AUTOMORPHISM 2-GROUP OF A WEIGHTED PROJECTIVE STACK

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Abstract. For a given sequence of positive integers $(n_0, \cdots, n_r)$ we define the *weighted projective general linear 2-group* $\mathrm{PGL}(n_0, \cdots, n_r)$ as a crossed-module in the category of schemes and show that it is a model for (i.e is naturally homotopy equivalent to) the *gr*-stack of self-equivalences of the weighted projective stack of weight $(n_0, \cdots, n_r)$. We also give an explicit description of the structure of $\mathrm{PGL}(n_0, \cdots, n_r)$.

1. Introduction

To study group actions on stacks is an intricate task. That is because most actions that arise in nature are *weak* actions. This means, given element g, h in our (discrete) group G, the action of gh on our stack $\mathcal{X}$ is not strictly equal to the composition of the actions of g and h, but only *2-isomorphic* to it, and such 2-isomorphisms are required to satisfy certain coherence conditions. In other words, an action of G on $\mathcal{X}$ is the same as a *weak* morphism of 2-groups $G \to \mathrm{Aut}\,\mathcal{X}$. (In fact the latter is a pseudo 2-group, but we will not get into that for the moment.) Furthermore, there is a notion of *transformation* between two such actions, and every two actions that are related by a transformation should be regarded as the "same". This makes the study of group actions on stacks quite tricky. Things get even more complicated when one wants to study actions of group schemes (or 2-group schemes) on stacks.

To bring the situation under control, one needs to use certain homotopy theoretic tools in order to strictify the weak actions in a computable manner. In the case of discrete group actions, a machinery for handling this problem has been developed in [No3].[1] In favorable situations, this machinery enables us to lift weak actions of a group G on a stack $\mathcal{X}$ to *strict* actions of an extension H of G on a scheme X sitting above $\mathcal{X}$.

To be able to run the machinery developed in [No3], the required input is a *crossed-module* (which is essentially the same thing as a strict 2-group) model for the pseudo 2-group of self-equivalences of the stack in question.

In the case of a weighted projective stacks over $\mathbb{C}$, such a model was introduced in [BeNo] and it was called a *weighted projective general linear 2-group*. Combined with the covering theory developed in [No1], it was used to give a classification of smooth Deligne-Mumford analytic curves by their uniformization types, and also to give an explicit presentation of such stacks as quotient stacks.

The purpose of this paper is to study self-equivalences of a weighted projective stack over an arbitrary base scheme and produce the corresponding crossed-module in full generality. We define a *weighted projective general linear 2-group scheme*

[1]The general case is the subject of a future paper.

over a given base scheme, and show that it models the *gr*-stack of self-equivalences of the corresponding weighted projective stack. We then go on to make explicit the structure of the weighted projective general linear 2-group schemes. This will enable one to study weak actions of group schemes (or 2-group schemes, for that matter) on weighted projective stacks in an explicit fashion.

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2. Notation and terminology

Our notation for 2-groups and crossed-modules is that of the papers [No2] and [No3], to which the reader is referred to for more on 2-group theory relevant to this work. In particular, we use mathfrak letters $\mathfrak{G}$, $\mathfrak{H}$ for 2-groups or crossed-modules.

For us a stack is a presheaf of groupoids (and not a category fibered in groupoids) over a Grothendieck site which satisfies the decent condition. We use mathcal letters $\mathcal{X}$, $\mathcal{Y}$,... to denote stacks.

Given a presheaf of groupoids $\mathcal{X}$ over a site, its stackification is denoted by $\mathcal{X}^a$. We use the same notation for the sheafification of a presheaf of sets (or groups).

The m^{th} general linear group scheme over $\mathrm{Spec}\, R$ is denoted by $\mathrm{GL}(m, R)$. When $R = \mathbb{Z}$, this is abbreviated to $\mathrm{GL}(m)$. The corresponding projectivized general linear group scheme is denoted by $\mathrm{PGL}(m)$; this notation does not conflict with the notation $\mathrm{PGL}(n_0, n_1, \cdots, n_r)$ for a weighted projective general linear 2-group (Section 5) because in the latter case we always assume $r \geq 1$.

3. Recall on 2-groups and crossed-modules

A (strict) *2-group* is a group object in the category of groupoids. Equivalently, a 2-group is a strict monoidal groupoid $\mathfrak{G}$ in which every object has a strict inverse. We define a *pseudo 2-group* to be a strict monoidal groupoid $\mathfrak{G}$ in which every object has a weak inverse; that is, multiplication by any object induces an equivalence of categories from $\mathfrak{G}$ to itself. This is of course much stronger than the conventional definition of a weak category, because we are assuming that associativity holds on the nose and there exists a strict identity element.

The set of isomorphism classes of objects in a (pseudo) 2-group $\mathfrak{G}$ is denoted by $\pi_1\mathfrak{G}$; this is a group. The automorphism group of the identity object $1 \in \mathrm{Ob}\,\mathfrak{G}$ is denoted by $\pi_2\mathfrak{G}$; this is an abelian group.

Pseudo 2-groups and strict monoidal functors between them form a category **Ps2Gp** which contains the category **2Gp** of (strict) 2-groups as a full subcategory. Morphisms in this category induce group homomorphisms on π_1 and π_2. A morphism between pseudo 2-groups is called an *equivalence* if the induced homomorphisms on π_1 and π_2 are isomorphisms.

The following lemma is straightforward.

Lemma 3.1. *Let $f\colon \mathfrak{H} \to \mathfrak{G}$ be a morphism of pseudo 2-groups. Then f, viewed as a morphism of underlying groupoids, is fully faithful if and only if $\pi_1 f\colon \pi_1\mathfrak{H} \to \pi_1\mathfrak{G}$ is injective and $\pi_2 f\colon \pi_2\mathfrak{H} \to \pi_2\mathfrak{G}$ is an isomorphism. It is an equivalence of groupoids if an only if both $\pi_1 f$ and $\pi_2 f$ are isomorphisms.*

A *crossed-module* $\mathfrak{G} = [\varphi\colon G_2 \to G_1]$ is a pair of groups G_1, G_2, a group homomorphism $\varphi\colon G_2 \to G_1$, and a (right) action of G_1 on G_2, denoted $-^a$, which lifts the conjugation action of G_1 on the image of φ and descends the conjugation action of G_2 on itself. The kernel of φ is a central (in particular abelian) subgroup of G_2 and is denoted by $\pi_2\mathfrak{G}$. The image of φ is a normal subgroup of G_1 whose cokernel is denoted by $\pi_1\mathfrak{G}$. A morphism of crossed-modules is a pair of group homomorphisms which commute with the φ maps and respect the actions. Such a morphism induces group homomorphisms on π_1 and π_2.

Crossed-modules and morphisms between them form a category, which we denote by **CrossedMod**. A morphism in this category is called an equivalence if it induces isomorphisms on π_1 and π_2.

There is a natural equivalence of categories **2Gp** $\simeq$ **CrossedMod**. This equivalence is compatible with the functors π_1 and π_2. This way, we can think of a crossed-module as a 2-group, and vice versa. For this reason, we will sometimes use the term 2-group for an object that is actually a crossed-module. We hope that this will not cause any confusion.

Definition 3.2. A *pseudo 2-group* is a strict monoidal category in which multiplication by any object is an equivalence of categories. A morphism of pseudo 2-groups is a strict monoidal functor.

In fact it is easy to show that the underlying category of a pseudo 2-group is a groupoid. So a pseudo 2-group is a monoid object in the category of groupoids. Recall that a 2-group is a monoidal object in the category of groupoids in which multiplication by any object is an isomorphism. The category **Ps2Gp** of pseudo 2-groups contains the category **2Gp** of 2-groups as a full subcategory. The functors π_1 and π_2 extend to **Ps2Gp** in the obvious way.

4. 2-groups over a site and gr-stacks

First a few words on terminology. For us a stack is presheaf of groupoids (and not a category fibered in groupoids) over a Grothendieck site. This may be a bit unusual for algebraic geometers, but it makes the exposition simpler. Of course, it is standard that this point of view is equivalent to the approach via categories fibered in groupoids. Just to recall how this equivalence works, to any category fibered in groupoids $\mathcal{X}$ one can associate a presheaf $\underline{\mathcal{X}}$ of groupoids over $\mathbf{C}$ which is defined as follows. By definition, $\underline{\mathcal{X}}$ is the presheaf that assigns to an object $U \in \mathbf{C}$ the groupoids $\underline{\mathcal{X}}(U) := \mathrm{Hom}(\underline{U}, \mathcal{X})$, where $\underline{U}$ stands for the presheaf of sets represented by U and Hom is computed in the category of stacks over $\mathbf{C}$. Conversely, to any presheaf of groupoids one associates a category fibered in groupoids defined by the Grothendieck construction. For more on this we refer the reader to [Ho], especially Section 5.2.

Let $\mathbf{C}$ be a Grothendieck site. Let $\mathbf{Ps2Gp_C}$ be the category of presheaves of pseudo 2-groups on $\mathbf{C}$; that is, the category of contravariant functors from $\mathbf{C}$ to $\mathbf{Ps2Gp}$. We define $\mathbf{2Gp_C}$ and $\mathbf{CrossedMod_C}$ analogously. The category $\mathbf{2Gp_C}$ is a full subcategory of $\mathbf{Ps2Gp_C}$. There is a natural equivalence of categories $\mathbf{2Gp_C} \simeq \mathbf{CrossedMod_C}$. In particular, we can think of a presheaf of crossed-modules as a presheaf of of 2-groups.

Let $\mathcal{X}$ be a presheaf of groupoids over $\mathbf{C}$. To $\mathcal{X}$ we associate a presheaf of pseudo 2-groups $\mathcal{A}ut\mathcal{X} \in \mathbf{Ps2Gp_C}$ which parameterizes autoequivalences of $\mathcal{X}$. By definition, $\mathcal{A}ut\mathcal{X}$ is the functor that associates to an object U in $\mathbf{C}$ the pseudo 2-group of self-equivalences of $\mathcal{X}_U$, where $\mathcal{X}_U$ is the restriction of $\mathcal{X}$ to the comma category $\mathbf{C}_U$. Notice that in the case where $\mathcal{X}$ is a stack, $\mathcal{A}ut\mathcal{X}$, viewed as a presheaf of groupoids, is also a stack. Indeed, $\mathcal{A}ut\mathcal{X}$ is almost a group object in the category of stacks over $\mathbf{C}$. To be more precise, $\mathcal{A}ut\mathcal{X}$ is a *gr*-stack in the sense of Definition 4.1 below.

Let $\underline{\mathfrak{G}} \in \mathbf{Ps2Gp_C}$ be a presheaf of pseudo 2-groups on $\mathbf{C}$. We define $\pi_1^{pre}\underline{\mathfrak{G}}$ to be the presheaf $U \mapsto \pi_1\big(\underline{\mathfrak{G}}(U)\big)$, and $\pi_1\underline{\mathfrak{G}}$ to be the sheaf associated to the presheaf $\pi_1^{pre}\underline{\mathfrak{G}}$. Similarly, $\pi_2^{pre}\underline{\mathfrak{G}}$ is defined to be the presheaf $U \mapsto \pi_2\big(\underline{\mathfrak{G}}(U)\big)$, and $\pi_2\underline{\mathfrak{G}}$ to be the sheaf associated to the presheaf $\pi_2^{pre}\underline{\mathfrak{G}}$.

We define $\pi_1\underline{\mathfrak{G}}$ and $\pi_2\underline{\mathfrak{G}}$ for a presheaf of crossed-modules $\underline{\mathfrak{G}} \in \mathbf{CrossedMod_C}$ in a similar manner. The equivalence of categories between $\mathbf{2Gp_C}$ and $\mathbf{CrossedMod_C}$ respects π_1^{pre}, π_1, π_2^{pre} and π_2. Lemma 3.1 remains valid in this setting if instead of π_1 and π_2 we use π_1^{pre} and π_2^{pre}.

Definition 4.1 ([Br], page 19)**.** Let $\mathbf{C}$ be a Grothendieck site. By a (strict) *gr-stack* over $\mathbf{C}$ we mean a stack $\mathcal{G}$ that is a (strict) monoid object in the category of stacks over $\mathbf{C}$ and for which weak inverses exist. All our *gr*-stacks will be strict, so from now on we will drop the adjective strict.

The condition on existence of weak inverses means that for every $U \in \operatorname{Ob}\mathbf{C}$ and every object a in the groupoid $\mathcal{G}(U)$, multiplication by a induces an equivalence of categories from $\mathcal{G}(U)$ to itself (or equivalently, an equivalence of stacks from $\mathcal{X}_U$ to itself). This condition is equivalent to saying that, for every $U \in \operatorname{Ob}\mathbf{C}$, $\mathcal{X}(U)$ is a pseudo 2-group. More compactly, it is equivalent to

$$\mathcal{G} \times \mathcal{G} \stackrel{(pr,mult)}{\longrightarrow} \mathcal{G} \times \mathcal{G}$$

being an equivalence of stacks.

A morphism of *gr*-stacks is by definition a morphism of stacks that (strictly) respects the monoidal structure. Let $\mathbf{grSt_C}$ be the category of *gr*-stacks and morphisms between them. There are natural functors

$$\mathbf{Ps2Gp_C} \to \mathbf{grSt_C} \quad \text{and} \quad \mathbf{CrossedMod_C} \to \mathbf{grSt_C}.$$

The former is simply the stackification functor that sends a presheaf of groupoids to the associated stack; note that since the stackification functor preserves products, we can carry over the monoidal structure from the presheaf of groupoids to its stackification. The latter functor is obtained from the former by using the natural functor $\mathbf{CrossedMod_C} \to \mathbf{Ps2Gp_C}$. Given a presheaf of crossed-modules

$[\varphi\colon \underline{G_2} \to \underline{G_1}]$, the associated *gr*-stacks has as the underlying stack the quotient stack $[\underline{G_1}/\underline{G_2}]$, where $\underline{G_2}$ acts on $\underline{G_1}$ by multiplication on the right (via φ).

Definition 4.2. Let $\mathcal{X}$ be a presheaf of groupoids over $\mathbf{C}$. We define $\pi^{pre}(\mathcal{X})$ to be the presheaf that sends an object U in $\mathbf{C}$ to the set of isomorphism classes in $\mathcal{X}(U)$. We denote the sheaf associated to $\pi^{pre}(\mathcal{X})$ by $\pi(\mathcal{X})$. For a global section e of $\mathcal{X}$, we define $\underline{\mathrm{Aut}}_{\mathcal{X}}(e)$ to be sheaf associated to the presheaf that sends an object U in $\mathbf{C}$ to the group of automorphisms, in the groupoid $\mathcal{X}(U)$, of the object e_U; note that when $\mathcal{X}$ is a stack this presheaf is already a sheaf and no sheafification is needed.

It is clear that π^{pre}, π and $\underline{\mathrm{Aut}}$ are functorial in $\mathcal{X}$.

Definition 4.3. Let $\mathcal{G}$ be a *gr*-stack. We define $\pi_1^{pre}(\mathcal{G}) := \pi^{pre}(\mathcal{G})$, and $\pi_2^{pre}(\mathcal{G}) := \underline{\mathrm{Aut}}(e)$, where e is the identity section of $\mathcal{G}$. We define $\pi_1(\mathcal{G})$ and $\pi_2(\mathcal{G})$ to be sheafifications of $\pi_1^{pre}(\mathcal{G})$ and $\pi_2^{pre}(\mathcal{G})$, respectively.

There are two ways of defining the notion of equivalence between *gr*-stacks. One way is to regard them as stacks and use the usual notion of equivalence of stacks. The other way is to regard them as presheaves of pseudo 2-groups and use π_1 and π_2. The next lemma shows that these two definitions agree.

Lemma 4.4. *Let $\mathcal{G}$ and $\mathcal{H}$ be* gr*-stacks, and let $f\colon \mathcal{H} \to \mathcal{G}$ be a morphism of* gr*-stacks. Then, the following are equivalent:*

- **i**. *f is an equivalence of stacks.*
- **ii**. *The induced maps $\pi_1(f)\colon \pi_1^{pre}(\mathcal{H}) \to \pi_1^{pre}(\mathcal{G})$ and $\pi_2(f)\colon \pi_2^{pre}(\mathcal{H}) \to \pi_2^{pre}(\mathcal{G})$ are isomorphisms of presheaves of groups.*
- **iii**. *The induced maps $\pi_1(f)\colon \pi_1(\mathcal{H}) \to \pi_1(\mathcal{G})$ and $\pi_2(f)\colon \pi_2(\mathcal{H}) \to \pi_2(\mathcal{G})$ are isomorphisms of sheaves of groups.*

Proof. The only non-trivial implication is (**iii**) $\Rightarrow$ (**ii**). In the proof we will use the following standard fact from closed model category theory.

> **Theorem** ([Hi], Theorem 3.2.13)**.** Let $\mathcal{M}$ be a closed model category, L a localizaing class of morphisms in $\mathcal{M}$, and $\mathcal{M}_L$ the localized model category. Let $\mathcal{X}$ and $\mathcal{Y}$ be fibrant objects (i.e. L-local objects) in $\mathcal{M}_L$, and let $f\colon \mathcal{Y} \to \mathcal{X}$ be a morphism in $\mathcal{M}$ that is a weak equivalence in the localized model structure $\mathcal{M}_L$ (that is, f is an L-local weak equivalence). Then, f is a weak equivalence in $\mathcal{M}$.

We will apply the above theorem with $\mathcal{M}$ being the model structure on the category $\mathbf{Gpd_C}$ of presheaves of groupoids on $\mathbf{C}$ in which weak equivalences are morphisms that induce isomorphisms (of presheaves of groups) on π_1^{pre} and π_2^{pre}, and fibrations are objectwise. We take L to be the class of hypercovers. The weak equivalences in the localized model structure will then be the ones inducing isomorphism (of sheaves of groups) on π_1 and π_2. The main reference for this is [Ho].

Let us now prove (**iii**) $\Rightarrow$ (**ii**). It is shown in [Ho] that $\mathcal{G}$ and $\mathcal{H}$ are L-local objects (see Section 5.2 and Section 7.3 of *loc. cit.*). By hypothesis, f induces isomorphisms (of sheaves) on π_1 and π_2, so it is a weak equivalence in the localized model structure. Therefore, since $\mathcal{G}$ and $\mathcal{H}$ are L-local, f is already a weak equivalence in the the non-localized model structure. This exactly means that $\pi_1(f)\colon \pi_1^{pre}(\mathcal{H}) \to \pi_1^{pre}(\mathcal{G})$ and $\pi_2(f)\colon \pi_2^{pre}(\mathcal{H}) \to \pi_2^{pre}(\mathcal{G})$ are isomorphisms of presheaves. $\square$

Lemma 4.5. *Let $\mathcal{X}$ be a presheaf of groupoids over $\mathbf{C}$ and $\varphi\colon \mathcal{X} \to \mathcal{X}^a$ its stackification. Then we have the following:*

- **i.** *The induced morphism $\pi\mathcal{X} \to \pi(\mathcal{X}^a)$ is an isomorphism of sheaves of sets.*
- **ii.** *For every global section e of $\mathcal{X}$, the natural map $\underline{\mathrm{Aut}}_{\mathcal{X}}(e) \to \underline{\mathrm{Aut}}_{\mathcal{X}^a}(e)$ is an isomorphism of sheaves of groups.*

Proof. This is a simple sheaf theory exercise. We include the prove of Part (**i**). Proof of Part (**ii**) is similar.

First we prove that $\pi\varphi\colon \pi\mathcal{X} \to \pi(\mathcal{X}^a)$ is injective. Let $U \in \mathrm{Ob}\,\mathbf{C}$, and let x, y be element in $\pi\mathcal{X}(U)$ such that $\pi\varphi(x) = \pi\varphi(y)$. We have to show that $x = y$. By passing to a cover of U, we may assume x and y lift to objects $\bar{x}$ and $\bar{y}$ in $\mathcal{X}(U)$. We will show that there is an open cover of U over which $\bar{x}$ and $\bar{y}$ become isomorphic. Since $\varphi(\bar{x})$ and $\varphi(\bar{y})$ become equal in $\pi(\mathcal{X}^a)$, there is a cover $\{U_i\}$ of U such that there is an isomorphism $\alpha_i\colon \varphi(\bar{x}|_{U_i}) \xrightarrow{\sim} \varphi(\bar{y}|_{U_i})$ in the groupoid $\mathcal{X}^a(U_i)$, for every i. By replacing $\{U_i\}$ with a finer cover, we may assume that α_i come from $\mathcal{X}(U_i)$. (More precisely, $\alpha_i = \varphi(\beta_i)$, where β_i is a morphism in the groupoid $\mathcal{X}(U_i)$.) This implies that, for every i, $\bar{x}|_{U_i}$ and $\bar{y}|_{U_i}$ are isomorphic as objects of the groupoid $\mathcal{X}(U_i)$. This is exactly what we wanted to prove.

Having proved the injectivity, to prove the surjectivity it is enough to show that every object x in $\pi(\mathcal{X}^a)(U)$ is in the image of φ, possibly after replacing U by an open cover. By choosing an appropriate cover, we may assume x lifts to $\mathcal{X}^a(U)$. Since $\mathcal{X}^a$ is the stackification of $\mathcal{X}$, we may assume, after refining our cover, that x is in the image of $\mathcal{X}(U) \to \mathcal{X}^a(U)$. The claim is now immediate. □

Lemma 4.6. *Let $\underline{\mathfrak{G}} = [\underline{G_2} \to \underline{G_1}]$ be a presheaf of crossed-modules, and let $\mathcal{G} = [\underline{G_1}/\underline{G_2}]$ be the corresponding* gr*-stack. Then, we have natural isomorphisms of sheaves of groups $\pi_i\underline{\mathfrak{G}} \xrightarrow{\sim} \pi_i\mathcal{G}$, $i = 1, 2$.*

Proof. Apply Lemma 4.5. □

We now specialize to the case where $\mathbf{C}$ is $\mathbf{Sch}_S$, the big site of schemes over a base scheme S, endowed with a subcanonical topology (say, étale, Zariski, fppf, fpqc,...). We define a *crossed-module in S-schemes* $[\varphi\colon G_2 \to G_1]$ to be a pair of S-group schemes G_1, G_2, an S-group scheme homomorphism $\varphi\colon G_2 \to G_1$, and a (right) action of G_1 on G_2 satisfying the axioms of a crossed-module. These are precisely the representable objects in $\mathbf{CrossedMod}_{\mathbf{Sch}_S}$; in other words, a crossed-module in schemes $[\varphi\colon G_2 \to G_1]$ gives rise to a presheaf of crossed-modules

$$U \mapsto [\varphi(U)\colon G_2(U) \to G_1(U)].$$

We will abuse the terminology and call a crossed-module in schemes over S simply a *2-group scheme over S*.

Proposition 4.7. *Let S be a base scheme. Let H be an abelian group scheme over S, acting on a S-scheme X, and let $\mathcal{X} = [X/H]$ be the quotient stack. Let G be the automorphisms of X that commute with the H action; this is a sheaf of groups on $\mathbf{Sch}_S$. We have the following:*

- **i.** *With the trivial action of G on H, the natural map $H \to G$ becomes a crossed-modules in $\mathbf{Sch}_S$-schemes (which is the same thing as a 2-group scheme).*

ii. *Let $\mathcal{G}$ be the* gr-*stack associated to $[H \to G]$. Then there is a natural morphism of* gr-*stacks $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$. Furthermore, this morphism induces an isomorphism of sheaves of groups $\pi_2\mathcal{G} \xrightarrow{\sim} \pi_2(\mathcal{A}ut\mathcal{X})$.*

iii. *Assume that $X \to S$ has geometrically connected fibers and $\mathcal{X} \to S$ is proper. Then $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$ is fully faithful (as a morphism of presheaves of groupoids). In particular, the induced map $\pi_1\mathcal{G} \xrightarrow{\sim} \pi_1(\mathcal{A}ut\mathcal{X})$ of sheaves of groups is injective.*

Proof. Let $\underline{\mathfrak{G}}$ denote the presheaf of 2-groups associated to $[H \to G]$. It is enough to construct a morphism of presheaves of 2-groups $\underline{\mathfrak{G}} \to \mathcal{A}ut\mathcal{X}$ and show that it has the required properties. Stackification of this map gives us the desired map (Lemma 4.6).

Construction of the morphism $\underline{\mathfrak{G}} \to \mathcal{A}ut\mathcal{X}$ is precisely as in ([BeNo], Lemma 8.2), and so are the proofs of (**i**) and (**ii**). Also, in the presence of the hypothesis on having geometrically connected fibers, the proof in *loc. cit.* of (**iii**) can be repeated word by word with $\mathbb{C}$ replaced by a connected scheme U. This implies that $\underline{\mathfrak{G}}(U) \to \mathcal{A}ut\mathcal{X}_U$ is fully faithful (Lemma 3.1) for an arbitrary U. From this it follows that $\underline{\mathfrak{G}} \to \mathcal{A}ut\mathcal{X}$ is fully faithful. It remains fully faithful after stackification (because $\mathcal{A}ut\mathcal{X}$ is a stack). □

5. Weighted projective general linear 2-groups

In this section we introduce our main objects of interest, the weighted projective general linear 2-group schemes, and prove that they model the self-equivalences of weighted projective stacks (Theorem 5.1). In fact, since the construction of the weighted projective stacks, and also of the weighted projective general linear 2-group schemes, commutes with base change, it will be enough to work over $\mathbb{Z}$.

We begin with some notation. We denote the multiplicative group scheme over $\operatorname{Spec}\mathbb{Z}$ by $\mathbb{G}_{m,\mathbb{Z}}$, or simply $\mathbb{G}_m$. The affine $(r+1)$-space over a base scheme S is denoted by $\mathbb{A}^{r+1}_S$; when the base scheme is $\operatorname{Spec} R$ it is denoted by $\mathbb{A}^{r+1}_R$, and when the base scheme is $\operatorname{Spec}\mathbb{Z}$ simply by $\mathbb{A}^{r+1}$. Since r will be fixed throughout this section, we will usually denote $\mathbb{A}^{r+1}_S - \{0\}$ by $\mathbb{U}_S$. We will abbreviate $\mathbb{U}_{\operatorname{Spec} R}$ and $\mathbb{U}_{\operatorname{Spec}\mathbb{Z}}$ to $\mathbb{U}_R$ and $\mathbb{U}$, respectively. We fix a Grothendieck topology on $\mathbf{Sch}_S$ that is not coarser than Zariski.

Let $n_0, n_1, \cdots, n_r$ be a sequence of positive integers, and consider the weight $(n_0, n_1, \cdots, n_r)$ action of $\mathbb{G}_m$ on $\mathbb{U} = \mathbb{A}^{r+1} - \{0\}$. (That is, for every scheme T, an element $t \in \mathbb{G}_m(T)$ acts on $\mathbb{U}_T$ by multiplication by $(t^{n_0}, t^{n_1}, \cdots, t^{n_r})$.) The quotient stack of this action is called the *weighted projective stack* of weight $(n_0, n_1, \cdots, n_r)$ and is denoted by $\mathcal{P}_{\mathbb{Z}}(n_0, n_1, \cdots, n_r)$, or simply by $\mathcal{P}(n_0, n_1, \cdots, n_r)$. The **weighted projective general linear 2-group scheme** $\mathrm{PGL}(n_0, n_1, \cdots, n_r)$ is defined to be the 2-group scheme associated to the crossed-module

$$[\varphi\colon \mathbb{G}_m \to G_{n_0,n_1,\cdots,n_r}],$$

where $G_{n_0,n_1,\cdots,n_r}$ is the group scheme, over $\mathbb{Z}$, of all $\mathbb{G}_m$-equivariant (for the above weighted action) automorphisms of $\mathbb{U}$. More precisely, the T-points of $G_{n_0,n_1,\cdots,n_r}$ are automorphisms

$$f\colon \mathbb{U}_T \to \mathbb{U}_T$$

that commute with the $\mathbb{G}_m$-action. The homomorphism $\varphi\colon \mathbb{G}_m \to G_{n_0,n_1,\cdots,n_r}$ is the one induced from the $\mathbb{G}_m$-action itself. We take the action of $G_{n_0,n_1,\cdots,n_r}$ on

$\mathbb{G}_m$ to be trivial. The associated *gr*-stack is denoted by $\mathcal{PGL}(n_0, n_1, \cdots, n_r)$, and is called the *projective general linear* gr-*stack* of weight $(n_0, n_1, \cdots, n_r)$.

The following theorem says that a weighted projective general linear 2-group scheme is a model for the *gr*-stack of self-equivalences of the corresponding weighted projective stack. A special case of this theorem (namely, the case where the base scheme is $\mathbb{C}$) was proved in ([BeNo], Theorem 8.1). We briefly sketch how the proof in *loc. cit.* can be modified to cover the general case.

Theorem 5.1. *Let* $\mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)$ *be the* gr-*stack of automorphisms of the weighted projective stack* $\mathcal{P}(n_0, n_1, \cdots, n_r)$*. Then, the natural map*

$$\mathcal{PGL}(n_0, n_1, \cdots, n_r) \to \mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)$$

is an equivalence of gr-*stacks. In particular, we have isomorphisms of sheaves of groups*

$$\pi_1\mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r) \cong \pi_1\mathcal{PGL}(n_0, n_1, \cdots, n_r) \cong \pi_1 \operatorname{PGL}(n_0, n_1, \cdots, n_r),$$

$$\pi_2\mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r) \cong \pi_2\mathcal{PGL}(n_0, n_1, \cdots, n_r) \cong \pi_2 \operatorname{PGL}(n_0, n_1, \cdots, n_r) \cong \mu_d,$$

where $d = \gcd(n_0, n_1, \cdots, n_r)$ *and* μ_d *stands for the multiplicative group scheme of* d^{th} *roots of unity.*

To prove the above theorem we use the following result.

Proposition 5.2. *Let* $\mathcal{P} = \mathcal{P}_S(n_0, n_1, \cdots, n_r)$*, where* $S = \operatorname{Spec} R$ *is the spectrum of a local ring. Then every line bundle on* $\mathcal{P}$ *is of the form* $\mathcal{O}(d)$ *for some* $d \in \mathbb{Z}$*.*

Proof. We only sketch the proof (due to A. Vistoli). Details can be found in [No4]. In the proof we use stacky versions of Grothendieck's base change and semicontinuity results ([Ha], III. Theorem 12.11). We will assume R is Noetherian.

In the case where R is a field, the assertion is easy to prove using the fact that the Picard group of $\mathcal{P}$ is isomorphic to the Weil divisor class group. To prove the general case, let x be the closed point of $S = \operatorname{Spec} R$. Let $\mathcal{L}$ be a line bundle on $\mathcal{P}$. After twisting with on appropriate $\mathcal{O}(d)$, we may assume $\mathcal{L}_x \cong \mathcal{O}$. We will show that $\mathcal{L}$ is trivial. We have $H^1(\mathcal{P}_x, \mathcal{L}_x) = H^1(\mathcal{P}_x, \mathcal{O}_x) = 0$. Hence, by semicontinuity, $H^1(\mathcal{P}_y, \mathcal{L}_y) = 0$ for every point y of S. Base change implies that $R^1 f_*(\mathcal{L}) = 0$, and that $R^0 f_*(\mathcal{L}) = f_*(\mathcal{L})$ is locally free (necessarily of rank 1). Therefore, $f_*(\mathcal{L})$ is free of rank 1 and, by base change, $H^0(\mathcal{P}_y, \mathcal{L}_y)$ is 1-dimensional as a $k(y)$-vector space, for every y in S. In fact, this is true for every tensor power $\mathcal{L}^{\otimes n}$, $n \in \mathbb{Z}$. So, $\mathcal{L}_y$ is trivial for every y in S. (Note that, when k is a field, $\dim_k H^0(\mathcal{P}_k(n_0, n_1, \cdots, n_r), \mathcal{O}(d))$ is equal to the number of solutions of the equation $a_1 n_0 + a_2 n_1 + \cdots + a_r n_r = d$ in non-negative integers a_i.)

Now let s be a generating section of $f_*(\mathcal{L}) \cong R$. It follows that $f^*(s)$ is a generating section of $\mathcal{L}$. So $\mathcal{L}$ is trivial. $\square$

Proof of Theorem 5.1. We apply Proposition 4.7 with $S = \operatorname{Spec} \mathbb{Z}$, $X = \mathbb{A}^{r+1} - \{0\}$, and $H = \mathbb{G}_m$. This implies that $\mathcal{PGL}(n_0, n_1, \cdots, n_r) \to \mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)$ is a fully faithful morphism of stacks. That is, for every scheme U, the morphism of groupoids $\mathcal{PGL}(n_0, n_1, \cdots, n_r)(U) \to \mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)(U)$ is fully faithful. All that is left to show is that it is essentially surjective. Since $\mathcal{PGL}(n_0, n_1, \cdots, n_r)$ and $\mathcal{A}ut\mathcal{P}(n_0, n_1, \cdots, n_r)$ are both stacks, it is enough to prove this for $U = \operatorname{Spec} R$, where R is a local ring. In this case, we know by Proposition 5.2 that $\operatorname{Pic} \mathcal{P}(n_0, n_1, \cdots, n_r) \cong \mathbb{Z}$. We can now proceed exactly as in ([BeNo], Theorem 8.1).

The isomorphisms stated at the end of the theorem follow from Lemma 4.4 and Lemma 4.6. ☐

6. Structure of $\mathrm{PGL}(n_0, n_1, \cdots, n_r)$

Throughout this section, the action of $\mathbb{G}_m$ on $\mathbb{U} = \mathbb{A}^{r+1} - \{0\}$ means the weight $(n_0, n_1, \cdots, n_r)$ action. To shorten the notation, we denote the group $G_{n_0,n_1,\cdots,n_r}$ by G. The rank m general linear group scheme over $\operatorname{Spec} R$ is denoted by $\mathrm{GL}(m, R)$. When $R = \mathbb{Z}$, this is abbreviated to $\mathrm{GL}(m)$. We always assume $r \geq 1$. The corresponding projectivized group scheme is denoted by $\mathrm{PGL}(m)$; this notation does not conflict with the notation $\mathrm{PGL}(n_0, n_1, \cdots, n_r)$ for a weighted projective general linear 2-group as in the latter case we have at least two variables.

We begin with a simple lemma.

Lemma 6.1. *Let R be an arbitrary ring, and let f be a global section of the structure sheaf of $\mathbb{U}_R = \mathbb{A}^{r+1}_R - \{0\}$, $r \geq 1$. Then f extends uniquely to a global section of $\mathbb{A}^{r+1}_R$.*

Proof. Let $U_i = \operatorname{Spec} R[x_0, \cdots, x_r, x_i^{-1}]$ and consider the covering $\mathbb{U}_R = \cup_{i=1}^{n} U_i$. We show that the restriction $f_i := f|_{U_i}$ is a polynomial for every i. To see this, observe that, except possibly for x_i, all variables occur with positive powers in f_i. To show that x_i also occurs with a positive power, pick some $j \neq i$ and use the fact that x_i occurs with a positive power in $f_j|_{U_i \cap U_j} = f_i|_{U_i \cap U_j}$.

Therefore, for every i, f_i actually lies in $R[x_0, \cdots, x_r, x_i^{-1}]$. Since $f_j|_{U_i} = f_i|_{U_j}$, it is obvious that all f_i are actually the same and provide the desired extension of f to $\mathbb{U}_R$. ☐

From now on, we will use a slightly different notation with indices. Namely, we assume that the weights are $m_1 < m_2 < \cdots < m_t$, with each m_i appearing exactly $r_i \geq 1$ times in the weight sequence (so in the previous notation we would have $r + 1 = r_1 + \cdots + r_t$). We denote the corresponding projective general linear 2-group by $\mathrm{PGL}(m_1 : r_1, m_2 : r_2, \cdots, m_t : r_t)$. We use the coordinates x^i_j, $1 \leq i \leq t$, $1 \leq j \leq r_i$, for $\mathbb{A}^{r+1}$. We think of x^i_j as a variable of degree m_i. We will usually abbreviate the sequence $x^i_1, \cdots, x^i_{r_i}$ to $\mathbf{x}^i$. Similarly, a sequence $F^i_1, \cdots, F^i_{r_i}$ of polynomials is abbreviated to $\mathbf{F}^i$.

Let R be a ring. The following proposition tells us how a $\mathbb{G}_{m,R}$-equivariant automorphisms of $\mathbb{U}_R$ looks like.

Proposition 6.2. *Let $F\colon \mathbb{U}_R \to \mathbb{U}_R$ be a $\mathbb{G}_m$-equivariant map. Then F is of the form $(\mathbf{F}^i)_{1 \leq i \leq t}$, where for every i, each component $F^i_j \in R[x^i_j; 1 \leq i \leq t, 1 \leq j \leq r_i]$ of $\mathbf{F}^i$ is a weighted homogeneous polynomial of weight m_i.*

Proof. The fact that components of F are polynomial follows from Lemma 6.1. The statement about homogeneity of F^i_j is a simple exercise in polynomial algebra and is left to the reader. ☐

In the above proposition, each F^i_j can be written in the form $F^i_j = L^i_j + P^i_j$, where L^i_j is linear in the variables $x^i_1, \cdots, x^i_{r_i}$, and P^i_j is a homogeneous polynomial of degree m_i in variables x^a_b with $a < i$. Let $L_F := (\mathbf{L}^i)_{1 \leq i \leq t}$ be the linear part of F. It is again a $\mathbb{G}_m$-equivariant endomorphism of $\mathbb{U}$.

Proposition 6.3. *Let F be as in the Proposition 6.2. The assignment $F \mapsto L_F$ respects composition of endomorphisms. In particular, if F is an automorphism, then so is L_F.*

Proof. This follows from direct calculation, or, alternatively, by using the fact that L_F is simply the derivative of F at the origin. □

Corollary 6.4. *There is a natural split homomorphism*

$$\phi\colon G \to \mathrm{GL}(r_1) \times \mathrm{GL}(r_2) \times \cdots \times \mathrm{GL}(r_t).$$

Next we give some information about the structure of the kernel U of ϕ. It consists of endomorphisms $F = (F_j^i)_{i,j}$, where F_j^i has the form

$$F_j^i = x_j^i + P_j^i.$$

Here, P_j^i is a homogeneous polynomial of degree m_i in variables x_b^a with $a < i$. Indeed, it is easily seen that, for an arbitrary choice of the polynomials P_j^i, the resulting endomorphism F is automatically invertible. So, to give such an $F \in U$ is equivalent to giving an arbitrary collection of polynomials $\{P_j^i\}_{1\leq i\leq t, 1\leq j\leq r_i}$ such that each P_j^i is a homogeneous polynomial of degree m_i in variables x_b^a with $a < i$. So, from now on we switch the notation and denote such an element of U by $(P_j^i)_{i,j}$.

Proposition 6.5. *For each $1 \leq a \leq t$, let $U_a \subseteq U$ be the set of those endomorphisms $F = (P_j^i)_{i,j}$ for which $P_j^i = 0$ whenever $i \neq a$. Let K_a denote the set of monomials of degree m_a in variables x_j^i, $i < a$, and let k_a be the cardinality of K_a. (In other words, k_a is the number of solutions of the equation*

$$\sum_{i=1}^{a-1} m_i \sum_{j=1}^{r_i} z_{i,j} = m_a$$

in non-negative integers $z_{i,j}$.) Then we have the following:

- **i**. *U_a is a subgroup of U and is canonically isomorphic to the vector group scheme $\mathbb{A}^{r_a} \otimes \mathbb{A}^{K_a} \cong \mathbb{A}^{r_a k_a}$. (Note: U_1 is trivial.)*
- **ii**. *If $a < b$, then U_a normalizes U_b.*
- **iii**. *The groups U_a, $1 \leq i \leq t$, generate U and we have $U_a \cap U_b = \{1\}$ if $a \neq b$.*

Proof of Part (**i**). The action of $(P_j^i)_{i,j} \in U_a$ on $\mathbb{A}^{r+1}$ is given by

$$(\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^t) \longmapsto (\mathbf{x}^1, \cdots, \mathbf{x}^a + \mathbf{P}^a, \cdots, \mathbf{x}^t).$$

So, if $\mathbb{A}^{K_a}$ stands for the vector group scheme on the basis K_a, there is a canonical isomorphism

$$U_a \cong \bigoplus_{i=1}^{r_a} \mathbb{A}^{K_a} \cong \mathbb{A}^{r_a} \otimes \mathbb{A}^{K_a}.$$

Proof of Part (**ii**). Let $G = (Q_j^i)_{i,j}$ be an element in U_a and $F = (P_j^i)_{i,j}$ an element in U_b. By (**i**), the inverse of G is $G^{-1} = (-Q_j^i)_{i,j}$. Let us analyze the effect of the composite $G \circ F \circ G^{-1}$ on $\mathbb{A}^{r+1}$:

$$\begin{aligned}
(\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^b, \cdots, \mathbf{x}^t) &\overset{G^{-1}}{\longmapsto} (\mathbf{x}^1, \cdots, \mathbf{x}^a - \mathbf{Q}^a, \cdots, \mathbf{x}^b, \cdots, \mathbf{x}^t)\\
&\overset{F}{\longmapsto} (\mathbf{x}^1, \cdots, \mathbf{x}^a - \mathbf{Q}^a, \cdots, \mathbf{x}^b + \mathbf{R}^b, \cdots, \mathbf{x}^t)\\
&\overset{G}{\longmapsto} (\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^b + \mathbf{R}^b, \cdots, \mathbf{x}^t).
\end{aligned}$$

Here the polynomial R^b_k, $1 \le k \le r_b$, is obtained from P^b_k by substituting the variables x^a_j with the polynomial $x^a_j - Q^a_j$.

Proof of Part (**iii**). Easy. □

Part (**ii**) implies that each U_a acts by conjugation on each of U_{a+1}, $U_{a+2}, \cdots$, U_t.[2] To fix the notation, in what follows we let the conjugate of an automorphism f by an automorphism g to be $g \circ f \circ g^{-1}$.

Notation. Let $\{U_a\}_{a=1}^t$ be a family of subgroups of a group U which satisfies the following properties: 1) Each U_a normalizes every U_b with $a < b$; 2) No two distinct U_a intersect; 3) The U_a generate U. In this case, we say that U is a successive semi-direct product of the U_a, and use the notation $U \cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1$.

The following is an immediate corollary of Proposition 6.5.

Corollary 6.6. *There is a natural decomposition of* U *as a semi-direct product*

$$U \cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1,$$

where $U_a \cong \mathbb{A}^{r_a k_a}$ *is the group introduced in Proposition 6.5. (Note that* U_1 *is trivial.)*

In the next theorem we use the notation $\mathbb{A}^m$ for two things. One that has already appeared is the affine group scheme of dimension m. When there is a group scheme G involved, we also use the notation $\mathbb{A}^m$ for the trivial representation of G on $\mathbb{A}^m$.

Theorem 6.7. *There is a natural decomposition of* G *as a semi-direct product*

$$G \cong U_t \rtimes \cdots \rtimes U_2 \rtimes U_1 \rtimes \big(\mathrm{GL}(r_1) \times \cdots \times \mathrm{GL}(r_t)\big),$$

where $U_a \cong \mathbb{A}^{r_a k_a}$ *and* k_a *is as in Proposition 6.5. (Note that* U_1 *is trivial.) Furthermore, for every* $1 \le a \le t$*, the action of* $\mathrm{GL}(r_a)$ *leaves each* U_b *invariant. We also have the following:*

- **i**. *When* $a > b$ *the induced action of* $\mathrm{GL}(r_a)$ *on* U_b *is trivial.*
- **ii**. *When* $a = b$ *the induced action of* $\mathrm{GL}(r_a)$ *on* U_a *is naturally isomorphic to the representation* $\rho \otimes \mathbb{A}^{K_a}$*, where* ρ *is the standard representation of* $\mathrm{GL}(r_a)$ *and* K_a *is as in Proposition 6.5. (Recall that* U_a *is canonically isomorphic to* $\mathbb{A}^{r_a} \otimes \mathbb{A}^{K_a}$*.)*
- **iii**. *When* $a < b$ *the action of* $\mathrm{GL}(r_a)$ *on* U_b *is naturally isomorphic to the representation*

$$\bigoplus_{0 \le l \le \lfloor \frac{m_b}{m_a} \rfloor} \mathbb{A}^{r_b d_l} \otimes \hat{\rho}^{\otimes l}.$$

Here $\hat{\rho}$ *stands for the inverse transpose of* ρ*, and* d_l *is the number of monomials of degree* m_b *in variables* x^i_j*,* $i < b$*,* $i \ne a$*; so* d_l *also depends on* a *and* b*. (In other words,* d_l *is the number of solutions of the equation*

$$\sum_{\substack{i=1 \\ i \ne a}}^{b-1} m_i \sum_{j=1}^{r_i} z_{i,j} = m_b - l m_a$$

in non-negative integers $z_{i,j}$*.)*

[2]All group actions in this section are assumed to be on the left.

Proof. Let $g \in \mathrm{GL}(r_a)$ and $F \in U_b$. As in the proof of Proposition 6.5.**i**, we analyze the effect of the composite $g \circ F \circ g^{-1}$ on $\mathbb{A}^{r+1}$. The element $g \in \mathrm{GL}(r_a)$ acts on $\mathbb{A}^{r+1}$ as follows: it leaves every component x_i^j invariant if $i \neq a$ and on the coordinates $x_1^a, \cdots, x_{r_a}^a$ it acts linearly (like the action of an $r_a \times r_a$ matrix on a column vector).

Proof of Part (**i**). The effect of $g \in \mathrm{GL}(r_a)$ only involves the variables $x_1^a, \cdots, x_{r_a}^a$ and does not see any other variable, whereas the effect of $F \in U_b$ only involves the variables x_j^i, $i \leq b$. Since $b < a$, these two are independent of each other. That is, F and g commute.

Proof of Part (**ii**). Assume $F = (P_j^i)_{i,j}$; so $P_j^i = 0$ if $i \neq a$. The effect of $g \circ F \circ g^{-1}$ can be described as follows:

$$\begin{array}{rcl}(\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^t) & \xmapsto{g^{-1}} & (\mathbf{x}^1, \cdots, \mathbf{y}^a, \cdots, \mathbf{x}^t) \\ & \xmapsto{F} & (\mathbf{x}^1, \cdots, \mathbf{y}^a + \mathbf{P}^a, \cdots, \mathbf{x}^t) \\ & \xmapsto{g} & (\mathbf{x}^1, \cdots, \mathbf{x}^a + \mathbf{Q}^a, \cdots, \mathbf{x}^t).\end{array}$$

Here, y_j^a is the linear combination of $x_1^a, \cdots, x_{r_a}^a$, the coefficients being the entries of the j^{th} row of the matrix g^{-1}. Similarly, Q_j^a is the linear combination of $P_1^a, \cdots, P_{r_a}^a$, coefficients being the entries of the j^{th} row of the matrix g.

Proof of Part (**iii**). Assume $F = (P_j^i)_{i,j}$; so $P_j^i = 0$ if $i \neq b$. Let $\mathbf{y}^a$ be as in (**ii**). The effect of $g \circ F \circ g^{-1}$ can be described as follows:

$$\begin{array}{rcl}(\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^b, \cdots, \mathbf{x}^t) & \xmapsto{g^{-1}} & (\mathbf{x}^1, \cdots, \mathbf{y}^a, \cdots, \mathbf{x}^b, \cdots, \mathbf{x}^t) \\ & \xmapsto{F} & (\mathbf{x}^1, \cdots, \mathbf{y}^a, \cdots, \mathbf{x}^b + \mathbf{R}^b, \cdots, \mathbf{x}^t) \\ & \xmapsto{g} & (\mathbf{x}^1, \cdots, \mathbf{x}^a, \cdots, \mathbf{x}^b + \mathbf{R}^b, \cdots, \mathbf{x}^t).\end{array}$$

Here the polynomials R_k^b, $1 \leq k \leq r_b$, are obtained from P_k^b by substituting the variable x_j^a with y_j^a.

Let λ be the representation of $\mathrm{GL}(r_a)$ on the space V of homogenous polynomials of degree m_b which acts as follows: it takes a polynomial $P \in V$ and substitutes the variables x_j^a, $1 \leq j \leq r_a$, with y_j^a. From the description above, we see that the representation of $\mathrm{GL}(r_a)$ on U_b is a direct sum of r_b copies of λ. We will show that

$$\lambda \cong \bigoplus_{0 \leq l \leq \lfloor \frac{m_b}{m_a} \rfloor} \mathbb{A}^{d_l} \otimes \hat{\rho}^{\otimes l}.$$

To obtain the above decomposition, simply note that a polynomial in V can be uniquely written in the form

$$\sum_{0 \leq l \leq \lfloor \frac{m_b}{m_a} \rfloor} S_l T_l,$$

where T_l is a homogenous polynomial of degree $l m_a$ in variables $x_1^a, \cdots, x_{r_a}^a$, and S_l is a homogenous polynomial of degree $m_b - l m_a$ in the rest of the variables. The action of $\mathrm{GL}(r_a)$ leaves S_l intact and acts on T_l by the l^{th} power of the inverse transpose of the standard representation. □

The actions of various pieces in the above semi-direct product decomposition, though explicit, are tedious to write down, except for small values of t. We give some examples.

Example 6.8. *Weight sequence $m < n, m \nmid n$.* In this case we have $t = 2$, and $r_1 = r_2 = 1$ and $k_1 = 0$. So $G \cong \mathbb{G}_m \times \mathbb{G}_m$.

Example 6.9. *Weight sequence $m < n, m \mid n$.* In this case we have $t = 2$, $r_1 = r_2 = 1$, and $k_1 = 1$. So we have

$$G \cong \mathbb{A} \rtimes (\mathbb{G}_m \times \mathbb{G}_m).$$

The action of an element $(\lambda_1, \lambda_2) \in \mathbb{G}_m \times \mathbb{G}_m$ on an element $a \in \mathbb{A}$ is given by

$$(\lambda_1, \lambda_2) \cdot a = \lambda_2 \lambda_1^{-\frac{n}{m}} a.$$

More explicitly, an element in G is map of the form

$$(x, y) \mapsto (\lambda_1 x, \lambda_2 y + a x^{\frac{n}{m}}).$$

Note the similarity with the group of 2×2 lower-triangular matrices.

Example 6.10. *Weight sequence $n = m$.* We obviously have $G \cong \mathrm{GL}(2)$.

Example 6.11. *Weight sequence* $1, 2, 3$. First we determine U. A typical element in U is of the form

$$(x, y, z) \mapsto (x, y + ax^2, z + bx^3 + cxy).$$

We have $U_2 = \mathbb{A}$ and $U_3 = \mathbb{A}^2$. The action of an element $a \in U_2$ on an element $(b, c) \in U_3$ is given by $(b - ac, c)$. That is, a acts on $U_3 = \mathbb{A}^2$ by the matrix

$$\begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$$

So, $U \cong \mathbb{A}^{\oplus 2} \rtimes \mathbb{A}$. Finally, we have

$$G \cong U \rtimes (\mathbb{G}_m)^3 = \mathbb{A}^{\oplus 2} \rtimes \mathbb{A} \rtimes (\mathbb{G}_m)^3,$$

where the action of an element $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{G}_m)^3$ on an element $(a, b, c) \in U$ is given by $(\lambda_1^{-2}\lambda_2 a, \lambda_1^{-3}\lambda_3 b, \lambda_1^{-2}\lambda_2^{-1}\lambda_3 c)$.

Example 6.12. *Weight sequence* $1, 2, 4$. An element in U has the general form

$$(x, y, z) \mapsto (x, y + ax^2, z + bx^4 + cx^2 y + dy^2).$$

We have $U_2 = \mathbb{A}$ and $U_3 = \mathbb{A}^3$. The action of an element $a \in U_2$ on an element $(b, c, d) \in U_3$ is given by the matrix

$$\begin{pmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{pmatrix}$$

So, $U \cong \mathbb{A}^{\oplus 3} \rtimes \mathbb{A}$.

Finally, we have

$$G \cong U \rtimes (\mathbb{G}_m)^3 = \mathbb{A}^{\oplus 3} \rtimes \mathbb{A} \rtimes (\mathbb{G}_m)^3,$$

where the action of an element $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{G}_m)^3$ on an element $(a, b, c, d) \in U$ is given by

$$(\lambda_1^{-2}\lambda_2 a, \lambda_1^{-4}\lambda_3 b, \lambda_1^{-2}\lambda_2^{-1}\lambda_3 c, \lambda_2^{-2}\lambda_3 d).$$

Next we look at $\mathrm{PGL}(m_1:r_1, m_2:r_2,\cdots,m_t:r_t)$. Recall that, as a crossed-module, this is given by $[\varphi\colon \mathbb{G}_m \to G]$, where φ is the obvious map coming from the action of $\mathbb{G}_m$ on $\mathbb{A}^{r+1}$, and the action of G on $\mathbb{G}_m$ is the trivial one.

Observe that the map φ factors though the component $\mathrm{GL}(r_1)\times\cdots\times\mathrm{GL}(r_t)$ of G. So, let us define L to be the cokernel of the following map:

$$\mathbb{G}_m \xrightarrow{(\overbrace{\lambda^{m_1},\ldots,\lambda^{m_1}}^{r_1},\ldots,\overbrace{\lambda^{m_t},\ldots,\lambda^{m_t}}^{r_t})} \mathrm{GL}(r_1)\times\cdots\times\mathrm{GL}(r_t).$$

From Theorem 6.7 we immediately obtain the following.

Proposition 6.13. *Let L be the group defined in the previous paragraph, and let $k=\gcd(m_1,\cdots,m_t)$. We have natural isomorphisms of group schemes*

$$\begin{aligned}\pi_1\,\mathrm{PGL}(m_1:r_1,m_2:r_2,\cdots,m_t:r_t) &\cong U_t\rtimes\cdots\rtimes U_2\rtimes U_1\rtimes L,\\ \pi_2\,\mathrm{PGL}(m_1:r_1,m_2:r_2,\cdots,m_t:r_t) &\cong \mu_k.\end{aligned}$$

Our final result is that, if all weights are distinct (that is, $r_i=1$), then the corresponding projective general linear 2-group is split.

Proposition 6.14. *Let $\{m_1,\cdots,m_t\}$ be distinct positive integers, and consider the projective general linear 2-group $\mathrm{PGL}(m_1,m_2,\cdots,m_t)$. Then, the projection map $G\to\pi_1\,\mathrm{PGL}(m_1,m_2,\cdots,m_t)$ splits. In particular, $\mathrm{PGL}(m_1,m_2,\cdots,m_t)$ is split. That is, it is completely classified by its homotopy group schemes:*

$$\begin{aligned}\pi_1\,\mathrm{PGL}(m_1,\cdots,m_t) &\cong U_t\rtimes\cdots\rtimes U_2\rtimes U_1\rtimes(\mathbb{G}_m)^{t-1},\\ \pi_2\,\mathrm{PGL}(m_1,\cdots,m_t) &\cong \mu_k.\end{aligned}$$

Proof. By Theorem 6.7 and Proposition 6.13 we know that $G\cong U_t\rtimes\cdots\rtimes U_2\rtimes U_1\rtimes(\mathbb{G}_m)^t$ and $\pi_1\,\mathrm{PGL}(m_1,m_2,\cdots,m_t)\cong U_t\rtimes\cdots\rtimes U_2\rtimes U_1\rtimes L$, where L is the cokernel of the map

$$\alpha\colon \mathbb{G}_m \xrightarrow{(\lambda^{m_1},\cdots,\lambda^{m_t})} (\mathbb{G}_m)^t.$$

So it is enough to show that the image of μ is a direct factor. Note that if we divide all the m_i by their greatest common divisor, the image of α does not change. So, we may assume $\gcd(m_1,\cdots,m_t)=1$. Let M be a $t\times t$ integer matrix whose determinant is 1 and whose first column is $(m_1,\cdots,m_t)$. The matrix M gives rise to an isomorphism $\mu\colon(\mathbb{G}_m)^t\to(\mathbb{G}_m)^t$ whose restriction to the subgroup $\mathbb{G}_m\times\{1\}^{t-1}\cong\mathbb{G}_m$ is naturally identified with α. The subgroup $\mu(\{1\}\times(\mathbb{G}_m)^{t-1})\subset(\mathbb{G}_m)^t$ is the desired complement of the image of α. $\square$

Corollary 6.15. *Let m, n be distinct positive integers, and let $k=\gcd(m,n)$. Then $\mathrm{PGL}(m,n)$ is a split 2-group. That is, it is classified by its homotopy groups:*

$$\begin{aligned}\pi_1\,\mathrm{PGL}(m,n) &\cong \begin{cases}\mathbb{G}_m, & \text{if } m<n, m\nmid n\\ \mathbb{A}\rtimes\mathbb{G}_m, & \text{if } m<n, m\mid n\end{cases}\\ \pi_2\,\mathrm{PGL}(m,n) &\cong \mu_k.\end{aligned}$$

(In the case $m\mid n$, the action of $\mathbb{G}_m$ on $\mathbb{A}$ in the cross product $\mathbb{A}\rtimes\mathbb{G}_m$ is simply the multiplication action.)

Proof. Everything is clear, except perhaps a clarification is in order regarding the parenthesized statement. Observe that the $\mathbb{G}_m$ appearing in the cross product $\mathbb{A} \rtimes \mathbb{G}_m$ is indeed the cokernel of the map

$$\alpha \colon \mathbb{G}_m \xrightarrow{(\lambda^m, \lambda^n)} (\mathbb{G}_m)^2,$$

which is naturally identified with the subgroup $\{1\} \times \mathbb{G}_m \subset (\mathbb{G}_m)^2$. Therefore, by the formula of Example 6.9, the action of an element $\lambda \in \mathbb{G}_m$ on an element $a \in \mathbb{A}$ is given by λa. $\square$

Finally, for the sake of completeness, we include the following.

Proposition 6.16. *The 2-group* $\mathrm{PGL}(k, k, \cdots, k)$*, k appearing t times, is given by the following crossed-module:*

$$[\mathbb{G}_m \xrightarrow{(\lambda^k, \cdots, \lambda^k)} \mathrm{GL}(t)].$$

We have $\pi_1 \mathrm{PGL}(k, \cdots, k) \cong \mathrm{PGL}(t)$ *and* $\pi_2 \mathrm{PGL}(k, \cdots, k) \cong \mu_k$*. In particular,* $\mathrm{PGL}(1, 1, \cdots, 1)$*, 1 appearing t times, is equivalent to the group scheme* $\mathrm{PGL}(t)$*.*

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