AN ADDITIVE PROBLEM WITH FRACTIONAL POWERS

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Abstract. For any $\alpha \in (1, 3/2)$ a sufficiently large positive ineteger x is representable in a form $x = [U^\alpha] + [V^\alpha]$ with positive integers U and V.

J.-M. Deshouillers[1] proved that for any $\alpha \in (1, 4/3)$ a sufficiently large positive ineteger x is representable in a form $x = [U^\alpha] + [V^\alpha]$ with nonnegative integers U and V. S. Gritsenko[2] extended the range of α possessing this property to $\alpha \in (1, 55/41)$. M. Z. Garaev and Ka-Lam Kueh[3] showed that for any $\alpha \in (1, 3/2]$ and $N \geq 2$ there are $\gg N$ numbers $x \leq N$ representable as $x = [U^\alpha] + [V^\alpha]$ with positive integers U and V.

In the present paper we prove the following result.

Theorem. *For any $\alpha \in (1, 3/2)$ a sufficiently large positive ineteger x is representable in a form $x = [U^\alpha] + [V^\alpha]$ with positive integers U and V.*

Proof. Take

$$\beta = (3/2 - \alpha)/2.$$

For positive integers k and n with $n/2 \leq k \leq n \leq x^\beta$ define positive numbers $u_0 = u_0(n,k)$ and $v_0 = v_0(n,k)$ in such a way that

$$u_0^\alpha + v_0\alpha = x \quad \text{and} \quad nu_0^{\alpha-1} = kv_0^{\alpha-1}. \tag{1}$$

So,

$$u_0 = \frac{k^{1/(\alpha-1)}}{L}x^{1/\alpha}, \quad v_0 = \frac{n^{1/(\alpha-1)}}{L}x^{1/\alpha}, \quad L = \left(k^{\alpha/(\alpha-1)} + n^{\alpha/(\alpha-1)}\right)^{1/\alpha}. \tag{2}$$

Note that

$$u_0 \asymp v_0 \asymp x^{1/\alpha}. \tag{3}$$

Denote $u = [u_0]$, $v = [v_0]$. By (3), we have

$$0 \leq u_0^\alpha - u^\alpha \ll u_0^{\alpha-1} \ll x^{(\alpha-1)/\alpha},$$

$$0 \leq v_0^\alpha - v^\alpha \ll x^{(\alpha-1)/\alpha}.$$

Therefore,

$$0 \leq x - u^\alpha - v^\alpha \ll x^{(\alpha-1)/\alpha}. \tag{4}$$

Our aim is to show that for suitable n, k, positive integers w, $U = u-wn$, $V = v+wk$ we have

$$x = [U^\alpha] + [V^\alpha]. \tag{5}$$

We consider the function

$$f(w) = (u - wn)^\alpha + (v + wk)^\alpha, \quad 0 \le w \le u/n.$$

Taking into account (3), we have

$$\begin{aligned}(u - wn)^\alpha &= u^\alpha - \alpha u^{\alpha-1} wn + \frac{\alpha(\alpha-1)}{2} u^{\alpha-2}(wn)^2 + O(u^{\alpha-3}(wn^3))\\ &= u^\alpha - \alpha u_0^{\alpha-1} wn + O(u_0^{\alpha-2} wn) + \frac{\alpha(\alpha-1)}{2} u^{\alpha-2}(wn)^2 + O(u^{\alpha-3}(wn^3))\\ &= u^\alpha - \alpha u_0^{\alpha-1} wn + \frac{\alpha(\alpha-1)}{2} u^{\alpha-2}(wn)^2 + O(x^{(\alpha-2)/\alpha} wn)\\ &\quad + O(x^{(\alpha-3)/\alpha}(wn)^3).\end{aligned} \tag{6}$$

By the similar way,

$$\begin{aligned}(v + wk)^\alpha = v^\alpha + \alpha v_0^{\alpha-1} wk + \frac{\alpha(\alpha-1)}{2} v^{\alpha-2}(wk)^2 + O(x^{(\alpha-2)/\alpha} wn)\\ + O(x^{(\alpha-3)/\alpha}(wn)^3).\end{aligned} \tag{7}$$

Here we have used that $k \asymp n$. Taking the sum of (6) and (7) and using (1) we get

$$\begin{aligned}f(w) = u^\alpha + v^\alpha + \frac{\alpha(\alpha-1)}{2} w^2(u^{\alpha-2}n^2 + v^{\alpha-2}k^2) + O(x^{(\alpha-2)/\alpha} wn)\\ + O(x^{(\alpha-3)/\alpha}(wn)^3).\end{aligned} \tag{8}$$

Now, take $wn = Ax^{1/(2\alpha)}$ with big A. We find from (8)

$$f(w) \ge u^\alpha + v^\alpha + \frac{\alpha(\alpha-1)}{2} A^2 u^{\alpha-2} x^{1/\alpha} + O(x^{(\alpha-3/2)/\alpha} A^3).$$

By (3), there is a constant $c = c(\alpha) > 0$ such that $\frac{\alpha(\alpha-1)}{2} u^{\alpha-2} \ge c x^{(\alpha-2)/\alpha}$. Hence,

$$f(w) \ge u^\alpha + v^\alpha + cA^2 x^{(\alpha-1)/\alpha} + O(x^{(\alpha-3/2)/\alpha} A^3) = u^\alpha + v^\alpha + (cA^2 + o(1)) x^{(\alpha-1)/\alpha}.$$

Now we can see from (4) that if we fix a big A then for sufficiently large n the inequality $f(w) > x + 1$ holds and, also, $f(0) < x + 1$. Thus, there is $w_0 > 0$ such that

$$(u - w_0 n)^\alpha + (v + w_0 k)^\alpha = f(w_0) = x + 1 \quad \text{and} \quad w_0 n \ll x^{1/(2\alpha)}. \tag{9}$$

To proceed further, we need the following estimate.

Lemma 1. *Let $z_0 > 0$, $z_1 \geq z_0/2$, $z_2 \geq z_0/2$. Then*

$$|z_1^\alpha - z_2^\alpha - \alpha(z_1 - z_2)z_0^{\alpha-1}| \ll |z_1 - z_2| \max_{z\in[z_1,z_2]} |z - z_0| z_0^{\alpha-2}.$$

Proof. We have

$$z_1^\alpha - z_2^\alpha = \int_{z_2}^{z_1} \alpha z^{\alpha-1} dz.$$

Therefore,

$$\begin{aligned}|z_1^\alpha - z_2^\alpha - \alpha(z_1 - z_2)z_0^{\alpha-1}| &\leq \int_{z_2}^{z_1} |\alpha z^{\alpha-1} - \alpha z_0^{\alpha-1}||dz| \\ &\leq |z_1 - z_2| \max_{z\in[z_1,z_2]} \alpha |z^{\alpha-1} - z_0^{\alpha-1}|.\end{aligned}$$

Note that, since $0 < \alpha - 1 < 1$, we have for $z \in [z_1, z_2]$

$$|z^{\alpha-1} - z_0^{\alpha-1}| \leq (\alpha - 1)|z - z_0|(\min(z, z_0))^{\alpha-2} \ll |z - z_0| z_0^{\alpha-2},$$

and the lemma follows.

Denote $w_1 = [w_0]$, $w_2 = [w_0] + 1$. We can write several useful inequalities based on Lemma 1.

Lemma 2. *There exists $\varepsilon = \varepsilon(\alpha, x)$ such that*

$$\varepsilon \ll n x^{(\alpha-3/2)/\alpha}$$

and the following inequalities hold:

$$|(u - w_j n)^\alpha - (u - w_0 n)^\alpha - \alpha(w_0 - w_j) n u_0^{\alpha-1}| \leq \varepsilon \quad (j = 1, 2), \tag{10}$$

$$|(v + w_j k)^\alpha - (v + w_0 k)^\alpha - \alpha(w_j - w_0) k v_0^{\alpha-1}| \leq \varepsilon \quad (j = 1, 2). \tag{11}$$

The proof is straightforward and follows immediately from Lemma 1 and (9). Note that, by the choice of n,

$$\varepsilon \ll x^\gamma, \quad \gamma = (\alpha - 3/2)(2 - \alpha)/(2\alpha) < 0. \tag{12}$$

The following lemma is the crucial part of the proof of the theorem.

Lemma 3. *If $\|\alpha n u_0^{\alpha-1}\| > 6\varepsilon$ then either $[(u - w_1 n)^\alpha] + [(v + w_1 k)^\alpha] = x$ or $[(u - w_2 n)^\alpha] + [(v + w_2 k)^\alpha] = x$.*

Proof. Taking the sum of (10) and (11) we have, by (1) and (9),

$$|(u - w_j n)^\alpha + (v + w_j k)^\alpha - x - 1| \leq 2\varepsilon \quad (j = 1, 2). \tag{13}$$

Consider the case if for some $j = 1, 2$ we have $\|(u - w_j n)^\alpha\| > 2\varepsilon$. Then

$$2\varepsilon < \{(u - w_j n)^\alpha\} + \{(v + w_j k)^\alpha\} < 2 - 2\varepsilon.$$

Hence, by (13),

$$x-1<[(u-w_jn)^\alpha]+[(v+w_jk)^\alpha]<x+1,$$

or $[(u-w_jn)^\alpha]+[(v+w_jk)^\alpha]=x$. Thus, the conclusion of the lemma holds unless

$$\|(u-w_1n)^\alpha\|\le 2\varepsilon,\quad \|(u-w_2n)^\alpha\|\le 2\varepsilon. \tag{14}$$

It suffices to study the case if (14) holds and, therefore,

$$\|(u-w_1n)^\alpha-(u-w_2n)^\alpha\|\le 4\varepsilon. \tag{15}$$

On the other hand, by (10),

$$|(u-w_1n)^\alpha-(u-w_2n)^\alpha-\alpha n u_0^{\alpha-1}|\le 2\varepsilon.$$

Thus, $\|\alpha n u_0^{\alpha-1}\|\le 6\varepsilon$. This completes the proof of the lemma.

Take $n=[x^\beta]$. To prove the theorem, it suffices to show the existence of an integer $k\in[n/2,n]$ such that

$$\|\alpha n u_0^{\alpha-1}\|>6\varepsilon, \tag{16}$$

where ε satisfies (12). Let

$$g(u)=\frac{\alpha u}{(1+u^{\alpha/(\alpha-1)})^{(\alpha-1)/\alpha}}.$$

By (2), the inequality (16) can be rewritten as

$$\left\|nx^{(\alpha-1)/\alpha}g(k/n)\right\|>6\varepsilon. \tag{17}$$

Fix a positive integer l so that $(l-1)\beta>(\alpha-1)/\alpha$ and an interval $[u_1,u_2]\subset(1/2,1)$ so that $u_2>u_1$ and $g^{(j)}(u)\ne 0$ for $u\in[u_1,u_2]$, $j=1,\dots,l$. Then for the exponential sum

$$S=\sum_{u_1n\le k\le u_2n} e\left(nx^{(\alpha-1)/\alpha}g(k/n)\right)$$

we have from [1, Theorem 2.9]

$$|S|\ll n(n^{-l}F)^{1/(2^l-2)}+nF^{-1},$$

where $F=nx^{(\alpha-1)/\alpha}$. Since $F\to\infty$ and, by the choice of l, $n^{-l}F\to 0$ as $n\to\infty$, we get $S=o(n)$ as $n\to\infty$. Therefore, for a large n there exists $k\in[u_1n,u_2n]$ such that

$$\left\|nx^{(\alpha-1)/\alpha}g(k/n)\right\|>0.2>6\varepsilon.$$

Indeed, if we assume the contrary, then

$$\Re S\ge\sum_{u_1n\le k\le u_2n}\cos(0.4\pi)\gg n,$$

but this inequality does not agree with $S = o(n)$.

So, (17) holds. The proof of the theorem is complete.

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