Max-Planck-Institut für Mathematik
Bonn

Cohomology representations of external and symmetric products of varieties

by

Laurentiu Maxim
Jörg Schürmann

Max-Planck-Institut für Mathematik
Preprint Series 2016 (12)

Cohomology representations of external and symmetric products of varieties

Laurentiu Maxim
Jörg Schürmann

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Dr
Madison, WI 53706-1388
USA

Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
Germany

MPIM 16-12

COHOMOLOGY REPRESENTATIONS OF EXTERNAL AND SYMMETRIC PRODUCTS OF VARIETIES

LAURENŢIU MAXIM AND JÖRG SCHÜRMANN

Abstract. We prove refined generating series formulae for characters of (virtual) cohomology representations of external products of suitable coefficients on (possibly singular) complex quasi-projective varieties, e.g., (complexes of) constructible or coherent sheaves, or (complexes of) mixed Hodge modules. These formulae generalize our previous results for symmetric and alternating powers of such coefficients, and apply also to other Schur functors. The proofs of these results are reduced via an equivariant Künneth formula to a more general generating series identity for abstract characters of tensor powers $\mathcal{V}^{\otimes n}$ of an element $\mathcal{V}$ in a suitable symmetric monoidal category. This abstract approach applies directly also in the equivariant context for varieties with additional symmetries (e.g., finite group actions, finite order automorphisms, resp., endomorphisms).

Contents

1. Introduction

All spaces in this paper are assumed to be complex quasi-projective varieties, though many constructions also apply to other categories of spaces (e.g., compact complex analytic manifolds or varieties over any base field of characteristic zero). In fact in Section 2 we explain our results from an abstract axiomatic viewpoint of the equivariant Künneth formula, which also covers cases like Zeta functions of constructible sheaves for the Frobenius endomorphism of varieties over finite fields (as in [28][Thm. on p.464] and [9][Thm.4.4 on p.174]).

Date: February 24, 2016.
2010 *Mathematics Subject Classification.* 55S15, 20C30, 19L20.
Key words and phrases. external and symmetric products of varieties, generating series, pre-lambda structure, Adams operation, symmetric groups, characters of representations, symmetric monoidal category, Schur functor.

1.1. **Generating series formulae.** In this paper, we obtain refined generating series formulae for characters of (virtual) cohomology representations of external products of suitable coefficients on (possibly singular) complex quasi-projective varieties, e.g., (complexes of) constructible or coherent sheaves, or (complexes of) mixed Hodge modules. These formulae generalize our previous results for symmetric products and configuration spaces from [20].

In more detail, we let $A(X)$ denote any of the following three categories of coefficients on a complex quasi-projective variety X:

(a) $D^b_c(X)$, the bounded derived category of (algebraically) constructible sheaf complexes of $\mathbb{C}$-vector spaces. Here constructibility also includes the assumption that all stalks are finite dimensional.

(b) $D^b_{coh}(X)$, the bounded derived category of complexes of $\mathcal{O}_X$-modules with coherent cohomology. In this case, we also assume that X is projective.

(c) $D^b\mathsf{MHM}(X)$, the bounded derived category of algebraic mixed Hodge modules on X.

All these categories of coefficients will be treated at once (in which case cohomology groups of such coefficients are regarded as finite dimensional $\mathbb{C}$-vector spaces), with the note that the case $A(X) = D^b\mathsf{MHM}(X)$ yields more refined results due to the additional mixed Hodge structures on the cohomology of X with mixed Hodge module coefficients. These more refined results will be stated separately.

For a fixed object $\mathcal{M} \in A(X)$, we consider the n-th self-external product $\mathcal{M}^{\boxtimes n}$ of $\mathcal{M}$ on the product X^n of n copies of X, with its induced Σ_n-action of the symmetric group Σ_n on n-elements. Then there is a Σ_n-equivariant Künneth isomorphism of finite dimensional vector spaces (resp. mixed Hodge structures if $A(X) = D^b\mathsf{MHM}(X)$), see [20] and the references therein, as well as [19] for the mixed Hodge module context:

$$H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n}) \simeq H^*_{(c)}(X, \mathcal{M})^{\otimes n}. \tag{1}$$

So, in particular, the (compactly supported) cohomology $H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})$ is a Σ_n-representation. Let $Rep_{\mathbb{C}}(\Sigma_n)$ be the Grothendieck group of (finite dimensional) complex representations of Σ_n. By associating to a representation its character, we get a group monomorphism (with finite cokernel):

$$tr_{\Sigma_n} : Rep_{\mathbb{C}}(\Sigma_n) \hookrightarrow C(\Sigma_n),$$

with $C(\Sigma_n)$ the free abelian group of $\mathbb{Z}$-valued class functions on Σ_n (recall that characters of a symmetric group are integer valued). Consider the generating *Poincaré polynomial for the characters* of the above Σ_n-representations, namely:

$$tr_{\Sigma_n}(H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})) := \sum_k tr_{\Sigma_n}(H^k_{(c)}(X^n, \mathcal{M}^{\boxtimes n})) \cdot (-z)^k \in C(\Sigma_n) \otimes \mathbb{Z}[z^{\pm 1}].$$

Aditionally, in the case when $A(X) = D^b\mathsf{MHM}(X)$, the cohomology groups $H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})$ carry mixed Hodge structures, and the associated graded vector spaces

$$H^{p,q,k}_{(c)}(X^n, \mathcal{M}^{\boxtimes n}) := Gr^p_F Gr^W_{p+q} H^k_{(c)}(X^n, \mathcal{M}^{\boxtimes n})$$

of the Hodge and resp. weight filtrations are also Σ_n-representations. So in this case we can also consider the following more refined generating *mixed Hodge polynomial for the characters* of the Σ_n-representations of these associated graded vector spaces, namely:

$$tr_{\Sigma_n}(H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})) := \sum_{p,q,k} tr_{\Sigma_n}(H^{p,q,k}_{(c)}(X^n, \mathcal{M}^{\boxtimes n})) \cdot y^p x^q (-z)^k \in C(\Sigma_n) \otimes \mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}].$$

While we use the same notation for the two types of generating polynomials (Poincaré and, resp., mixed Hodge), the reader should be able to distinguish their respective meaning from the context. Note that by forgetting the grading with respect to the mixed Hodge structure (i.e., by letting $y = x = 1$), the mixed Hodge polynomial (defined for mixed Hodge module coefficients) specializes to the Poincaré polynomial for the underlying constructible sheaf complex.

To simplify the notations and statements even further, we let $\mathbb{L}$ denote any of the two Laurent polynomial rings $\mathbb{Z}[z^{\pm 1}]$ and, respectively, $\mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}]$. Once again, its meaning in the results below should be clear from the context.

In this paper, we aim to calculate the generating series:

$$\sum_{n\geq 0} tr_{\Sigma_n}(H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})) \cdot t^n \in \bigoplus_n C(\Sigma_n) \otimes \mathbb{L}[[t]]$$

in terms of the corresponding *Poincaré polynomial*

$$P_{(c)}(X, \mathcal{M})(z) := \sum_k b^k_{(c)}(X, \mathcal{M}) \cdot (-z)^k \in \mathbb{L} := \mathbb{Z}[z^{\pm 1}],$$

and, respectively, *mixed Hodge polynomial*

$$h_{(c)}(X, \mathcal{M})(y, x, z) := \sum_{p,q,k} h^{p,q,k}_{(c)}(X, \mathcal{M}) \cdot y^p x^q (-z)^k \in \mathbb{L} := \mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}]$$

of $\mathcal{M}$ in the mixed Hodge module setting. Here,

$$b^k_{(c)}(X, \mathcal{M}) := \dim_{\mathbb{C}} H^k_{(c)}(X, \mathcal{M})$$

and

$$h^{p,q,k}_{(c)}(X, \mathcal{M}) := h^{p,q}(H^k_{(c)}(X, \mathcal{M})) := \dim_{\mathbb{C}} Gr^p_F Gr^W_{p+q} H^k_{(c)}(X, \mathcal{M})$$

denote the Betti and, respectively, mixed Hodge numbers of the (compactly supported) cohomology $H^*_{(c)}(X, \mathcal{M})$ of $\mathcal{M}$.

After composing with the Frobenius character homomorphism [17][Ch.1,Sect.7]:

$$ch_F : C(\Sigma) \otimes \mathbb{Q} := \bigoplus_n C(\Sigma_n) \otimes \mathbb{Q} \xrightarrow{\simeq} \Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[p_i, i \geq 1],$$

the generating series

$$\sum_{n\geq 0} tr_{\Sigma_n}(H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})) \cdot t^n$$

can be regarded as an element in the $\mathbb{Q}$-algebra $\mathbb{L} \otimes \mathbb{Q}[p_i, i \geq 1][[t]]$. Here, Λ is the graded ring of $\mathbb{Z}$-valued symmetric functions in infinitely many variables x_m ($m \in \mathbb{N}$), with $p_i = \sum_m x^i_m$ the i-th power sum function.

The first main result of this note is the following:

Theorem 1.1. *For any object $\mathcal{M} \in A(X)$, the following generating series identity for the Poincaré polynomials of characters of external products of $\mathcal{M}$ holds in the $\mathbb{Q}$-algebra $\mathbb{Q}[p_i, i \geq 1, z^{\pm 1}][[t]]$:*

$$\sum_{n\geq 0} tr_{\Sigma_n}(H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})) \cdot t^n = \exp\left(\sum_{r\geq 1} p_r \cdot P_{(c)}(X, \mathcal{M})(z^r) \cdot \frac{t^r}{r}\right). \tag{2}$$

Moreover, in the case when $A(X) = D^b MHM(X)$, the following refined generating series identity for the mixed Hodge polynomials of characters of external products of $\mathcal{M}$ holds in the $\mathbb{Q}$-algebra $\mathbb{Q}[p_i, i \geq 1, y^{\pm 1}, x^{\pm 1}, z^{\pm 1}][[t]]$:

$$\sum_{n \geq 0} tr_{\Sigma_n}(H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})) \cdot t^n = \exp\left(\sum_{r \geq 1} p_r \cdot h_{(c)}(X, \mathcal{M})(y^r, x^r, z^r) \cdot \frac{t^r}{r} \right). \tag{3}$$

The results of Theorem 1.1 can be specialized in several different ways, e.g., (i) for specific values of the parameters x, y, z; (ii) for special choices of the coefficients $\mathcal{M}$; and (iii) for special values of the Frobenius parameters p_r. All of these special cases will be discussed below.

(i) By letting $z = 1$ in (2) we obtain a generating series identity for the characters of the virtual cohomology representations

$$[H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})] := \sum_k (-1)^k [H^k_{(c)}(X^n, \mathcal{M}^{\boxtimes n})] \in Rep_{\mathbb{C}}(\Sigma_n),$$

with $P_{(c)}$ on the right-hand side of (2) being replaced by the corresponding (compactly supported) *Euler characteristic*

$$\chi_{(c)}(X, \mathcal{M}) := \sum_k (-1)^k \cdot b^k_{(c)}(X, \mathcal{M}) \in \mathbb{Z}.$$

Similarly, by letting $z = 1$ in (3), we get a generating series formula for the characters of graded parts (with respect to both filtrations) of the virtual cohomology representations

$$\sum_{k,p,q} (-1)^k \cdot [Gr^p_F Gr^W_{p+q} H^k_{(c)}(X^n, \mathcal{M}^{\boxtimes n})] y^p x^q \in Rep_{\mathbb{C}}(\Sigma_n)[y^{\pm 1}, x^{\pm 1}],$$

where $h_{(c)}$ on the right-hand side of (3) gets replaced by its specialization to the *E-polynomia*l $E_{(c)}$. In this case we recast Getzler's generating series for the E-polynomial [10][Prop.5.4]. Finally, by letting $x = z = 1$ in (3), we get a generating series formula for the characters of graded parts (with respect to the Hodge filtration) of the virtual cohomology representations

$$\sum_{k,p} (-1)^k \cdot [Gr^p_F H^k_{(c)}(X^n, \mathcal{M}^{\boxtimes n})] y^p \in Rep_{\mathbb{C}}(\Sigma_n)[y^{\pm 1}],$$

where $h_{(c)}$ on the right-hand side of (3) gets replaced by its specialization to the Hodge polynomial (or *Hirzebruch characteristic*) $\chi^{(c)}_{-y}$.

Remark 1.2. If X is *projective*, some of these special cases of Euler characteristic-type generating series have been derived in [21][Eqn.(7),(8)] by taking degrees of suitable equivariant characteristic class formulae. Note that in the mixed Hodge context, these characteristic class formulae only take into account the Hodge filtration, so the E-polynomial version discussed above, as well as Theorem 1.1 cannot be deduced as degree formulae. Moreover, if X is a projective manifold, the specialization χ_y mentioned above becomes the classical *Hirzebruch χ_y-genus*. This is also the reason why we choose y to be the parameter corresponding to the Hodge filtration (hence the unusual ordering y, x, z of parameters in the definition of mixed Hodge polynomial).

(ii) For the convenience of the reader, let us now specialize Theorem 1.1 to important concrete examples of coefficients $\mathcal{M} \in A(X)$, e.g., the constant sheaf $\mathbb{C}_X$ for $A(X) = D^b_c(X)$, the structure sheaf $\mathcal{O}_X$ for $A(X) = D^b_{coh}(X)$ and, respectively, the constant Hodge module (complex) $\mathbb{Q}^H_X$ for $A(X) = D^b\mathsf{MHM}(X)$.

Corollary 1.3. *Let X be a complex quasi-projective variety, which is moreover assumed to be projective in the coherent context. Then the following generating series identities hold:*

$$\sum_{n\geq 0} tr_{\Sigma_n}(H^*_{(c)}(X^n,\mathbb{C}))\cdot t^n = \exp\left(\sum_{r\geq 1} p_r \cdot P_{(c)}(X,\mathbb{C})(z^r)\cdot \frac{t^r}{r}\right), \tag{4}$$

$$\sum_{n\geq 0} tr_{\Sigma_n}(H^*(X^n,\mathcal{O}))\cdot t^n = \exp\left(\sum_{r\geq 1} p_r \cdot P(X,\mathcal{O})(z^r)\cdot \frac{t^r}{r}\right), \tag{5}$$

$$\sum_{n\geq 0} tr_{\Sigma_n}(H^*_{(c)}(X^n,\mathbb{Q}^H))\cdot t^n = \exp\left(\sum_{r\geq 1} p_r \cdot h_{(c)}(X,\mathbb{Q}^H)(y^r,x^r,z^r)\cdot \frac{t^r}{r}\right). \tag{6}$$

Note that in formula (6), the mixed Hodge structures on the (compactly supported) cohomology $H^*_{(c)}(X,\mathbb{Q}^H)$ coincides with Deligne's mixed Hodge structure on the rational vector spaces $H^*_{(c)}(X,\mathbb{Q})$.

Another distinguished choice of coefficients on a pure-dimensional variety X is the (shifted) *intersection cohomology Hodge module*

$$IC'^H_X := IC^H_X[-\dim(X)] \in D^b\mathsf{MHM}(X),$$

with underlying constructible sheaf complex $IC'_X := IC_X[-\dim(X)] \in D^b_c(X)$. The (compactly supported) cohomology groups $H^*_{(c)}(X, IC'^H_X)$ endow the (compactly supported) intersection cohomology groups of X, that is,

$$IH^*_{(c)}(X) := H^*_{(c)}(X, IC'_X),$$

with mixed Hodge structures. Thus, as a special case of (3) we get a generating series formula for the characters of graded parts (with respect to both filtrations) of *intersection cohomology representations* of cartesian products of X, namely:

$$\sum_{n\geq 0} tr_{\Sigma_n}(IH^*_{(c)}(X^n))\cdot t^n = \exp\left(\sum_{r\geq 1} p_r \cdot h_{(c)}(X,IC'^H_X)(y^r,x^r,z^r)\cdot \frac{t^r}{r}\right). \tag{7}$$

By letting $y = x = 1$ in (7), we obtain a generating series for the corresponding Poincaré-type polynomials of characters of intersection cohomology representations of cartesian products of X.

(iii) For suitable values of the Frobenius parameters p_r in Theorem 1.1, formulae (2) and (3) also generalize several generating series identities from [20] for the Betti numbers (respectively, mixed Hodge numbers) of *symmetric powers* $\mathcal{M}^{(n)}$ and *alternating powers* $\mathcal{M}^{\{n\}}$ of elements $\mathcal{M} \in A(X)$ (respectively, $\mathcal{M} \in D^b\mathsf{MHM}(X)$) on symmetric products $X^{(n)} := X^n/\Sigma_n$ of a quasi-projective variety X. (See [20] or Section 2.3 for a precise definition of symmetric and alternating powers of coefficients.)

By making $p_r = 1$ for all r, we recover from (2) the generating series for the Poincaré polynomials and Betti numbers of symmetric powers $\mathcal{M}^{(n)}$ of $\mathcal{M} \in A(X)$, namely:

$$\sum_{n \geq 0} P_{(c)}(X^{(n)}, \mathcal{M}^{(n)})(z) \cdot t^n = \exp\left(\sum_{r \geq 1} P_{(c)}(X, \mathcal{M})(z^r) \cdot \frac{t^r}{r} \right). \tag{8}$$

If, moreover, $\mathcal{M} \in D^b\mathrm{MHM}(X)$, then we recast from (3) the following generating series for the mixed Hodge numbers $h^{p,q,k}_{(c)}(X^{(n)}, \mathcal{M}^{(n)})$ of the symmetric powers of $\mathcal{M}$, i.e.,

$$\sum_{n \geq 0} h_{(c)}(X^{(n)}, \mathcal{M}^{(n)})(y, x, z) \cdot t^n = \exp\left(\sum_{r \geq 1} h_{(c)}(X, \mathcal{M})(y^r, x^r, z^r) \cdot \frac{t^r}{r} \right). \tag{9}$$

Let us recall here from [19] that for $\mathcal{M} = \mathbb{Q}^H_X \in D^b\mathrm{MHM}(X)$, the corresponding symmetric powers are computed by the formula

$$(\mathbb{Q}^H_X)^{(n)} = \mathbb{Q}^H_{X^{(n)}}, \tag{10}$$

so in this case (9) specializes to Cheah's generating series formula [6] for the mixed Hodge numbers of symmetric products of X. Similarly, (8) specializes for the choice of the constant sheaf coefficients $\mathbb{C}_X \in D^b_c(X)$ to Macdonald's generating series formula [18] for the Poincaré polynomials and Betti numbers of the symmetric products of X (see also formula (65) at the end of this paper). Furthermore, if X is projective and we let $\mathcal{M} = \mathcal{O}_X \in D^b_{coh}(X)$, then (8) yields the Poincaré polynomial generalization of Moonen's generating series formula [23][Cor.2.7,p.161] for the arithmetic genus of symmetric products of a projective variety. For $\mathcal{M} = IC'^H_X \in D^b\mathrm{MHM}(X)$, it is shown in [19] that the corresponding symmetric powers yield the (shifted) intersection cohomology modules on the symmetric products of X, i.e.,

$$(IC'^H_X)^{(n)} = IC'^H_{X^{(n)}}, \tag{11}$$

so (8) and (9) reduce in this case to generating series identities for the (compactly supported) intersection cohomology Betti numbers and mixed Hodge numbers, respectively. For more applications and special cases of formulae (8) and (9), the reader is advised to consult our previous work [20].

By making $p_r = (-1)^{r-1}$ for all r, we obtain a generating series formula for the Betti numbers $b^k_{(c)}(X^{(n)}, \mathcal{M}^{\{n\}})$ and, respectively, mixed Hodge numbers $h^{p,q,k}_{(c)}(X^{(n)}, \mathcal{M}^{\{n\}})$ if $\mathcal{M} \in D^b\mathrm{MHM}(X)$, of the alternating powers of $\mathcal{M}$. If, moreover, the underlying constructible complex of $\mathcal{M}$ is just a sheaf (placed in degree zero), then the alternating powers $\mathcal{M}^{\{n\}}$ of $\mathcal{M}$ are supported on the configuration space $X^{\{n\}} \subset X^{(n)}$ of n-tuples of distinct unordered points on X (see [20]), so we recover in this case the generating series formula for the Poincaré polynomial of Betti numbers $b^k_c(X^{\{n\}}, \mathcal{M}^{\{n\}})$:

$$\sum_{n \geq 0} P_c(X^{\{n\}}, \mathcal{M}^{\{n\}})(z) \cdot t^n = \exp\left(\sum_{r \geq 1} -P_c(X, \mathcal{M})(z^r) \cdot \frac{(-t)^r}{r} \right). \tag{12}$$

and, respectively, mixed Hodge numbers $h_c^{p,q,k}(X^{\{n\}}, \mathcal{M}^{\{n\}})$ if $\mathcal{M} \in D^b\mathsf{MHM}(X)$:

$$\sum_{n\geq 0} h_c(X^{\{n\}}, \mathcal{M}^{\{n\}})(y,x,z) \cdot t^n = \exp\left(\sum_{r\geq 1} -h_c(X,\mathcal{M})(y^r,x^r,z^r) \cdot \frac{(-t)^r}{r}\right). \tag{13}$$

For concrete examples and special cases of these formulae (e.g., for $\mathcal{M} = \mathbb{C}_X \in D^b_c(X)$ and resp. $\mathcal{M} = \mathbb{Q}^H_X \in D^b\mathsf{MHM}(X)$), see [20] and also [10].

The specialization $p_1 \mapsto 1$ and $p_r \mapsto 0$ if $r \geq 2$ corresponds to forgetting the Σ_n-action, up to the Frobenius-type factor $\frac{1}{n!}$. So, as a consequence of Theorem 1.1, we get the following:

Corollary 1.4. *For a complex quasi-projective variety X and a fixed coefficient $\mathcal{M} \in A(X)$ the following generating series holds in $\mathbb{Q}[z^{\pm 1}][[t]]$:*

$$\sum_{n\geq 0} P_{(c)}(X^n, \mathcal{M}^{\boxtimes n})(z) \cdot \frac{t^n}{n!} = \exp\left(P_{(c)}(X,\mathcal{M})(z)\cdot t\right). \tag{14}$$

Moreover, in the case when $\mathcal{M} \in D^b$MHM(X), the following refined generating series identity holds in the $\mathbb{Q}$-algebra $\mathbb{Q}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}][[t]]$:

$$\sum_{n\geq 0} h_{(c)}(X^n, \mathcal{M}^{\boxtimes n})(y,x,z) \cdot \frac{t^n}{n!} = \exp\left(h_{(c)}(X,\mathcal{M})(y,x,z)\cdot t\right). \tag{15}$$

Formulae (14) and (15) can also be obtained directly from the Künneth isomorphism (1).

1.2. **Twisting by symmetric group representations.** Additionally, for a fixed n, one can consider the coefficient of t^n in the generating series for the characters of cohomology representations $H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})$ of all exterior powers $\mathcal{M}^{\boxtimes n}$. Moreover, in this case, one can twist the coefficients $\mathcal{M}^{\boxtimes n}$ by a rational Σ_n-representation V (see Remark 2.12), to get a Σ_n-equivariant object $V \otimes \mathcal{M}^{\boxtimes n}$ in $A(X^n)$, and compute the corresponding characters for the twisted cohomology Σ_n-representations $H^*_{(c)}(X^n, V\otimes\mathcal{M}^{\boxtimes n})$ via the equivariant Künneth formula

$$H^*_{(c)}(X^n, V\otimes \mathcal{M}^{\boxtimes n}) \simeq V \otimes H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n}) \simeq V \otimes H^*_{(c)}(X,\mathcal{M})^{\otimes n}. \tag{16}$$

Here, in the Hodge context, we regard V as a pure Hodge structure of type $(0,0)$. By the multiplicativity of characters, we then have:

$$tr_{\Sigma_n}(H^*_{(c)}(X^n, V\otimes\mathcal{M}^{\boxtimes n})) = tr_{\Sigma_n}(V)\cdot tr_{\Sigma_n}(H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})). \tag{17}$$

Expanding the exponential series of Theorem 1.1, together with the above multiplicativity, we have the following identity in $\mathbb{Q}[p_i, i\geq 1, z^{\pm 1}]$:

$$tr_{\Sigma_n}(H^*_{(c)}(X^n, V\otimes\mathcal{M}^{\boxtimes n})) = \sum_{\lambda=(k_1,k_2,\cdots)\dashv n} \frac{p_\lambda}{z_\lambda}\chi_\lambda(V)\cdot \prod_{r\geq 1}\left(P_{(c)}(H^*(X;\mathcal{M})(z^r)\right)^{k_r}, \tag{18}$$

and, for $A(X) = D^b\mathsf{MHM}(X)$, the following refined formula holds in $\mathbb{Q}[p_i, i\geq 1, y^{\pm 1}, x^{\pm 1}, z^{\pm 1}]$:

$$tr_{\Sigma_n}(H^*_{(c)}(X^n, V\otimes\mathcal{M}^{\boxtimes n})) = \sum_{\lambda=(k_1,k_2,\cdots)\dashv n} \frac{p_\lambda}{z_\lambda}\chi_\lambda(V)\cdot \prod_{r\geq 1}\left(h_{(c)}(H^*(X;\mathcal{M})(y^r,x^r,z^r)\right)^{k_r}. \tag{19}$$

Here, for a partition $\lambda = (k_1, k_2, \cdots)$ of n (i.e., $\sum_{r\geq 1} r\cdot k_r = n$) corresponding to a conjugacy class of an element $\sigma \in \Sigma_n$, we denote by $z_\lambda := \prod_{r\geq 1} r^{k_r}\cdot k_r!$ the order of the stabilizer of σ, by $\chi_\lambda(V) = trace_\sigma(V)$ the corresponding trace, and we set $p_\lambda := \prod_{r\geq 1} p_r^{k_r}$.

Interesting new specializations (besides those already discussed above) arise for different choices of the representation V. For example, by choosing $V = \mathrm{Ind}_K^{\Sigma_n}(triv)$, the representation induced from the trivial representation of a subgroup K of Σ_n, and for $\mathcal{M} = \mathbb{C}_X \in D_c^b(X)$ the constant sheaf, formulae (18) and (19) specialize for $p_r = 1$ (for all r) to Macdonald's Poincaré polynomial formula [18][p.567] for the quotient X^n/K, i.e.,

$$(20) \qquad P_{(c)}(X^n/K, \mathbb{C})(z) = \sum_{\lambda=(k_1,k_2,\cdots)\dashv\, n} \frac{1}{z_\lambda}\chi_\lambda(\mathrm{Ind}_K^{\Sigma_n}(triv)) \cdot \prod_{r\geq 1} \left(P_{(c)}(H^*(X;\mathbb{C})(z^r)\right)^{k_r},$$

resp., to the corresponding formula for the mixed Hodge polynomial $h_{(c)}(X^n/K, \mathbb{Q}^H)(y,x,z)$, see (58). If X is projective, similar identities hold for the Poincaré polynomial of the coherent structure sheaf $\mathcal{O}_X$.

Similarly, for $V = V_\mu \simeq V_\mu^*$ the (self-dual) irreducible representation of Σ_n corresponding to a partition μ of n, (18) and (19) specialize for $p_r = 1$ (for all r) to formulae for the Poincaré resp. mixed Hodge polynomials of the corresponding *Schur-type objects* $S_\mu(\mathcal{M}) \in A(X^{(n)})$ associated to $\mathcal{M} \in A(X)$ (see Section 2.4 for a definition):

$$(21) \qquad P_{(c)}(X^{(n)}, S_\mu(\mathcal{M}))(z) = \sum_{\lambda=(k_1,k_2,\cdots)\dashv\, n} \frac{1}{z_\lambda}\chi_\lambda(V_\mu) \cdot \prod_{r\geq 1} \left(P_{(c)}(H^*(X;\mathcal{M})(z^r)\right)^{k_r},$$

and for $\mathcal{M} \in D^b\mathrm{MHM}(X)$:

$$(22) \quad h_{(c)}(X^{(n)}, S_\mu(\mathcal{M}))(y,x,z) = \sum_{\lambda=(k_1,k_2,\cdots)\dashv\, n} \frac{1}{z_\lambda}\chi_\lambda(V_\mu) \cdot \prod_{r\geq 1} \left(h_{(c)}(H^*(X;\mathcal{M})(y^r,x^r,z^r)\right)^{k_r}.$$

Note that at the cohomology level, we have the isomorphisms:

$$(23) \qquad H^*_{(c)}(X^{(n)}, S_\mu(\mathcal{M})) \cong \left(V_\mu \otimes H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes})\right)^{\Sigma_n},$$

and similarly for the graded pieces with respect to the Hodge and weight filtrations in the Hodge context. These Schur-type objects $S_\mu(\mathcal{M})$ generalize the symmetric and alternating powers of $\mathcal{M}$, which correspond to the trivial and resp. sign representation. Moreover, they can be used to get an alternative description of the characters of cohomology representations $H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})$ in terms of the *Schur functions* $s_\mu := ch_F(V_\mu) \in \Lambda \subset \mathbb{Q}[p_i, i \geq 1]$, see [17][Ch.1, Sect.3 and Sect.7], namely we have for any $\mathcal{M} \in A(X)$:

$$(24) \qquad tr_{\Sigma_n}(H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})) = \sum_{\mu \dashv\, n} s_\mu \cdot P_{(c)}(X^{(n)}, S_\mu(\mathcal{M}))(z),$$

with $P_{(c)}(X^{(n)}, S_\mu(\mathcal{M}))(z)$ computed as in (21). A similar formula holds for $\mathcal{M} \in D^b\mathrm{MHM}(X)$ by using instead the Hodge polynomials.

As a concrete example, for X pure dimensional with $\mathcal{M} = IC'^H_X \in D^b\mathrm{MHM}(X)$, the corresponding Schur-type object $S_\mu(IC'^H_X)$ is given by the (shifted) twisted intersection cohomology Hodge module $IC'^H_{X^{(n)}}(V_\mu)$, with twisted coefficients corresponding to the local system on the configuration space $X^{\{n\}} \subset X^{(n)}$ of unordered n-tuples of distinct points in X, induced from V_μ by the group homomorphism $\pi_1(X^{\{n\}}) \to \Sigma_n$ (compare [20][p.293] and [22][Prop.3.5]). So, formula (22) reduces in this case to the calculation of Hodge polynomials of twisted intersection cohomology

$$IH^*_{(c)}(X^{(n)}, V_\mu) := H^*_{(c)}(X^{(n)}; {IC'}^H_{X^{(n)}}(V_\mu)),$$

namely,

(25)

$$h_{(c)}(X^{(n)}, {IC'}^H_{X^{(n)}}(V_\mu))(y,x,z) = \sum_{\lambda=(k_1,k_2,\cdots)\dashv n} \frac{1}{z_\lambda}\chi_\lambda(V_\mu)\cdot \prod_{r\geq 1}\left(h_{(c)}(X; {IC'}^H_X)(y^r,x^r,z^r)\right)^{k_r}.$$

A special case of this formula, for the χ_{-y}-polynomial, has been recently obtained by the authors in [21][Eqn.(21)], by taking degrees of a certain characteristic class identity.

1.3. **Abstract generating series formulae and applications.** Theorem 1.1 is a direct application of a generating series formula for abstract characters cl_n of tensor powers $\mathcal{V}^{\otimes n}$ of an element $\mathcal{V}$ in a suitable symmetric monoidal category $(A, \otimes)$, which in our case will be

$$\mathcal{V} = H^*_{(c)}(X, \mathcal{M}), \quad \text{resp.,} \;\; \mathcal{V} = Gr^*_F Gr^W_* H^*_{(c)}(X, \mathcal{M}),$$

as an element in the abelian tensor category of finite dimensional (multi-)graded vector spaces. Note that the functor $Gr^*_F Gr^W_*$ is an exact tensor functor on the category of mixed Hodge structures, so it is compatible with the Künneth isomorphism (1).

In more detail, let A be a pseudo-abelian (or Karoubian) $\mathbb{Q}$-linear additive category which is also symmetric monoidal, with tensor product $\otimes$ $\mathbb{Q}$-linear additive in both variables. Then the corresponding Grothendieck ring $K_0(A)$ is a pre-lambda ring with a pre-lambda structure defined by (see [13]):

$$\sigma_t : K_0(A) \to K_0(A)[[t]]\,, \;\; [\mathcal{V}] \mapsto 1 + \sum_{n\geq 1} [(\mathcal{V}^{\otimes n})^{\Sigma_n}]\cdot t^n\,, \tag{26}$$

for $(-)^{\Sigma_n}$ the functor defined by taking the Σ_n-invariant part. Recall that a pre-lambda structure on a commutative ring R with unit 1 is a group homomorphism

$$\sigma_t : (R, +) \to (R[[t]], \cdot)\,; \;\; r \mapsto 1 + \sum_{n\geq 1} \sigma_n(r)\cdot t^n$$

with $\sigma_1 = id_R$, where "$\cdot$" on the target side denotes the multiplication of formal power series.

Let A_{Σ_n} be the additive category of the Σ_n-equivariant objects in A, as in [20][Sect.4], with corresponding Grothendieck group $K_0(A_{\Sigma_n})$. Then one has the following decomposition (e.g., see [20][Eqn.(45)] and Section 2.1):

$$K_0(A_{\Sigma_n}) \simeq K_0(A) \otimes_{\mathbb{Z}} Rep_{\mathbb{Q}}(\Sigma_n), \tag{27}$$

with $Rep_{\mathbb{Q}}(\Sigma_n)$ the ring of rational representations of Σ_n. We next denote by cl_n the composition:

$$cl_n : K_0(A_{\Sigma_n}) \simeq K_0(A) \otimes_{\mathbb{Z}} Rep_{\mathbb{Q}}(\Sigma_n) \xrightarrow{id\otimes tr_{\Sigma_n}} K_0(A) \otimes_{\mathbb{Z}} C(\Sigma_n).$$

Fix now an object $\mathcal{V} \in A$, and consider the generating series:

$$\sum_{n\geq 0} cl_n([\mathcal{V}^{\otimes n}])\cdot t^n \in K_0(A)\otimes C(\Sigma)[[t]].$$

After composing (in the second tensor factor) with the Frobenius character homomorphism

$$ch_F : C(\Sigma)\otimes\mathbb{Q} = \bigoplus_n C(\Sigma_n)\otimes \mathbb{Q} \xrightarrow{\simeq} \mathbb{Q}[p_i, i\geq 1], \tag{28}$$

the generating series $\sum_{n\geq 0} ch_F\left(cl_n([\mathcal{V}^{\otimes n}])\right)\cdot t^n$ is an element in the formal power series ring of the $\mathbb{Q}$-algebra $K_0(A)\otimes\mathbb{Q}[p_i, i\geq 1]$.

In the above notations, the main abstract formula of this note can now be stated as follows:

Theorem 1.5. *For any $\mathcal{V}\in A$, the following generating series identity holds in the $\mathbb{Q}$-algebra $\left(K_0(A)\otimes\mathbb{Q}[p_i, i\geq 1]\right)[[t]] = \left(\mathbb{Q}[p_i, i\geq 1]\otimes K_0(A)\right)[[t]]$:*

$$\sum_{n\geq 0} ch_F\left(cl_n([\mathcal{V}^{\otimes n}])\right)\cdot t^n = \exp\left(\sum_{r\geq 1}\psi_r([\mathcal{V}])\otimes p_r\cdot\frac{t^r}{r}\right), \tag{29}$$

with ψ_r the r-th Adams operation of the pre-lambda ring $K_0(A)$.

Note that by setting $p_r = 1$ for all r, formula (29) specializes to the well-known pre-lambda ring identity (e.g., see [16] or [17][Ch.1,Rem.2.15]):

$$\sigma_t\left([\mathcal{V}]\right) = 1 + \sum_{n\geq 1}\left[(\mathcal{V}^{\otimes n})^{\Sigma_n}\right]\cdot t^n = \exp\left(\sum_{r\geq 1}\psi_r([\mathcal{V}])\cdot\frac{t^r}{r}\right)\in K_0(A)\otimes_{\mathbb{Z}}\mathbb{Q}[[t]], \tag{30}$$

relating the pre-lambda structure to the corresponding Adams operations. Formula (30) was the main tool used for proving our results in [20] (and see also [10]). In this paper, we use a more general equivariant approach, which does not rely on the theory of pre-lambda rings.

Similarly, formulae (18) and (19) for twisted coefficients can be derived from the following abstract twisting formula (see Theorem 2.4 of Sect.2.1):

Theorem 1.6. *For V a rational representation of Σ_n and $\mathcal{V}\in A$, the following identity holds in $\mathbb{Q}[p_i, i\geq 1]\otimes K_0(A)$:*

$$ch_F\left(cl_n(V\otimes\mathcal{V}^{\otimes n})\right) = \sum_{\lambda=(k_1,k_2,\cdots)\dashv n}\frac{p_\lambda}{z_\lambda}\chi_\lambda(V)\otimes\prod_{r\geq 1}\left(\psi_r([\mathcal{V}])\right)^{k_r}, \tag{31}$$

where $\chi_\lambda(V) = trace_\sigma(V)$, for $\sigma\in\Sigma_n$ of cycle-type corresponding to the partition λ of n.

In Section 3, we indicate further applications of the above abstract setup to suitable equivariant versions of (characters of) Poincaré and mixed Hodge polynomials of *equivariant* coefficients. For simplicity, we illustrate here such equivariant formulae just for the constant coefficients $\mathcal{M} = \mathbb{Q}^H$ in the Hodge context, and for Macdonald-type generating series of symmetric products (i.e., with all Frobenius variables p_r set to 1).

Let (a) G be a finite group acting algebraically on X, (b) g be an algebraic automorphism of X of finite order, or (c) $g: X\to X$ be a (proper) algebraic endomorphism. Due the algebraic nature of the action, the (compactly supported) cohomology $H^*_{(c)}(X;\mathbb{Q})$ gets an induced pullback action of G, of the cyclic group $\langle g\rangle$, or, resp., of g, compatible with the mixed Hodge structures (with the assumption that g is proper if $H_c(-)$ is considered). It follows that the graded pieces $H^{p,q,k}_{(c)}(X;\mathbb{C})$ carry a similar action. So we can define a corresponding *equivariant mixed Hodge polynomial* $h^G_{(c)}(X;\mathbb{Q})$, $h^{\langle g\rangle}_{(c)}(X;\mathbb{Q})$, and resp. $h^g_{(c)}(X;\mathbb{Q})$ in this equivariant context as follows:

(a) If G is a finite group,

$$h^G_{(c)}(X,\mathbb{Q}^H)(y,x,z) := \sum_{p,q,k} tr_G(H^{p,q,k}_{(c)}(X,\mathbb{C}))\cdot y^p x^q(-z)^k\in C(G)\otimes\mathbb{C}[y,x,z],$$

with $C(G)\otimes\mathbb{C}$ the complex valued class-functions of G, and tr_G the usual character map.

(b) If g is an algebraic automorphism of X of finite order, we let

$$\chi^{p,q,k}_{\langle g\rangle}(X) := \sum_{\lambda\in\widehat{\mu}} \dim_{\mathbb{C}}(H^{p,q,k}_{(c)}(X,\mathbb{C})_{\lambda})\cdot(\lambda) \in \mathbb{Z}[\widehat{\mu}],$$

with $\mathbb{Z}[\widehat{\mu}]$ the group ring of the abelian group $\widehat{\mu}$ of roots of unity in $\mathbb{C}$ (with respect to multiplication), and $H^{p,q,k}_{(c)}(X,\mathbb{C})_{\lambda}$ denoting the corresponding λ-eigenspace of g. Then we set

$$h^{\langle g\rangle}_{(c)}(X,\mathbb{Q}^H)(y,x,z) := \sum_{p,q,k} \chi^{p,q,k}_{\langle g\rangle}(X)\cdot y^p x^q(-z)^k \in \mathbb{Z}[\widehat{\mu}]\otimes\mathbb{C}[y,x,z],$$

(c) If $g: X\to X$ is a (proper) algebraic endomorphism, then we set

$$h^{g}_{(c)}(X,\mathbb{Q}^H)(y,x,z) := \sum_{p,q,k} trace_g(H^{p,q,k}_{(c)}(X;\mathbb{C}))\cdot y^p x^q(-z)^k \in \mathbb{C}[y,x,z].$$

The external products X^n get an induced diagonal action of G, $\langle g\rangle$ or resp. g, commuting with the symmetric group action. Therefore, the symmetric products $X^{(n)}$ inherit a similar action of G, $\langle g\rangle$ or resp. g, so the corresponding invariants as above are also defined for each $X^{(n)}$.

We can now formulate the following Macdonald-type generating series result (for more general statements, see Theorem 3.3):

Theorem 1.7. (a) *If G is a finite group acting algebraically on X, then:*

$$\sum_{n\geq 0} h^G_{(c)}(X^{(n)},\mathbb{Q}^H)\cdot t^n = \exp\left(\sum_{r\geq 1}\psi_r(h^G_{(c)}(X,\mathbb{Q}^H))\cdot\frac{t^r}{r}\right) \in C(G)\otimes\mathbb{C}[y,x,z][[t]], \tag{32}$$

with $\psi_r(h^G_{(c)}(X,\mathbb{Q}^H)(y,x,z))(g) := h^G_{(c)}(X,\mathbb{Q}^H)(y^r,x^r,z^r)(g^r)$, for all $g\in G$.

(b) *If g is an algebraic automorphism of X of finite order, then:*

$$\sum_{n\geq 0} h^{\langle g\rangle}_{(c)}(X^{(n)},\mathbb{Q}^H)\cdot t^n = \exp\left(\sum_{r\geq 1}\psi_r(h^{\langle g\rangle}_{(c)}(X,\mathbb{Q}^H))\cdot\frac{t^r}{r}\right) \in \mathbb{Z}[\widehat{\mu}]\otimes\mathbb{C}[y,x,z][[t]], \tag{33}$$

with $\psi_r((\lambda)\cdot h(y,x,z)) := (\lambda^r)\cdot h(y^r,x^r,z^r)$, for $\lambda\in\widehat{\mu}$ and $h(y,x,z)\in\mathbb{C}[y,x,z]$.

(c) *If $g: X\to X$ is a (proper) algebraic endomorphism of X, then*

$$\sum_{n\geq 0} h^{g}_{(c)}(X^{(n)},\mathbb{Q}^H)(y,x,z)\cdot t^n = \exp\left(\sum_{r\geq 1} h^{g^r}_{(c)}(X,\mathbb{Q}^H)(y^r,x^r,z^r)\cdot\frac{t^r}{r}\right) \in \mathbb{C}[y,x,z][[t]]. \tag{34}$$

Let us finally compare special cases of Theorem 1.7 with other results available in the literature.

(a) By specializing to $z=1$, our invariant $h^G_{(c)}$ becomes the corresponding *equivariant E- (or Hodge-Deligne) polynomial* $E^G_{(c)}$. By further specializing also y and x to the value 1, this reduces to the more classical *equivariant Euler characteristic* $\chi^G_{(c)}\in C(G)\otimes\mathbb{C}$. Then (32) becomes a variant of [11][Lemma 1], which is formulated in terms of the Burnside ring $A(G)$ of G, instead of class functions.

(b) By specializing to $z=1$, our invariant $h^{\langle g \rangle}_{(c)}$ becomes the corresponding *equivariant E- (or Hodge-Deligne) polynomial* $E^{\langle g \rangle}_{(c)}$. Then formula (33) reduces in case of compact supports to [8][Theorem 1], which is formulated in terms of the power structure on the pre-lambda ring $\mathbb{Z}[\widehat{\mu}] \otimes \mathbb{C}[y,x]$. By further specializing to $x=1$, this equivariant E-polynomial reduces to the well-studied *Hodge spectrum* of a finite order automorphism.

(c) The right-hand side of formula (34) is a Hodge version of the classical *Lefschetz Zeta function*, to which it reduces by specializing the variables y, x, z to the value 1. Similarly, as $g^r = id_X$ for all r in case $g = id_X$, the graded (resp. Hodge) version of the classical *Lefschetz Zeta function* specializes in this case to (Cheah's Hodge version [6] of) *Macdonald's generating series formula* [18] for the Poincaré polynomials and Betti numbers of the symmetric products of X (see formula (65) and also Theorem 3.4 for the corresponding graded version of the Lefschetz Zeta function).

Remark 1.8. The interested reader should compare our results also with [15][Prop.15.5] and resp. [4][Thm.3.12], for an abstract analog of (34) and resp. (32) in the context of an automorphism resp. of a finite group action for a *dualizable* object in a suitable tensor category, with a corresponding notion of a trace.

The specialization at $z=1$ of Theorem 1.7 (a) resp. (b) above can also be reformulated (compare also with [8, 11]) by saying that

(35) $$E^G_c : K^G_0(var/\mathbb{C}) \to C(G) \otimes \mathbb{C}[y,x] \quad \text{resp.} \quad E^{\langle g \rangle}_c : K^{\langle g \rangle}_0(var/\mathbb{C}) \to \mathbb{Z}[\widehat{\mu}] \otimes \mathbb{C}[y,x]$$

is a morphism of pre-lambda rings, with the pre-lambda structure of the corresponding equivariant Grothendieck group of complex algebraic varieties (with respect to the scissor relation) defined via the *Kapranov Zeta function*

$$[X] \mapsto [pt] + \sum_{n \geq 1} [X^{(n)}] \cdot t^n \,.$$

Similar considerations apply for the variant

(36) $$h^G_{(c)} : \bar{K}^G_0(var/\mathbb{C}) \to C(G) \otimes \mathbb{C}[y,x,z] \quad \text{resp.} \quad h^{\langle g \rangle}_{(c)} : \bar{K}^{\langle g \rangle}_0(var/\mathbb{C}) \to \mathbb{Z}[\widehat{\mu}] \otimes \mathbb{C}[y,x,z]$$

on the corresponding equivariant Grothendieck group of complex algebraic varieties (with respect to disjoint unions) as studied in [20][Sec.2.2] in the non-equivariant context.

In future works, the equivariant context for a finite group action will be extended to *wreath products*, needed, e.g., in the abstract framework for the study of the *plethysm* action on the lambda ring $K_0(A) \otimes \mathbb{Q}[p_i, i \geq 1]$ and the composition of Schur- resp. polynomial functors (as e.g. in [17][I, App. A]), as well as for *orbifold versions* of our results, and in the study of *configuration spaces* and their *Fulton-MacPherson compactifications* (as considered for example in [10]; compare also with [14] in the context of algebraic varieties over finite fields).

Acknowledgements. *L. Maxim was partially supported by grants from NSF, NSA, by a grant of the Ministry of National Education, CNCS-UEFISCDI project number PN-II-ID-PCE-2012-4-0156, and by a fellowship from the Max-Planck-Institut für Mathematik, Bonn. J. Schürmann was supported by the SFB 878 "groups, geometry and actions".*

2. Abstract generating series identities and Applications

In this section, we derive Theorem 1.1 from the Introduction as a consequence of a more general generating series for abstract characters of tensor powers $\mathcal{V}^{\otimes n}$ of an element $\mathcal{V}$ in a suitable symmetric monoidal category.

2.1. **Symmetric monoidal categories.** Let A be a pseudo-abelian (or Karoubian) $\mathbb{Q}$-linear additive category which is also symmetric monoidal, with the tensor product $\otimes$ $\mathbb{Q}$-linear additive in both variables. Let $K_0(A)$ denote the corresponding Grothendieck ring. Similarly, let A_{Σ_n} be the additive category of the Σ_n-equivariant objects in A, as in [20][Sect.4], with corresponding Grothendieck group $K_0^{\Sigma_n}(A) := K_0(A_{\Sigma_n})$. Then one has the following decomposition (e.g., see [20][Eqn.(45)]):

$$K_0^{\Sigma_n}(A) \simeq K_0(A) \otimes_{\mathbb{Z}} Rep_{\mathbb{Q}}(\Sigma_n) \simeq Rep_{\mathbb{Q}}(\Sigma_n) \otimes_{\mathbb{Z}} K_0(A), \tag{37}$$

with $Rep_{\mathbb{Q}}(\Sigma_n)$ the ring of rational representations of Σ_n. In fact, this follows directly from the corresponding decomposition of $\mathcal{Y} \in A_{\Sigma_n}$ by *Schur functors* $S_\mu : A_{\Sigma_n} \to A$, $\mathcal{Y} \mapsto (V_\mu \otimes \mathcal{Y})^{\Sigma_n}$ (e.g., see [7, 13]):

$$\mathcal{Y} \simeq \sum_{\mu \vdash n} V_\mu \otimes S_\mu(\mathcal{Y}) \,, \tag{38}$$

with $V_\mu \simeq V_\mu^*$ the (self-dual) irreducible $\mathbb{Q}$-representation of Σ_n corresponding to the partition μ of n. Here, the Karoubian $\mathbb{Q}$-linear additive structure of A is used to defined the Σ_n-invariant part functor by the projector

$$(-)^{\Sigma_n} := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \sigma_*,$$

with σ_* denoting the action of $\sigma \in \Sigma_n$.

As in the classical representation theory, the rings $K_0^{\Sigma_n}(A)$ have product, induction and restriction functors compatible with (37), induced from the corresponding functors on A_{Σ_n}, see [7][Sect.1], [13][Sect.4.1]:

(a) the product:

$$\otimes : K_0^{\Sigma_n}(A) \otimes K_0^{\Sigma_m}(A) \to K_0^{\Sigma_n \times \Sigma_m}(A)$$

induced from

$$\otimes : A_{\Sigma_n} \otimes A_{\Sigma_m} \to A_{\Sigma_n \times \Sigma_m}.$$

(b) induction functor:

$$\mathrm{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} : K_0^{\Sigma_n \times \Sigma_m}(A) \to K_0^{\Sigma_{n+m}}(A)$$

induced from the additive functor

$$\mathrm{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} : A_{\Sigma_n \times \Sigma_m} \to A_{\Sigma_{n+m}}, \quad \mathcal{Y} \mapsto (\mathbb{Q}[\Sigma_{n+m}] \otimes \mathcal{Y})^{\Sigma_n \times \Sigma_m}.$$

(c) the restriction functor

$$\mathrm{Res}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} : K_0^{\Sigma_{n+m}}(A) \to K_0^{\Sigma_n \times \Sigma_m}(A)$$

induced from the obvious restriction functor: $\mathrm{Res}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} : A_{\Sigma_{n+m}} \to A_{\Sigma_n \times \Sigma_m}$.

We next denote by cl_n the composition:

$$cl_n : K_0^{\Sigma_n}(A) \simeq K_0(A) \otimes_{\mathbb{Z}} Rep_{\mathbb{Q}}(\Sigma_n) \xrightarrow{id \otimes tr_{\Sigma_n}} K_0(A) \otimes_{\mathbb{Z}} C(\Sigma_n).$$

By the above considerations, cl_n is compatible with the product, induction and restriction functors, with the corresponding classical notions for the character group. Therefore, we get an induced graded ring homomorphism (which becomes an isomorphism after tensoring with $\mathbb{Q}$)

$$cl := \sum_n cl_n : \bigoplus_n K_0^{\Sigma_n}(A) \longrightarrow K_0(A) \otimes_{\mathbb{Z}} \left(\bigoplus_n C(\Sigma_n) \right) = K_0(A) \otimes_{\mathbb{Z}} C(\Sigma).$$

Here the commutative induction product on both sides is given by:

$$\odot := \mathrm{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} (\cdot \otimes \cdot).$$

Fix now an object $\mathcal{V} \in A$, and consider the generating series:

$$\sum_{n \geq 0} cl_n([\mathcal{V}^{\otimes n}]) \cdot t^n \in K_0(A) \otimes C(\Sigma)[[t]].$$

Remark 2.1. Note that the total power maps

$$\mathcal{V} \mapsto \sum_{n \geq 0} [\mathcal{V}^{\otimes n}] \cdot t^n \mapsto \sum_{n \geq 0} cl_n([\mathcal{V}^{\otimes n}]) \cdot t^n$$

only depend on the Grothendieck class $[\mathcal{V}] \in K_0(A)$, see [20][Prop.3.2].

After composing with the Frobenius character homomorphism

$$ch_F : C(\Sigma) \otimes \mathbb{Q} = \bigoplus_n C(\Sigma_n) \otimes \mathbb{Q} \xrightarrow{\simeq} \mathbb{Q}[p_i, i \geq 1], \tag{39}$$

the generating series $\sum_{n \geq 0} ch_F \left(cl_n([\mathcal{V}^{\otimes n}]) \right) \cdot t^n$ is an element in the formal power series ring of the $\mathbb{Q}$-algebra $K_0(A) \otimes \mathbb{Q}[p_i, i \geq 1]$. Note that the homomorphisms

$$K_0(A) \otimes \left(\bigoplus_n Rep_{\mathbb{Q}}(\Sigma_n) \right) [[t]] \to K_0(A) \otimes C(\Sigma)[[t]] \to K_0(A) \otimes C(\Sigma) \otimes \mathbb{Q}[[t]]$$

are injective if $K_0(A)$ is $\mathbb{Z}$-torsion-free, so no information is lost in this case after tensoring with $\mathbb{Q}$, or after applying the Frobenius character homomorphism. For example, this is the case if A is the tensor category of finite dimensional multi-graded vector spaces, or the category of (polarizable) mixed Hodge structures.

We can now state our main abstract generating series formula:

Theorem 2.2. *For any $\mathcal{V} \in A$, the following generating series identity holds in the $\mathbb{Q}$-algebra $(K_0(A) \otimes \mathbb{Q}[p_i, i \geq 1])\,[[t]] = (\mathbb{Q}[p_i, i \geq 1] \otimes K_0(A))\,[[t]]$:*

$$\sum_{n \geq 0} ch_F \left(cl_n([\mathcal{V}^{\otimes n}]) \right) \cdot t^n = \exp \left(\sum_{r \geq 1} \psi_r([\mathcal{V}]) \otimes p_r \cdot \frac{t^r}{r} \right), \tag{40}$$

with ψ_r the r-th Adams operation of the pre-lamda ring $K_0(A)$.

Proof. This formula can be seen as a special case of Theorem 3.1 from our previous work [21]. However, here we give a direct proof based on the calculus of symmetric functions, adapted to the context of this section.

For $\sigma \in \Sigma_n$, we denote by

$$cl_n([\mathcal{V}^{\otimes n}])(\sigma) \in K_0(A)$$

the element obtained from $cl_n([\mathcal{V}^{\otimes n}])$ by evaluating the character at σ. Then, if $\sigma \in \Sigma_n$ has cycle-type $(k_1, k_2, \cdots)$, by using the fact that cl_n commutes with the restriction and product functors it follows that the following multiplicativity property holds:

$$cl_n([\mathcal{V}^{\otimes n}])(\sigma) = \otimes_r \left(cl_r([\mathcal{V}^{\otimes r}])(\sigma_r)\right)^{k_r}, \tag{41}$$

where $\sigma_r \in \Sigma_r$ is a cycle of length r. For any $r \geq 1$, let us now set

$$b_r := cl_r([\mathcal{V}^{\otimes r}])(\sigma_r) \in K_0(A).$$

By the definition of the Frobenius character [17][Ch.1, Sect.7], we have:

$$ch_F\left(cl_n([\mathcal{V}^{\otimes n}])\right) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} cl_n([\mathcal{V}^{\otimes n}])(\sigma) \otimes \psi(\sigma) \in K_0(A) \otimes \mathbb{Q}[p_i, i \geq 1], \tag{42}$$

where

$$\psi(\sigma) = \prod_r p_r^{k_r} = p_\lambda$$

for $\sigma \in \Sigma_n$ in the conjugacy class corresponding to the partition $\lambda := (k_1, k_2, \cdots)$ of n (i.e., $\sum_r rk_r = n$). Then by (41), formula (42) can be re-written as:

$$ch_F\left(cl_n([\mathcal{V}^{\otimes n}])\right) = \sum_{\lambda \dashv n} \frac{p_\lambda}{z_\lambda} \otimes \prod_r b_r^{k_r} \in K_0(A) \otimes \mathbb{Q}[p_i, i \geq 1], \tag{43}$$

with $z_\lambda := \prod_r r^{k_r} \cdot k_r!$ the order of the stabilizer in Σ_n of an element of cycle-type λ. So, we have as in [17][p.25] (see also [15][p.554]):

$$\begin{aligned}
\exp\left(\sum_{r \geq 1} b_r \otimes p_r \cdot \frac{t^r}{r}\right) &= \prod_{r \geq 1} \exp\left(b_r \otimes p_r \cdot \frac{t^r}{r}\right) \\
&= \prod_{r \geq 1} \sum_{k_r = 0}^{\infty} \frac{(b_r \otimes p_r)^{k_r}}{r^{k_r} \cdot k_r!} \cdot t^{rk_r} \\
&= \sum_n \left(\sum_{\lambda \dashv n} \frac{p_\lambda}{z_\lambda} \otimes \prod_r b_r^{k_r}\right) \cdot t^n \\
&\overset{(43)}{=} \sum_n ch_F\left(cl_n([\mathcal{V}^{\otimes n}])\right) \cdot t^n.
\end{aligned} \tag{44}$$

To conclude the proof of the theorem, recall from [20][Sect.3] that the r-th Adams operation on $K_0(A)$ can be given as

$$\psi_r([\mathcal{V}]) = cl_n([\mathcal{V}^{\otimes r}])(\sigma_r) =: b_r,$$

for σ_r a cycle of length r in Σ_r (as originally introduced by Atiyah in the context of topological K-theory [2]). □

Remark 2.3. Formula (42) also explains that the specialization $p_1 \mapsto 1$ and $p_r \mapsto 0$ if $r \geq 2$ used in Corollary 1.4 corresponds to forgetting the Σ_n-action, up to the Frobenius-type factor $\frac{1}{n!}$.

We next explain in this abstract setting the twisting construction used in the Introduction (see Section 1.2). Let $Vect_{\mathbb{Q}}(\Sigma_n)$ be the category of finite dimensional rational Σ_n-representations. We define a pairing

$$Vect_{\mathbb{Q}}(\Sigma_n) \times A_{\Sigma_n} \xrightarrow{\otimes} A_{\Sigma_n};\ (V, \mathcal{Y}) \mapsto V \otimes \mathcal{Y} \tag{45}$$

by the composition

$$Vect_{\mathbb{Q}}(\Sigma_n) \times A_{\Sigma_n} \xrightarrow{\otimes} A_{\Sigma_n \times \Sigma_n} \xrightarrow{\mathrm{Res}} A_{\Sigma_n},$$

with the underlying tensor product $\otimes$ defined via the $\mathbb{Q}$-linear additive structure of A (as in [7]) together with its induced Σ_n-action on each factor, and Res denoting the restriction functor for the diagonal subgroup $\Sigma_n \hookrightarrow \Sigma_n \times \Sigma_n$. This induces a pairing

$$Rep_{\mathbb{Q}}(\Sigma_n) \times K_0^{\Sigma_n}(A) \xrightarrow{\otimes} K_0^{\Sigma_n}(A) \tag{46}$$

on the corresponding Grothendieck groups such that

$$cl_n([V \otimes \mathcal{Y}]) = tr_{\Sigma_n}(V) \cdot cl_n([\mathcal{Y}]) \in K_0(A) \otimes_{\mathbb{Z}} C(\Sigma_n) \simeq C(\Sigma_n) \otimes_{\mathbb{Z}} K_0(A), \tag{47}$$

for V a rational Σ_n-representation and $\mathcal{Y} \in A_{\Sigma_n}$, with multiplication $\cdot$ induced by the usual multiplication of class functions.

By using formula (43), together with the above multiplicativity (47), we obtain (after composing with the Frobenius character homomorphism ch_F) the following:

Theorem 2.4. *In the above notations, the following identity holds in* $\mathbb{Q}[p_i, i \geq 1] \otimes K_0(A)$*:*

$$ch_F\left(cl_n(V \otimes \mathcal{V}^{\otimes n})\right) = \sum_{\lambda = (k_1, k_2, \cdots) \dashv n} \frac{p_\lambda}{z_\lambda} \chi_\lambda(V) \otimes \prod_{r \geq 1} \left(\psi_r([\mathcal{V}])\right)^{k_r}, \tag{48}$$

where $\chi_\lambda(V) = trace_\sigma(V)$*, for* $\sigma \in \Sigma_n$ *of cycle-type corresponding to the partition* λ *of* n*, and* ψ_r *the* r*-th Adams operation on* $K_0(A)$ *as before.*

We next make the following

Definition 2.5. Let V be a finite dimensional rational Σ_n-representation. The associated *Schur* (or *homogeneous polynomial*) *functor* $S_V : A \to A$ is defined by

$$S_V(\mathcal{V}) := (V \otimes \mathcal{V}^{\otimes n})^{\Sigma_n}.$$

If $V = V_\mu \simeq V_\mu^*$ is the (self-dual) irreducible representation of Σ_n corresponding to a partition μ of n, we denote by $S_\mu := S_{V_\mu}$ the corresponding Schur functor.

Remark 2.6. The Schur functor S_V associated to V induces a corresponding pairing ($\mathbb{Q}$-linear and additive only in the first factor)

$$Rep_{\mathbb{Q}}(\Sigma_n) \times K_0(A) \longrightarrow K_0(A)$$

on Grothendieck groups, defined via the composition:

$$Rep_{\mathbb{Q}}(\Sigma_n) \times K_0(A) \xrightarrow{id \times (-)^{\otimes n}} Rep_{\mathbb{Q}}(\Sigma_n) \times K_0^{\Sigma_n}(A) \xrightarrow{\otimes} K_0^{\Sigma_n}(A) \xrightarrow{(-)^{\Sigma_n}} K_0(A),$$

where the n-th power map

$$(-)^{\otimes n} : K_0(A) \to K_0^{\Sigma_n}(A);\ [\mathcal{V}] \mapsto [\mathcal{V}^{\otimes n}]$$

is well-defined by [20][Prop.3.2], $\otimes$ is the pairing defined above, and $K_0^{\Sigma_n}(A) \stackrel{(-)^{\Sigma_n}}{\longrightarrow} K_0(A)$ is induced from the corresponding additive projection functor.

By specializing (48) to $p_r = 1$ for all r (which, by the Schur functor decomposition (38), corresponds to taking the Σ_n-invariant part), we obtain a computation of the Grothendieck class $[S_V(\mathcal{V})] = S_V([\mathcal{V}])$ of the Schur (or polynomial) functor associate to V in terms of Adams operations. More precisely,

Corollary 2.7. *In the above notations, we have:*

$$S_V([\mathcal{V}]) = \sum_{\lambda=(k_1,k_2,\cdots)\dashv\, n} \frac{1}{z_\lambda}\chi_\lambda(V) \otimes \prod_{r\geq 1} (\psi_r([\mathcal{V}]))^{k_r} \in K_0(A)\otimes \mathbb{Q}. \tag{49}$$

Finally, the Schur functor decomposition (38) yields (after composing with the Frobenius character homomorphism ch_F), the following identity for any $\mathcal{V} \in A$:

$$ch_F\left(cl_n([\mathcal{V}^{\otimes n}])\right) = \sum_{\mu \dashv\, n} s_\mu \otimes S_\mu([\mathcal{V}]) \in \Lambda \otimes K_0(A), \tag{50}$$

with $s_\mu := ch_F(V_\mu) \in \Lambda \subset \mathbb{Q}[p_i, i \geq 1]$ the corresponding *Schur functions*, see [17][Ch.1, Sect.3 and Sect.7]. Note that the Frobenius character ch_F induces an isomorphism of graded rings

$$ch_F : Rep_{\mathbb{Q}}(\Sigma) := \bigoplus_n Rep_{\mathbb{Q}}(\Sigma_n) \stackrel{\simeq}{\longrightarrow} \Lambda \subset \mathbb{Q}[p_i, i \geq 1].$$

Remark 2.8. The non-degenerate pairing $Rep_{\mathbb{Q}}(\Sigma_n) \times Rep_{\mathbb{Q}}(\Sigma_n) \longrightarrow \mathbb{Z}$, given by $(V, W) \mapsto \dim_{\mathbb{Q}}(V \otimes W)^{\Sigma_n}$ induces a duality isomorphism

$$D : Rep_{\mathbb{Q}}(\Sigma_n) \simeq \mathrm{Hom}_{\mathbb{Z}}(Rep_{\mathbb{Q}}(\Sigma_n), \mathbb{Z}) =: Rep_{\mathbb{Q}}(\Sigma_n)_*$$

identifying the Schur functor $S_V : K_0(A) \to K_0(A)$ with the corresponding *operation* on $K_0(A)$ defined by $D(V)$, as in [20][Sect.3] (where we followed Atiyah's approach to K-theory operations). Summing over all n, we get isomorphisms of commutative graded rings

$$\Lambda \xleftarrow[\sim]{ch_F} Rep_{\mathbb{Q}}(\Sigma) \xrightarrow[\sim]{D} Rep_{\mathbb{Q}}(\Sigma)_*$$

identifying their respective operations on $K_0(A)$ (see also [4][Lem.2.6] and [29][Cor.5.2]). Here,

(1) Λ acts as a universal lambda ring on $K_0(A)$, as in [10].
(2) $Rep_{\mathbb{Q}}(\Sigma)$ acts via direct sums of Schur functors (also called polynomial functors), as considered in the present paper.
(3) $Rep_{\mathbb{Q}}(\Sigma)_*$ acts via operations as in [20][Sect.3].

2.2. **From abstract to concrete identities.** Let us now explain how to derive our Theorem 1.1, as well as formula (17) from the Introduction from the above abstract generating series formula. We start with the proof of formula (3) in the mixed Hodge context.

For an additive tensor category $(Ab, \otimes)$, let $Gr^-(Ab)$ denote the additive tensor category of bounded graded objects in Ab, i.e., functors $G : \mathbb{Z} \to Ab$, with $G_n := G(n) = 0$ except for finitely many $n \in \mathbb{Z}$. Here,

$$(G \otimes G')_n := \oplus_{i+j=n} G_i \otimes G_j \ ,$$

with the Koszul symmetry isomorphism (indicated by the $-$ sign in Gr^-):

$$(-1)^{i\cdot j} s(G_i, G_j) : G_i \otimes G_j \simeq G_j \otimes G_i \ .$$

If $(Ab, \otimes)$ is a $\mathbb{Q}$-linear Karoubian (or abelian) symmetric monoidal category, then the same is true for $Gr^{-}(Ab)$. This applies for example to the category mHs of mixed Hodge structures. Note that in the Künneth formula (1), we have to view $H^{*}_{(c)}(X, \mathcal{M})$ as an element in the $Gr^{-}(mHs)$ with tensor product $\otimes$ defined via the above Koszul rule.

Let $Gr^{*}_{F}Gr^{W}_{*} : mHs \to Gr^{2}(\mathsf{vect}_{f}(\mathbb{C}))$ be the functor of taking the associated bigraded finite dimensional $\mathbb{C}$-vector space:

$$V \mapsto \oplus_{p,q}\, Gr^{p}_{F}Gr^{W}_{p+q}(V \otimes_{\mathbb{Q}} \mathbb{C}) \in Gr^{2}(\mathsf{vect}_{f}(\mathbb{C}))\, .$$

This is an exact tensor functor of such abelian tensor categories, if we use the induced symmetry isomorphism without any sign changes for the abelian category $Gr^{2}(\mathsf{vect}_{f}(\mathbb{C}))$ of bigraded finite dimensional complex vector spaces. The transformation $Gr^{*}_{F}Gr^{W}_{*}$ is compatible with the Künneth isomorphism (1). Similarly, $Gr^{*}_{F}Gr^{W}_{*}$ is compatible with the abstract pairing (46), as well as taking invariant subobjects. Moreover, for $A = Gr^{-}(mHs)$, the abstract pairing gets identified with the tensor product on A, as used in (16), after regarding a rational representation as a pure Hodge structure of type $(0, 0)$ placed in degree zero.

Recall next that the ring homomorphism

$$h : K_{0}(Gr^{-}(Gr^{2}(\mathsf{vect}_{f}(\mathbb{C})))) \to \mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}]$$

given by

$$[\oplus (V^{p,q})^{k}] \mapsto \sum_{p,q,k} dim((V^{p,q})^{k}) \cdot y^{p}x^{q}(-z)^{k}\, ,$$

with k the degree with respect to the grading in Gr^{-} is an isomorphism of pre-lambda rings, see [20][Prop.2.4]. The pre-lambda structure on $K_{0}(Gr^{-}(Gr^{2}(\mathsf{vect}_{f}(\mathbb{C}))))$ is defined as in (26), whereas the pre-lambda structure on the Laurent polynomial ring $\mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}]$ corresponds to the Adams operations

$$\psi_{r}(p(y, x, z)) = p(y^{r}, x^{r}, z^{r}).$$

The sign choice of numbering by $(-z)^{k}$ in the definition of h is needed for the compatibility with these pre-lambda structures.

Finally, we have an equality

$$(h \otimes id) \circ cl_{n} = tr_{\Sigma_{n}} : K^{\Sigma_{n}}_{0}(Gr^{-}Gr^{2}(\mathsf{vect}_{f}(\mathbb{C}))) \to \mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}] \otimes C(\Sigma_{n}),$$

as can be easily checked on generators given by a Σ_{n}-representation placed in a single multi-degree.

Formula (3) follows now by applying the ring homomorphism

$$(h \circ Gr^{*}_{F}Gr^{W}_{*}) \otimes id : K_{0}(Gr^{-}(mHs)) \otimes \mathbb{Q}[p_{i}, i \geq 1] \to \mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}] \otimes \mathbb{Q}[p_{i}, i \geq 1]$$

to formula (40) of Theorem 2.2, with $\mathcal{V} := H^{*}_{(c)}(X, \mathcal{M}) \in A := Gr^{-}(mHs)$. Similarly, formula (17) follows by applying this ring homomorphism to the identity (47).

For the proof of the generating series (2) and the multiplicativity (17) for the Poincaré-type polynomials, we consider similarly the isomorphism of pre-lambda rings

$$P : K_{0}(Gr^{-}(\mathsf{vect}_{f}(\mathbb{C}))) \to \mathbb{Z}[z^{\pm 1}]$$

defined by taking the dimension counting Laurent polynomial

$$[\oplus V^{k}] \mapsto \sum_{k} dim(V^{k}) \cdot (-z)^{k}\, ,$$

with k the degree with respect to the grading in Gr^-. Here, $\mathrm{vect}_f(\mathbb{C})$ is the abelian tensor category of finite dimensional complex vector spaces, and the Adams operation on $\mathbb{Z}[z^{\pm 1}]$ is given by $\psi_r(p(z)) = p(z^r)$. Similarly, we have an equality

$$(P \otimes id) \circ cl_n = tr_{\Sigma_n} : K_0^{\Sigma_n}(Gr^-(\mathrm{vect}_f(\mathbb{C}))) \to \mathbb{Z}[z^{\pm 1}] \otimes C(\Sigma_n).$$

Then formula (2) follows by applying the ring homomorphism

$$P \otimes id : K_0(Gr^-(\mathrm{vect}_f(\mathbb{C}))) \otimes \mathbb{Q}[p_i, i \geq 1] \to \mathbb{Z}[z^{\pm 1}] \otimes \mathbb{Q}[p_i, i \geq 1]$$

to formula (29), with $\mathcal{V} := H^*_{(c)}(X, \mathcal{M}) \in A := Gr^-(\mathrm{vect}_f(\mathbb{C}))$. Similarly, formula (17) follows by applying this ring homomorphism to the identity (47).

2.3. **Pseudo-functors.** In this section we explain the connection of Theorem 1.1 with our previous results from [20] about generating series of symmetric and alternating powers of suitable coefficients, e.g., (complexes of) constructible or coherent sheaves, or (complexes of) mixed Hodge modules. In fact, all of this can be discussed in the abstract setting of suitable pseudo-functors, as in [20], which we now recall.

Let $(-)_*$ be a (covariant) pseudo-functor on the category of complex quasi-projective varieties (with proper morphisms), taking values in a pseudo-abelian (or Karoubian) $\mathbb{Q}$-linear additive category $A(-)$, e.g., see [20][Sect.4.1]. In fact, our abstract axiomatic approach would also work for a suitable (small) category of *spaces* with finite products and a terminal object pt (corresponding to the empty product, see [20][Appendix] for more details). Assume, moreover, that the following properties are satisfied:

(i) For any quasi-projective variety X and all n there is a multiple external product

$$\boxtimes^n : \times^n A(X) \to A(X^n),$$

equivariant with respect to a permutation action of the symmetric group Σ_n, i.e., $M^{\boxtimes n} \in A(X^n)$ is a Σ_n-equivariant object, for all $M \in A(X)$.

(ii) $A(pt)$ is endowed with a $\mathbb{Q}$-linear tensor structure $\otimes$, which makes it into a symmetric monoidal category.

(iii) For any quasi-projective variety X, $M \in A(X)$ and all n, there is a Σ_n-equivariant isomorphism

$$k_*(M^{\boxtimes n}) \simeq (k_* M)^{\otimes n},$$

with k the constant morphism to a point pt. Here, the Σ_n-action on the left-hand side is induced from (i), whereas the one on the right-hand side comes from (ii).

For example, the above properties are fullfilled for $A(X) = D^b\mathrm{MHM}(X)$, the bounded derived category of algebraic mixed Hodge modules on X, viewed as a pseudo-functor with respect to either of the push-forwards $(-)_*$ or $(-)_!$, as well as for the derived categories $D^b_c(X)$ and $D^b_{coh}(X)$ of bounded complexes with constructible and resp. coherent cohomology, see [20] for more details. In the coherent setting, we restrict to projective varieties X, so that in this context $(-)_* = (-)_!$.

Remark 2.9. Property (iii) is the abstract analogue of the Künneth isomorphism (1).

Let $\pi_n : X^n \to X^{(n)}$ be the natural projection onto the n-th symmetric product $X^{(n)} := X^n/\Sigma_n$. By property (i), for any $M \in A(X)$ the exterior product $M^{\boxtimes n}$ is a Σ_n-equivariant object in $A(X^n)$, i.e., it is an element of $A_{\Sigma_n}(X^n)$, e.g., see [20][Sect.4.2]. Then the push down $\pi_{n*}M^{\boxtimes n}$ to the

n-th symmetric product is a Σ_n-equivariant object on $X^{(n)}$. Since Σ_n acts trivially on $X^{(n)}$, the Σ_n-action on $\pi_{n*}M^{\boxtimes n}$ corresponds to a group homomorphism

$$\Psi : \Sigma_n \to Aut_{A(X^{(n)})}(\pi_{n*}M^{\boxtimes n}).$$

Moreover, since $A(X^{(n)})$ is a $\mathbb{Q}$-linear additive category, we can define the symmetric projector

$$(-)^{\Sigma_n} := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \Psi_\sigma$$

onto the Σ_n-invariant part, and, respectively, the alternating projector

$$(-)^{sign-\Sigma_n} := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} sign(\sigma)\Psi_\sigma,$$

for $sign : \Sigma_n \to \{\pm 1\}$ the sign character, and Ψ_σ denoting the σ-action $\Psi(\sigma)$. Using the Karoubian structure, we can then associate to an object $M \in A(X)$ its *n-th symmetric power*

$$M^{(n)} := \left(\pi_{n*}M^{\boxtimes n}\right)^{\Sigma_n}$$

and, respectively, its *n-th alternating power*

$$M^{\{n\}} := \left(\pi_{n*}M^{\boxtimes n}\right)^{sign-\Sigma_n},$$

as objects in $A(X^{(n)})$. As in [20][Sect.2], we then have the identities (with k denoting in this paper the constant map from any space to a point):

$$k_*(M^{(n)}) \simeq \left((k_*M)^{\otimes n}\right)^{\Sigma_n} \quad \text{and} \quad k_*(M^{\{n\}}) \simeq \left((k_*M)^{\otimes n}\right)^{sign-\Sigma_n}. \tag{51}$$

which allow the calculation of invariants of $k_*M^{(n)}$ and $k_*M^{\{n\}}$, respectively, only in terms of those for $k_*M \in A(pt)$ and the symmetric monoidal structure $\otimes$, see [20] for more details. Here we are interested in representation-theoretic refinements of such formulae from [20] expressed in terms of abstract generating series identities for the Σ_n-equivariant objects ($n \geq 0$):

$$k_*M^{\boxtimes n} \simeq (k_*M)^{\otimes n} \in A_{\Sigma_n}(pt).$$

In this section $A(pt) =: A$ plays the role of the underlying symmetric monoidal category used in Section 2.1.

Let $\bar{K}_0(-)$ denote the Grothendieck group of an additive category viewed as an exact category by the split exact sequences corresponding to direct sums $\oplus$, i.e., the Grothendieck group associated to the abelian monoid of isomorphism classes of objects with the direct sum. Here we do not use the notation $K_0(-)$ as before, because if A is a triangulated category (e.g., $D^b\mathrm{MHM}(pt)$, $D^b_c(pt)$ or $D^b_{coh}(pt)$), then $K_0(-)$ usually denotes the Grothendieck group of this triangulated category. Of course, the two notions coincide for the abelian tensor category of multi-graded finite dimensional vector spaces. As in Section 2.1, $\bar{K}_0(A(pt))$ becomes a pre-lambda ring.

By Theorem 2.2, applied to

$$\mathcal{V} := k_*\mathcal{M} \in A(pt),$$

with $M \in A(X)$, we obtain by property (iii) of the pseudo-functor $(-)_*$ the following equivariant generalization of [20][Thm.1.7]:

Theorem 2.10. *For any $M \in A(X)$, the following generating series identity holds in the $\mathbb{Q}$-algebra $\left(\bar{K}_0(A(pt)) \otimes \mathbb{Q}[p_i, i \geq 1]\right)[[t]] = \left(\mathbb{Q}[p_i, i \geq 1] \otimes \bar{K}_0(A(pt))\right)[[t]]$:*

$$\sum_{n\geq 0} ch_F\left(cl_n([k_*M^{\boxtimes n}])\right) \cdot t^n = \exp\left(\sum_{r\geq 1} p_r \otimes \psi_r([k_*M]) \cdot \frac{t^r}{r}\right), \tag{52}$$

with ψ_r the corresponding r-th Adams operation of the pre-lambda ring $\bar{K}_0(A(pt))$.

Specializing to $p_r = 1$ for all r corresponds via the composed homomorphism $ch_F \circ cl_n$ to the functor induced on Grothendieck groups by taking the Σ_n-invariant part

$$(-)^{\Sigma_n} = \frac{1}{n!}\sum_{\sigma\in\Sigma_n} \Psi_\sigma : A_{\Sigma_n}(pt) \longrightarrow A(pt).$$

Indeed, this reduces via the decomposition

$$\bar{K}_0^{\Sigma_n}(A(pt)) \simeq \bar{K}_0(A(pt)) \otimes Rep_{\mathbb{Q}}(\Sigma_n)$$

to the corresponding classical formula for the represention ring $Rep_{\mathbb{Q}}(\Sigma_n)$. So, by letting $p_r = 1$ for all r in Theorem 2.10, one obtains by the isomorphism

$$k_*(M^{(n)}) \simeq \left((k_*M)^{\otimes n}\right)^{\Sigma_n}$$

the following generating series from [20][Thm.1.7]:

$$1 + \sum_{n\geq 1} [k_*M^{(n)}] \cdot t^n = \exp\left(\sum_{r\geq 1} \psi_r([k_*M]) \cdot \frac{t^r}{r}\right) \in \bar{K}_0(A(pt)) \otimes_{\mathbb{Z}} \mathbb{Q}[[t]]\,. \tag{53}$$

Similarly, by specializing to $p_r = (-1)^{r-1} = sign(\sigma_r)$ for all r (with σ_r denoting as before an r-cycle in Σ_r) corresponds via the composed homomorphism $ch_F \circ cl_n$ to the functor induced on Grothendieck groups by taking the projector onto the alternating part of the Σ_n-action:

$$(-)^{sign-\Sigma_n} = \frac{1}{n!}\sum_{\sigma\in\Sigma_n} sign(\sigma)\Psi_\sigma : A_{\Sigma_n}(pt) \longrightarrow A(pt).$$

So, by letting $p_r = (-1)^{r-1}$ for all r in Theorem 2.10, one obtains by the isomorphism

$$k_*(M^{\{n\}}) \simeq \left((k_*M)^{\otimes n}\right)^{sign-\Sigma_n}$$

the following generating series from [20][Thm.1.7]:

$$1 + \sum_{n\geq 1} [k_*M^{\{n\}}] \cdot t^n = \exp\left(-\sum_{r\geq 1} \psi_r([k_*M]) \cdot \frac{(-t)^r}{r}\right) \in \bar{K}_0(A(pt)) \otimes_{\mathbb{Z}} \mathbb{Q}[[t]]\,. \tag{54}$$

We conclude by showing how to derive our concrete formulae of Theorem 1.1 from the Introduction by usingTheorem 2.10 of this section. The virtue of this second proof is that it also explains the connection of Theorem 1.1 with our previous results from [20] about generating series of symmetric and alternating powers of suitable coefficients.

Consider the homomorphism of pre-lambda rings

$$h : \bar{K}_0(D^b\mathsf{MHM}(pt)) \to \mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}]$$

defined via the commutative diagram as in [20][p.301]:

$$
(55)\qquad
\begin{array}{ccccc}
\bar{K}_0(D^b\mathsf{MHM}(pt)) & \xrightarrow{H^*} & \bar{K}_0(Gr^-(\mathsf{MHM}(pt))) & \xrightarrow{\sim} & \bar{K}_0(Gr^-(mHs^p)) \\
h\downarrow & & \downarrow & & \downarrow \mathrm{forget} \\
\mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}] & \xleftarrow[h]{} & \bar{K}_0(Gr^-(Gr^2(\mathsf{vect}_f(\mathbb{C})))) & \xleftarrow{Gr_F^* Gr_*^W} & \bar{K}_0(Gr^-(mHs))\,.
\end{array}
$$

The bottom row was already explained in the previous section. Additionally, the following notations are used:

(a) $H^* : D^b\mathsf{MHM}(pt) \to Gr^-(\mathsf{MHM}(pt))$ is the total cohomology functor $\mathcal{V} \mapsto \oplus_n H^n(\mathcal{V})$. Note that this is a functor of additive tensor categories (i.e., it commutes with direct sums $\oplus$ and tensor products $\otimes$), if we choose the Koszul symmetry isomorphism on $Gr^-(\mathsf{MHM}(pt))$. In fact, $D^b\mathsf{MHM}(pt)$ is a triangulated category with bounded t-structure satisfying [3][Def.4.2], so that the claim follows from [3][Thm.4.1, Cor.4.4].

(b) The isomorphism $\mathsf{MHM}(pt) \simeq mHs^p$ is Saito's identification of the abelian tensor category of mixed Hodge modules over a point space with Deligne's abelian tensor category of polarizable mixed Hodge structures.

(c) $\mathrm{forget} : mHs^p \to mHs$ is the functor of forgetting that the corresponding $\mathbb{Q}$-mixed Hodge structure is graded polarizable.

Remark 2.11. The fact that the total cohomology functor $H^* : D^b\mathsf{MHM}(pt) \to Gr^-(\mathsf{MHM}(pt))$ is a tensor functor corresponds to the Künneth formula

$$H^*(\mathcal{V}^{\otimes n}) \simeq (H^*(\mathcal{V}))^{\otimes n}\ , \quad \text{for } \mathcal{V} \in D^b\mathsf{MHM}(pt).$$

For $\mathcal{V} = k_*M$, this implies by Property (iii) the important Künneth isomorphism (1) from the Introduction. For a more direct approach to Künneth formulae, see [27][eq.(1.17), Cor.2.0.4] and [19][Sect.3.8] for the constructible context, [5][Thm.2.1.2] for the coherent context and resp. [19][Thm.1] for the mixed Hodge module context.

Formula (3) follows now by applying the ring homomorphism

$$h \otimes id : \bar{K}_0(D^b\mathsf{MHM}(pt)) \otimes \mathbb{Q}[p_i, i \geq 1] \longrightarrow \mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}] \otimes \mathbb{Q}[p_i, i \geq 1]$$

to formula (52) of Theorem 2.10.

By exactly the same method one also gets the following homomorphism of pre-lambda rings:

$$P : \bar{K}_0(D^b_c(pt)) \xrightarrow{H^*} \bar{K}_0(Gr^-(\mathsf{vect}_f(\mathbb{C}))) \xrightarrow{P} \mathbb{Z}[z^{\pm 1}]$$

and, resp.,

$$P : \bar{K}_0(D^b_{coh}(pt)) \xrightarrow{H^*} \bar{K}_0(Gr^-(\mathsf{vect}_f(\mathbb{C}))) \xrightarrow{P} \mathbb{Z}[z^{\pm 1}]\,,$$

with $P : \bar{K}_0(Gr^-(\mathsf{vect}_f(\mathbb{C}))) \to \mathbb{Z}[z^{\pm 1}]$ the Poincaré polynomial homomorphism given by taking the dimension counting Laurent polynomial

$$[\oplus V^k] \mapsto \sum_k dim(V^k) \cdot (-z)^k\ ,$$

for k the degree with respect to the grading in Gr^-. Then formula (2) follows by applying

$$P \otimes id : \bar{K}_0(A(pt)) \otimes \mathbb{Q}[p_i, i \geq 1] \to \mathbb{Z}[z^{\pm 1}] \otimes \mathbb{Q}[p_i, i \geq 1]$$

to formula (52), where $A(pt)$ is either $D^b_c(pt)$ or $D^b_{coh}(pt)$.

2.4. **Pseudo-functors and twisting.** In the context of twisting by representations, we need to require the pseudo-functor $(-)_*$ with values in the category $A(-)$ to satisfy an additional property:

(iv) For any quasi-projective variety X, there exists a pairing
$$\otimes : A(pt) \times A(X) \longrightarrow A(X),$$
which is additive, $\mathbb{Q}$-linear and functorial in each variable, as well as functorial with respect to $(-)_*$. Moreover, if $X = pt$ is a point, this pairing coincides with the tensor structure on $A(pt)$ of property (ii).

The pairing of (iv) induces similar ones on the corresponding equivariant categories, as well as on the (equivariant) Grothendieck groups. These pairings are bilinear and functorial with respect to the pseudo-functor $(-)_*$. Note that this additional property is fullfilled for all examples of pseudo-functors considered in this paper, i.e.., $A(X) = D^b\mathrm{MHM}(X)$, $D^b_c(X)$ or $D^b_{coh}(X)$, where it is given as a special case of the exterior product $\boxtimes$, with
$$\otimes := k_*(- \boxtimes -) : A(pt) \times A(pt) \to A(pt)$$
for $k : pt \times pt \simeq pt$. As before, in the coherent setting we restrict to projective varieties X.

Remark 2.12. In the context of our Examples, the category $Vect_{\mathbb{Q}}(\Sigma_n)$ is a tensor subcategory of $A_{\Sigma_n}(pt)$, where in the Hodge context we regard a representation as a pure Hodge structure of type $(0,0)$ placed in degree zero, together with Saito's identification $\mathrm{MHM}(pt) \simeq mHs^p$. Property (iv) yields now the pairing mentioned in Sect.1.2:
$$\otimes : Vect_{\mathbb{Q}}(\Sigma_n) \times A_{\Sigma_n}(X) \to A_{\Sigma_n}(X),$$
which is induced from the composition:
$$A_{\Sigma_n}(pt) \times A_{\Sigma_n}(X) \xrightarrow{\otimes} A_{\Sigma_n \times \Sigma_n}(X) \xrightarrow{\mathrm{Res}} A_{\Sigma_n}(X),$$
with Res the restriction functor for the diagonal subgroup $\Sigma_n \subset \Sigma_n \times \Sigma_n$. Moreover, if $X = pt$ is a point space, this pairing coincides with the abstract pairing (45) defined via the $\mathbb{Q}$-linear structure.

Remark 2.13. By the functoriality of the above pairing, we have the following projection formula for a morphism $f : X \to X'$, $V \in Vect_{\mathbb{Q}}(\Sigma_n)$ and $\mathcal{M} \in A_{\Sigma_n}(X)$:

(56)
$$f_*(V \otimes \mathcal{M}) = V \otimes f_*(\mathcal{M}),$$

using the identification $id_{pt} \times f = f$. Applying this formula for f the constant map $k : X^n \to pt$, together with the tensor property of the total cohomology functor H^* as in Remark 2.11, we get the first isomorphism of the equivariant Künneth formula (16).

Definition 2.14. For $V \in Vect_{\mathbb{Q}}(\Sigma_n)$ a rational Σ_n-representation, the *Schur-type object* $S_V(\mathcal{M}) \in A(X^{(n)})$ associated to $\mathcal{M} \in A(X)$ is defined by

(57)
$$S_V(\mathcal{M}) := \left(V \otimes \pi_{n*}(M^{\boxtimes n})\right)^{\Sigma_n}.$$

If $V = V_\mu \simeq V_\mu^*$ is the (self-dual) irreducible representation of Σ_n corresponding to a partition μ of n, we denote the corresponding Schur functor by $S_\mu := S_{V_\mu}$.

Note that for V the trivial (resp. sign) representation of Σ_n, the corresponding Schur functor coincides with the symmetric (resp. alternating) n-th power of $\mathcal{M}$. Moreover, by using the projection formula for the constant map k to a point, we have that
$$k_* S_V(\mathcal{M}) := k_*\left(V \otimes \pi_{n*}(\mathcal{M}^{\boxtimes n})\right)^{\Sigma_n} \simeq \left(V \otimes k_*(\mathcal{M}^{\boxtimes n})\right)^{\Sigma_n} \simeq S_V(k_*\mathcal{M}),$$

with the last identification following from Property (iii) of the pseudo-functor $(-)_*$. Together with the tensor property of the total cohomology functor H^* as in Remark 2.11, this yields formula (23) from the Introduction.

Another important example of a Schur functor S_V is obtained by choosing $V = \mathrm{Ind}_K^{\Sigma_n}(triv)$, the representation induced from the trivial representation of a subgroup K of Σ_n. Then, if $\pi : X^n \longrightarrow X^n/K$ and $\pi' : X^n/K \to X^{(n)}$ are the projections factoring π_n, we have:

$$(\pi_{n*}(V \otimes \mathcal{M}^{\boxtimes n}))^{\Sigma_n} \simeq (\mathrm{Ind}_K^{\Sigma_n}(triv) \otimes \pi_{n*}(\mathcal{M}^{\boxtimes n}))^{\Sigma_n} \simeq (\pi_{n*}(\mathcal{M}^{\boxtimes n}))^K \simeq \pi'_* \left((\pi_*(\mathcal{M}^{\boxtimes n}))^K \right),$$

for $\mathcal{M} \in A(X)$. As an example, if $\mathcal{M} = \mathbb{Q}_X^H \in D^b\mathrm{MHM}(X)$ is the constant Hodge module on X, we get

(58)
$$h_{(c)}(X^n/K, \mathbb{Q}^H)(y,x,z) = \sum_{\lambda=(k_1,k_2,\cdots)\dashv\, n} \frac{1}{z_\lambda}\chi_\lambda(\mathrm{Ind}_K^{\Sigma_n}(triv)) \cdot \prod_{r\geq 1} \left(h_{(c)}(H^*(X;\mathbb{C})(y^r,x^r,z^r) \right)^{k_r},$$

and similarly for the Poincaré polynomials as in (20).

3. Further applications

In this last section, we indicate further applications of the abstract setup of the previous sections to suitable equivariant versions of (characters of) Poincaré and mixed Hodge polynomials of *equivariant* coefficients. More precisely, we consider the following situations (with $A(X)$ any of our three main examples of coefficients: $D^b\mathrm{MHM}(X)$, $D^b_c(X)$ and $D^b_{coh}(X)$, and with all spaces projective in the coherent context):

(a) G is a fixed finite group acting algebraically on X, with $\mathcal{M} \in A_G(X)$ a G-equivariant object in $A(X)$ (as in [20][Appendix]).

(b) g is a finite order algebraic automorphism acting on X, with $\mathcal{M} \in A_{\langle g\rangle}(X)$ a $\langle g\rangle$-equivariant object (in particular, $\mathcal{M} \in A(X)$ is endowed with an isomorphism $\Psi_g : \mathcal{M} \to g_*\mathcal{M}$ in $A(X)$). Here the order of the cyclic group $\langle g\rangle$ can depend on $\mathcal{M}$ (i.e., this order could exceed that of the action on X).

(c) (g, Ψ_g) is an endomorphism in the category of pairs $(X, \mathcal{M})$, with $\mathcal{M} \in A(X)$ (i.e., $(X, \mathcal{M}) \in A^{op}/space(X)$ in the sense of [20][Appendix]). This means that $g : X \to X$ is an algebraic morphism, together with a morphism $\Psi_g : \mathcal{M} \to g_{*(!)}\mathcal{M}$ in $A(X)$. Here, we use g_* (resp. $g_!$) when considering (compactly supported) cohomology $H^*_{(c)}(X;\mathcal{M})$ with the endomorphism induced from Ψ_g. Note that $g_! = g_*$ if g is proper, e.g., an automorphism.

Any of the above situations can be viewed in the context of a (semi-)group action of G, with $G := \mathbb{Z}$ for (b) and $g = 1 \in \mathbb{Z}$ acting with finite order, and resp. $G := \mathbb{N}_0$ for (c). Examples of such G-equivariant coefficients on a G-space X include the constant (Hodge) sheaf and the structure sheaf in the coherent context, where in case (c) g is required to be proper if compactly supported cohomology is considered. Here Ψ_g is induced by the adjunction map $id \longrightarrow g_*g^*$ corresponding to the usual pullback in cohomology (as used in Theorem 1.7). Similarly, in cases (a) and (b) one can use the intersection cohomology (Hodge) sheaf if X is pure dimensional.

For a G-equivariant object $\mathcal{M} \in A(X)$ as above, the external products $\mathcal{M}^{\boxtimes n} \in A(X^n)$ and their pushforwards $\pi_{n*}(\mathcal{M}^{\boxtimes n}) \in A(X^{(n)})$ get an induced diagonal G-action commuting with the action of the symmetric group Σ_n as before, so that for V a Σ_n-representation (with trivial G-action), the (twisted) cohomology $H^*_{(c)}(X^n; V \otimes \mathcal{M}^{\boxtimes n})$ has an induced action of $G \times \Sigma_n$. Moreover, the

Schur objects $S_V(\mathcal{M})$ and their cohomology $H^*_{(c)}(X^{(n)}; S_V(\mathcal{M}))$ get an induced G-action.

All our concrete results from Sections 1.1 and 1.2 can be now formulated in this equivariant context, once we redefine the Poincaré and resp. mixed Hodge polynomials, and the corresponding characters tr_{Σ_n}, as follows:

- *G-Poincaré polynomials*:
$$P^G_{(c)}(X, \mathcal{M})(z) := \sum_k [H^k_{(c)}(X, \mathcal{M})] \cdot (-z)^k \in Rep_{\mathbb{C}}(G)[z^{\pm 1}],$$
- *G-mixed Hodge polynomials*:
$$h^G_{(c)}(X, \mathcal{M})(y, x, z) := \sum_{p,q,k} [H^{p,q,k}_{(c)}(X, \mathcal{M})] \cdot y^p x^q (-z)^k \in Rep_{\mathbb{C}}(G)[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}]$$
- *G-equivariant characters*:
$$tr^G_{\Sigma_n}(H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})) := \sum_k tr^G_{\Sigma_n}(H^k_{(c)}(X^n, \mathcal{M}^{\boxtimes n})) \cdot (-z)^k \in C(\Sigma_n) \otimes Rep_{\mathbb{C}}(G) \otimes \mathbb{L},$$

  with $\mathbb{L} = \mathbb{Z}[z^{\pm 1}]$, and resp., $\mathbb{L} = \mathbb{Z}[y^{\pm 1}, x^{\pm 1}, z^{\pm 1}]$ in the Hodge context.

Here $Rep_{\mathbb{C}}(G) := K_0(A_G)$ denotes the Grothendieck ring of the following $\mathbb{C}$-linear Karoubian (even abelian) tensor categories A_G, corresponding to each of our situations above:

(a) $Vect_{\mathbb{C}}(G)$, the category of finite-dimensional complex G-representations.
(b) $Vect^f_{\mathbb{C}}(G)$, the category of finite-dimensional complex G-representations, with $g = 1 \in G := \mathbb{Z}$ acting with finite order.
(c) $End_{\mathbb{C}}$ the category of endomorphisms of finite-dimensional $\mathbb{C}$-vector spaces.

The tensor structure on A_G is induced from the tensor product of the underlying complex vector spaces with induced diagonal action. Then a $G \times \Sigma_n$-action on a finitely dimensional vector space V is the same a Σ_n-action on V regarded as an object in A_G. By the Schur functor decomposition (38) applied to A_G, we get the isomorphism
$$K_0^{\Sigma_n}(A_G) \simeq Rep_{\mathbb{Q}}(\Sigma_n) \otimes K_0(A_G) = Rep_{\mathbb{Q}}(\Sigma_n) \otimes Rep_{\mathbb{C}}(G).$$
Then the G-equivariant characters $tr^G_{\Sigma_n} : K_0^{\Sigma_n}(A_G) \to C(\Sigma_n) \otimes Rep_{\mathbb{C}}(G)$ above are defined by taking the Σ_n-character in the first tensor factor.

Remark 3.1. If G is the trivial group, then $A_G = Vect_{\mathbb{C}}$ is the category of finite-dimensional $\mathbb{C}$-vector spaces, and $\dim : Rep_{\mathbb{C}}(G) \simeq \mathbb{Z}$, so the above G-equivariant Poincaré and mixed Hodge polynomials, resp. G-characters reduce in this case to the classical notions from Sections 1.1 and 1.2 of the Introduction.

Analogues results to those presented in Sections 1.1 and 1.2 can now be formulated for these modified notions of invariants in the G-equivariant context, with the corresponding Adams operations
$$\psi_r : Rep_{\mathbb{C}}(G) \otimes \mathbb{L} \to Rep_{\mathbb{C}}(G) \otimes \mathbb{L}$$
defined as the tensor product of the Adams operations on the tensor factors (with $Rep_{\mathbb{C}}(G)$ a pre-lambda ring by [13]). Moreover, their proofs follow as before from the Theorems 1.5 and 1.6 in the abstract context, but using the category A_G in place of A as the underlying $\mathbb{Q}$-linear Karoubian tensor category, provided that the derived Künneth formula of Property (iii) holds G-equivariantly as in the Remark below, as it is the case in the three main situations considered here

(see [20][Appendix] for the constructible and coherent context, and [19][Sect.1.12] for the Hodge context). Here, the required compability follows from the *equivariance of the multiple Künneth formula*:

$$(k^{\times n})_*(\boxtimes_{i=1}^n(-)) = \boxtimes_{i=1}^n(k_*(-)) : A(X)^{\times n} \to A(pt^{\times n}) ,$$

together with

$$\otimes_{i=1}^n(-) = k_*(\boxtimes_{i=1}^n(-)) : A(pt)^{\times n} \to A(pt) .$$

The corresponding G-equivariance in the twisting defined via Property (iv) follows already from the required functorialities.

Remark 3.2. In the abstract context of a pseudo-functor, this G-equivariance of the derived Künneth formula can be formulated as the following property of the pseudo-functor $(-)_*$ with values in the category $A(-)$:

(v) For $g : X \to X$ an algebraic (iso)morphism and $\mathcal{M} \in A(X)$ with a(n) (iso)morphism $\Psi_g : \mathcal{M} \longrightarrow g_*\mathcal{M}$ given by the G-action, we have an isomorphism

$$(g^{\times n})_*(\mathcal{M}^{\boxtimes n}) \simeq (g_*\mathcal{M})^{\boxtimes n} \tag{59}$$

such that the (iso)morphism

$$k_*\Psi_g^{\boxtimes n} : k_*\mathcal{M}^{\boxtimes n} \to k_*\mathcal{M}^{\boxtimes n}$$

induced by pushing down to a point (via k_*) the (iso)morphism

$$\Psi_g^{\boxtimes n} : \mathcal{M}^{\boxtimes n} \longrightarrow (g_*\mathcal{M})^{\boxtimes n} \simeq (g^{\times n})_*(\mathcal{M}^{\boxtimes n})$$

agrees under the identification $k_*\mathcal{M}^{\boxtimes n} \simeq (k_*\mathcal{M})^{\otimes n}$ of Property (iii) with the endomorphism

$$(k_*\Psi_g)^{\otimes n} : (k_*\mathcal{M})^{\otimes n} \longrightarrow (k_*\mathcal{M})^{\otimes n}.$$

In the case (a) of a finite group action, we ask this compability for all $g \in G$ (in such a way that the corresponding G-actions via $k_*\Psi_g^{\boxtimes n}$ and $(k_*\Psi_g)^{\otimes n}$ are identified under (iii)).

Let us illustrate such formulae analogous to (2) and (18) in the G-equivariant context for the case of Poincaré polynomial invariants. Similar results for the mixed Hodge context, as well as various specializations of the variables are left to the reader. For the special case of symmetric products (i.e., by setting all p_r equal to 1) and constant coefficients in the Hodge context, see Theorem 1.7 from the Introduction.

Theorem 3.3. *Let $\mathcal{M} \in A_G(X)$ be a G-equivariant object in $A(X)$. Then:*

$$\sum_{n\geq 0} tr^G_{\Sigma_n}(H^*_{(c)}(X^n, \mathcal{M}^{\boxtimes n})) \cdot t^n = \exp\left(\sum_{r\geq 1} p_r \otimes \psi_r(P^G_{(c)}(X,\mathcal{M})(z)) \cdot \frac{t^r}{r}\right) \tag{60}$$

holds in the graded $\mathbb{Q}$-algebra $Rep_{\mathbb{C}}(G) \otimes \mathbb{Q}[p_i, i \geq 1, z^{\pm 1}][[t]]$, and, respectively,

$$tr^G_{\Sigma_n}(H^*_{(c)}(X^n, V \otimes \mathcal{M}^{\boxtimes n})) = \sum_{\lambda=(k_1,k_2,\cdots)\dashv\, n} \frac{p_\lambda}{z_\lambda}\chi_\lambda(V) \otimes \prod_{r\geq 1}\Big(\psi_r(P^G_{(c)}(X,\mathcal{M})(z))\Big)^{k_r} , \tag{61}$$

holds in $Rep_{\mathbb{C}}(G) \otimes \mathbb{Q}[p_i, i \geq 1, z^{\pm 1}]$ for a given $V \in Rep_{\mathbb{Q}}(\Sigma_n)$.

The concrete formulae of Sections 1.1 and 1.2 of the Introduction can be recovered from their above G-equivariant versions by applying the ring homomorphism $\dim : Rep_{\mathbb{C}}(G) \longrightarrow \mathbb{Z}$, which corresponds to forgetting the G-symmetry. Similarly, Theorem 1.7 can be recovered from the Hodge version of (60) for the constant coefficients $\mathcal{M} = \mathbb{Q}_X^H$, by specializing all p_r's to 1, and by applying suitable ring homomorphisms $sp : Rep_{\mathbb{C}}(G) \to R$ to a commutative ring R. More concretely, in the three situations (a)-(c) considered at the beginning of this section (resp., at the end of Introduction), examples of such specializations $sp : Rep_{\mathbb{C}}(G) \to R$ are given as follows:

(a) for a finite group G, we take the complex characters of G-representations, i.e., apply the pre-lambda ring homomorphism

$$tr_G : Rep_{\mathbb{C}}(G) \longrightarrow C(G) \otimes \mathbb{C},$$

with Adams operations ψ_r on $C(G) \otimes \mathbb{C}$ given by $\psi_r(\alpha(g)) := \alpha(g^r)$, for $g \in G$.

(b) for $G = \mathbb{Z}$, with $g = 1 \in \mathbb{Z}$ acting with finite order, we have a pre-lambda ring isomorphism

$$sp : Rep_{\mathbb{C}}(G) \simeq \mathbb{Z}[\widehat{\mu}],$$

with $\widehat{\mu}$ the abelian group of roots of unity in $\mathbb{C}$ (with respect to multiplication), given by $[\chi_\lambda] \mapsto (\lambda)$, where χ_λ is the one-dimensional representation with $1 \in \mathbb{Z}$ acting by multiplication with λ. The r-th Adams operations ψ_r on $\mathbb{Z}[\widehat{\mu}]$ is defined by $(\lambda) \mapsto (\lambda^r)$, for all $\lambda \in \widehat{\mu}$ (i.e., it is induced from the group homomorphism $\widehat{\mu} \to \widehat{\mu}; \lambda \mapsto \lambda^r$ of the abelian group of roots of unity $(\widehat{\mu}, \cdot)$).

(c) for the endomorphism category $End_{\mathbb{C}}$, consider the usual ring homomorphism

$$trace : K_0(End_{\mathbb{C}}) \longrightarrow \mathbb{C}$$

defined by taking the trace of the endomorphism, with

$$trace\,(\psi_r(g : V \to V)) = trace(g^r : V \to V). \tag{62}$$

The identity (62) can be obtained as follows: we first factor $trace$ through the projection from $K_0(End_{\mathbb{C}})$ to the usual Grothendieck group of the abelian tensor category $End_{\mathbb{C}}$, which is a pre-lambda ring homomorphism (cf. [20][Lemma 2.1]), then reduce via short exact sequences to the case of one-dimensional representations (given by eigenspaces). Note that $trace$ is not a pre-lambda ring homomorphism. Pre-lambda ring homomorphisms relevant to this situation are: the *characteristic polynomial*:

$$\lambda_t : K_0(End_{\mathbb{C}}) \longrightarrow W_{rat}(\mathbb{C}) := \{P(t)/Q(t) \mid P(t), Q(t) \in 1 + t\mathbb{C}[t]\} \subset \mathbb{C}(t),$$

$$[V, g] \mapsto \lambda_t(V, g) := \det(1 + tg) = \sum_{i \geq 0} trace_{\Lambda^i g}(\Lambda^i V) \cdot t^i$$

given by the traces of the induced endomorphisms of the alternating powers of V, and respectively, the L-*function*:

$$[V, g] \mapsto L(V, g)(t) := \det(1 - tg)^{-1} = \sum_{i \geq 0} trace_{Sym^i g}(Sym^i V) \cdot t^i$$

given by the traces of the induced endomorphisms of the symmetric powers of V. Here, $W_{rat}(\mathbb{C})$ is the subring of *rational* elements (as in [24][Prop.6]) in the *big Witt ring* $W(\mathbb{C}) := (1 + t\mathbb{C}[[t]], \cdot)$, with a suitable ring structure as in [1, 12, 25], and whose underlying additive structure is the multiplication of rational functions resp. normalized formal powers series.

As a final example, let us formulate the *graded version of the classical Lefschetz Zeta function*, i.e., the specialization of formula (34) from the end of Introduction to $y = x = 1$, corresponding to the Poincaré polynomial version (and corresponding to the use of the trace homomorphism in the context (c) as above, for the constant constructible sheaf, with all Fobenius parameters $p_r = 1$):

Theorem 3.4. *If $g : X \to X$ is a (proper) algebraic endomorphism of X, then the following equalities hold in $\mathbb{C}[z][[t]]$:*

$$
\begin{aligned}
\sum_{n\geq 0} P^{g}_{(c)}(X^{(n)},\mathbb{C})(z)\cdot t^n &= \exp\left(\sum_{r\geq 1} P^{g^r}_{(c)}(X,\mathbb{C})(z^r)\cdot \frac{t^r}{r}\right)\\
&= \exp\left(\sum_{k\geq 0}(-1)^k\left(\sum_{r\geq 1} trace_{g^r}(H^k_{(c)}(X,\mathbb{C}))\cdot \frac{(z^kt)^r}{r}\right)\right)\\
&= \prod_{k\geq 0}\left(\sum_{i\geq 0} trace_{Sym^i g}(Sym^i(H^k_{(c)}(X,\mathbb{C})))\cdot (z^kt)^i\right)^{(-1)^k}\\
&= \prod_{k\geq 0}\left(L(H^k_{(c)}(X,\mathbb{C}),g)(z^kt)\right)^{(-1)^k}
\end{aligned}
\tag{63}
$$

Note that this formula (63) specializes for $z = 1$ to the usual *Lefschetz Zeta function* of the (proper) endomorphism $g : X \to X$:

$$
\sum_{n\geq 0} \chi^{g}_{(c)}(X^{(n)},\mathbb{C})(z)\cdot t^n = \prod_{k\geq 0}\left(L(H^k_{(c)}(X,\mathbb{C}),g)(t)\right)^{(-1)^k} . \tag{64}
$$

On the other hand, for $g = id_X$ the identity of X, formula (63) reduces to *Macdonald's generating series formula* [18] for the Poincaré polynomials and Betti numbers of the symmetric products of X:

$$
\sum_{n\geq 0} P_{(c)}(X^{(n)},\mathbb{C})(z)\cdot t^n = \exp\left(\sum_{r\geq 1} P_{(c)}(X,\mathbb{C})(z^r)\cdot \frac{t^r}{r}\right) = \prod_{k\geq 0}\left(\frac{1}{1-z^kt}\right)^{(-1)^k\cdot b^k_{(c)}(X)} , \tag{65}
$$

with $b^k_{(c)}(X) := dim_{\mathbb{C}} H^k_{(c)}(X,\mathbb{C})$, which for $z = 1$ specializes (also as the particular case of (64) for $g = id_X$) to:

$$
\sum_{n\geq 0} \chi_{(c)}(X^{(n)},\mathbb{C})\cdot t^n = (1-t)^{-\chi_{(c)}(X,\mathbb{C})} . \tag{66}
$$

Remark 3.5. For the counterpart of (64) in the context of the Zeta function of a constructible sheaf for the Frobenius endomorphism of varieties over finite fields, see also [28][Thm. on p.464] and [9][Thm.4.4 on p.174]. For a similar counterpart of (63) taking a weight filtration into account, see [24][Prop.8(i)].

Finally, the product $*$ on the big Witt ring $W(\mathbb{C})$ (or its subring $W_{rat}(\mathbb{C})$) corresponds under the ring homomorphisms $\lambda_t, L(t) : End_{\mathbb{C}} \to W_{rat}(\mathbb{C}) \subset W(\mathbb{C})$ to the tensor product of endomorphisms. By the specialization above of the (graded version of the) Lefschetz Zeta function to Macdonald's generating series formula for the Poincaré polynomials and Betti numbers of the symmetric products of X, it should not come as a surprise that the Witt multiplication $*$ naturally arises if one attempts to express these generating functions for a product space $X \times X'$ in terms of the corresponding generating functions of the factors X and X' (as further discussed in [25, 26]).

References

[1] G. Almkvist, *The Grothendieck ring of the category of endomorphisms*, J. Algebra **28** (1974), 375–388.
[2] M. F. Atiyah, *Power operations in K-theory*, Quart. J. Math. **17** (1966), 165–193.
[3] S. Biglari, *A Künneth formula in tensor triangulated categories*, J. Pure Appl. Algebra **210** (2007), 645–650.
[4] S. Biglari, *On lambda operations on mixed motives*, J. K-Theory **12** (2013), 381–404.
[5] J. Bonsdorff, *A Fourier transformation for Higgs bundles*, J. Reine Angew. Math. **591** (2006), 21–48.
[6] J. Cheah, *On the cohomology of Hilbert schemes of points*, J. Algebraic Geometry **5** (1996), 479–511.
[7] P. Deligne, *Catégories tensorielles*, Mosc. Math. J. **2** (2002), 227–248.
[8] W. Ebeling, S.M. Gusein-Zade, *Higher order spectra, equivariant Hodge-Deligne polynomials and Macdonald type equations*, arXiv:1507.08088.
[9] E. Freitag, R. Kiehl, *Étale cohomology and the Weil conjecture*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), **13**. Springer-Verlag, Berlin, 1988.
[10] E. Getzler, *Mixed Hodge structures on configuration spaces*, arXiv:math/9510018.
[11] S.M. Gusein-Zade, I. Luengo, A. Melle-Hernández, *On equivariant versions of higher order orbifold Euler characteristics*, arXiv:1504.07467.
[12] M. Hazewinkel, *Operations in the K -theory of endomorphisms*, J. Algebra **84** (1983), 285–304.
[13] F. Heinloth, *A note on functional equations for Zeta functions with values in Chow motives*, Annales de L'Inst. Fourier **57** (2007), 1927–1945.
[14] A. Henderson, *Species over a finite field*, J. Algebraic Combin. **21** (2005), 147–161.
[15] B. Kahn, *Zeta functions and motives*, Pure Appl. Math. Q. **5** (2009), 507–570.
[16] D. Knutson, *λ-Rings and the Representation Theory of the Symmetric Group*, Lecture Notes in Mathematics, Vol. **308**. Springer-Verlag, Berlin-New York, 1973.
[17] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1979.
[18] I. G. Macdonald, *The Poincaré polynomial of a symmetric product*, Proc. Cambridge Philos. Soc. **58**, 1962, 563 - 568.
[19] L. Maxim, M. Saito, J. Schürmann, *Symmetric products of mixed Hodge modules*, J. Math. Pures Appl. **96** (2011), 462–483.
[20] L. Maxim, J. Schürmann, *Twisted genera of symmetric products*, Selecta Math. (N.S.) **18** (2012), 283–317.
[21] L. Maxim, J. Schürmann, *Equivariant characteristic classes of external and symmetric products of varieties*, arXiv:1508.04356.
[22] S. Meinhardt, M. Reineke, *Donaldson-Thomas invariants versus intersection cohomology of quiver moduli*, arXiv:1411.4062.
[23] B. Moonen, *Das Lefschetz-Riemann-Roch-Theorem für singuläre Varietäten*, Bonner Mathematische Schriften **106** (1978), viii+223 pp.
[24] N. Naumann, *Algebraic independence in the Grothendieck ring of varieties*, Trans. Amer. Math. Soc. **359** (2007), 1653–1683.
[25] N. Ramachandran, *Zeta functions, Grothendieck groups, and the Witt ring*, Bull. Sci. Math. **139** (2015), 599–627.
[26] N. Ramachandran, G. Tabuada, *Exponentiation of motivic measures*, arXiv:1412.1795.
[27] J. Schürmann, *Topology of singular spaces and constructible sheaves*, Monografie Matematyczne **63** (New Series), Birkhäuser, Basel, 2003.
[28] SGA 5, *Sém. géométrie algébrique (1965–68). Cohomologie l-adique et Fonction L*, Lect. Notes Math. **589**. Springer, Berlin Heidelberg (1973)
[29] H. Uehara, R.A. DiVall, *Hopf algebra of class functions and inner plethysms*, Hiroshima Math. J. **12** (1982), 225–244.

L. Maxim: Department of Mathematics, University of Wisconsin-Madison, 480 Lincoln Drive, Madison, WI 53706, USA.

E-mail address: maxim@math.wisc.edu

J. Schürmann : Mathematische Institut, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany.

E-mail address: jschuerm@math.uni-muenster.de