \longrightarrow A_{k-d}(P)$ with $P := \mathbb{P}(V \oplus \mathbb{1})$ (see 1.1 (3)).

1.3 <u>Refined intersections and enumerative geometry</u>. In this section I will sketch how refined intersections can be used to solve the enumerative problems in 'Example 1 revisited', b), on p. 14. The references are [6] and [F], 9.1. So let X be a smooth, quasi- projective variety, and let $U_i \subseteq X$, $i = 1,\dots,r$ be subvarieties of dimension k_i, regularly embedded, then the refined intersection construction above applied to the diagram

(1) $$\begin{array}{ccc} U_1 \cap \dots \cap U_r & \longrightarrow & X \\ \downarrow & & \downarrow \delta \\ U_1 \times \dots \times U_r & \longrightarrow & \underbrace{X \times \dots \times X}_{r} \end{array},$$

with δ the diagonal, gives rise to a refined intersection class $U_1 \circ \dots \circ U_r \in A_p(U_1 \cap \dots \cap U_r)$, $p := n - \sum_{i=1}^{r} k_i$, $n = \dim X$. If the Z_λ are unions of connected components and a partition of $U_1 \cap \dots \cap U_r$, $A_p(U_1 \cap \dots \cap U_r)$ decomposes as $\bigoplus_\lambda A_p(Z_\lambda)$, and correspondingly $U_1 \circ \dots \circ U_r$ decomposes as $\sum_\lambda (U_1 \circ \dots \circ U_r)^{Z_\lambda}_Z$ for uniquely defined classes $(U_1 \circ \dots \circ U_r)^{Z_\lambda} \in A_p(Z_\lambda)$, called the equivalences of Z_λ for the intersection $U_1 \circ \dots \circ U_r$.

For a complete variety Y there is the degree map $\deg : A_o(Y) \longrightarrow \mathbb{Z}$ mapping a o-cycle to its sum of coefficients. We consider the case $\sum_{i=0}^{r} k_i = n$, so $p = 0$, and X projective. Define the intersection number

$$U_1 \cdot \dots \cdot U_r \;:=\; \deg(U_1 \circ \dots \circ U_r) ,$$

then it decomposes as

$$U_1 \cdot \dots \cdot U_r \;=\; \sum_\lambda (U_1 \cdot \dots \cdot U_r)^{Z_\lambda}$$

for uniquely defined numbers $(U_1 \cdot \dots \cdot U_r)^{Z_\lambda}$. Now, if $U_1 \cap \dots \cap U_r$ decomposes as a union of connected components Z common to all the U_i and isolated points P, we get the formula

$$\text{(1a)} \qquad \sum_P i(P) \;=\; U_1 \cdot \dots \cdot U_r \;-\; (U_1 \cdot \dots \cdot U_r)^Z$$

where $i(P) = i(P, U_1 \dots U_r; X)$ is the intersection multiplicity of the U_i at P (here we assume, for simplicity, that the ground field is algebraically closed). Now, $U_1 \cdot \dots \cdot U_r$ is often known for global reasons since it can be computed in $A_*(X)$ (e.g. by BÉZOUT), so if one can determine $(U_1 \cdot \dots \cdot U_r)^Z$, one gets at least a formula for the weighted number of points in $U_1 \cap \dots \cap U_r - Z$, and in favourable cases, where the U_i meet transversally at the isolated points P (i.e. $i(P) = 1$ for all P), a formula for $\#(U_1 \cap \ldots \cap U_r - Z)$.

Now, in order to isolate the equivalence $(U_1 \circ \ldots \circ U_r)^Z$ and to compute $(U_1 \cdot \ldots \cdot U_r)^Z := \deg(U_1 \ldots U_r)^Z$, one expresses the intersection product by means of characteristic classes, thus being able to use the powerful manipulating machinery of characteristic classes and to perform actual computations.

Digression on characteristic classes: CHERN and SEGRE classes

In what follows, X can be a quasi- projective variety, an arbitrary variety or even an algebraic scheme over a field;*) references for the following discussion are [12], [5], [6] for the quasi- projective case and F for the general case; for the manipulations with characteristic classes in the smooth case, see also the article [9].

Let $n := \dim X$. X has CHOW homology groups $A_k(X)$, CHOW cohomology groups $A^\ell(X)$ and a pairing, the cap product

$$(2) \qquad \cap : A^\ell(X) \otimes A_k(X) \longrightarrow A_{k-\ell}(X)$$

between them. Furthermore, $A^+(X)$ is a graded ring; there is a cup product

*) we need only the quasi - projective case

(2a) $$\cup \; : \; A^k(X) \otimes A^\ell(X) \longrightarrow A^{k+\ell}(X) \quad .$$

The fundamental class $[X] \in A_n(X)$ defines the POINCARE duality map $D_X \; : \; A^k(X) \xrightarrow[\cap[X]]{} A_{n-k}(X)$, which is an isomorphism for X non - singular; in fact, $A^k(X) \; = \; A_{n-k}(X)$ for smooth quasi - projective X by construction. Under this identification, both (2) and (2a) correspond to the intersection product $\circ$, so we will in general denote $\cap$ and $\cup$ also by $\circ$, the latter also simply by $\cdot$.

The $A_k(-)$ are covariant functors for proper maps. The $A^k(-)$ are contravariant:

Let $f \; : \; X' \longrightarrow X$ be a map of smooth varieties X' , X of dimension n' , n . Then there are defined GYSIN homomorphisms

(3) $$f^* \; : \; A^k(X) \longrightarrow A^k(X')$$

for each k by putting, for a k -codimensional subvariety $W \subsetneq X$

$$f^*[W] := \; p_*([\Gamma_f]\circ[X'\times W])$$

where $\Gamma_f \subseteq X' \times X$ is the graph of f and $p \; : \; X' \times X \longrightarrow X'$ the first projection. In the singular case, there also is f^* , and, moreover when f is flat, there is

(3a) $$f^* \; : \; A_i(X) \longrightarrow A_{n'-n+i}(X')$$

corresponding to (2) under duality if X , X' are smooth (see Part B, 1) .

Let $E \longrightarrow X$ be a vectorbundle of rank r . Then E has CHERNclasses $c_i(E) \in A^i(X)$, $i = 0,1,2,\ldots$ with the following properties:

1) <u>Normalization</u> a) $c_o(E) = 1$, $c_i(E) = 0$ for $i > r$.

So, if $E = L$ a line bundle, $c_i(L) = 0$ for $i > 1$. Then

b) If $L \cong L_D$, the line bundle associated to a divisor D (see Part B, 2.3),

$$c_1(L_D) \frown [X] = i_* [D]$$

where $i : D \hookrightarrow X$ is the inclusion and $[D]$ the fundamental class of D (see Part B , 2.2).

We put $c(E) := 1 + c_1(E) + c_2(E) + \ldots \in A^*(X)$ and call it the total CHERNclass of E .

2) <u>Functoriality</u> If $f : X' \longrightarrow X$ is any morphism,

$$c_i(f^*E) = f^*(c_i(E))$$

for all i .

3) <u>WHITNEY sum formula</u> If

$$0 \longrightarrow E'' \longrightarrow E \longrightarrow E' \longrightarrow 0$$

is an exact sequence of vectorbundles on X ,

$$c(E) = c(E') \circ c(E'') \quad .$$

1) and 2) settle uniqueness of CHERN classes by the so-called splitting principle. A slick construction can be given by means os SEGRE classes; see [F] and (8) below.

We need the following generalization of 1) b) : Let $s : X \longrightarrow E$ embed X via a section. If X , and hence E , is smooth, there is

$$s^* \; : \; A_k(E) \longrightarrow A_{k-r}(X)$$

by (3) . If X is singular, s^* is not defined via (3a) since s need not be flat. However, even then s^* can be canonically defined(see 2.1 below; this possibility is the key to constructing an intersection theory without a moving lemma). Then

4) <u>'Selfintersection formula'</u> : For all $\alpha \in A_*(X)$

$$c_r(E) \cap \alpha = s^* s_*(\alpha) \quad .$$

We now compute the refined intersection class. First a general remark: If $E \longrightarrow X$ is a rank r vectorbundle and $p : \mathbb{P}(E \oplus \mathbb{1}) \longrightarrow X$ the projective completion, there is the tautological line bundle $\mathcal{O}_{E \oplus \mathbb{1}}(-1) \subseteq p^*(E \oplus \mathbb{1})$ (see Part B , 3.2), and so the exact sequence

$$(4) \qquad 0 \longrightarrow \mathcal{O}_{E \oplus \mathbb{1}}(-1) \longrightarrow p^*(E \oplus \mathbb{1}) \longrightarrow Q \longrightarrow 0$$

defines a rank r bundle on $p^*(E \oplus \mathbb{1})$, called the canonical quotient bundle Q . The embedding $\mathbb{1} \hookrightarrow E \oplus \mathbb{1}$ defines a map $\mathbb{1} \longrightarrow p^*(E \oplus \mathbb{1})$ and hence a section of $p^*(E \oplus \mathbb{1}) \longrightarrow \mathbb{P}(E \oplus \mathbb{1})$ vanishing precisely on $X \hookrightarrow P(E \oplus \mathbb{1})$ embedded via the zero section $X \hookrightarrow E$, and from 4) results

$$(5) \qquad c_r(Q) \cap [\mathbb{P}(E \oplus \mathbb{1})] \;=\; [X] \quad \text{in} \quad A_*(\mathbb{P}(E \oplus \mathbb{1})) \,.$$

(This is in fact the only case of 4) we need and can be given a direct proof with the setup of Part B , 3.2 , see [F] , 3.3). Turning to the situation (6) in 1.2, we get by (7) of 1.2 and (5) above:

$$U \circ V \;=\; p_*(c_d(Q) \cap j_*[\overline{C_W U}])$$

where $j : \overline{C_W U} \hookrightarrow \mathbb{P}(\nu \oplus \mathbb{1})$ the inclusion, and Q the rank d quotient bundle on $P(\nu \oplus \mathbb{1})$. Consider the following commutative diagram:

$$\begin{array}{ccc} \overline{C_W U} & \overset{j}{\hookrightarrow} & \mathbb{P}(\nu \oplus \mathbb{1}) \\ & {\scriptstyle \bar{\pi}} \searrow \quad \swarrow {\scriptstyle p} & \\ & W & \end{array}$$

and conclude, since $c_d(Q) \cap j_*[\overline{C_W U}] = j_*(j^* c_d(Q) \cap [\overline{C_W U}])$,

$$U \circ V = \bar{\pi}_*(j^* c_d(Q) \cap [\overline{C_W U}])$$

$$= \bar{\pi}_*(c_d(j^* Q) \cap [\overline{C_W U}]) \text{ .}$$

The exact sequence (4) for $E = \nu$ restricts via j to the exact sequence

$$0 \to \mathcal{O}_{\overline{C_W U}}(-1) \to \bar{\pi}^* \nu \oplus \mathbb{1} \to j^* Q \to 0 \text{ ,}$$

thus

$$c_d(j^* Q) = (c(\bar{\pi}^* \nu \oplus \mathbb{1}) \cdot c(\mathcal{O}_{\overline{C_W U}}(-1))^{-1})_d \text{ ,}$$

where $(\ )_d$ denotes taking the piece in graduation d .
Thus

$$(6) \qquad U \circ V = \bar{\pi}_*(\bar{\pi}^* c(\nu \oplus \mathbb{1}) \cdot c(\mathcal{O}_{\overline{C_W U}}(-1))^{-1})_{k-d}$$

$$= (c(\nu) \cap \bar{\pi}_*(c(\mathcal{O}_{\overline{C_W U}}(-1))^{-1} \cap [\overline{C_W U}]))_{k-d} \text{ .}$$

The class on the right hand side depends only on the cone $C_W U$ and leaads to the following definition.

<u>Definition</u> Let $\pi : C \longrightarrow X$ be a cone on X . The SEGRE class $s(C) \in A_*(X)$ is defined to be

$$s(C) := \bar{\pi}_*(c(\mathcal{O}_{\bar{C}}(-1))^{-1} \cap [\bar{C}])$$

where $\bar{\pi} : \bar{C} \longrightarrow X$ is the projective completion of π .

These classes have the following properties:

(i) For a vectorbundle E :

(7) $$c(E)^{-1} \cap [X] = s(E)$$

It follows that for smooth X this completely determines $c(E)$ via $s(E)$, and, in fact, is a good <u>definition</u> for CHERN classes. For singular X , one may define refined SEGRE classes $s^*(E)$ characterized by the properties

$$s^*(f^*E) = f^*s^*(E)$$

for all morphisms $f : X' \longrightarrow X$ and all bundles E on X , and

(8) $$s^*(E) \cap \alpha = \bar{\pi}_*(c(\mathcal{O}_{\mathbb{P}(E \oplus 1)}(-1))^{-1} \cap \bar{\pi}^* \alpha)$$

$$= p_*(c(\mathcal{O}_{\mathbb{P}(E)}(-1))^{-1} \cap p^* \alpha)$$

for all $\alpha \in A_*(X)$, where $p : \mathbb{P}(E) \longrightarrow X$ is the projective bundle of E .

Now the properties 1) and 2) of the CHERN classes define CHERN classes for line bundles and hence refined SEGRE classes for vectorbundles. Then put

$$(8) \qquad c(E) := s^*(E)^{-1} \quad .$$

This definition then gives all CHERN classes, and (7) remains true in the singular case.

(ii) If C is pure dimensional and $p : \mathbb{P}(C) \longrightarrow X$ is surjective, we have in generalization of the second equality in (8)

$$(9) \qquad s(C) = p_*(c(\mathcal{O}_{\mathbb{P}(C)}(-1))^{-1} \cap [\mathbb{P}(C)])$$

$$= \sum_{i \geqslant 0} p_*(c_1(\mathcal{O}_{\mathbb{P}(C)}(1))^i \cap [\mathbb{P}(C)])$$

(iii) Let Y be a subscheme of X. Then the SEGRE class $s(X,Y) \in A_*(Y)$ of Y in X is defined to be

$$s(X,Y) := s(C_Y X) \quad ,$$

where $C_Y X$ is the normal cone of Y in X. It follows:

a) If the embedding $Y \hookrightarrow X$ is regular (especially if Y is a submanifold of the manifold X) with normal bundle ν ,

$$(10) \qquad s(X,Y) = c(\nu)^{-1} \cap [Y] \quad .$$

b) In general, if X is a variety:

(11)
$$s(X,Y) = \sum_{i\geq 0} p_*(c_1(\mathcal{O}_{\mathbb{P}C_YX}(1))^i \cap [\mathbb{P}C_YX])$$

with $p : \mathbb{P}C_YX \longrightarrow Y$ the projective normal cone. It follows that the SEGRE classes can be computed by blowing up: Let $\pi : \tilde{X} \longrightarrow X$ be the blowup of X along Y, $E := \pi^{-1}(Y) \subseteq \tilde{X}$ be the exceptional divisor. Then $\pi|E : E \longrightarrow Y$ is just $p : \mathbb{P}C_YX \longrightarrow Y$, and $\mathcal{O}_{\mathbb{P}C_YX}(-1) = \mathcal{O}(E)$, or in previous notation L_E, the line bundle associated to E. Put $e := c_1(\mathcal{O}_{\mathbb{P}C_YX}(1)) \in A^1(E)$. The refined intersection construction applied to the diagram

$$\begin{array}{ccc} E & = & E \\ \| & & \downarrow \\ E & \hookrightarrow & X \end{array}$$

gives $E^2 := E \circ E \in A_{n-1}(E)$, $n = \dim X$, and by (6) $E \circ E = -e \cap [E]$. Hence, by iterating, we get for the k - fold selfintersection $E^k := E \circ \ldots \circ E \quad A_{n-k}(X)$ the formula $E^k = (-1)^{k-1} e^k \cap [E]$. There results

(12)
$$s(X,Y) = p_*(c(\mathcal{O}(-E))^{-1} \cap [E])$$

$$= \sum_{i\geq 0} (\pi|E)_*(e^i \cap [E])$$

$$= \sum_{i \geq 0} (-1)^{i-1} (\pi|E)_*([E^i]) \quad .$$

c) SEGRE classes are covariant for birational maps: If $Y \subseteq X$, $f : X' \longrightarrow X$ is birational , and $Y' := f^{-1}(Y)$, then $s(X,Y) = g_* s(X',Y')$, with $g = f\,Y'$.

Since we need this result, I give the proof (see [6]):

1) If Y is a divisor, one has by (11)

$$s(X,Y) = \sum_{i \geq 0} c_1(\mathcal{O}(Y))^i \cap [Y] \ .$$

Then, if both Y and Y' are divisors, $\mathcal{O}(Y') = f^* \mathcal{O}(Y)$, and $g_*[Y'] = [Y]$, so $g_*(c_1(\mathcal{O}(Y'))^i \cap [Y']) = g_*(f^* c_1(\mathcal{O}(Y))^i \cap [Y']) = c_1(\mathcal{O}(Y))^i \cap g_*[Y']$, which implies the claim .

2) If f is the blowup of X along Y , it is just (11) .

In general, form the diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\pi'} & X' \\ {\scriptstyle \tilde{f}} \downarrow & & \downarrow {\scriptstyle f} \\ \tilde{X} & \xrightarrow[\pi]{} & X \end{array}$$

where π blows up Y and π' blows up Y' ; $\tilde{f}$ is the unique morphism making the diagram commute existing by the universality of the blowup π . The result then follows from 1) and 2) .

The definition of SEGRE classes thus being exploited, we return to (6) and get the

Main intersection formula In the situation of 1.2 , (6) :
(13) $\qquad U \circ V \quad = \quad (c(\mathcal{V}) \cap s(U,W))_{k-d}$

This formula may now be used to manipulate intersection products and compute the equivalence of the VERONESE in the examples above. Before doing so, we end this digression by making some calculations for projective space used below.

So let $\mathbb{P}^N$ be given as the space of lines in $\mathbb{A}^{N+1}$, and let $\mathcal{O}(-1) \longrightarrow \mathbb{P}^N$ be the canonical (tautological) line bundle. Then its dual $\mathcal{O}(1)$ is the line bundle $\mathcal{O}(H)$:= L_H associated to the divisor of a hyperplane H , i.e. to the divisor defined by a linear equation. Put, for an integer d , $\mathcal{O}(d) := \mathcal{O}(1)^{\otimes d}$ for positive and $\mathcal{O}(-1)^{\otimes(-d)}$ for the negative d ; then $\mathcal{O}(d) \cong \mathcal{O}(V)$ for any hypersurface defined by a homogeneous polynomial of degree d , $d > 0$. One has $A^*(\mathbb{P}^N) = \mathbb{Z}[t]/(t^{N+1})$ as a ring under the correspondence $h \longleftrightarrow t$, where $h := c_1(\mathcal{O}(1)) \in A^1(\mathbb{P}^N)$ and so $h = [H] \in A_{N-1}(\mathbb{P}^N)$ for any hyperplane H*). It follows that $c_1(\mathcal{O}(V)) = dh$ for V a hypersurface defined by an equation of degree d . The usual exact sequence

*) therefore h is called the hyperplane class

$$0 \longrightarrow \mathbb{1} \longrightarrow \mathcal{O}(1)^{\oplus(N+1)} \longrightarrow T\mathbb{P}^N \longrightarrow 0$$

gives

(14) $$c(T\mathbb{P}^N) = (1 + h)^{N+1}$$

Secondly, let $V \subseteq \mathbb{P}^N$ be a subvariety of codimension k. Then $V = dh^k$ in $A_{N-k}(\mathbb{P}^N)$ for a unique integer d, since that group is freely generated by h^k. The integer d is called the degree $\deg(V)$ of V; one has

$$\deg(V) = \deg(h^k \cap [V])$$

with $h^k \cap [V] \in A_o(\mathbb{P}^N)$ and $\deg : A_o(\mathbb{P}^N) \longrightarrow \mathbb{Z}$ the degree mapping (augmentation). Thus

$$\deg(V) = \deg([H]^k \circ V) ,$$

so by 1.1 $\deg(V)$ has the interpretation of the number of intersection points of V with a k - codimensional hyperplane intersecting properly, and properly counted. E.g. if V is a hyperplane defined by an equation of degree d, this says that V has d intersection points with a line, which surely was to be expected and accounts for the name degree of a variety. In this case, the normal bundle of V in $\mathbb{P}^N$ is the line bundle $\mathcal{O}(V)|V =: \nu$, and so

(15) $$c(\nu) = 1 + dh .$$

We now are in a position to compute the SEGRE class we are interested in. So consider, thirdly, the d - fold VERONESE embedding $v_d : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$, $N := \binom{n+d}{d}$, which is defined by mapping a point with homogeneous coordinates $z_o,\dots,z_N$ to the point whose coordinates are all monomials in the z_i of degree d . Since any linear equation lifts under v_d to an equation of degree d there follows $v_d^*(h_N) = dh_n$, where $h_N \in A^1(\mathbb{P}^N)$ and $h_n \in A^1(\mathbb{P}^n)$ are the hyperplane classes. Identifying $\mathbb{P}^n$ with its image V in $\mathbb{P}^N$ (the 'd - fold VERONESE') and putting $h := h_n$ there results from the exact sequence

$$0 \longrightarrow TV \longrightarrow T\mathbb{P}^N|V \longrightarrow \nu \longrightarrow 0$$

where ν is the normal bundle of V in $\mathbb{P}^N$:

(16)
$$\begin{aligned} s(\mathbb{P}^N,V) &= c(\nu)^{-1} \\ &= v_d^* c(T\mathbb{P}^N)^{-1} \cdot c(V) \\ &= (1+dh)^{-(N+1)}(1+h)^{n+1} . \end{aligned}$$

E.g. for the VERONESE embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ there results, using the formal identity $(1-x)^{-(k+1)} = \sum_{n\geq 0}\binom{n+k}{k} x^n$ and $h^3 = 0$:

(17)
$$\begin{aligned} s(P^5,V) &= (1+2h)^{-6}(1+h)^3 \\ &= (1-12h+84h^2)(1+3h+3h^2) \\ &= 1-9h+51h^2 . \end{aligned}$$

<u>Remark</u> It is amusing to note that these computations could have been carried out classically using the theory of complete conics and the computation (12) of SEGRE classes using the blowup. In fact, as explained in the appendix, the blowup $\pi : X \longrightarrow \mathbb{P}^5$ of $\mathbb{P}^5$ along V identifies X with the space of complete conics in the plane in the classical sense. As explained further there, $A^*(X)$ is generated by two classes m , $n \in A^1(X)$, where m is the lift of h_5 ; and if $E := \pi^{-1}(V)$ is the exceptional divisor, $l := i_*([E]) \in A_4(X) = A^1(X)$, where $i : E \hookrightarrow X$ is the inclusion, one has the formulae

$$(18) \qquad 2m = n + l$$

$$(19) \quad m^5=n^5=1 \ ; \ m^4n=mn^4=2 \ ; \ m^3n^2=m^2n^3=4 \quad \text{in } A^5 = A_0 = \mathbb{Z} \ .$$

These are modern interpretations of classical formulae (see appendix (13) and (25) , and (13a) and p.); (18) relates to the number of coincidences of a certain correspondence on a line, and the numbers $m^i n^{5-i}$ are the number of (complete) conics passing through i general points and touching $5-i$ general lines. Now, by (12) :

$$s(P^5,V) = \sum_{k=0}^{2} \left((-1)^k \left((\pi|E)_*([E^{3+k}])\right) \cdot h^{2-k}\right) h^k$$

and, since $2h = h_5|V$, one has

$$(\pi|E)_*(E^{3+k}) \, h^{2-k} = 1/2^{2-k} \, l^{3+k} m^{2-k} \ .$$

Plugging $l = 2m - n$ by (18) into this, one should have

$$(2m-n)^3 m^2 = 4$$

$$(2m-n)^4 m = 18$$

$$(2m-n)^5 = 51 ,$$

which can be checked by (19) .

This ends our digression on characteristic classes, and we now solve our enumerative problems via refined intersections. Consider the intersection diagram (1) : The main intersection formula (13) gives for the equivalence $(U_1 \circ \dots \circ U_r)^Z$ if Z is a union of connected components of $U_1 \cap \dots \cap U_r$ and ν_i the normal bundle of U_i in X :

$$(20) \qquad (U_1 \circ \dots \circ U_r)^Z = \left(\prod_i c(\nu_i | Z) \cap s(X,Z) \right)_p$$

We now first consider the case of five general lines ℓ_i , $i = 1,\dots,5$ in the plane, defining the five hypersurfaces H_i of conics tangent to ℓ_i , which are hypersurfaces of degree 2 in the $\mathbb{P}^5$ parametrizing the conics. In order to compute and decompose $s(\mathbb{P}^5,W)$, where W is the scheme- theoretic intersection $H_1 \cap \dots \cap H_5$, we apply the birational invariance of SEGRE classes ((iii),c)) :

$$(21) \qquad s(\mathbb{P}^5,W) = (\pi|\theta)_* s(X, \pi^{-1}W) ,$$

where $\pi : X \longrightarrow \mathbb{P}^5$ is the blowup of $\mathbb{P}^5$ along the VERONESE V. We have

$$\pi^{-1}W = \pi^{-1}H_1 \cap \dots \cap \pi^{-1}H_5$$

Now, if $H \subseteq \mathbb{P}^5$ is a hypersurface, we get an equation of divisors

$$\pi^{-1}H = \tilde{H} + \lambda \cdot E$$

where $E = \pi^{-1}V$ is the exceptional divisor, λ a suitable natural number, and $\tilde{H} := \overline{\pi^{-1}(H - V)}$ is the strict transform of H ; this corresponds just to writing locally on X $f = e^{\lambda} \cdot \tilde{f}$, where f is a local equation of H on $\mathbb{P}^5$ lifted to X , e a local equation of E and $\tilde{f}$ a local equation of $\tilde{H}$. In our case, we have $H = H_i$, and then $\lambda = 1$; this can be checked,once one has an equation for H , by using the equations of blowing up a submanifold. To obtain an equation of H , choose coordinates $[x:y:z]$ on $\mathbb{P}^2$ as to have $\ell = \ell_i$ being given by $z = 0$; then the conic $ax^2+bxy+cxz+dy^2+eyz+fz^2 = 0$ with coordinates $[a:b:c:d:e:f]$ touches ℓ if and only if $b^2-4ad = 0$, which is an equation for H .

It follows from this that, if $f_1,\dots,f_5$ are local equations for $H_1,\dots, H_5$, $\pi^{-1}W$ is locally defined by the ideal $(f_1,\dots,f_5) = (e)\cdot(\tilde{f}_1,\dots,\tilde{f}_5)$, and so, if e,, $f_1,\dots,f_5$ are coprime, by the ideal $(e)\cap(f_1,\dots,f_5)$, so that in this case E is a scheme theoretic component of $\pi^{-1}W$. Now it can be shown, as explained in the appendix, that E may be interpreted as the set of lines (corresponding to the conics which are double lines) together with two points on it called foci (which may coincide), and a point in $E \cap H_i$ is such a line with ℓ_i passing through one of the foci. Thus, if no three ℓ_i pass through one point, $E \cap H_1 \cap .. \cap H_5 = \emptyset$, and this same condition guarantees that no degenerate conic consisting of two distinct intersecting lines is tangent to all ℓ_i, because for those 'being tangent to ℓ_i' means 'ℓ_i passing through the point of intersection'. Thus all points on $\tilde{H}_1 \cap \ldots \cap \tilde{H}_5 = H_1 \cap .. H_5 - V$ represent nonsingular conics, and since the projective group PGL(3) operates transitively on the nonsingular conics, it follows by a general theorem of KLEIMAN that by suitably moving the H_i these points are isolated points (even of transversal intersection). So then we are in the situation of (1a), and W decomposes as V and isolated points P. So Z = V in (20), and we get

$$(H_1 \circ \ldots \circ H_5)^V = \left(\prod_i c(\nu_i|V) \cap s(\mathbb{P}^5,V)\right)_0$$

$$= (((1+2h_5)^5|V)(1-9h+51h^2)_0 \quad ((15),(17))$$

$$= ((1+4h)^5(1-9h+51h^2))_0$$

$$= ((1+20h+160h^2)(1-9h+51h^2))_0$$

$$= 160h^2 - 180h^2 + 51h^2 \quad .$$

So $(H_1 \cdot \ldots \cdot H_5)^V = 31$, and, since by BÉZOUT, $H_1 \cdot \ldots \cdot H_5 = 2^5 = 32$, there follows by (1a)

$$\sum_P i(P) = 32 - 31$$

$$= 1 \; ,$$

leaving exactly one point for the true solution (One may show the above condition is already sufficient for having only isolated solutions outside V).

The case of conics touching five given conics C_i, $i = 1, \ldots, 5$, in the plane is similar. We use the same notations as above, with the $H_i := H_{C_i}$, the hypersurface of conics touching C_i, now of degree 6 ; an explicit equation is given in [3]. We then have

$$\pi^{-1}H = \tilde{H} + 2E$$

for $H = H_i$; this can either be shown using the eplicit equation, or by appealing to the following fact coming from the theory of complete conics: If the hypersurface $H \subseteq \mathbb{P}^5$ represents a simple condition with characteristics α, β, then $\lambda = \beta$ in (21) . This allows as to write locally for the ideal defining W on X $(e^2) \cap (\tilde{f}_1, \ldots, \tilde{f}_5)$ as soon as $E \cap H_1 \cap \ldots \cap H_5 = \emptyset$. This is the case under the following conditions coming again from the description of E above:

(i) No three C_i pass through a point

(ii) there is no line with two points on it such that each C_i is either tangent to the line or passes through one of the points .

So in this case there is a connected scheme theoretic component Z of W , supported on V , with $\pi^{-1}Z = 2E$ as a divisor, and its contribution to $s(\mathbb{P}^5,W)$ is, analogous to (21),

$$s(\mathbb{P}^5,Z) = (\pi|E)_* s(X, \pi^{-1}Z)$$

$$= (\pi|E)_*(c(\mathcal{O}(-2E))^{-1}\cap[E]) .$$

Since $s(\mathbb{P}^5,V) = (\pi|E)_*(c(\mathcal{O}(-E))^{-1}\cap[E])$, we have, as $c_1(\mathcal{O}(2E))^i = 2^i c_1(\mathcal{O}(E))^i$, for the components of degree k

$$s_k(\mathbb{P}^5,Z) = 2^{3+k}s_k(\mathbb{P}^5,V) ,$$

and so

$$s(\mathbb{P}^5,Z) = 2^3\cdot 1 - 2^4\cdot 9\cdot h + 2^5\cdot 51\cdot h^2 .$$

Thus $$(H_1\circ\ldots\circ H_5)^Z = ((1+6\cdot 2h)^5(8-144h+1632h^2))_0$$

$$= ((1+60h+1440h^2)(8-144h+1632h^2))_0$$

$$= 1632h^2-60\cdot 144h^2+8\cdot 1440h^2$$

$$= 1632h^2+20\cdot 144h^2$$

$$= 1632h^2+2880h^2$$

$$= 4512h^2 \quad .$$

Thus, by (1a) , if $H_1 \cap \dots \cap H_5 - V$ consists of isolated points only,

$$\sum_P i(P) = 6^5 - 4512$$

$$= 7776 - 4512$$

$$= 3264 \quad .$$

In fact, by the same transversality arguments as above, the P will represent nonsingular conics with $i(P) = 1$ for $(C_1,\dots,C_5)$ in a nonempty ZARISKI- open subset of $(\mathbb{P}^5)^5$, so that generically there are exactly 3264 different conics touching five given conics, and they all are non singular. Precise conditions are, in addition to (i) and (ii) the following (see [6] and [F], example 9.1.9)

(iii) no two of the five conics are tangent ,

(iv) the pairs of lines , each one of them tangent to two of the conics, do not intersect on the third ;

then the points in $H_1 \cap \dots \cap H_5 - V$ are isolated and represent only nonsingular conics; one has, if P represents the conic C :

$$i(P) = \prod_i (4 - \#(C \cap C_i)) \quad .$$

We finally comment on the equality $\pi^{-1}H = \tilde{H} + \beta E$ when H is a hypersurface in $\mathbb{P}^5$ representing a simple condition with characteristics (α , β) ; for more details, I refer to the appendix. This uses the fact that $A^1(X)$, X being the blowup of $\mathbb{P}^5$ along V, is freely generated by $m := \pi^* h$, where $h = h_5 \in A^1(\mathbb{P}^5)$ is the hyperplane class, and $l := [E]$, the class of the exceptional divisor.

First, let H be the hyperplane of conics tangent to a line. Then we have seen above that $\pi^{-1}H = H + E$ as divisors and so $\pi^*[H] = [\tilde{H}] + l$ in $A^1(X)$. On the other hand, H has degree 2 , and so $\pi^*[H] = 2m$. If we put $n := [\tilde{H}]$, we have $2m = n + l$, and m and n also freely generate $A^1(X)$.

Secondly, let H be arbitrary, of degree d . Then the characteristics are given by writing $[\tilde{H}] = \alpha m + \beta n$. On the other hand, $[H] = dh$ in $A^1(\mathbb{P}^5)$ and so $\pi^*[H] = dm$. Putting this into $\pi^*[H] = [\tilde{H}] + \lambda l$, using $2m = n + l$ and equating coefficients gives $\lambda = \beta$, as desired, and, in addition, $d = \alpha + 2\beta$. So if H is the hypersurface of conics tangent to a line, $\alpha = 0$, $\beta = 1$, and H is of degree $2 = 0 + 2\cdot 1$, which checks, and if H is the hypersurface of conics to a conic, H has characteristics $\alpha = 2$, $\beta = 2$, and is of degree $6 = 2 + 2\cdot 2$, which also checks.

2. Intersection theory revisited

2.1. Redefining intersections. The above discussion of refined intersections shows upon a little thinking that the only things needed in the construction of the refined intersection class are:

a) a theory of rational equivalence (CHOWgroups) on possibly singular varieties

b) the possibility of, given a vectorbundle (of rank r, say) $E \to X$ over a possibly singular variety, 'intersecting with the zero section', that means, given a class $\alpha \in A_k(E)$, to produce a class $X \circ \alpha \in A_{k-r}(X)$ in a canonical way.

Establishing a) and b) is the technical heart of the intersection construction in [F] and will be exposed rigorously (though, of course, not in complete detail) in Part B below. As to b), the underlying philosophy is as follows: since $\pi : E \to X$ is flat, it induces the flat pullback

$$\pi^* : A_{k-r}(X) \longrightarrow A_k(E)$$

by lifting the local equations of a $(k-r)$ - dimensional subvariety V on X to local equations on E via π, thus getting a closed subscheme W of E, and then mapping V to the class represented by W in $A_k(E)$ (see Part B, 1.2 and 1.6). Then one would like to have:

$$(1) \qquad X \circ \pi^*(\xi) = \xi$$

for any $\xi \in A_{k-r}(X)$. Now there is the easy (see Part B, 3.1)

<u>Lemma</u> ([F], Proposition 1.9) π^* <u>is always surjective</u>.

So, if (1) is to hold, π^* should better be an isomorphism; and that it is, is a central result for the theory. In fact we have the

<u>THOM isomorphism theorem</u> ([F], Theorem 3.3). <u>For any vectorbundle</u> $E \xrightarrow{\pi} X$ <u>of rank</u> r, <u>the flat pullback</u>

$$\pi^* : A_{k-r}(X) \longrightarrow A_k(E)$$

<u>is an isomorphism</u>.

Here, there is no assumption on quasi- projectivity of X. The proof will be given in Part B.

Corollary. *Let* $E \xrightarrow{\pi} X$ *be as in the theorem, and let* $s : X \longrightarrow E$ *embed* X *as the zero section. Then there is, for any* k *, a unique homomorphism*

$$s^* : A_k(E) \longrightarrow A_{k-r}(X)$$

with $s^* \circ \pi^* = id_{A_{k-r}(X)}$. s^* *is called 'intersecting with the zero section'.*

We have $X \circ \alpha = s^*(\alpha)$ by (2) .

Thus, granting a) and b) , intersection theory proceeds as follows. Recall the situation of the refined intersection problem in 1.2 : U , V subvarieties of the smooth, quasi-projective n- dimensional variety X of dimensions k , ℓ , and with V smooth. This gives the cartesian square

$$\begin{array}{ccc} U \cap V & \hookrightarrow & U \\ \downarrow & & \downarrow \\ V & \hookrightarrow & X \end{array} ,$$

and we produced an intersection class $U \circ V \in A_{k+\ell-n}(U \cap V)$. This situation is generalized as follows: Let there be given a cartesian square

$$\begin{array}{ccc} f^{-1}V = W & \overset{j}{\hookrightarrow} & U \\ g \downarrow & & \downarrow f \\ V & \underset{i}{\hookrightarrow} & X \end{array}$$

of schemes (all schemes are algebraic over a given field) with:

1) i is a regular embedding of codimension d, i.e. the ideal subsheaf of $\mathcal{O}_X$ defining V is locally generated by a regular sequence of d elements;

2) U is pure dimensional, of dimension k, say;

3) f is any morphism.

We then have the following basic construction.

<u>Main intersection construction</u> ([F], 6.1) : The intersection class

$$U \circ V \in A_{k-d}(W) \quad ,$$

$d = \text{codim } i$, $k = \dim U$, is constructed as follows: Let the $\mathcal{O}_X$ - ideal $\mathcal{J}$ define i and $(\mathcal{J}/\mathcal{J}^2)^\vee = \nu_V^X$ be the normal bundle of V in X (here we identify a vectorbundle with the locally free sheaf of its sections), a d- dimensional vectorbundle on V . Let $\nu := g^* \nu^X_V$ be the lift of ν^X_V to W . Then

$$C_W U \hookrightarrow \nu \quad ,$$

this inclusion corresponding to the epimorphism of graded $\mathcal{O}_W$ - algebras

$$\oplus\, g^*(\mathcal{I}^k/\mathcal{I}^{k+1}) \twoheadrightarrow \oplus\, \mathcal{J}^k/\mathcal{J}^{k+1}$$

where $\mathcal{J} := f^{-1}\mathcal{I} = \mathcal{I}\cdot\mathcal{O}_U$ is the ideal sheaf generated by $\mathcal{I}$ in $\mathcal{O}_U$ via f which defines W in U . Let s be the zero section of ν . Then define the intersection product $U \circ V$ of U and V to be

$$U \circ V \;:=\; W \circ [C_W U] \;=\; s^*[C_W U] \in A_{k-d}(W)$$

(where the square brackets denote rational equivalence classes in the CHOWgroups).

<u>Remark</u> 1 This construction passes to rational equivalence: Given a cartesian square of schemes

$$\begin{array}{ccc} V' & \hookrightarrow & X' \\ \downarrow & & \downarrow f \\ V & \underset{i}{\hookrightarrow} & X \end{array}$$

with i a regular embedding of codimension d and f any morphism, there is a refined GYSINhomomorphism

(2) $$i^{!} : A_k(X') \longrightarrow A_{k-d}(V')$$

given by $i^{!}[U] := [U \circ V]$ for any k - dimensional subvariety $U \subseteq X'$. It generalizes (1) and (2) of 2.4 in Part B below. For details and functorial properties see [F], Chap. 6. In case $f = id_X$, it will be denoted

(3) $$i^{*} : A_k(X) \longrightarrow A_{k-d}(V) .$$

and called the GYSINhomomorphism of the inclusion $V \overset{i}{\hookrightarrow} X$. We also call it 'intersecting with V' and write $i^{*}(\alpha) = V \circ \alpha$.

Remark 2 Note that for the main construction no moving lemma and hence no assumption on quasi- projectivity is needed.

2.2 CHOWring and intersection multiplicities for nonsingular varieties Suppose X is a smooth (not necessarily quasi- projective) variety, of dimension n. Given subvarieties U, V of dimension k, ℓ, the cartesian square

$$\begin{array}{ccc} U \cap V & \hookrightarrow & U \times V \\ \downarrow & & \downarrow \\ X & \overset{\delta}{\hookrightarrow} & X \times X \end{array}$$

with δ the diagonal directly gives via the main construction in 2.1 the refined intersection class of 1.2 :

$$U \circ V \in A_{k+\ell-n}(U \cap V) \quad .$$

Explicitely: $\nu_X^{X \times X} = TX$, the tangent bundle, and

$$U \circ V = (U \cap V) \circ \left[C_{U \cap V}(U \times V) \right]$$

the intersection in $TX|U \cap V$ of the normal cone of the embedding $U \cap V \underset{\delta}{\hookrightarrow} U \times V$ with the zero section of $TX|U \cap V$. This construction passes to rational equivalence and gives us the intersection product

$$\circ \ : \ A_k(X) \otimes A_\ell(X) \longrightarrow A_{k+\ell-n}(X)$$

of 1.1 (1) as the composite

$$A_k(X) \otimes A_\ell(X) \xrightarrow{\times} A_{k+\ell}(X \times X) \xrightarrow{\delta^*} A_{k+\ell-n}(X) \quad ,$$

where δ^* is the Gysinhomomorphism according to (3) above.

Intersection multiplicities arise as follows. Let Y be a scheme. Then any two rationally equivalent cycles on Y contain a given irreducible component of Y with the same multiplicity; hence, for any $\alpha \in A_*(Y)$ and Z an irreducible component of Y , there is a well- defined multi plicity

$e(Z,\alpha;Y)$:= coefficient with which Z appears in any cycle representing α .

Definition. If Z is a proper component of $U \cap V$, i.e. $\dim Z = k + \ell - n$, then

$$i(Z, U \circ V; X) := e(Z, U \circ V; U \cap V)$$

is called the intersection multiplicity of U and V along Z in X .

In fact, one has

$$i(Z, U \circ V; X) = e(TX\ Z, C_{U \cap V} U \times V; TX|U \cap V) .$$

Theorem([F] , 7.1.1., 7.1.2). This intersection multiplicity agrees with the multiplicity defined by SAMUEL, and hence with those of WEIL, CHEVALLEY, SERRE.

2.3. Applications of the theory (see [F]) . Here I just list some topics which can be treated rigorously and satisfactorily with the methods and results of the general theory:

1) Theory of CHERN classes for vectorbundles on singular varieties ([F] , Chapter 3)

2) Theory of excess intersections; double point formula for maps with no codimension restriction on the double

point locus ([F] , Chapter 9).

3) Rigorous formulation of the 'principle of conservation of number' and rigorous solution of some classical enumerative problems ([F] , Chapter 10).

4) Theory of 'dynamic' intersection (in contrast to the above approach, which classically would be qualified as being 'static') ([F] , Chapter 11).

5) Formulae for degeneracy loci of sections of vector-bundles ('THOM - PORTEOUS - formulae') with no restrictions on the codimension of the loci ([F] , Chapter 14).

Appendix

On CHASLES'S and SCHUBERT'S theory of enumeration

General references are:

Steven KLEIMAN: Chasles's enumerative theory of conics: A historical introduction, Aarhus University Preprint Series 1975/76, No. 32, and in: Studies in Algebraic Geometry, MAA Studies in Math. 20, A. Seidenberg editor (1980), 117-138)

(This article contains a wealth of information about the history and the modern rigorous solution of the STEINER-BISCHOFF- problem on the number of conics in the plane tangent to five others, as well as all necessary bibliographical items for the work of STEINER, BISCHOFF, CHASLES, DE JONQUIÈRES, SCHUBERT and others.)

Herrmann SCHUBERT: Kalkül der abzählenden Geometrie, Reprint of the1879 Teubner edition with an introduction by Steven KLEIMAN, Springer 1979

William FULTON, Steven KLEIMAN and Robert MacPHERSON: About the enumeration of contacts, Proceedings of the conference on open questions in algebraic geometry, Ravello 1982, SLN

In 1864, CHASLES asserted that the number N of conics in a 1- parameter- system submitted to a simple condition was of the form

$$(1) \qquad N = \alpha \cdot \mu + \beta \cdot \nu ,$$

with α and β depending only on the condition, called the <u>characteristics of the condition</u>, and μ and ν depending only on the family, called the <u>characteristics of the family</u>, in fact

$$(2) \qquad \mu = \#\{\text{members of the family passing through a general point}\}$$

and, dually

$$(3) \qquad \nu = \#\{\text{members of the family touching a general line}\} .$$

Here, a condition imposed on a class of geometric figures parametrized by the points of an algebraic variety is called 'simple' if the points representing the figures satisfying the given condition form a hypersurface in the parameterspace.

CHASLES did not give a rigorous proof but verified it in some 200 examples; in 1.1, example 2 b) we have seen formula (1) holds for the condition 'being tangent to a general curve of degree d ', for any family of curves, not only conics, with

(2a) $\qquad \alpha \quad = \# \{$tangents to C through a general point$\}$

$\qquad\qquad = \quad$ the class $d^{\vee} = d(d-1)$ of C

and dually

(3a) $\qquad \beta \quad = \# \{$points of C on a general line$\}$

$\qquad\qquad = \quad$ the degree d of C ,

where C is the given curve of degree d . Recall that the method of proof consisted of appealing to the PCN (Principle of Conservation of Number) and moving C to a special, degenerate position where the problem becomes amenable to analysis.

CHASLES then proceeded to determine the number of curves subject to several simple conditions at the same time. For instance, consider a 2- parameter- system of curves in the plane and two independent conditions c_1 and c_2 . CHASLES would argue as follows:

The curves in the 2- parameter- system satisfying c_1 form a 1- parameter- system, call it (c_1) , which has the characteristics μ , ν , say, and then by (1) the number $N(c_1,c_2)$ of curves satifying both c_1 and c_2 is the number of members of (c_1) satisfying c_2 and thus by

(1) given as

(4) $$N(c_1,c_2) = \alpha_2 \cdot \mu + \beta_2 \cdot \nu ,$$

where α_2 and β_2 are the characteristics of the condition c_2, and μ, ν still to be determined. For this determination, let p denote the condition of passing through a general point p , and ℓ denote the condition of being tangent to a general line ℓ . Then, by (2) , $\mu = N(p,c_1)$, the number of members of (c_1) passing through p , and so by (4) applied to p and c_1 instead of c_1 and c_2 :

(5) $$\mu = \alpha_1 \cdot \mu^2 + \beta_1 \cdot \mu^1 \nu^1 ,$$

with μ^2 := number of members of the system passing through 2 general points, $N(p,p)$; and $\mu^1 \nu^1$:= number of members of the system passing through a general point and touching a general line, $N(p,\ell)$. Similarly:

(6) $$\nu = \alpha_1 \cdot \nu^1 \mu^1 + \beta_1 \cdot \nu^2$$

with $\nu^1 \mu^1 := N(\ell,p)$ and $\nu^2 := N(\ell,\ell)$.

Plugging (5) and (6) into (4) gives CHASLES'S formula for $N(c_1,c_2)$, which can be generalized and neatly expressed in a way discovered by HALPHEN (1872) :

Obviously, the argument above iterates; thus, if $c_1,\dots,c_r$ are r independent conditions, the number $N(c_1,\dots,c_r)$ of members in an r- parameter family of plane curves satisfying the conditions is given by

$$(7) \qquad N(c_1,\dots,c_r) = \prod_{i=1}^{r} (\alpha_i \cdot \mu + \beta_i \cdot \nu) \quad ,$$

where

$$(8) \qquad (\alpha_i, \beta_i) = \text{characteristics of the condition } c_i \, ,$$

μ , ν are indeterminates, and (7) is to be read as follows: expand the right hand side formally with respect to the powers $\mu^i \nu^{r-i}$ and put

$$(9) \qquad \mu^i \nu^{r-i} := N(\underbrace{p,\dots,p}_{i},\underbrace{\ell,\dots,\ell}_{r-i})$$

number of members in the family passing through i general points and touching $r-i$ general lines.

The $\mu^i \nu^{r-i}$ depend only on the family and are called the <u>characteristics of the family</u>. Thus the enumerative problem has been reduced to the determination of the characteristics of the individual simple conditions involved and to the determination of the $\mu^i \nu^{r-i}$, which is an improvement, because for them the general conditions have been replaced by <u>linear</u> ones, namely involving only passing through points and touching lines.

SCHUBERT generalized this to higher dimensions; thus, given an r- parameter- system of varieties in $\mathbb{P}^N$ and independent simple conditions $(c_1,\dots,c_r)$, then

$$(10) \qquad N(c_1,\dots,c_r) = \prod_{i=1}^{r} m_{c_i}$$

is the number of members in the family satisfying the conditions $c_1,\dots,c_r$, where

$$m_{c_i} = \alpha_o(c_i)\mu_o + \dots + \alpha_{N-1}(c_i)\mu_{N-1}$$

is the so-called <u>module of the condition c_i</u>, the μ_j being indeterminates and the $\alpha_j(c_i)$, depending only on c_i , the characteristics of c_i , and the prescription for evaluating (10) is as before, with

(11) $\mu_o^{j_o} \dots \mu_{N-1}^{j_{N-1}}$:= number of varieties in the family simultaneously touching j_o general o- planes, j_1 general 1- planes etc.

So again the enumerative problem has been reduced to the determination of the characteristics of the c_i and the corresponding enumerative problems for the given family with respect to linear conditions.

Consider, for instance the important case of r varieties V_i in general position and c_i to be the condition (also called V_i) 'to be tangent to V_i'. In this case, SCHUBERT would prove (10) analogously to (7), namely reducing to the case $r = 1$ as above, then deforming V_1 continuously to a special configuration and appealing to the PCN. In this case, there comes

(12) $\alpha_j(V_i)$ = degree of the polar locus of V_i of dimension j

=: j- th class of V_i

:= number of tangent planes (and limits of such) to a general section of $(V_i)_{reg}$ with an j- codimensional plane meeting a general $N-j-2$- plane,

and so the corresponding enumerative problem has been reduced to the computation of the characteristics (11), which remains, in general, a hard problem. In fact, the only principle used by the classical geometers to derive (7) respectively (10) has been the PCN, and for the determination of (11) in concrete cases one has to appeal to a second classical principle, the <u>correspondence principle</u> CP. Roughly, this states: Let T be an (α, β) - correspondence between the points of a line, i.e. to each point P there correponds a set $T(P)$ of β points, and to each point Q a set $T^{-1}(Q)$ of α points. Then there are $\alpha+\beta$ coincidences of a point P with a point in $T(P)$, or ∞ many. (The argument was like this: The graph of T is defined in $\mathbb{P}^1 \times \mathbb{P}^1$ by an equation homogeneous of degree α in one variable and of degree β in the other. Setting the variables equal gives an equation of degree $\alpha+\beta$ or the zero equation. SCHUBERT based his whole book [11] on the PCN and CP. For a modern treatment of CP and an explanation of SCHUBERT'S ideas see [8].

We are now in a position to derive CHASLES'S number 3264 as the solution of the STEINER- BISCHOFF-problem. So consider (7) for $r = 5$ and c_i meaning 'being tangent to the general conic C_i'. In this case, $\alpha_i = \beta_i = 2$ by (2a) and (3a), since $d = 2$. So the number N of plane conics tangent to the C_i is

$$N = (2\mu + 2\nu)^5$$

$$= 2^5(\nu^5 + 5\mu^4\nu + 10\mu^3\nu^2 + 10\mu^2\nu^3 + 5\mu\nu^4 + \nu^5) .$$

Obviously, $\mu^i\nu^{r-i} = \mu^{r-i}\nu^i$ by duality, and $\mu^5 = \nu^5 = 1$ by elementary analytic geometry (this checks, of course, with the more sophisticated discussion

We thus must compute μ^4 and $\mu^3\nu^2$. This was classically done as follows. Consider a 1- parameter- system of conics with characteristics μ and ν. Let T be the correspondence on a given general line ℓ given as

$$(x,y) \in T \quad := \quad \text{there is a member of the system passing through } x \text{ and } y .$$

By (2), T is a (μ,μ) - correspondence. So, by the CP, the number of coincidences is 2μ. On the other hand, a coincidence of T is either a member of the system tangent to ℓ, or a double line in the system. There results the formula

$$(13) \qquad 2\mu = \nu + \lambda ,$$

where λ is the number of double lines in the system. Considering now the 1- parameter- system $(C_1C_2C_3C_4)$ of conics tangent to $C_1,\ldots,C_4$, there results $2\mu^5 = \nu\mu^4 + \lambda\mu^4$. $\lambda\mu^4 = 0$, since no double line passes to 3, so a forteriori 4, general points, and, since $\mu^5 = 1$, there results $\nu\mu^4 = 2$. The same reasoning gives $2\mu^4\nu = \nu^2\mu^3 + \lambda\nu\mu^3$ and so $\mu^3\nu^2 = 4$. The result is

$$(13a) \quad \mu^5 = \nu^5 = 1 ; \quad \mu^4\nu = \mu\nu^4 = 2 ; \quad \mu^3\nu^2 = \mu^2\nu^3 = 4 ;$$

and so

$$N = 2^5(1 + 5\cdot 2 + 10\cdot 4 + 10\cdot 4 + 5\cdot 2 + 1)$$

$$= 32\cdot 2\cdot(1 + 10 + 40)$$

$$= 64\cdot 51$$

$$= 3264 .$$

SCHUBERT has the following spectacular numbers based on the formulae (10), (11) and (12):

As a first example, consider the quadric surfaces in 3- space. Since, in homogeneous coordinates $z_o,\ldots,z_3$, they have the equation $\sum a_{j_o\ldots j_3} z_o^{j_o}\ldots z_3^{j_3} = 0$ with $j_o+\ldots+j_3 = 2$ and not all $a_{j_o\ldots j_3} = 0$, they are parametrized by $\mathbb{P}^9$, since there are $\binom{4+2-1}{2} = \binom{5}{2} = 10$ monomials of degree 2 in 4 variables. Thus they form a 9-parameter- family. Let Q_i, $i = 1,\ldots,9$ be 9 quadrics in general position. Then, by (12), $\alpha_o(Q_i) = \alpha_1(Q_i) = \alpha_2(Q_i) = 2$, and the number N of quadrics in 3- space touching 9 quadrics is by (10)

$$N = (2\mu + 2\nu + 2\rho)^9$$

$$= 2^9(\mu + \nu + \rho)^9$$

(where we have accepted SCHUBERT'S notation $\mu = \mu_o$, $\nu =$

μ_1 and $\rho = \mu_2$). The $\binom{9+3-1}{9} = 55$ numbers $\mu^i \nu^j \rho^{9-i-j}$ are worked out in the table on p.105 of [11] and give

(13b) $$N = 666,841,088 .$$

As a second example, consider the twisted cubic space curves. They form a 12- parameter- family. Therefore, the number of them touching 12 quadric surfaces in general position is

$$N = (2\nu + 2\rho)^{12}$$

$$2^{12}(\nu + \rho)^{12}$$

(again SCHUBERT's notation). The 13 numbers $\nu^i \rho^{12-i}$ are computed on p.178 of [11] and give ([11] , p. 184):

(13c)

$$\begin{aligned} N = {} & 2^{12}(80\,160 + 12\cdot 134\,400 + \tbinom{12}{2}\cdot 209\,760 + \tbinom{12}{3}\cdot 297\,280 \\ & + \tbinom{12}{4}\cdot 375\,296 + \tbinom{12}{5}\cdot 415\,360 + \tbinom{12}{6}\cdot 401\,920 + \tbinom{12}{7}\cdot 343\,360 \\ & + \tbinom{12}{8}\cdot 264\,320 + \tbinom{12}{9}\cdot 188\,256 + \tbinom{12}{10}\cdot 128\,160 + \tbinom{12}{11}\, 85\,440 \\ & + 56\,960) \\ = {} & 5,819,539,783,680 . \end{aligned}$$

This determination won SCHUBERT a gold medal from the Royal Danish Academy 1875 .

Of course, the classical derivation of (1) and its generalizations (7) and (10) does not meet our today- standards of rigour, since the PCN is hardly to be made precise without a fully developed intersection theory and theory of rational or numerical equivalence (see [F], 10.2.. for a modern account). Soon, counterexamples to the expression (1) were discovered (see [10]), and so the need arises to explore the range of the validity of (1) and its generalizations.

In modern terms, the deformation proof of (1) should be thought of as determining the rational equivalence class of the cycle representing the figures obeying the given condition in the CHOWring of the parameterspace, and the PCN then will appear in the guise of the principle that intersection numbers are stable under rational equivalence. Thus, informally, one should think of (1) as follows:

Suppose the figures considered are parametrized by the points of a complete (i.e. compact, if the field of definition is C) variety X. An i - fold condition c is to be considered as a class in $A_{n-i}(X)$ (passing to rational equivalence classes includes already the freedom to move), where $n = \dim X$. Suppose further that $A_{n-i}(X)$ is generated (modulo torsion, which can be neglected for enumerative purposes) by finitely many classes $m_1,\ldots,m_r$, so that modulo torsion c can be written

$$c = \sum_j \alpha_j m_j$$

and the m_j can be thought of as basic conditions, into which c degenerates with multiplicities given by the α_j when moving it into special position. Consider the degree homomorphism

$$\deg : A_0(X) \twoheadrightarrow \mathbb{Z}$$

mapping a 0 - cycle to the sum of its coefficients (if X is irreducible, it is isomorphic). One has

$$\alpha_j = \deg(c \circ a^j) \quad ,$$

when the a^j form, modulo torsion, a dual basis to the m_j under intersection; i.e.

$$\deg(m_i \circ a^j) = \delta_i^j$$

the a^j being called the dual conditions.

Now suppose given an i - parameter - family; this defines a class $f \in A_i(X)$, and the number $N(c)$ of members of the family satisfying c is

$$N(c) = \deg(c \circ f)$$

$$= \sum_j \alpha_j \deg(m_j \circ f) \quad .$$

We thus arrive at the formula

(14) $$\boxed{N(c) = \sum_j \alpha_j \mu_j}$$

with

(15) $$\mu_j = \deg(m_j \circ f)$$

$= \# \{$members of the family f satisfying the condition $c_j\}$

and

(16) $$\alpha_j = \deg(c \circ a^j)$$

$= \# \{$members of the family a^j satisfying the condition $c\}$.

(14) - (15) is the modern interpretation of (1) - (3) . (14) shows that CHASLES's expression (1) cannot be based on the conics being parametrized by points of $\mathbb{P}^5$: for then BÉZOUT's theorem would supply the following result. We have, as a ring, $A^*(\mathbb{P}^N) \doteq \mathbb{Z}[t]/(t^{N+1})$ under the correspondence $h \leftrightarrow t$, where $h = [H] \in A^1(\mathbb{P}^N) = A_{N-1}(\mathbb{P}^N)$ is the class of a hyperplane H (see the Structure Theorem in Part B , 3.2). A 1- parameter- system of conics would be a class $m \in A_1(\mathbb{P}^5)$, and a simple condition a class $c \in A^1(\mathbb{P}^5) = A_4(\mathbb{P}^5)$, and there would result

$$N(c) = \alpha\mu$$

with α = deg c and μ = deg m , a result already arrived erroneously at by DE JONQUIÈRES in 1861 (see [10] and the Enzyklopädie article of ZEUTHEN).

So something more subtle must be behind (1) . The idea behind it is to choose a <u>compactification</u> X of the space of nonsingular conics <u>different from $\mathbb{P}^5$</u> by adding certain degenerate configurations (already explicit in SCHUBERT's book [11] on p. 92) so that the pathology that every double line represented by a point in the VERONESE $V \subseteq \mathbb{P}^5$ is tangent to every conic cannot occur. (This pathology is caused by the fact that two conics C and D are said to be tangent if they intersect in less than four points; from this follows that every double line is tangent to every conic.) The degenerate configurations added were

(i) two lines together with their point of intersection

(ii) a double line together with two points on it, called 'foci' (in modern terminology: a double line together with a divisor of degree two on it)

The condition of being tangent to a conic C for these degenerate configurations was then

For (i) : C is tangent to one of the lines in the usual sense, or passes through the point of intersection

For (ii) : C passes through one of the foci, or is tangent to the line in the usual sense.

The space X parametrizing the nonsingular conics and the degenerate configurations (i) and (ii) was called the space of complete conics, after STUDY 1886 and VAN DER WAERDEN (Zur Algebraischen Geometrie XV, Math. Ann. 115(1938)). If one thinks of configurations in (i) as two lines together with the pencil of lines through the point of intersection*), and of configurations in (ii) as a double line together with two pencil of lines through two points on it**), the configurations (i) and (ii) correspond under duality. Thus there is a well- defined duality map

$$\bar{\delta} : X \longrightarrow X$$

which maps a nonsingular conic to its usual dual and interchanges (i) and (ii), extending the rational map

$$\delta : \mathbb{P}^5 \dashrightarrow \check{\mathbb{P}}^5 \tag{17}$$

defined by sending a conic to its dual, which is only defined outside the VERONESE. To be precise, if one let a

*) called a δ in [11] **) called an η in [11]

conic be represented by a 3×3 symmetric nonzero matrix up to a scalar multiple, δ maps a matrix to its adjoint matrix of 2×2 - minors,*) so it is not defined on the subscheme of $\mathbb{P}^5$ defined by the vanishing of all 2×2 - minors, which scheme-theoretically is just the VERONESE V . Thus X is given as

$$(18) \qquad X := \overline{\text{graph}\,\delta} \subseteq \mathbb{P}^5 \times \check{\mathbb{P}}^5 ,$$

the scheme- theoretic closure of the graph of the map (17) . (see the description of X in V.D.WAERDEN, loc.cit. p.646-647). This shows:

(19) The first projection $p : X \longrightarrow \mathbb{P}^5$ realizes X as the blowup of $\mathbb{P}^5$ along the VERONESE V .

This shows further that X can be interpreted as the space of pairs $(C,\check{C})$ where C is a nonsingular conic, together with its dual, and pairs $(\lim_{\lambda\to 0} C_\lambda , \lim_{\lambda\to 0} \check{C}_\lambda)$ where $(C_\lambda)_{\lambda\in\Lambda}$ is a degenerating 1- parameter- family of conics, which is the way classical geometers use to think about complete conics. For instance, a configuration in (i) can be thought of as the limit position of the 1- parameter- family of hyperbolas (in affine coordinates) $b^2x^2-a^2y^2 = \lambda^2$, with the degenerate line- pair the asymptotes, and configurations in (ii) as the limit of the confocal family of ellipses $b^2x^2+(b^2+\lambda^2)y^2 = b^2(b^2+\lambda^2)$, with the two points being the two foci of the ellipses.

*) If the symmetric matrix A represents the nonsingular conic C , it suffices to show A^{-1} represents $\check{C}$. But the projective automorphisms act transitively and intertwine δ , so it suffices to prove this for $A = \mathbb{1}$, which represents the conic $\sum z_i^2 = 0$; this has tangent $\sum z_i^\circ z_i = 0$ at the point of $\mathbb{P}^2$ with coordinates z_i° , so the tangent has coordinates z_i° in the dual space $\check{\mathbb{P}}^2$. Qed.

Now, with the interpretation (19) of the appropriate parameterspace X of complete conics at hand, the CHASLES expression (1) can be derived.

The CHOWring of a blowup is known (see [F] , 6.7). One has: Let $p : X \longrightarrow \mathbb{P}^5$ and $q : X \longrightarrow \check{\mathbb{P}}^5$ be the maps given by the first and second projection. Then $A^1(X)$ has a base consisting of m , n *) with

$$(20) \qquad m := p^*(h) ,$$

where $h = [H] \in A^1(\mathbb{P}^5)$ is the class represented by a hyperplane, which can be thought of as the linear system of all conics passing through a general point; so m is the condition of passing through a general point; in a similar vein

$$(21) \qquad n := q^*(\check{h})$$

where $\check{h} \in A^1(\check{\mathbb{P}}^5)$ is the class represented by the system of all conics whose duals pass through a general point in the dual plane, hence n is the condition of being tangent to a general line.

If now $c \in A^1(X)$ is a simple condition and $f \in A_1(X)$ a 1- parameter- family of complete conics, (14) gives the CHASLES formula (1) , and (15) confirms (2) and (3) .

---

*) $A^1(X)$ has a distinguished base, namely m and l , where l is the class of the exceptional divisor of the blowup p . That m and n form a base then follows from the formula (25) proved below.

Secondly, let $c_i \in A^1(X)$, $i = 1,\ldots,r$ be r independent simple conditions. Then the composite condition c of satisfying simultaneously the c_i is just the intersection product $c = \prod_i c_i \in A^r(X)$, and the number of members of an r - parameter- family $f \in A_r(X)$ of complete conics satisfying c is given as

$$N(c) = \deg(\prod_i c_i \circ f)$$

But $c_i = \alpha_i m + \beta_i n$, and $\deg(m^i \circ n^{r-i} \circ f)$ is just the number $\mu^i \nu^{r-i}$ in the interpretation of (9) , which proves HALPHEN's formula (7) .

Finally, we verify the reasoning based on (13) leading to the characteristics (13a) $\mu^i \nu^{5-i}$ for the plane conics, thus completing the verification of the number 3264 .

For this, we contemplate on the description (18) of X . Generally, if Y is a variety , and D is a divisor on Y , there results a rational map

$$r_D : Y \longrightarrow \check{\mathbb{P}}(V)$$

where V is the vectorspace $H^0(Y, L_D)$ of sections of the line bundle L_D associated to D , by mapping a point y in Y to the hyperplane in V of the sections of L_D vanishing at y . Forming, in generalization of (18),

$$X := \overline{\text{graph } r_D} \subseteq Y \times \check{\mathbb{P}}(V) \quad ,$$

. the first projection $p : X \longrightarrow Y$ exhibits X as the blowup of Y along the locus B of indeterminacy of r_D (i.e. the base locus of the linear system defined by D), and the second projection $q : X \longrightarrow \check{\mathbb{P}}(V)$ extends r_D to an everywhere defined map on X . This is a classical construction, and is such that a hyperplane $\check{H} \subseteq \check{\mathbb{P}}(V)$ pulls back to D under r_D . Let $p^{-1}D$ be the divisor on X obtained by lifting the equations of D , i.e. the scheme-theoretic inverse image of D ; then $p^{-1}D$ is called the <u>total transform of D</u> . Furthermore, $p \mid X - p^{-1}B \ : X - p^{-1}B \longrightarrow Y - B$ is isomorphic, and $\tilde{D} := \overline{p^{-1}(D - B)}$ is called the <u>strict transform of D</u> . The following equation holds in $\mathrm{Div}(X)$:

$$(22) \qquad p^{-1}D = \tilde{D} + \lambda E \qquad \text{for some } \lambda \in \mathbb{N} \quad ,$$

where $E := p^{-1}B$ is the exceptional divisor of the blowup p (just check local equations) . We thus obtain

$$(23) \qquad p^*[D] = q^*([\check{H}]) + \lambda [E]$$

in $A_{n-1}(X)$ ($n := \dim Y = \dim X$) .

We now apply this to $\delta : \mathbb{P}^5 \rightarrow \check{\mathbb{P}}^5$. By the description of the map δ given on the top of p. , we see that it is just the map $r_D : \mathbb{P}^5 \longrightarrow \check{\mathbb{P}}^5$ given by a divisor of degree two (e.g. take any of the equations defined by the vanishing of a 2×2 - minor of a 3×3 - symmetric matrix). Thus we have, with the notation (20) :

$$p^*[D] = 2\,m \tag{24}$$

Thus, by (23) and (21), since $\lambda = 1$:

$$2\,m = n + l \quad , \tag{25}$$

where $l := [E] \in A^1(X)$ represents the simple condition 'to be a degenerate configuration of type (ii) ' , i.e. 'to be a double line with a degree two divisor on it'. This is the modern interpretation of (13) . Multiplying with m^4 gives $2m^5 = m^4 \circ n + m^4 \circ l$; $\deg m^5 = 1$, since p pushes m^5 down to a generator of $A^5(\mathbb{P}^5) = \mathbb{Z}$, and $m^4 \circ l = 0$, since no four general points lie on a line. So $\mu^5 = 1$, $\mu^4 \nu = 2$. The rest of (13a) is verified analogously. Thus, to complete the verification of the number 3264 , one has to compute $\alpha = \beta = 2$. This can be done in various ways. One way is the following: Let $H \subseteq \mathbb{P}^5$ be the hypersurface of conics tangent to a general conic; then use the explicit equation (see [3]) to compute $\deg H = 6$ and $\pi^* H = \tilde{H} + 2E$; the claim follows from the facts $\pi^* H = H + \beta E$ and $\alpha + 2\beta = \deg H$, which are explained on p. . Another way is to use special families for which N is known (recall α and β depend only on the condition) , see [10] , p. 127 .

Of course, there remains the question whether the 3264 conics are all distinct and nonsingular if the five given conics are in general position. This is not settled by the above considerations. In fact, the answer is yes; this can be shown by a closer analysis of the strict transforms of $H_1, \dots, H_5$ (see p. 50) or by invoking KLEIMAN's theorem on the transversality of general translates under the transitive operation of an algebraic group (note that PGL(3) operates transitively on the non singular conics) ; see [6] and [10] .

I would like to conclude this description of the classical method with a short discussion of its relation to the method of refined intersections used in 1.3 . The gist of CHASLES's method may be summarized in that it consists of replacing a simple condition as represented by a hypersurface in $\mathbb{P}^5$ by its strict transforms under the blowup $\pi : X \longrightarrow \mathbb{P}^5$ of $\mathbb{P}^5$ along the VERONESE,*) and then computing intersection numbers in X . So if $H_1, \dots, H_5$ are five simple conditions, CHASLES's method gives for the number of conics fulfilling these conditions the intersection number

$$\tilde{H}_1 \cdot \ldots \cdot \tilde{H}_5 = \deg([\tilde{H}_1] \circ \ldots \circ [\tilde{H}_5]) .$$

The difference between BÉZOUT's number $H_1 \cdot \ldots \cdot H_5 = \deg(\pi^*[H_1] \circ \ldots \circ \pi^*[H_5])$ and $\tilde{H}_1 \cdot \ldots \cdot \tilde{H}_5$ can be expressed using the fact (explained on p. 84) $\pi^*[H_i] = [\tilde{H}_i] + \beta_i 1$,

*) as to get rid of the unwanted excess contribution of V

and may be shown to give just the expression for the equivalence of $H_1 \circ \ldots \circ H_5$ for the component of $H_1 \cap \ldots \cap H_5$ supported by V (H_i is to have characteristics (α_i, β_i)).

For instance, in the case of the STEINER - BISCHOFF - problem, all $\beta_i = 2$; we get for the difference

$$\deg(\pi^*[H_1] \circ \ldots \circ \pi^*[H_5]) - \deg((\pi^*[H_1] - 2l) \circ \ldots \circ (\pi^*[H_5] - 2l))$$

Since deg can be computed on $\mathbb{P}^5$ by pushing down via the projection formula, we get the degree of the class

$$\binom{5}{3} \cdot 6^2 \cdot m^2 (2l)^3 - \binom{5}{4} \cdot 6 \cdot m(2l)^4 + (2l)^5$$

which is easily seen to be exactly $(\prod_i c(\nu_i | Z) \cap s(\mathbb{P}^5, Z))_0$ as calculated in 1.3 . One may also use the values of the numbers $m^2 l^3$, ml^4, l^5 given on p. to evaluate this expression to be

$$
\begin{aligned}
10 \cdot 36 \cdot 8 \cdot 4 - 5 \cdot 6 \cdot 16 \cdot 18 + 32 \cdot 51 &= 320 \cdot 36 - 240 \cdot 36 + 32 \cdot 51 \\
&= 80 \cdot 36 + 32 \cdot 51 \\
&= 2880 + 1632 \\
&= 4512
\end{aligned}
$$

as desired.

The derivation of (1) and its generalization (7) given depends on the compactification of the parameter space of non-singular curves which is peculiar for conics, so it is restricted to this case. In fact, (7) holds for arbitrary curves and can be given a different proof without using 'non standard compactifications'. This proof is due to FULTON and MacPHERSON and given in full detail in [F], 10.4, so I have not repeated it here. It is based on the geometry of the incidence correspondence

$$I := \{(P,L) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2 \mid P \in L\}$$

i.e. on the structure of $A_*(I)$. This proof generalizes to the general case of an r- parameter- system of varieties in $\mathbb{P}^N$ and the condition of touching r varieties in general position and so to a proof of (10) in this important case, which is given in [3] : If $V \subseteq \mathbb{P}^N$ is a variety, define the <u>conormal variety</u> CV to be

$$CV := \overline{\{(P,H) \in \mathbb{P}^N \times \check{\mathbb{P}}^N \mid P \in V_{reg} \text{ and } T_P V \subseteq H\}}$$

the closure taken in $\mathbb{P}^N \times \check{\mathbb{P}}^N$. Then, if

$$I := \{(P,H) \in \mathbb{P}^N \times \check{\mathbb{P}}^N \mid P \in H\}$$

is the incidence correspondence, the following formula holds in $A_*(I)$ (see [7], (2.23)):

$$(26) \qquad \boxed{[CV] = \alpha_0(V)[CH_0] + \dots + \alpha_{N-1}(V)[CH_{N-1}]} ,$$

where the $\alpha_j(V)$ are given by (12), and H_j is a linear subspace of dimension j. This can be thought of as another modern interpretation of CHASLES's expression (1) in arbitrary dimensions. Using the definition

$$V \text{ touches } W \ :\Leftrightarrow\ CV \cap CW \neq \emptyset ,$$

the formula (10) results, as in the case of conics, by multiplying and extracting degrees (see [7] , (2.22)).

In [7] , various proofs for (26) are given. One variant (loc.cit. p.175 ff) just makes precise SCHUBERT's approach of degenerating the given varieties V_i to be touched in a family of rational equivalence into linear spaces by exposing them to a continuous family of special mappings, called 'Homographien' by SCHUBERT (cf. [11] , p.91) , and 'homolographies' in [7] . This is also the spirit of proof in [F] . Especially, the classical procedure sketched in Part A , p. 7 for proving (1) can thus be made rigorous.

Besides this, [7] contains a thorough discussion when the contacts are proper and what the multiplicities are with which the solutions have to be counted.

Another beautiful generalization of CHASLES'S theory has been developed by DE CONCINI and PROCESI (see [1] and [2]). To motivate it, consider the case of quadric hypersurfaces in $\mathbb{P}^n$ (which is also classical, see the references in [1] and [2]). The starting point is the observation that the space

$$\begin{aligned} X &:= \{ Q \mid Q \text{ a smooth quadric in } \mathbb{P}^n \} \\ &= \mathbb{P}(\mathrm{Sym}(\mathbb{C}^{n+1})) \\ &= \mathbb{P}^{q(n)} \quad , \quad q(n) = (n+1)(n+2)/2 - 1 \end{aligned}$$

is a homogeneous space $PGL(n+1) \,/\, PO(n+1)$ of a special type, which DE CONCINI and PROCESI call 'symmetric varieties'. In general, they define a symmetric variety to be a quotient $X := G/H$, where G is a semisimple algebraic group of adjoint type, and H the fixed group G^{σ} of an involution $\sigma : G \rightarrow G$. If we think of X as a parameter space, whose points represent geometric objects, and of cycles $a \in Z^k(X)$ as r- dimensional conditions on these objects, the 'naive' intersection product

$$(27) \qquad \circ \; : \; \begin{array}{ccc} Z^r(X) \otimes Z_r(X) & \longrightarrow & \mathbb{Z} \\ a \; , \; b & \longmapsto & \#(a' \cap b') \end{array}$$

where a , b are represented by irreducible varieties and a' and b' are generic G- translates, need not be well- defined due to the noncompactness of X . On the other hand, if one chooses a 'bad' equivariant compactification X' of X so that a and b have a G - orbit at infinity in common (which happened to be the case for $X = \mathbb{P}^5$ in the case of plane conics), the intersection product in X' does not compute the number of generic points in X satisfying both conditions a and b .

DE CONCINI and PROCESI construct a canonical smooth G - equivariant compactification $\overline{X}$ of X . Among their findings are the following properties:

(i) $\overline{X} - X$ is a union of smooth hypersurfaces. All intersections are orbit closures, with incidence strucure given by explicit combinatorial data.

(ii) For any condition $a \in Z^r(X)$ there is a smooth equivariant compactification X' lying over $\overline{X}$ such that $\overline{a}$ *) meets all the closures of G - orbits in X' properly. X' arises from $\overline{X}$ by blowing up codimension 2 subvarieties.

From this they conclude that the intersection product (27) can be computed in the CHOWring of a suitable blowup X' of $\overline{X}$.

In favourite situations (e.g. in the case of quadrics, see below), it suffices to consider $\overline{X}$. In this case, it suffices to know the numbers $\mu_0^{j_0} \cdots \mu_{N-1}^{j_{N-1}}$ of A , 1.2 (2) , which, in this setting, are the characteristic numbers $\int_{\overline{X}} x_1 \ldots x_d$ for $x_1, \ldots, x_d \in H^2(\overline{X}) = A^1(X)$, $d = \dim \overline{X}$. They produce an explicit algorithm which allows, in principle, to compute all these numbers in any particular case.

In the case of quadrics, $X = PGL(n+1) / PO(n+1)$, the results are as follows. $\overline{X}$ is constructed in the following way: consider the PGL(n+1) - equivariant embedding

---

*) here $\overline{a}$ denotes the closure of a in X' .

$$j : X \hookrightarrow \mathbb{P}(\mathrm{Sym}(\Lambda^i \mathbb{C}^{n+1}))$$
$$Q \longmapsto (\Lambda^1 Q, \Lambda^2 Q, \ldots, \Lambda^n Q) ,$$

and put $\overline{X} := \overline{j(X)}$. Then:

1) X is nonsingular.

2) $\overline{X} - X = S_1 \cup \ldots \cup S_n$ for smooth hypersurfaces $S_1, \ldots, S_n$. There is a bijection

$$\mathcal{P}\{1,\ldots,n\} \longrightarrow \{PGL(n+1) \text{ - orbits in } \overline{X}\}$$
$$I \longmapsto \mathcal{O}_I$$

such that $\overline{\mathcal{O}_I} = \bigcap_{i \in I} S_i$ and $\mathcal{O}_I = \overline{\mathcal{O}_I} - \bigcup_{i \notin I} S_i$.

3) $\mathcal{O}_{\{1,\ldots,n\}}$ is isomorphic to the flag variety $\mathcal{F}$ of flags in $\mathbb{P}^n$.

4) $H^2(\overline{X};\mathbb{Z}) = A^1(\overline{X}) = \bigoplus_{i=0}^{n-1} \mathbb{Z}\lambda_i$, where $\lambda_i := [\overline{D(\pi_i)}]$, and $D(\pi_i) := \{Q \in X \mid Q$ is tangent to $\pi_i\}$ for a chosen flag $\pi_0 \subset \pi_1 \subset \ldots \subset \pi_{n-1} \subset \mathbb{P}^n$.

5) The computation of numbers $\int_{\overline{X}} \lambda_{i_1} \cdots \lambda_{i_{q(n)}}$ can be reduced to evaluations of numbers $\int_{\mathcal{F}} \lambda_{j_1} \cdots \lambda_{j_{q(n)-n}}$.

One may then prove:

<u>Theorem</u>. Let $V \subset \mathbb{P}^n$ be a fixed subvariety, $D := D(V) := \{Q \in X \mid Q$ tangent to $V\}$. Then

(i) $\bar{D}$ does not contain a PGL(n+1) - orbit.

(ii) If V is a quadric, $[\bar{D}] = 2\sum_{i=0}^{n-1} \lambda_i$ in $A^1(X)$.

<u>Corollary.</u> (i) SCHUBERT's formula (10) holds for smooth quadrics tangent to subvarieties in general position.

(ii) In particalur, the number of smooth quadrics touching h_0 points, h_1 lines,..., h_{n-1} hyperplanes and h_n quadrics in general position, where $h_0+\ldots+h_n = q(n)$, is given by

$$N(h_0,\ldots,h_n) = 2^{q(n)} \int_X \lambda_0^{h_0} \cdots \lambda_{n-1}^{h_{n-1}} \left(\sum_{i=0}^{n-1} \lambda_i\right) .$$

From this, one gets SCHUBERT's number (see [1]) :

N(0,0,0,9) = 666,841,088 .

More numbers have been computed in [3]. For example:

N(0,0,0,0,14) = 48,942,189,946,470,000

N(0,0,0,0,0,20) = 641,211,464,734,373,953,791,690,014,720 .

Thus it appears that the basic formula (10), from which SCHUBERT derives his numbers (13b) and (13c) has finally, after more than 100 years, been put on a firm basis.

Part B: The construction

In this part, I will give the details needed to perform the main intersection construction, namely, as formulated in 2.1 of Part A , develop the theory of rational equivalence on singular schemes, and construct the operation of intersecting with the zero section in vector-bundles. The exposition consists of three steps:

1. Construction of the CHOW homology groups.

2. Intersecting with divisors (= intersecting with the zero section in line bundles = theory of 1st CHERN class of line bundles).

3. Intersecting with the zero section in vector-bundles (= THOM isomorphism theorem).

In what follows, all schemes will be algebraic schemes over a fixed field k , that is, be ringed spaces covered by finitely many parts of the type spec A , where A is a finitely generated k - algebra, i.e. of the form $k[X_1,,\ldots,X_N]$ / I , with the X_i indeterminates and I a polynomial ideal . A variety is an integral (= reduced and irreducible) scheme, that is, it is just a variety in the classical sense (disregarding the non closed points) if k is algebraically closed. If V is a variety, R(V) denotes the field of rational (or meromorphic) functions.

1. CHOW homology

1.1 Cycles Let X be a scheme; put

$$Z(X) := \text{free abelian group on generators the (closed) subvarieties of } X$$

$Z(X)$ has two $\mathbb{Z}$ - graduations:

$$Z(X) = \bigoplus_k Z_k(X) \qquad \text{(by dimension)}$$

$$= \bigoplus_k Z^k(X) \qquad \text{(by codimension)} .$$

If X is of pure dimension n, then $Z^k = Z_{n-k}$.

Denote the image of a subvariety V in $Z(X)$ by $[V]$.
X has a fundamental cycle $[X] \in Z(X)$ defined as follows:

$$[X] := \sum_\lambda \ell(\mathcal{O}_{X,X_\lambda}) \cdot [X_\lambda] ,$$

the sum ranging over the irreducible components X_λ of X, and $\ell(\mathcal{O}_{X,X_\lambda})$ denoting the length of the ARTINian ring $\mathcal{O}_{X,X_\lambda}$.

More generally, if $Y \subseteq X$ is a closed subscheme, put

$$[Y] \in Z(Y) \subseteq Z(X) .$$

If X is of pure dimension n, $[X] \in Z_n(X)$.

1.2 <u>Functoriality</u> a) If $f : X \longrightarrow Y$ is proper, it induces $f_* : Z_k(X) \longrightarrow Z_k(Y)$ via

$$f_*[V] := \begin{cases} \deg(f|V)\cdot[f(V)] & \text{if } \dim V = \dim f(V) \\ 0 & \text{else} \end{cases}$$

where $V \subseteq X$ is a k - dimensional subvariety, and $\deg(f \mid V) := [R(V):R(f(V))]$, the degree of the function field extension $R(f(V)) \subseteq R(V)$ (if k is algebraically closed, this is just the geometric mapping degree of $f|V$, namely the number of distinct preimages of a general point on $f(V)$).

This makes Z_k a covariant functor for proper morphisms. f_* is called the <u>pushforward</u> (by f) .

b) If $f : X \longrightarrow Y$ is flat, it induces $f^* : Z^k(Y) \longrightarrow Z^k(X)$ via

$$f^*[V] := [f^{-1}V] ,$$

for any k - dimensional subvariety $V \subseteq X$, where $f^{-1}V$ is the schematic preimage of V . Hence $f^*[Z] = [f^{-1}Z]$ for any subscheme $Z \subseteq X$. This makes Z^k contravariant for flat morphisms. f^* is called the <u>flat pullback</u> (by f). If f is flat of relative dimension d , one has

$$f^* : Z_k(Y) \longrightarrow Z_{k+d}(X) .$$

c) If

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \longrightarrow & Y \end{array}$$

is cartesian with f proper and g flat (hence f' proper and g' flat), one has

$$g^* f_* = f'_* g'^* \ .$$

1.3 <u>Meromorphic functions</u> If X is a variety of dimension $k+1$, any meromporphic function $r \in R(X)^*$ defines a k-cycle, denoted $[div(r)]$: View r as a dominant (and thus flat) morphism $r : X \longrightarrow \mathbb{P}^1$ and put

$$[div(r)] := [r^{-1}(0)] - [r^{-1}(\infty)]$$

$$= r^*([0] - [\infty]) \ .$$

In more classical terms: For any codimension one subvariety V of X and any $f \in \mathcal{O}_{X,V}{}^* = \mathcal{O}_{X,V} - \{0\}$ define the <u>order of vanishing of f along V</u> by

$$ord_V(f) := \ell(\mathcal{O}_{X,V} / (f))$$

and extend it to a homomorphism $\mathrm{ord}_V : R(X)^* \longrightarrow \mathbb{Z}$ by

$$\mathrm{ord}_V(f/g) \;:=\; \mathrm{ord}_V(f) - \mathrm{ord}_V(g) \quad .$$

Then

$$[\mathrm{div}(r)] \;=\; \sum_{\substack{V \subseteq X \\ \mathrm{codim}\, V = 1}} \mathrm{ord}_V(r) \cdot [V] \quad .$$

There is the following

<u>Key proposition</u> ([F] , Proposition 1.4) <u>If</u> $f : X \longrightarrow Y$ <u>is a proper surjective morphism, then</u>

$$f_* \, [\mathrm{div}(r)] = \begin{cases} \mathrm{div}(N(r)) & \dim X = \dim Y \\ 0 & \text{else} \end{cases}$$

<u>where</u> N <u>denotes the norm in the field extension</u> $R(X) \subseteq R(Y)$ (<u>that is</u>, $N(r) = \det_{R(Y)}(r)$, r <u>acting by multiplication in the</u> R(Y) - <u>vectorspace</u> R(X)).

<u>Sketch of proof</u> The idea is to reduce, by base change, to properties of length for one- dimensional local domains.
We consider only the most important case where $f : X \longrightarrow Y$ is a finite surjective morphism of affine varieties, referring for the full proof to [F] .

Let $A := k[X]$, $B = k[Y]$ be the coordinate rings of X, Y, and $K = R(X)$, $L = R(Y)$ be the corresponding quotient fields of meromorphic functions. f corresponds to an integral ring extension $B \subseteq A$.

The claim is: For all codimension one subvarieties W of Y and $a \in K$

$$(1) \qquad \mathrm{ord}_W(N(a)) = \sum_{\substack{V \subseteq X \\ f(V)=W}} \mathrm{ord}_V(a) \cdot [R(V):R(W)] \ .$$

W corresponds to a prime ideal $\mathfrak{q}$ of height one; $B_{\mathfrak{q}} = \mathcal{O}_{Y,W}$ is a one- dimensional local domain, and $A_{\mathfrak{q}} = A \otimes_B B_{\mathfrak{q}}$ (A viewed as a B - module) is a finite $B_{\mathfrak{q}}$ - algebra whose maximal ideals correspond to the V mapping to W. Let $a \in A$. Then (1) translates into the statement

$$(2) \qquad \ell_{B_{\mathfrak{q}}}(B_{\mathfrak{q}}/N(a)) = \sum_{\mathfrak{p} \text{ prime of } A_{\mathfrak{q}}} \ell_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/a) \cdot [A_{\mathfrak{p}}:B_{\mathfrak{q}}]$$

and it suffices to prove (2), with $a \in A$ (since $K = \mathrm{Quot}(A)$). But now one has the

<u>Lemma</u>([F] , A.3) <u>Let</u> D <u>be a one dimensional local domain</u>, $L := \mathrm{Quot}(D)$, M <u>a finitely generated</u> D - <u>module</u>, <u>and</u> $\varphi : M \longrightarrow M$ <u>a monomorphism. Then</u>

$$(3) \qquad \ell_D(D/\det(\varphi \otimes \mathrm{id}_L)) = \ell_D(M/\varphi M) \ .$$

Putting $D = B_{\mathfrak{q}}$, $M = A_{\mathfrak{q}}$ in (3) gives (2), since

$$\sum_{\mathfrak{P} \text{ prime of } A_{\mathfrak{q}}} \ell_{A_{\mathfrak{P}}}(A_{\mathfrak{P}}/a)\cdot[A_{\mathfrak{P}}:B_{\mathfrak{q}}] \;=\; \ell_{B_{\mathfrak{q}}}(A_{\mathfrak{q}}/a) \text{ . Q.e.d.}$$

1.4 <u>Rational equivalence</u> Let $p : X \longrightarrow C$ be a flat map from a scheme to a smooth curve. If V is a k - dimensional subvariety of X, p restricts to V, thus defining, for given $t \in C$, a specialization morphism

$$\sigma_t : Z_k(X) \longrightarrow Z_{k-1}(X_t) \quad ,$$

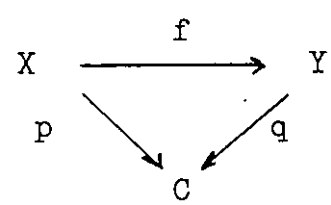
where $X_t := p^{-1}(t)$ is the schematic fibre of p over t, as follows: Let $V_t := (p|V)^{-1}(t)$ and put

$$\sigma_t[V] := \begin{cases} [V_t] & \text{if } p|V \text{ is dominant} \\ 0 & \text{else} \end{cases} .$$

(Note that, if $p|V$ is dominant, it is flat, and then $[V_t] = (p|V)^*[t]$).

One should think of σ_t as 'intersecting with X_t'.

These specialization morphisms enjoy the following functorial properties:

<u>Proposition</u> <u>Let</u> 

<u>be a commutative</u>

<u>triangle , where</u> p <u>and</u> q <u>are flat morphisms to the smooth curve</u> C .

a) <u>If</u> f <u>is proper, the diagram</u>

$$\begin{array}{ccc} Z_k(X) & \xrightarrow{f_*} & Z_k(Y) \\ \sigma_t \downarrow & & \downarrow \sigma_t \\ Z_{k-1}(X_t) & \xrightarrow[f_{t*}]{} & Z_{k-1}(Y_t) \end{array}$$

<u>commutes, where</u> $f_t := f \mid X_t$.

b) <u>If</u> f <u>is flat of relative dimension</u> d , <u>the diagram</u>

$$\begin{array}{ccc} Z_k(Y) & \xrightarrow{f^*} & Z_{k+d}(X) \\ \sigma_t \downarrow & & \downarrow \sigma_t \\ Z_{k-1}(Y_t) & \xrightarrow[f_t^*]{} & Z_{k+d-1}(X_t) \end{array}$$

<u>commutes</u>.

<u>Sketch of proof</u> Ad a): (Intuitively, the assertion is that, if $f : X \longrightarrow Y$ has degree d , it also has degree d on the fibres of p and q .) We may assume that f is surjective, X , Y are varieties, and p is dominant (hence so is q , since $\overline{p(X)} = \overline{qf(X)}$ and $f(X) = Y$).

(i) dim X = dim Y : The commutativity of the triangle asserts

$$f_* [X_t] = \deg(f) \cdot [Y_t] .$$

Restricting to an affine neighbourhood of t , assume $C = \mathbb{A}^1$, that is, p and q are regular functions. It thus suffices to show

$$f_* [\mathrm{div}(f^* r)] = \deg(f) \cdot [\mathrm{div}(r)]$$

for $r \in R(Y)^*$, which follows from the key proposition in 1.3 , since $N(f^* r) = r^{\deg(f)}$.

(i) dim X > dim Y : Then $\dim X_t > \dim Y_t$.

Ad b) (this is essentially covariance of the flat pull-back.) Let $W \subsetneq Y$ be a subvariety of dimension k . Suppose

$$[f^{-1} W] = \sum_i m_i [V_i] ,$$

where the V_i are the irreducible components of $f^{-1} W$. Then:

$$\sigma_t f^* [W] = \sum_i m_i \sigma_t [V_i] = \sum_i m_i [(p|V_i)^{-1}(t)] .$$

Now there is the

Lemma([F] , 1.7.2) Suppose X is a purely n - dimensional scheme, $D \subseteq X$ a closed subscheme, locally generated by a non- zero- divisor. Let $X = \sum_i m_i [X_i]$, where the X_i are the irreducible components of X , and let $D_i := D \cap X_i$. Then

$$[D] = \sum_i m_i [D_i] \quad \text{in} \quad Z_{n-1}(X) .$$

Apply this lemma to $X := f^{-1}W$ and $D := f^{-1}W_t$; there follows

$$[f^{-1}W_t] = \sum_i m_i [f^{-1}W_t \cap V_i]$$

in $Z_{k+d-1}(f^{-1}W)$, and so in $Z_{k+d-1}(X)$. But, by definition:

$$\begin{aligned} f^{-1}W_t \cap V_i &= (qf \mid V_i)^{-1}(t) \\ &= (p \mid V_i)^{-1}(t) , \end{aligned}$$

hence

$$\begin{aligned} [f^{-1}W_t] &= \sum_i m_i [(p \mid V_i)^{-1}(t)] \\ &= \sigma_t f^* [W] . \end{aligned}$$

The assertion now follows, since $[f^{-1}W_t] = f_t^*(\sigma_t[W])$.
Q.e.d.

If now p is trivial fibration with fibre F , we have for each $t \in C$ GYSINhomomorphisms (cf. (3) in Part A, 2.1)

$$(1) \qquad i_t^* \ : \ Z_k(X) \xrightarrow{\sigma_t} Z_{k-1}(X_t) \cong Z_{k-1}(F) \ ,$$

where $i_t : F \hookrightarrow X$ denotes the inclusion $f \mapsto (f,t)$.
A cycle $\beta \in Z_k(X)$ then defines a family $(\beta(t))_{t \in C}$ of cycles $\beta(t) \in Z_{k-1}(F)$ via

$$\beta(t) \ := \ i_t^*(\beta) \ ,$$

called <u>specializations of β</u> . And we call two cycles α_o , $\alpha_1 \in Z_{k-1}(F)$ <u>equivalent with respect to p</u> if they both are specializations of the same cycle, i.e. if there is $\beta \in Z_k(X)$, t_o , $t_1 \in C$, with $\alpha_o = \beta(t_o)$ and $\alpha_1 = \beta(t_1)$.

<u>Theorem - Definition</u> <u>Let</u> X <u>be a scheme. For two cycles</u> α_o , $\alpha_1 \in Z_k(X)$, <u>the following are equivalent</u>:

(i) α_o <u>and</u> α_1 <u>are equivalent with respect to the flat projection</u> $X \times \mathbb{P}^1 \xrightarrow{p} \mathbb{P}^1$.

(ii) <u>There are finitely many</u> $(k+1)$- <u>dimensional sub-varieties</u> W_i <u>of</u> $X \times \mathbb{P}^1$ <u>such that</u> $p \mid W_i$ <u>is dominant for each</u> i <u>and</u> $\alpha_o - \alpha_1 = \sum_i ([W_i](0) - [W_i](\infty))$.

(iii) <u>There are finitely many</u> $(k+1)$- <u>dimensional sub-</u>

varieties V_i of X and $r_i \in R(V_i)^*$ for each i such that $\alpha_0 - \alpha_1 = \sum_i [\mathrm{div}(r_i)]$.

Such cycles are said to be rationally equivalent.

The proof is easy and left to the reader (see [F], 1.6).

We denote rational equivalence by $\alpha_0 \sim \alpha_1$. For given k , the cycles rationally equivalent to zero form a subgroup $R_k(X)$ of $Z_k(X)$; this follows most easily by the description (iii) of rational equivalence in the theorem, since $\mathrm{div}(1/r) = -\mathrm{div}(r)$.

1.5 CHOW homology There is but one definition to make:

Definition $A_k(X) := Z_k(X) / R_k(X)$ is called the k -th CHOW homology group of X .

1.6 Functorial properties of CHOW homology

Theorem a) A_* is covariant (of degree 0)) for proper morphisms.

b) A_* is contravariant (of degree d) for flat morphisms (of relative dimension d) .

<u>Proof</u> Ad a): We use the description (i) in the theorem above of rational equivalence: If β is a cycle of equivalence for α_0 and α_1, $(f \times id_{\mathbb{P}^1})_*(\beta)$ is a cycle of equivalence for $f_*(\alpha_0)$ and $f_*(\alpha_1)$ by a) of the proposition in 1.4 .
Or use (iii) and the key proposition of 1.3 .

Ad b): We use again (i); $(f \times id_{\mathbb{P}^1})^*(\beta)$ is a cycle of equivalence for $f^*(\alpha_0)$ and $f^*(\alpha_1)$ by b) of the proposition in 1.4 . Q.e.d.

1.7 <u>An exact sequence</u> Let Y be a closed subscheme of a scheme X, $U := X - Y$, $i : Y \hookrightarrow X$, $j : U \hookrightarrow X$ the inclusions. Then, for all k, is the sequence

$$(1) \qquad A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \longrightarrow 0$$

exact. This is straightforward from the definitions and the fact that any subvariety of U extends to X by taking its schematic closure.

2. <u>Intersecting with divisors</u>

2.1 <u>Divisors</u> Let X be a scheme. The sheaf $\mathcal{M}_X$ of germs of meromorphic functions on X is defined to be the sheaf associated to the presheaf

$$U \longmapsto \mathcal{S}(U)^{-1} \mathcal{O}_X(U) \quad , U \subseteq X \text{ open} ,$$

the localization of $\mathcal{O}_X(U)$ being with respect to the subset $\mathcal{S}(U) \subseteq \mathcal{O}_X(U)$ of those f which restrict , in each local ring $\mathcal{O}_{X,x}$, $x \in U$, to a non- zero- divisor. If X is locally NOETHERian, which is the case in our considerations since we suppose all schemes to be algebraic over a fixed field, the stalk $\mathcal{M}_{X,x}$ at $x \in X$ is just the total ring of quotients $\mathrm{Quot}(\mathcal{O}_{X,x})$, and if $U \subseteq X$ is affine, one has

$$\mathcal{M}_X(U) = \mathrm{Quot}(\mathcal{O}_X(U)) \quad ,$$

and so the presheaf

$$U \longmapsto \mathrm{Quot}(\mathcal{O}_X(U))$$

restricted to the open affines of X is a sheaf defining via the usual extension process the sheaf $\mathcal{M}_X$.

This understood, a (CARTIER) divisor on X is defined as a section of the sheaf $\mathcal{M}_X^* / \mathcal{O}_X^*$, and, by the above remarks, can be thought of as being defined by a collection $\{(U_\alpha, f_\alpha)\}$, where the U_α form an affine open cover of X, $f_\alpha \in R(U_\alpha)^*$ (with, for U open affine, $R(U) := \mathcal{M}_X(U) := \mathrm{Quot}(\mathcal{O}_X(U)))$, such that $f_\alpha / f_\beta \in \mathcal{O}_X(U_\alpha \cap U_\beta)^*$ for all α , β . The divisors form an additive group $\mathrm{Div}(X)$. Let $\mathrm{Div}^+(X)$ denote the submonoid of effective divisors, i.e. which can be given by $f_\alpha \in \mathcal{O}_X(U_\alpha)$; note that, conceptually, an effective divisor is just the same thing as a closed subscheme locally defined by a non- zero- divisor, i.e. defined by an invertible ideal sheaf.

2.2 The cycle of a divisor A WEIL divisor on X is defined to be an element of $Z^1(X)$ (our terminology will be such that 'divisor' always means 'CARTIER divisor'). A divisor D on X has an associated WEIL divisor cyc(D) or $[D]$, also called the associated cycle defined via the homomorphism

$$\mathrm{cyc} : \mathrm{Div}(X) \longrightarrow Z^1(X) ,$$

this homomorphism being well- defined by the requirements

(i) cyc is compatible with restriction to open subschemes

(ii) if $D \in \mathrm{Div}^+(X)$, then

$$\mathrm{cyc}(D) = \text{fundamental cycle of the closed subscheme } D \subseteq X .$$

If X is a variety of dimension n , it is easy to see, then, by chasing the definitions, that

$$[D] = \sum_{\substack{V \subseteq X \\ \mathrm{codim} V \, 1}} \mathrm{ord}_V(D) \cdot [V] \in Z_{n-1}(X) ,$$

with $\mathrm{ord}_V(D) := \mathrm{ord}_V(f)$ for f any local defining equation of D in any open U with $U \cap V \neq \emptyset$, this 'order of vanishing of D along V' being well- defined since f is locally well- defined up to units. Especially, if D is given by a single meromorphic function $r \in R(X)^*$ (such a divisor is called a principal divisor, denoted (r) or $\mathrm{div}(r)$), then $[D] = [\mathrm{div}(r)]$ as defined in 1.3 .

2.3 <u>Pseudodivisors</u> Let $D \in \mathrm{Div}(X)$ be a divisor on the scheme X . The <u>support of D</u> , supp D , is defined to be the (closed) subset of those $x \in X$ such that $f_x \notin \mathcal{O}_{X,x}^*$ for the germ of some (and hence any) local equation f of D at x . We also write $|D| := \mathrm{supp}\, D$, and

$$|D| = \bigcup_{\substack{V \subseteq X \text{ subvariety} \\ \mathrm{codim} V = 1 \\ \mathrm{ord}_V(D) \neq 0}} V .$$

Now let V be a k - dimensional subvariety of X . Our intention will be to construct the <u>intersection cycle</u>

$$D \circ [V] \in Z_{k-1}(|D| \cap V)$$

as follows: Restrict D to V and put

$$D \circ [V] := [D|V] \quad \text{, the WEIL divisor associated to } D|V \text{ by 2.2 .}$$

Unfortunately, D restricts to V only if $V \nsubseteq D$. But there is the following

<u>Key observation</u>: Line bundles always restrict.

Now, to a divisor D is canonically associated a line bundle L_D (also called $\mathcal{O}_X(D)$) together with a meromorphic section s_D ; if D is given by the collection $\{(U_\alpha, f_\alpha)\}$, L_D has transition functions $g_{\alpha\beta} := f_\alpha / f_\beta$ in $U_\alpha \cap U_\beta$, and s_D is given by f_α in U_α . If we call pairs (L,s) and (L',s') of line bundles and meromorphic sections isomorphic if and only if there is an isomorphism $L \cong L'$ of line bundles carrying s into s' , the assignment $D \mapsto (L_D, s_D)$ identifies divisors with isomorphism classes of line bundles together with a nontrivial section.

This considerations, together with the key observation, motivate the

__Definition__ __A pseudodivisor__ D __on a scheme__ X __is a triple__ (L,Z,s) , __where__ L __is a line bundle on__ X , Z __a closed subset of__ X , __and__ s __a nonvanishing section of__ $L \mid X - Z$, __up to obvious isomorphism__.

Put $D := Z$, $L_D := \mathcal{O}_X(D) := L$, $s_D := s$.

Now let X be a variety. Then any line bundle is the line bundle of a divisor (in other words, any line bundle has a nontrivial meromorphic section, and this follows from $H^1(X; \mathcal{M}_X^*) = 0$, which is due to the fact that $\mathcal{M}_X$ is a constant sheaf). Furthermore, any two nontrivial sections define linearly equivalent divisors, whose associated WEIL divisors consequently are rationally equivalent.

This considerations show the following: Let D be a pseudodivisor on the variety X . Choose a divisor C with $L_D \cong L_C$ and $s_D = s_C \mid X - D$ under this isomorphism (such a C is said to represent D). Then $[C]$ is well-defined in $A_{n-1}(|D|)$ ($n = \dim X$) , i.e. does not depend on the choice of C , and is called the <u>WEIL divisor class of the pseudodivisor D</u> , denoted $[D] \in A_{n-1}(|D|)$.

2.4 <u>Intersecting with pseudodivisors</u> The notion of pseudodivisor has been stated in such a way as to make restriction to closed subschemes always possible. We now are in a position to perform the

<u>Main intersection construction for pseudodivisors</u>

If X is a scheme, D a pseudodivisor on X , $\alpha \in Z_k(X)$, there is a well-defined intersection class $D \circ \alpha \in A_{k-1}(|D| \cap |\alpha|)$, where $|\alpha| := \operatorname{supp} \alpha$ is the support of α , defined by $|\alpha| := \bigcup_{m_\lambda \neq 0} V_\lambda$ when $\alpha = \sum_\lambda m_\lambda [V_\lambda]$. It is given by

$$D \circ [V] := [D \mid V] \text{ , the WEIL divisor class of the pseudodivisor } D|V \text{ on } V$$

for V a k - dimensional subvariety of X .

Notation Write $D \circ \alpha \in A_{k-1}(Y)$ for any Y with $|\alpha| \subseteq Y \subseteq X$.

This construction has the expected properties: biadditivity, projection formula, flat pullback, etc.; see [F] , 2.3 . E.g. the projection formula is: If $f : X \longrightarrow Y$ is proper, then

$$D \circ f_*(\alpha) = f_*(f^*D \circ \alpha)$$

in the appropriate CHOW groups. (Note that pseudodivisors always pull back.)

Theorem([F], 2.4) Intersecting passes to rational equivalence.

Idea of proof The main point is to show: X a variety, $D , D' \in \mathrm{Div}(X) \Rightarrow D \circ [D'] = D' \circ [D]$ in $A_{n-2}(|D| \cap |D'|)$. (Note that the corresponding assertion on the cycle level is false: If $\pi : X \longrightarrow \mathbb{A}^2$ blows up the origin 0 , put $D := \pi^{-1}(\mathbb{A} \times 0)$ and $D' := \pi^{-1}(0 \times \mathbb{A})$.)

Then one has $D \circ [\mathrm{div}(r)] = \mathrm{div}(r) \circ [D]$, and intersecting with a principal divisor is zero by construction. The conclusion follows by the description (iii) of rational equivalence in 1.4 .

Consequence For any closed subscheme Y of a scheme X ,

D a pseudodivisor on X, there is a well- defined homomorphism

$$(1) \qquad D \circ - \; : \; A_k(Y) \longrightarrow A_{k-1}(|D| \cap Y) \quad . \\ [V] \longmapsto D \circ [V]$$

with the 'usual' properties, called <u>intersecting with D</u>. Especially, if D is an effective divisor, and i the inclusion $D \hookrightarrow X$ of the closed subscheme D, $Y := X$, we call it the GYSINhomomorphism induced by i and denote it (cf. (3) in Part A , 2.1)

$$(2) \qquad i^* \; : \; A_k(X) \longrightarrow A_{k-1}(D) \quad .$$

2.5 <u>CHERN class of a line bundle</u> The fact that for any line bundle on a variety the corresponding WEIL divisor class as defined in 2.3 is uniquely defined can be formalized conveniently by the usual apparatus of the first CHERN class: If L is a line bundle on a scheme X, define the <u>first CHERN operators</u>

$$c_1(L) \cap - \; : A_k(X) \longrightarrow A_{k-1}(X)$$

by $c_1(L) \cap [V] :=$ WEIL divisor class of $L|V$ in $A_{k-1}(X)$ for any k - dimensional subvariety V. If D is any pseudodivisor with $L_D = L$, then

$$c_1(L)\cap - \; = \; D \circ -$$

as defined in 2.4 . The operations $c_1(L)\cap -$ are endomorphisms of $A_*(X)$ with the usual properties (projection formula, pullback, additivity with respect to $\otimes$, etc.). Their behaviour with respect to the GYSINhomomorphisms i^* of an effective divisor $D \overset{i}{\hookrightarrow} X$ defined at the end of 2.4 is as follows:

(i) $\quad i_*i^*(\alpha) \; = \; c_1(L_D)\cap\alpha$

(ii) $\quad i^*i_*(\alpha) \; = \; c_1(\nu_D)\cap\alpha$, where $\nu_D := L_D|D$ is the normal line bundle of D in X ('self intersection formula') .

3. Intersecting with the zero section

3.1 Decomposable bundles If $L \xrightarrow{\pi} X$ is a line bundle, the zero section $s : X \longrightarrow L$ embeds X as a divisor D_L on L , which, by trivializing L locally over the covering $\{U_\alpha\}$ of X by means of $\pi^{-1}U_\alpha \underset{\varphi_\alpha}{\cong} U_\alpha \times \mathbb{A}$, is given by the collection $\{(\pi^{-1}U_\alpha, f_\alpha)\}$, where f_α is $\pi^{-1}U \underset{\varphi_\alpha}{\cong} U \times \mathbb{A} \xrightarrow[pr_2]{} \mathbb{A}$; an explicit isomporphism of schemes $D_L \cong X$ is given by $\pi|D_L$. Define the GYSIN-homomorphism

$$s^* : A_k(L) \longrightarrow A_{k-1}(X)$$

via

$$\alpha \longmapsto (\pi | D_L)_*(D_L \circ \alpha)$$

with $D_L \circ \alpha \in A_{k-1}(|D_L|)$ defined as in 2.4 (so identifiying X with D_L, this is just the GYSINhomomorphism of the inclusion $s : X \hookrightarrow L$ as defined in 2.4). Since the local equations of D_L, restricted to $\varphi_\alpha^{-1}(U_\alpha \times \{0\})$, define $U_\alpha \times \{0\}$, one sees, applying the definitions:

$$s^*\pi^* = \mathrm{id}_{A_{k-1}(X)} \quad ,$$

hence π^* is injective, s^* surjective.

If now $E \xrightarrow{\pi} X$ is a decomposable vectorbundle of rank r, this construction can be iterated and yields

(1) $$s^* : A_k(X) \longrightarrow A_{k-r}(X)$$

with

$$s^*\pi^* = \mathrm{id}_{A_{k-r}(X)} \quad .$$

In fact, π^* is isomorphic with inverse s^* :

<u>Lemma</u> <u>If</u> $E \xrightarrow{\pi} X$ <u>is any vectorbundle of rank</u> r, $\pi^* : A_{k-r}(X) \longrightarrow A_k(E)$ <u>is surjective</u>.

<u>Corollary</u> (THOM isomorphism theorem for decomposable

bundles) *If* E *is decomposable (i.e. a direct sum of line bundles)* , π^* *is isomorphic with inverse* s^* .

Proof of the lemma Let $Y \subseteq X$ be closed and consider the commutative diagram

$$\begin{array}{ccccccc}
A_k(E|Y) & \longrightarrow & A_k(E) & \longrightarrow & A_k(E|X - Y) & \longrightarrow & 0 \\
\uparrow \pi^* & & \uparrow \pi^* & & \uparrow \pi^* & & \\
A_{k-r}(Y) & \longrightarrow & A_{k-r}(X) & \longrightarrow & A_{k-r}(E|X - Y) & \longrightarrow & 0
\end{array} ,$$

whose rows are exact by 1.7 . This diagram allows to reduce to the case that $E \longrightarrow X$ is trivial, that is, the bundle $X \times \mathbb{A}^r \xrightarrow{pr_1} X$. Since pr_1 factorizes as

$$X \times \mathbb{A}^r \longrightarrow X \times \mathbb{A}^{r-1} \longrightarrow \ldots \longrightarrow X \times \mathbb{A}^1 \longrightarrow X ,$$

induction on r reduces to the case $r = 1$, where a direct argument can be applied ([F], Proposition 1.9) . Q.e.d.

3.2 *General bundles* In this section, we complete the main intersection construction in Part A , 2.1 , by proving the

Theorem (THOM isomorphism theorem, [F] , theorem 3.3)
For any vectorbundle E X *on a scheme* X *of rank* r , *the flat pullback* π^* : $A_{k-r}(X) \longrightarrow A_k(E)$ *is an iso-*

morphism.

Hence, we may define the intersection with the zero section s to be the homomorphism

$$s^* := (\pi^*)^{-1} : A_k(E) \longrightarrow A_{k-r}(X) .$$

In case E is a decomposable bundle, this agrees with 3.1,(1).

Proof of the THOM isomorphism theorem In view of the lemma in 3.1 , we must prove that π^* is injective.

For any vectorbundle $V \xrightarrow{\pi} X$, we let $\mathbb{P}(V) \xrightarrow{p} X$ denote the associated projective bundle, whose fibre over $x \in X$ is the projective space of lines in V_x . Over $\mathbb{P}(V)$ lies the line bundle

$$\mathcal{O}_V(-1) \longrightarrow \mathbb{P}(V)$$

whose fibre over a point in $\mathbb{P}(V)$ is the line represented by that point. Let $\mathcal{O}_V(1)$ be its dual; it has a representation as a quotient of $p^*V^\vee$ via the exact sequence

$$0 \longrightarrow H \longrightarrow p^*V^\vee \longrightarrow \mathcal{O}_V(1) \longrightarrow 0 ,$$

where H is the hyperplane bundle on $\mathbb{P}(V)$, whose fibre over a point ξ in $\mathbb{P}(V)$, i.e. a line in V_x with $x := p(\xi)$, is just that line considered as a hyperplane of the dual space $V_x^\vee$. We then have the

<u>Structure theorem</u> <u>Let</u> V <u>have rank</u> $r = e+1$. <u>For any</u> k , <u>the map</u>

$$\theta_V : \bigoplus_{j=0}^{e} A_{k-e+j}(X) \longrightarrow A_k(\mathbb{P}(V))$$

$$(\alpha_0,\dots,\alpha_e) \longmapsto \sum_{j=0}^{e} c_1(\mathcal{O}_V(1))^j \cap p^*(\alpha_j)$$

<u>is a natural isomorphism</u>.

We defer the proof of this structure theorem to the end of this section; granting the theorem we turn to the proof of the THOM isomorphism theorem. For this, we consider the <u>projective closure</u> $\mathbb{P}(E \oplus \mathbb{1})$ of E ; the name refers to the fact that the inclusion $E \hookrightarrow E \oplus \mathbb{1}$ induces $\mathbb{P}(E) \overset{i}{\hookrightarrow} \mathbb{P}(E \oplus \mathbb{1})$ which embeds $\mathbb{P}(E)$ as an effective divisor, called the 'hyperplane section at ∞' , and $\mathbb{P}(E\oplus\mathbb{1}) - \mathbb{P}(E) = E$ in a canonical fashion. The projection $E \oplus \mathbb{1} \to \mathbb{1}$ gives a map on the total space E which is linear on the fibres, hence a section of $(E \oplus \mathbb{1})^\vee$ which can be pushed down to a section of the quotient $\mathcal{O}_{E\oplus\mathbb{1}}(1)$ of $(E\oplus\mathbb{1})^\vee$ vanishing precisely on $\mathbb{P}(E)$; in fact, for any $\beta \in A_k(\mathbb{P}(E\oplus\mathbb{1}))$:

$$(1) \qquad c_1(\mathcal{O}_{E\oplus\mathbb{1}}(1)) \cap \beta = \mathbb{P}(E) \circ \beta$$

$$= i_* i^* \beta \quad .$$

Denoting by $q : P(E) \longrightarrow X$, $p : \mathbb{P}(E \oplus \mathbb{1}) \longrightarrow X$ the projections, there is a diagram

$$\begin{array}{ccccccc} A_k(\mathbb{P}(E)) & \underset{i^*}{\overset{i_*}{\rightleftarrows}} & A_k(\mathbb{P}(E \oplus \mathbb{1})) & \longrightarrow & A_k(E) & \longrightarrow & 0 \\ & {\scriptstyle q^*} \nwarrow & \uparrow {\scriptstyle p^*} \circlearrowright & \nearrow {\scriptstyle \pi^*} & & & \\ & & A_{k-r}(X) & & & & \end{array}$$

with an exact row, from which follows

$$\pi^*(\alpha) = 0 \iff p^*(\alpha) \in \operatorname{im} i_* \quad .$$

But, by the structure theorem for $\mathbb{P}(E)$, im i_* consists of elements of the form

$$i_*\Big(\sum_{j=0}^{e} c_1(\mathcal{O}_E(1))^j \cap q^*(\alpha_j)\Big) =$$

$$\sum_{j=0}^{e} c_1(\mathcal{O}_{E\oplus\mathbb{1}}(1))^j \cap i_* i^* p^*(\alpha_j)$$ (by the projection formula, since $i^*\mathcal{O}_{E\oplus\mathbb{1}}(1) = \mathcal{O}_E(1)$)

$$=$$

$$\sum_{j=1}^{r} c_1(\mathcal{O}_{E\oplus\mathbb{1}}(1))^j \cap p^*(\alpha_j)$$ (by (1)) .

So, if $p^*(\alpha)$ is of this form, we conclude $\alpha = 0$, since, by the structure theorem for $\mathbb{P}(E\oplus\mathbb{1})$, this form must be unique. Q.e.d.

<u>Proof of the structure theorem</u> (i) Θ_V is surjective: As in the proof of the lemma in 3.1 we may assume V is

the trivial bundle by NOETHERian induction. Hence we can split V as $V'\oplus 1\!\!1$. Consider the following commutative diagram with an exact row:

$$\begin{array}{ccccccc}
A_k(\mathbb{P}(V')) & \xrightarrow{i_*} & A_k(\mathbb{P}(V)) & \xrightarrow{j^*} & A_k(V') & \longrightarrow & 0 \\
 & & & \nwarrow^{p^*} \quad \circlearrowright & \uparrow \pi'^* & & \\
 & & & & A_{k-e}(X) & &
\end{array}$$

with $i : \mathbb{P}(V') \hookrightarrow \mathbb{P}(V)$ and $j : V' \hookrightarrow \mathbb{P}(V)$ the inclusions and $\pi' = \pi | V'$ the projection of V'. If $\beta \in A_k(V')$, $j^*(\beta) = \pi'^*(\alpha)$ for an $\alpha \in A_{k-e}(X)$ by the lemma in 3.1, and then $\beta - p^*(\alpha) \in \mathrm{im}\, i_*$ by exactness of the row. The claim then follows by induction.

(ii) Θ_V is injective: Assume a nontrivial representation

$$0 = \sum_{j=0}^{e} c_1(\mathcal{O}_V(1))^j \cap p^*(\alpha_j) =: \beta$$

and let ℓ be the largest j with $\alpha_j \neq 0$.

<u>Lemma</u> <u>For any</u> $\alpha \in A_k(X)$:

$$p_*(c_1(\mathcal{O}_V(1))^j \cap p^*(\alpha)) = \begin{cases} 0 & j < e \\ \alpha & j = e \end{cases}$$

in $A_{k+e-j}(X)$.

<u>Proof of the lemma</u> We may restrict to the case $\alpha = [U]$, where U is a k - dimensional variety in X. By considering $p_U := V|U \longrightarrow U$ and naturality, one further is reduced to the case $U = X$, $\alpha = [X]$.

1) $j < e$: $A_{k+e-j}(X) = 0$.

2) $j = e$: $A_{k+e-j}(X) = A_k(X) = \mathbb{Z}\cdot[X]$, hence

$$p_*(c_1(\mathcal{O}_V(1)) \cap p^*(\alpha)) = m \cdot [X]$$

for some integer m, and to show is $m = 1$. Now $p^*(\alpha) = [\mathbb{P}(V)]$. Restricting to an open $U \subseteq X$ with $V|U$ trivial, we may assume V is trivial and so splits as $V' \oplus \mathbb{1}$ just as is (i) above. Then, by (1):

$$p_*(c_1(\mathcal{O}_V(1)) \cap [\mathbb{P}(V)]) = p_*(i_* [\mathbb{P}(V')])$$

$$= q_*([\mathbb{P}(V')]) ,$$

notations as in (i) above. Iterating:

$$p_*(c_1(\mathcal{O}_V(1))^e \cap [\mathbb{P}(V)]) = [X] , \text{ q.e.d. lemma .}$$

Now compute $0 = p_*(c_1(\mathcal{O}_V(1))^{e-\ell} \cap \beta) = \alpha_\ell$, a contradiction. Q.e.d.

This completes the main intersection construction.

References

[F] FULTON,W.: Intersection theory. Springer 1984

[1] DE CONCINI, C. and PROCESI, C.: Complete symmetric varieties. SLN 996 .

[2] ______________________________: Complete symmetric varieties II . In: Advanced studies in pure mathematics 6, ed. R. HOTTA, Kinokuniya - North-Holland 1985, 481-513.

[3] DE COCINI, C. , GIANNI, P. and TRAVERSO, C.: Computation of new Schubert tables for quadrics and projectivities. Successor to 2 , 515-523.

[4] FULTON, W.: Introduction to intersection theory in algebraic geometry. AMS 1984, Regional Conference Series in Mathematics, Number 54.

[5] _________ and MACPHERSON, R.: Intersecting cycles on an algebraic variety. In: Real and complex singularities, Oslo 1976, ed. P. HOLM. Sijthoff and Noordhoff 1977, 179-197.

[6] ____________________________: Defining algebraic intersections. SLN 687 , 1-30.

[7] ________, KLEIMAN,S. and MACPHERSON, R.: On the enumeration of contacts. SLN 997, 156-196.

[8] GRAYSON, D. R.: Coincidence formulas in enumerative geometry. Comm. in Algebra 7(16), 1979, 1685-1711.

[9] KLEIMAN, S.: The enumerative theory of singularities. Same reference as 6 , 297-396.

[10] __________: Chasles's enumerative theory of conics. A historical introduction. In: Studies in algebraic geometry. Math. Ass. Am. Stud. Math. 20(1980), 117-138.

[11] SCUBERT, H.: Kalkül der abzählenden Geometrie. Reprint of the 1879 Teubner edition with an introduction by Steven Kleiman. Springer 1979.

[12] FULTON, W.: Rational equivalence on singular varieties. Publ. IHES 45(1975), 147-167.