Submultiplicativity of Boutet de Monvel's Algebra for Boundary Value Problems

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Abstract. We show that the algebra of operators of order and type zero in Boutet de Monvel's calculus for boundary value problems is a submultiplicative Fréchet subalgebra of $\mathcal{L}(L^2(X))$, where X is either the half-space $\mathbf{R}^n_+$ or a compact manifold with boundary.

Key Words: Boundary value problems, Boutet de Monvel's calculus, Fréchet algebras, submultiplicativity.

AMS subject classification: 58G20, 46H35, 35S15, 47D25

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Contents

Introduction

A Fréchet algebra $\mathcal{A}$ is called submultiplicative, if there is a defining system for the topology of $\mathcal{A}$, say $\{q_k : k \in \mathbf{N}\}$, such that

$$q_k(ab) \leq q_k(a)q_k(b). \tag{1}$$

Property (1) has attracted our attention in connection with recent results in the theory of Fréchet algebras, Gramsch [10], particularly on non-commutative cohomology and Oka principle; for applications see Gramsch & Kaballo [13].
Already in 1991, C. Phillips stressed the importance of submultiplicativity for the construction of a K–theory for Fréchet algebras.
For the Fréchet algebra $\mathcal{A}$ of operators of order and type zero in Boutet de Monvel's calculus it has been shown in Schrohe [30], cf. [34], that the group of invertible elements is open; $\mathcal{A}$ even is a Ψ^*–algebra in the sense of Gramsch [9]. This made Boutet de Monvel's algebra accessible to the results on perturbation theory by Gramsch [9] and Gramsch & Kaballo [12], on Jordan operators [25], on J.L. Taylor's multidimensional functional calculus [24] and on nonlinear functional analysis [22]. In several contributions of Ali Mehmeti, [1], [2], [3], the interaction operator on nets and ramified spaces with transmission provides classes of Ψ^*–algebras. This can be seen by applying the methods of [15].
Spectral invariance and Ψ^*–algebras also play an interesting role in the articles of Bony & Chemin [4], Sjöstrand [40], and Helffer [19]. Some aspects of differential geometry, e.g. periodic geodesics, in special Fréchet manifolds [14] depend on the notion of a Ψ^*–algebra, and there is a functional analytic approach to the propagation of singularities using special Fréchet operator algebras [11]. For further work on Ψ^*–algebras see the introduction of [15].
Recently, Gramsch, Ueberberg, and Wagner [15] showed that the algebras $\Psi^0_{\rho,\delta}, 0 \leq \delta \leq \rho \leq 1, \delta < 1$ of zero order pseudodifferential operators on $\mathbf{R}^n$ with symbols in the Hörmander classes $S^0_{\rho,\delta}$ are submultiplicative. Their argument relies on Beals' theorem on the characterization of pseudodifferential operators by the mapping properties of their iterated commutators with multipliers and vector fields on Sobolev spaces:

Theorem: *A continuous operator $A : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}'(\mathbf{R}^n)$ belongs to $\Psi^0_{\rho,\delta}$ if and only if for all multi-indices α, β and all $s \in \mathbf{R}$ there is a bounded extension*

$$\mathrm{ad}^\alpha(-ix)\mathrm{ad}^\beta(D_x)A : H^s(\mathbf{R}^n) \to H^{s+\rho|\alpha|-\delta|\beta|}(\mathbf{R}^n).$$

In fact, they showed that one can construct a submultiplicative system of semi-norms for every algebra that is defined in terms of the behavior of its elements under the application of derivations and order shifts.
Our proof of the submultiplicativity of Boutet de Monvel's algebra on $\mathbf{R}^n_+$ is based on two similar results.
The first is a characterization of the zero order pseudodifferential operators satisfying the (uniform two-sided) transmission condition, Schrohe [33]. Extending a result by Grubb and Hörmander [18], it was shown that the transmission condition in Boutet de Monvel's sense [6] can be described in terms of the behavior of the commutators with multipliers and vector fields tangential to the boundary on the wedge Sobolev spaces introduced by Schulze, cf. [36], section 3.1.

More specifically, an operator $A \in \Psi^0_{1,0}$ satisfies the transmission condition if, for all multi-indices α, β and all $s, \sigma, \tau \in \mathbf{R}$, it has a bounded extension

$$\mathrm{ad}^{\alpha}(-ix')\mathrm{ad}^{\beta}(D_{x'})A : \mathcal{W}^s(\mathbf{R}^{n-1}, H^{\sigma,\tau}_{\{0\}}(\mathbf{R}_+)) \to \mathcal{W}^{s+|\alpha|}(\mathbf{R}^{n-1}, H^{\sigma+|\alpha|,\tau}_{\{0\}}(\mathbf{R}_+)),$$

and the same extension holds for the L^2-adjoint A^*.
Here, $H^{\sigma,\tau}_{\{0\}}$ denotes the usual weighted Sobolev space $H^{\sigma,\tau}$ for $\sigma \geq 0$, and the space $H^{\sigma,\tau}_0$ for $\sigma < 0$. In both cases, the weight is $\langle x_n \rangle^{\tau}$.
Similarly, the singular Green operators of order -1 and type zero have a characterization via the behavior of their iterated commutators with tangential vector fields and multipliers on wedge Sobolev spaces, Schrohe [32]. This theorem was motiviated by a result of Schulze of 1992, identifying the singular Green operators of type zero with elements of certain operator-valued symbol classes, cf. theorem 3.1 in [38].
Technically, we concentrate on the upper left corner in Boutet Monvel's algebra. We may write it as a non-direct sum of two Fréchet spaces, namely the pseudodifferential operators of type zero and the singular Green operators of order -1 and type zero. To each part we now apply the construction of Gramsch, Ueberberg, and Wagner. For the 'mixed' terms in the products we use a simple functional analytic argument.
In the case of a compact manifold with boundary, the proof is similar. However, we now use a more subtle decomposition of Boutet de Monvel's algebra into a non-direct sum of four Fréchet spaces:

(1) the zero order pseudodifferential operators with the transmission property acting close to the boundary,
(2) the singular Green operators of order -1 and type zero acting close to the boundary,
(3) the pseudodifferential operators of order zero acting in the interior, and
(4) the regularizing operators.

To the first three algebras we may apply the construction of [15]; the fourth is easily seen to be submultiplicative. Decomposing the product of two elements correspondingly produces 21 terms. Treating the 'mixed' products is essentially similar to the method above. It does, however, require a little more attention; for example one has to make sure the semi-norms for (1) and (3) are compatible.
It seems to be an open problem, cf. [15], whether every Ψ^*-algebra is submultiplicative. A Ψ^*-algebra trivially has an open group of invertible elements. Very recently, Żelazko [44] constructed an example of a (non-commutative) Fréchet algebra which is *not* submultiplicative, but has an open group of invertible elements.
On the other hand, Turpin has shown that every commutative Fréchet algebra with an open group of invertible elements is submultiplicative, cf. [42], p.123.

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1 Pseudodifferential Operators, the Transmission Property, Singular Green Operators, and Wedge Sobolev Spaces

1.1 Definition. (a) For $m \in \mathbf{R}, S^m_{1,0} = S^m_{1,0}(\mathbf{R}^k \times \mathbf{R}^n)$ denotes the set of all smooth functions p on $\mathbf{R}^k \times \mathbf{R}^n, k, n \in \mathbf{N}$, satisfying the estimates

$$|D^\alpha_\xi D^\beta_x p(x,\xi)| \leq C_{\alpha\beta} \langle\xi\rangle^{m-|\alpha|} \tag{1}$$

for all $x \in \mathbf{R}^k, \xi \in \mathbf{R}^n$. Here, $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$. The choice of best constants in (1) gives the Fréchet topology for $S^m_{1,0}$.
In general, the symbols will take values in matrices over $\mathbf{C}$, but for the purposes here it will be sufficient to deal with scalar functions.
(b) A symbol $p \in S^m_{1,0}$ defines a pseudodifferential operator $\operatorname{Op} p$ by

$$[\operatorname{Op} p\, u](x) = (2\pi)^{-n/2} \int e^{ix\xi} p(x,\xi)\hat{u}(\xi)d\xi, \tag{2}$$

where u is a rapidly decreasing function and the hat denotes the Fourier transform.
(c) For $s \in \mathbf{R}, H^s(\mathbf{R}^n)$ denotes the usual Sobolev space on $\mathbf{R}^n$, cf. [23], ch. 3, definition 2.1. For $s,t \in \mathbf{R}$, let

$$H^{s,t}(\mathbf{R}^n) = \{\langle x\rangle^{-t} u : u \in H^s(\mathbf{R}^n_x)\}.$$

$H^{s,t}(\mathbf{R}^n, E), E$ a Hilbert space, denotes the vector-valued analog.

1.2 Notation on the half-space. We will write $\mathbf{R}^n_+ = \{(x_1,\ldots,x_n) : x_n > 0\}$ and $x = (x', x_n), \xi = (\xi', \xi_n)$ with $x' = (x_1,\ldots,x_{n-1}), \xi' = (\xi_1,\ldots,\xi_{n-1})$.
(a) For a function or distribution f on $\mathbf{R}^n$ let $\mathrm{r}^+ f$ denote its restriction to $\mathbf{R}^n_+$; for a function g on $\mathbf{R}^n_+$ denote by $\mathrm{e}^+ g$ its extension to $\mathbf{R}^n$ by zero. Similarly define r^- and e^-.
(b) Let $\mathcal{S}(\mathbf{R}^n_+) = \{\mathrm{r}^+ f : f \in \mathcal{S}(\mathbf{R}^n)\}$, and $H^{s,t}(\mathbf{R}^n_+) = \{\mathrm{r}^+ f : f \in H^{s,t}(\mathbf{R}^n)\}, s,t \in \mathbf{R}$. $H^{s,t}_0(\mathbf{R}^n_+)$ is the closure of $C^\infty_0(\mathbf{R}^n_+)$ in the topology of $H^{s,t}(\mathbf{R}^n)$.
It will be very convenient to use the following (nonstandard) notation:

$$H^{\sigma,\tau}_{\{0\}}(\mathbf{R}_+) = H^{\sigma,\tau}(\mathbf{R}_+) \quad \text{for } \sigma \geq 0, \qquad = H^{\sigma,\tau}_0(\mathbf{R}_+) \quad \text{for } \sigma < 0. \tag{1}$$

For $\tau = 0$ we shall simply omit the superscript τ.
(c) Let $H = H^+ \oplus H^-_0 \oplus H'$, where

$$H^+ = \{(\mathrm{e}^+ f)^\wedge : f \in \mathcal{S}(\mathbf{R}_+)\}, \qquad H^-_0 = \{(\mathrm{e}^- f)^\wedge : f \in \mathcal{S}(\mathbf{R}_-)\},$$

and H' denotes the space of all polynomials. For $d \in \mathbf{N}_0$ denote by H_d the subspace of H consisting of all functions $f(t)$ that are $\mathrm{O}(\langle t\rangle^{d-1})$.

There are several notions of the transmission property in the literature. Not all are equivalent. A detailed discussion was given in [33]. We will be using the following.

1.3 Definition. A symbol $p \in S^m_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ has the transmission property if for every $k \in \mathbf{N}_0$,

$$\partial^k_{x_n} p(x', x_n, \xi', \langle\xi'\rangle\, \xi_n)|_{x_n=0} \in S^m_{1,0}(\mathbf{R}^{n-1}_{x'} \times \mathbf{R}^{n-1}_{\xi'}) \hat{\otimes}_\pi H_{d,\xi_n},$$

where $d = \max\{\text{entier}(m) + 1, 0\}$, cf. [28].

1.4 Definition. Let $\mu \in \mathbf{R}$. The class $\tilde{\mathcal{B}}^{\mu,0}$ consists of all smooth functions g on $\mathbf{R}^{n-1}_{x'} \times \mathbf{R}^{n-1}_{\xi'} \times \mathbf{R}_{+x_n} \times \mathbf{R}_{+y_n}$ (symbol kernels) satisfying the estimates

$$\|x_n^k D^{k'}_{x_n} y_n^m D^{m'}_{y_n} D^\alpha_{\xi'} D^\beta_{x'} g(x', \xi', x_n, y_n)\|_{L^2(\mathbf{R}_{+x_n} \times \mathbf{R}_{+y_n})} = O(\langle\xi'\rangle^{\mu+1-k+k'-m+m'-|\alpha|}) \qquad (1)$$

for every fixed choice of $k, k', m, m', \alpha, \beta$, with constants independent of x'.
Such a symbol kernel g induces the singular Green operator $\operatorname{Op}_G g$ by

$$[\operatorname{Op}_G g(f)](x) = (2\pi)^{\frac{n-1}{2}} \iint_0^\infty e^{ix'\xi'} g(x', \xi', x_n, y_n)(\mathcal{F}_{x' \to \xi'} f)(\xi', y_n)\, dy_n\, d\xi', \qquad (2)$$

$f \in \mathcal{S}(\mathbf{R}^n_+)$; g is called the symbol kernel of $\operatorname{Op}_G g$.
$\operatorname{Op}_G g$ then is called a singular Green operator of order $\mu + 1$ (!) and type zero.

1.5 Definition. An operator of order and type zero in Boutet de Monvel's calculus on $\mathbf{R}^n_+$ is an operator of the form

$$A = \begin{bmatrix} P_+ + \mathrm{G} & \mathrm{K} \\ \mathrm{T} & \mathrm{S} \end{bmatrix} : \begin{matrix} C_0^\infty(\bar{\mathbf{R}}^n_+) \\ \oplus \\ C_0^\infty(\mathbf{R}^{n-1}) \end{matrix} \longrightarrow \begin{matrix} C^\infty(\bar{\mathbf{R}}^n_+) \\ \oplus \\ C^\infty(\mathbf{R}^{n-1}) \end{matrix},$$

where P is a pseudodifferential operator with the transmission property of order zero, $P_+ = \mathrm{r}^+ P \mathrm{e}^+$, G is a singular Green operator of order and type zero, K is a potential (or Poisson) operator, T a trace operator, and S is a pseudodifferential operator with a symbol in $S^0_{1,0}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1})$.

The most interesting part within this setting is the algebra

$$\mathcal{A} = \{A : A = P_+ + G\}$$

of the elements in the upper left corner, and we shall from now on focus on it. Details about Poisson and trace operators may be found in [6], [16], or [28].

1.6 The ad−notation. For multi-indices $\alpha, \beta \in \mathbf{N}^n_0$ and an operator T acting on functions or distributions on $\mathbf{R}^n$, let

$$\operatorname{ad}^\alpha(-ix)\operatorname{ad}^\beta(D_x)T = \operatorname{ad}^{\alpha_1}(-ix_1)\cdots\operatorname{ad}^{\alpha_n}(-ix_n)\operatorname{ad}^{\beta_1}(D_{x_1})\cdots\operatorname{ad}^{\beta_n}(D_{x_n})T.$$

Here, $\operatorname{ad}^0(-ix_j)T = T$, and $\operatorname{ad}^k(-ix_j)T = [-ix_j, \operatorname{ad}^{k-1}(-ix_j)T], k = 1, 2, \ldots$; the iterated commutators $\operatorname{ad}^{\beta_j}(D_{x_j})T$ are defined correspondingly. We are assuming for the moment that all compositions involved make sense.

Wedge Sobolev spaces were introduced by B.-W. Schulze, cf. [36], section 3.1.

1.7 Definition. Let E be a Banach space and suppose that $\{\kappa_\lambda : \lambda \in \mathbf{R}_+\}$ is a strongly continuous group of operators on E, i.e. $\lambda \mapsto \kappa_\lambda \in C(\mathbf{R}_+, \mathcal{L}_\sigma(E))$, and $\kappa_\lambda\kappa_\rho = \kappa_{\lambda\rho}$.
The *wedge Sobolev space* modelled on E, $\mathcal{W}^s(\mathbf{R}^q, E), s \in \mathbf{R}, q \in \mathbf{N}_0$, is defined as the completion of $\mathcal{S}(\mathbf{R}^q, E) = \mathcal{S}(\mathbf{R}^q)\hat{\otimes}_\pi E$ with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbf{R}^q,E)} = \left(\int \langle\eta\rangle^{2s} \|\kappa_{\langle\eta\rangle^{-1}}\mathcal{F}_{y\to\eta}u(\eta)\|_E^2 d\eta\right)^{\frac{1}{2}}.$$

Here, $\mathcal{F}_{y\to\eta}u$ denotes the Fourier transform of the E-valued function or distribution u,

$$\mathcal{F}_{y\to\eta}u(\eta) = (2\pi)^{-q/2}\int e^{-iy\eta}u(y)dy.$$

In general, the wedge Sobolev space will depend on the choice of the group action on E. Here, however, we will only deal with the usual weighted Sobolev spaces on $\mathbf{R}_+$, cf. 1.2(b), and we will always use the group defined by

$$(\kappa_\lambda f)(t) = \lambda^{\frac{1}{2}}f(\lambda t).$$

Let $\{E_k : k \in \mathbf{N}\}$ be a sequence of Banach spaces with $E_{k+1} \hookrightarrow E_k, E = \text{proj-lim } E_k$, and suppose that the group action coincides on all spaces. Then

$$\mathcal{W}^s(\mathbf{R}^q, E) = \text{proj-lim } \mathcal{W}^s(\mathbf{R}^q, E_k).$$

Vice versa, if $E_k \hookrightarrow E_{k+1}, E = \text{ind-lim } E_k$, and the group action is the same for all spaces, then

$$\mathcal{W}^s(\mathbf{R}^q, E) = \text{ind-lim } \mathcal{W}^s(\mathbf{R}^q, E_k).$$

We will use this last notation particularly in connection with the projective and inductive limits $\mathcal{S}(\mathbf{R}_+)$ and $\mathcal{S}'(\mathbf{R}_+)$.

1.8 Remark. The following identities are useful:

(a) $\mathcal{S}(\mathbf{R}^n_+) = \text{proj-lim}_{\,s,t\to\infty} H^{s,t}(\mathbf{R}^n_+)$.

(b) $\mathcal{S}'(\mathbf{R}^n_+) = \text{ind-lim}_{\,s,t\to\infty} H_0^{-s,-t}(\mathbf{R}^n_+)$.

(c) $\mathcal{W}^s(\mathbf{R}^q, H^s(\mathbf{R}_+)) = H^s(\mathbf{R}^{q+1}_+), s \geq 0$,

(d) $\mathcal{W}^s(\mathbf{R}^q, H_0^s(\mathbf{R}_+)) = H_0^s(\mathbf{R}^{q+1}_+), s \leq 0$.

(e) $\left(\mathcal{W}^s(\mathbf{R}^q, H^{\sigma,\tau}_{\{0\}}(\mathbf{R}_+))\right)' = \mathcal{W}^{-s}(\mathbf{R}^q, H^{-\sigma,-\tau}_{\{0\}}(\mathbf{R}_+))$

For (c) and (d), cf. [36], section 3.1.1, (17) and (18), for (e) [36], section 3.1.2, proposition 10. The duality is based on an extension of the usual $L^2(\mathbf{R}^n_+)$ duality.

The singular Green operators in Boutet de Monvel's calculus can be characterized in terms of the behavior of their iterated commutators. The following theorem was motivated by a result of B.–W. Schulze, [38] theorem 3.1.

1.9 Theorem. (Schrohe [32]) *Let $G : \mathcal{S}(\mathbf{R}^n_+) \longrightarrow \mathcal{S}'(\mathbf{R}^n_+)$ be a continuous linear operator. Then the following are equivalent:*
(i) *$G = \text{Op}_G g$ for some $g \in \tilde{\mathcal{B}}^{-1,0}$.*
(ii) *For all multi-indices $\alpha, \beta \in \mathbf{N}_0^{n-1}$, all $s \in \mathbf{R}$, the operator $\text{ad}^\alpha(-ix')\text{ad}^\beta(D_{x'})G$ has a continuous extension*

$$\text{ad}^\alpha(-ix')\text{ad}^\beta(D_{x'})G : \mathcal{W}^s(\mathbf{R}^{n-1}, \mathcal{S}'(\mathbf{R}_+)) \longrightarrow \mathcal{W}^{s+|\alpha|}(\mathbf{R}^{n-1}, \mathcal{S}(\mathbf{R}_+)). \qquad (1)$$

Also the pseudodifferential operators with the transmission property fit into the concept of wedge Sobolev spaces:

1.10 Theorem. (Schrohe [33]) *Let $P : \mathcal{S}(\mathbf{R}^n) \longrightarrow \mathcal{S}'(\mathbf{R}^n)$ be a continuous operator. Then the following assertions are equivalent.*

(i) $P = \operatorname{Op} p$ *for some* $p \in S^0_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ *with the transmission property of* 1.3.

(ii) P *has the following properties*

(α) *for all multi-indices* $\alpha, \beta \in \mathbf{N}^n$ *and all* $s \in \mathbf{R}, \mathrm{ad}^\alpha(-ix)\mathrm{ad}^\beta(D_x)P$ *has a bounded extension*

$$\mathrm{ad}^\alpha(-ix)\mathrm{ad}^\beta(D_x)P : H^s(\mathbf{R}^n) \longrightarrow H^{s+|\alpha|}(\mathbf{R}^n).$$

(β) *for all multi-indices* $\alpha', \beta' \in \mathbf{N}^{n-1}$, *all* $s, \sigma, \tau \in \mathbf{R}$, $\mathrm{ad}^{\alpha'}(-ix')\mathrm{ad}^{\beta'}(D_{x'})P_+$ *has a bounded extension*

$$\mathrm{ad}^{\alpha'}(-ix')\mathrm{ad}^{\beta'}(D_{x'})P_+ : \mathcal{W}^s(\mathbf{R}^{n-1}, H^{\sigma,\tau}_{\{0\}}(\mathbf{R}_+)) \longrightarrow \mathcal{W}^{s+|\alpha'|}(\mathbf{R}^{n-1}, H^{\sigma+|\alpha'|,\tau}_{\{0\}}(\mathbf{R}_+)).$$

(γ) *The properties in* (β) *also hold for the formal* L^2 *adjoint* $P_+{}^* = P^*{}_+$ *of* P.

2 Submultiplicativity of Boutet de Monvel's Algebra on the Half-Space

We will now show that the algebra of Green operators of order and type zero in Boutet de Monvel's calculus on the half-space $\mathbf{R}^n_+$ is submultiplicative. The proof of theorem 2.3, below, relies on a construction of submultiplicative norms for algebras of operators characterized by commutators and order shifts given by Gramsch, Ueberberg, and Wagner in [15]. They showed the submultiplicativity of the algebra of zero order pseudodifferential operators with symbols in $S^0_{\rho,\delta}(\mathbf{R}^n \times \mathbf{R}^n)$, $0 \le \delta \le \rho \le 1, \delta < 1$, cf. [15].

2.1 Definition. Let $\mathcal{A}$ be a Fréchet algebra. We shall say that $\mathcal{A}$ is *submultiplicative*, if there is a defining system $\{a_k : k = 1, 2, \ldots\}$ of semi-norms for the topology of $\mathcal{A}$ such that

$$a_k(AB) \le a_k(A)a_k(B) \tag{1}$$

for all $A, B \in \mathcal{A}, k = 1, 2, \ldots$.
$\mathcal{A}$ is called *essentially submultiplicative* if there is a defining system $\{\tilde{a}_k : k = 1, 2, \ldots\}$ of semi-norms and constants $C_k \ge 0$ with

$$\tilde{a}_k(AB) \le C_k \tilde{a}_k(A)\tilde{a}_k(B). \tag{2}$$

Occasionally, we shall also say that the corresponding semi-norm is submultiplicative or essentially submultiplicative.

2.2 Remark. (a) An essentially submultiplicative Fréchet algebra with unit is submultiplicative: Define

$$a_k(A) = \sup\{\tilde{a}_k(AC)/\tilde{a}_k(C) : C \in \mathcal{A}, \tilde{a}_k(C) \neq 0\}.$$

It is easily checked that a_k is equivalent to $\tilde{a}_k$ and submultiplicative.
(b) If $\mathcal{A}$ is a Fréchet algebra with unit e and we are given a countable defining system $\{a_k\}$ of submultiplicative semi-norms satisfying $a_k(e) = 1$ for all k, we may define $p_k(x) = \max\{a_1(x), \ldots, a_k(x)\}$, and will obtain an equivalent submultiplicative system with $p_k = 1$ for all k.

The rest of this section is devoted to the proof of the theorem, below.

2.3 Theorem. *The algebra*

$$\mathcal{A} = \left\{P_+ + G : P \in \mathrm{Op}\ S^0_{1,0,tr}(\mathbf{R}^n \times \mathbf{R}^n), G \in \mathrm{Op}_G\tilde{\mathcal{B}}^{-1,0}\right\}$$

of Green operators of order and type zero on $\mathbf{R}^n_+$ *is a submultiplicative Fréchet algebra with unit* $e = Id$.

2.4 Remark. Here we consider $\mathcal{A}$ as the topological subalgebra of $\mathcal{L}(L^2(\mathbf{R}^n_+))$ endowed with the topology induced from the respective symbol topologies on $S^0_{1,0,tr}(\mathbf{R}^n \times \mathbf{R}^n)$ and $\tilde{\mathcal{B}}^{-1,0}$, respectively, modulo the quotient of symbols inducing the same operators. For details cf. [30], [34].
The topology on $S^0_{1,0,tr}(\mathbf{R}^n \times \mathbf{R}^n)$ is given by the best constants in the estimates in definition 1.1(1) plus the semi-norms induced from the fact that the symbol has the transmission property, cf. 1.3.
The topology of $\tilde{\mathcal{B}}^{-1,0}$ is defined by taking the best constants in 1.4(1).

2.5 Definition. Write $\mathcal{P} = \{\mathrm{Op}\, p : p \in S^0_{1,0,tr}(\mathbf{R}^n \times \mathbf{R}^n)\}$, $\mathcal{P}_+ = \{P_+ : P \in \mathcal{P}\}$, $\mathcal{G} = \mathrm{Op}_G\tilde{\mathcal{B}}^{-1,0}$. In this notation,

$$\mathcal{A} = \mathcal{P}_+ + \mathcal{G} \subseteq \mathcal{L}(L^2(\mathbf{R}^n_+)).$$

An important step towards the proof of 2.3 is the lemma, below.

2.6 Lemma. *In order to prove theorem 2.3 it is enough to show the following:*
There are defining systems of semi-norms

$$p_1 \le p_2 \le \ldots \ \textit{for}\ \mathcal{P}$$

and

$$g_1 \le g_2 \le \ldots \ \textit{for}\ \mathcal{G}$$

with the following properties

(i) $(\mathcal{P}, \{p_k\})$ *is essentially submultiplicative.*

(ii) $(\mathcal{G}, \{g_k\})$ *is essentially submultiplicative.*

(iii) *There is an increasing function* $f : \mathbf{N} \to \mathbf{N}$ *and constants* C_k *such that*

$$
\begin{aligned}
g_k(P_+G) &\leq C_k\, p_{f(k)}(P)\, g_k(G), &(1)\\
g_k(GP_+) &\leq C_k\, p_{f(k)}(P)\, g_k(G), \text{ and} &(2)\\
g_k(L(P,Q)) &\leq C_k\, p_{f(k)}(P)\, p_{f(k)}(Q). &(3)
\end{aligned}
$$

Here, $L(P,Q)$ *denotes the singular Green left-over term* $L(P,Q) = P_+Q_+ - (PQ)_+$.

Proof. By replacing $\{p_k\}$ by $\{p_{f(k)}\}$ we may assume that $f(k) = k$. Without loss of generality let C_k also be the constants for the essential submultiplicativity of the semi-norms for $\mathcal{P}$ and $\mathcal{G}$.
On $\mathcal{A}$ define the semi-norm system $\{a_k : k = 1, 2, \ldots\}$ by

$$a_k(A) = \inf\{p_k(P) + g_k(G) : P \in \mathcal{P}, G \in \mathcal{G}, P_+ + G = A\}.$$

Now show

$$a_k(AB) \leq 2C_k\, a_k(A)\, a_k(B). \tag{4}$$

As we know from 2.2, this implies the submultiplicativity since $\mathcal{A}$ is unital.
Given $\epsilon > 0$, we may find $P, Q \in \mathcal{P}, G, H \in \mathcal{G}$ with $A = P_+ + G, B = Q_+ + H$, and

$$
\begin{aligned}
a_k(A) &\geq p_k(P) + g_k(G) - \epsilon,\\
a_k(B) &\geq p_k(Q) + g_k(H) - \epsilon.
\end{aligned}
$$

Then

$$
\begin{aligned}
a_k(AB) &\leq p_k(PQ) + g_k(L(P,Q)) + g_k(P_+H) + g_k(GQ_+) + g_k(GH)\\
&\leq 2C_k[p_k(P) + g_k(G)][p_k(Q) + g_k(H)]\\
&\leq 2C_k(a_k(A) + \epsilon)(a_k(B) + \epsilon)
\end{aligned}
$$

Since ϵ was arbitrary, (4) is established.

The following two lemmas are obvious.

2.7 Lemma. *Let* $\mathcal{F}$ *be an algebra,* p *a semi-norm on* $\mathcal{F}$. *Suppose* $\mathcal{F}$ *has an involution '*', i.e.* $(ab)^* = b^*a^*, (a+b)^* = a^* + b^*, (\lambda a)^* = \bar{\lambda} a^*, a^{**} = a$. *Then*

$$q(a) := p(a^*) \tag{1}$$

defines a semi-norm on $\mathcal{F}$. *If* p *is (essentially) submultiplicative, then so is* q. *If* $\{p_k\}$ *is an increasing system, then the system* $\{q_k\}$ *defined by* (1) *also is increasing.*

2.8 Lemma. *Let* $\{p_k\}, \{q_k\}$ *be two semi-norm systems on the algebra* $\mathcal{F}$. *Then* $\{r_k = \max\{p_k, q_k\}\}$ *is a semi-norm system which is increasing or (essentially) submultiplicative whenenver the others are.*

2.9 Proposition. *On* $\mathcal{P}$ *define three systems of semi-norms:*

$$
\begin{aligned}
p_k^{[1]}(P) &= \max_{\substack{|\alpha|+|\beta|\le k\\ -k\le s\le k-|\alpha|}} \|ad^\alpha(-ix)ad^\beta(D_x)P\|_{\mathcal{L}(H^s(\mathbf{R}^n),H^{s+|\alpha|}(\mathbf{R}^n))}\\
p_k^{[2]}(P) &= \max \|ad^\alpha(-ix')ad^\beta(D_{x'})P_+\|_{\mathcal{L}(\mathcal{W}^s(\mathbf{R}^{n-1},H^{\sigma,\tau}_{\{0\}}(\mathbf{R}_+)),\mathcal{W}^{s+|\alpha|}(\mathbf{R}^{n-1},H^{\sigma+|\alpha|,\tau}_{\{0\}}(\mathbf{R}_+)))},
\end{aligned}
$$

where the maximum is taken over all multi-indices $\alpha, \beta,$ *and integers* s, σ, τ *such that* $|\alpha| + |\beta| \le k, -k \le s, \sigma \le k - |\alpha|, -k \le \tau \le k$.

$$
p_k^{[3]}(P) = p_k^{[2]}(P^*).
$$

Then

(a) *Each of these systems is essentially submultiplicative.*

(b) *Together, these three semi-norm systems define the topology of* $\mathcal{P}$.

(c) *Letting*

$$
p_k(P) = \max\left\{p_k^{[1]}, p_k^{[2]}, p_k^{[3]}\right\}
$$

we obtain an increasing system of semi-norms which defines the topology and is essentially submultiplicative.

Proof. (a) For $p^{[1]}$, this is shown in [15], proposition 3.5. For $p^{[2]}$ the situation is analogous. For $p^{[3]}$, use 2.7.
(b) This is a consequence of the closed graph theorem: $\mathcal{P}$ is known to be a Fréchet space. By theorem 1.10, it coincides with the set of all operators in $\mathcal{L}(\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n))$ such that all these norms are finite. This space obviously is also Fréchet. Moreover, the symbol topology is stronger than the topology induced by these norms, so both topologies agree.
(c) now follows from 2.8.

2.10 Proposition. *For every* $k \in \mathbf{N}_0$ *define the norm* g_k *on* $\mathcal{G}$ *by*

$$
g_k(G) = \max_{\substack{|\alpha|+|\beta|\le k\\ -k\le s\le k-|\alpha|}} \|ad^\alpha(-ix')ad^\beta(D_{x'})G\|_{\mathcal{L}(\mathcal{W}^s(\mathbf{R}^{n-1},H^{-k,-k}_{\{0\}}(\mathbf{R}_+)),\mathcal{W}^{s+|\alpha|}(\mathbf{R}^{n-1},H^{k,k}_{\{0\}}(\mathbf{R}_+)))}.
$$

Then

(a) *The system* $\{g_k : k \in \mathbf{N}\}$ *is increasing and essentially submultiplicative.*

(b) *It is defining for the topology of* $\mathcal{G}$.

Proof. (a) is again the result of [15], theorem 3.5.
(b) Follows as in 2.9(ii) from the closed graph theorem, the fact that $\mathcal{G}$ is Fréchet and theorem 1.9.

2.11 Proposition. *Endow $\mathcal{P}$ and $\mathcal{G}$ with the systems of norms in* 2.9 *and* 2.10, *respectively. Then* $(\mathcal{P}, p_k), (\mathcal{G}, g_k)$ *have the properties* (i), (ii), *and* (iii) *required in lemma* 2.6.

Proof. Properties 2.6(i) and 2.6(ii) were already checked in 2.9 and 2.10, respectively. Let us check relations (1), (2), and (3) in 2.6(iii).
ad (1): For fixed k, multi-indices α, β with $|\alpha| + |\beta| \le k$, and $-k \le s \le k - |\alpha|$ we have to estimate

$$\|ad^{\alpha}(-ix')ad^{\beta}(D_{x'})(P_+G)\|_{\mathcal{L}(\mathcal{W}^s(\mathbf{R}^{n-1}, H^{-k,-k}_{\{0\}}(\mathbf{R}_+)), \mathcal{W}^{s+|\alpha|}(\mathbf{R}^{n-1}, H^{k,k}_{\{0\}}(\mathbf{R}_+)))}$$

in terms of $g_k(G)$ and $p_{f(k)}(P)$.
By Leibniz' rule

$$ad^{\alpha}(-ix')ad^{\beta}(D_{x'})(P_+G) = \sum_{\substack{\alpha_1+\alpha_2=\alpha \\ \beta_1+\beta_2=\beta}} c_{\alpha_1\alpha_2\beta_1\beta_2} ad^{\alpha_1}(-ix')ad^{\beta_1}(D_{x'})(P_+)ad^{\alpha_2}(-ix')ad^{\beta_2}(D_{x'})(G).$$

Now $|\alpha_1| + |\alpha_2| = |\alpha| \le k$. So for all s with $-k \le s \le k - |\alpha|$, we have

$$-k \le s \le k - |\alpha_2| \quad \text{and}$$
$$-k \le s + |\alpha_2| \le k - |\alpha_1|.$$

This implies that

$$\|ad^{\alpha_2}(-ix')ad^{\beta_2}(D_{x'})(G)\|_{\mathcal{L}(\mathcal{W}^s(\mathbf{R}^{n-1}, H^{-k,-k}_{\{0\}}(\mathbf{R}_+)), \mathcal{W}^{s+|\alpha_2|}(\mathbf{R}^{n-1}, H^{k,k}_{\{0\}}(\mathbf{R}_+)))} \le g_k(G)$$

and

$$\|ad^{\alpha_1}(-ix')ad^{\beta_1}(D_{x'})(P_+)\|_{\mathcal{L}(\mathcal{W}^{s+|\alpha_2|}(\mathbf{R}^{n-1}, H^{k,k}_{\{0\}}(\mathbf{R}_+)), \mathcal{W}^{s+|\alpha|}(\mathbf{R}^{n-1}, H^{k,k}_{\{0\}}(\mathbf{R}_+)))} \le p_k(P).$$

This shows the assertion.
ad (2): The proof is the same with the order reversed.
ad (3): This is a simple functional analytic argument. Let $\{\tilde{p}_k : k \in \mathbf{N}\}, \{\tilde{g}_k; k \in \mathbf{N}\}$ be increasing sets of norms for the symbol topology of $\mathcal{P}$ and $\mathcal{G}$, respectively. The systems $\{p_k\}$ and $\{\tilde{p}_k\}$ are equivalent by proposition 2.9(c), and so are $\{g_k\}$ and $\{\tilde{g}_k\}$ by proposition 2.10(c).
Moreover, we know that the mapping

$$L(\cdot,\cdot) : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{G}, \qquad (P,Q) \mapsto L(P,Q),$$

is continuous with respect to the symbol topology. Hence there are constants $\alpha_k, \beta_k, \gamma_k$ and increasing functions f_1, f_2, f_3 such that

$$\begin{aligned} g_k(L(P,Q)) &\le \alpha_k \tilde{g}_{f_1(k)}(L(P,Q)) \\ &\le \alpha_k\beta_k\ \tilde{p}_{f_2f_1(k)}(P)\ \tilde{p}_{f_2f_1(k)}(Q) \\ &\le \alpha_k\beta_k\gamma_k\ p_{f_3f_2f_1(k)}(P)\ p_{f_3f_2f_1(k)}(Q). \end{aligned}$$

For $f = f_3f_2f_1$ we obtain the assertion.

3 Boutet de Monvel's Algebra on a Compact Manifold

Let X be a compact $n-$dimensional manifold with boundary Y, embedded in its 'double' Ω. Let $\mathcal{A}$ denote the algebra of operators of order and type zero in Boutet de Monvel's calculus for the manifold X - a concise definition will be given below. Again, we shall concentrate on the "upper left corner" and scalar-valued operators.
We consider $\mathcal{A}$ as a subalgebra of $\mathcal{L}(L^2(X))$, and we shall prove the following theorem.

3.1 Theorem. *$\mathcal{A}$ is a submultiplicative Fréchet algebra.*

3.2 The set-up. In order to describe the algebra, first identify a neighborhood of Y in Ω with $Y \times \left(-\frac{3}{2}, \frac{3}{2}\right)$. Then cover Ω by finitely many coordinate charts Ω_j such that either Ω_j does not intersect $Y \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ ("interior charts") or $\Omega_j = \Omega_j' \times [-1, 1]$ with an open subset Ω_j' of Y ("boundary charts"). A cover $\{X_j\}$ of X is then obtained by letting $X_j = \Omega_j \cap X$.

Now choose a partition of unity $\{\varphi_j\}$ subordinate to the cover Ω_j, and choose cut-off functions $\{\psi_j\}$ supported in Ω_j with $\varphi_j \psi_j = \varphi_j$.

3.3 Definition. A linear operator $A : C^\infty(\overline{X}) \to \mathcal{D}'(X)$ is an operator of order and type zero in Boutet de Monvel's calculus for the manifold X, if it has the following properties. Writing

$$A = \sum \varphi_j A \psi_j + R, \tag{1}$$

(i) R is a regularizing operator, i.e. it has a continuous extension

$$R : H_0^{-s}(X) \to H^t(X)$$

for all $s, t \in \mathbf{R}_+$. Equivalently, $R : \mathcal{D}'(X) \to C^\infty(\overline{X})$ is continuous, or R is an integral operator with a smooth kernel density.

(ii) If Ω_j is a boundary chart, then the operator $(\varphi_j A \psi_j)_*$ - defined on $\mathbf{R}^n_+$ from $\varphi_j A \psi_j$ via the transport by coordinate charts – is an operator of order and type zero in Boutet de Monvel's calculus on $\mathbf{R}^n_+$.

(iii) If Ω_j is an interior chart, then $(\varphi_j A \psi_j)_*$ is a pseudo-differential operator of order zero.

Similarly, $A : C^\infty(\Omega) \to \mathcal{D}'(\Omega)$ is a pseudodifferential operator (...with the transmission property) of order zero on Ω, if the terms in (ii) and (iii) are pseudodifferential operators (...with the transmission property) of order zero on $\mathbf{R}^n$.
$A : C^\infty(\overline{X}) \to \mathcal{D}'(X)$ is a singular Green operator of order and type zero in Boutet de Monvel's calculus on X if the terms in (ii) are singular Green operators of order and type

zero on $\mathbf{R}^n_+$ and the terms in (iii) are regularizing operators on $\mathbf{R}^n_+$. In particular, an operator A of order and type zero in Boutet de Monvel's calculus on X can be written

$$A = P_+ + G,$$

where $P_+ = \mathrm{r}^+ P \mathrm{e}^+$; e^+ denotes extension by zero from X to Ω, r^+ restriction to X, P is a pseudodifferential operator with the transmission property on Ω, and G is a singular Green operator of order and type zero on X.
The composition and mapping results for Boutet de Monvel's calculus on $\mathbf{R}^n_+$ imply that the algebra $\mathcal{A}$ of elements of order and type zero on X is indeed an algebra and a subalgebra of $\mathcal{L}(L^2(X))$. Moreover, it can be given the Fréchet topology induced from the algebra on the half-space.

3.4 Proposition. *For $s, \sigma \in \mathbf{R}$, the wedge Sobolev space $\mathcal{W}^s(Y, H^\sigma_{\{0\}}(\mathbf{R}_+))$ is well-defined in a neighborhood of Y in X.*

More precisely, suppose $u \in \mathcal{D}'(X)$ is a distribution supported in the open set identified with $Y \times (0,1)$. Then we may multiply by the functions in the partition of unity, φ_j, and transfer the product to $\mathbf{R}^n_+$ using the coordinate charts. Denoting this by $(\varphi_j u)_*$, let

$$\|u\|^2_{\mathcal{W}^s(Y, H^\sigma_{\{0\}}(\mathbf{R}_+))} = \left(\sum \left\|(\varphi_j u)_*\right\|^2_{\mathcal{W}^s(\mathbf{R}^{n-1}, H^\sigma_{\{0\}}(\mathbf{R}_+))}\right)^{1/2},$$

where the summation is over all boundary charts. Up to equivalence, the norm is independent of the choice of the partition of the unity in view of the lemma, below.

3.5 Lemma. *Let $u \in \mathcal{W}^s(\mathbf{R}^{n-1}, H^\sigma_{\{0\}}(\mathbf{R}_+))$, and let $\kappa : \mathbf{R}^n_+ \to \mathbf{R}^n_+$ be a diffeomorphism. Suppose κ and κ^{-1} have bounded derivatives of all order. Then the operator $T_\kappa : u \mapsto u \circ \kappa$, first defined as a mapping from $\mathcal{S}(\mathbf{R}^n_+)$ to $\mathcal{S}(\mathbf{R}^n_+)$, extends to a bounded map*

$$T_\kappa : \mathcal{W}^s(\mathbf{R}^{n-1}, H^\sigma_{\{0\}}(\mathbf{R}_+)) \to \mathcal{W}^s(\mathbf{R}^{n-1}, H^\sigma_{\{0\}}(\mathbf{R}_+))$$

Proof. The adjoint of T_κ with respect to the usual $L^2(\mathbf{R}^n_+)$ inner product is given as $|\det D\kappa^{-1}|\, T_{\kappa^{-1}}$, which is essentially of the same kind. In view of the duality

$$\mathcal{W}^s(\mathbf{R}^{n-1}, H^\sigma_{\{0\}}(\mathbf{R}_+))' = \mathcal{W}^{-s}(\mathbf{R}^{n-1}, H^{-\sigma}_{\{0\}}(\mathbf{R}_+))$$

and Hirschmann's interpolation results [20], theorem 6.4, it is sufficient to show that T_κ is bounded on $\mathcal{W}^s(\mathbf{R}^{n-1}, H^s(\mathbf{R}_+))$ and on $\mathcal{W}^0(\mathbf{R}^{n-1}, H^s(\mathbf{R}_+))$, $s \in \mathbf{N}_0$.
Now $\mathcal{W}^s(\mathbf{R}^{n-1}, H^s(\mathbf{R}_+)) = H^s(\mathbf{R}^n_+)$, and in this case the result is well-known. On the other hand, $u \in \mathcal{W}^0(\mathbf{R}^{n-1}, H^s(\mathbf{R}_+))$ if and only if $\partial^j_{x_n} u \in H^{-j}(\mathbf{R}^{n-1}, L^2(\mathbf{R}_+))$ for $j = 0, \ldots, s$; the corresponding topologies coincide, cf. [33]. It follows from the differentiation rules that $\partial^j_{x_n}(u \circ \kappa)$ belongs to $H^{-j}(\mathbf{R}^{n-1}, L^2(\mathbf{R}_+))$, $j = 0, \ldots, s$, provided the same is true for $\partial^j_{x_n} u$.

3.6 Definition. We shall say that an operator $T : C^\infty(\overline{X}) \to \mathcal{D}'(X)$ is supported by a set K, if

$$T = \varphi T = T \varphi$$

for every smooth function φ on $\overline{X}$ with $\varphi \equiv 1$ on K.

3.7 Lemma. *Fix functions* $\omega, \omega', \omega'' \in C^\infty(X)$ *such that*

$$\omega \equiv 1 \text{ on } Y \times \left(0, \frac{3}{4}\right], \quad \operatorname{supp} \omega \subseteq Y \times (0,1),$$
$$\omega' \equiv 1 \text{ on } Y \times \left(0, \frac{2}{3}\right], \quad \operatorname{supp} \omega' \subseteq Y \times \left(0, \frac{3}{4}\right),$$
$$\omega'' \equiv 1 \text{ on } Y \times \left(0, \frac{1}{3}\right], \quad \operatorname{supp} \omega'' \subseteq Y \times \left(0, \frac{1}{2}\right),$$

so that in particular $\omega\omega' = \omega'$. *Then*

(a) *Each* $A \in \mathcal{A}$ *can be written*

$$A = P_+ + G + Q + R,$$

where

- $P_+ = r^+Pe^+$, *and* P *is a pseudodifferential operator of order zero with the transmission property on* Ω, *supported by* $Y \times [-1,1]$,
- G *is a singular Green operator of order and type zero on* X, *supported by* $Y \times [0,1]$,
- Q *is a zero order pseudodifferential operator on* Ω, *supported by* $X \setminus \left\{Y \times (0, \frac{1}{2})\right\}$, *and*
- R *is a regularizing operator.*

(b) *We may express the product in* $\mathcal{A}$ *correspondingly. Write, as in* (a) *for* $A' \in \mathcal{A}$

$$A' = P'_+ + G' + Q' + R'.$$

Then

$$\begin{aligned} AA' &= [PP']_+ + [(PQ')_b]_+ + [(QP')_b]_+ \\ &\quad + L(P,P') + P_+G' + GP'_+ + GG' \\ &\quad + (PQ')_i + (QP')_i + QQ' \qquad (1) \\ &\quad + (PQ')_{r_+} + (QP')_{r_+} + GQ' + QG' + P_+R' + RP'_+ \\ &\quad + QR' + RQ' + GR' + RG' + RR' \\ &= P''_+ + G'' + Q'' + R'', \end{aligned}$$

where P''_+, G'', Q'' *and* R'' *are the operators in lines* 1, 2, 3 *and* $4f$ *of* (1). *We have used the abbreviations*

$$\begin{aligned} (PQ')_b &= (1-\omega'')PQ'\omega \\ (PQ')_i &= (1-\omega')PQ'(1-\omega) \\ (PQ')_r &= \omega'PQ'(1-\omega) + \omega''PQ'\omega \\ (QP')_b &= \omega QP'(1-\omega'') \\ (QP')_i &= (1-\omega)QP'(1-\omega') \\ (QP')_r &= (1-\omega)QP'\omega' + \omega QP'\omega''. \end{aligned}$$

Proof. (a) is obvious, cf. the decomposition in 3.3(1).
(b) The decomposition (1) is straightforward. Let us check that all terms have the asserted form. In view of the fact that the symbol of Q' vanishes near Y, extension by zero is a trivial operation on its range. So $PQ'\omega$ is a pseudodifferential operator with the transmission property of order zero and supported by $Y \times [-1,1]$. $(1-\omega')PQ'(1-\omega)$ is a pseudodifferential operator of order zero supported by $X\backslash\{Y \times (0,1/2)\}$. The operator $\omega' PQ'(1-\omega)$ is regularizing, since $\omega'(1-\omega) = 0$. Similarly, $\omega'' PQ'$ is regularizing, since $Q' = Q'\varphi$ for a function $\varphi, \varphi \equiv 1$ on $X\backslash\{Y \times (0,1/2)\}$ and vanishing on $\operatorname{supp}\omega''$. The analysis of QP' is almost the same. Finally note that $GQ' = G(1-\omega'')Q'$ and $QG' = Q(1-\omega'')G'$ are regularizing: the calculus implies that $G(1-\omega'')$ and $(1-\omega'')G$ are both regularizing singular Green operators of type zero and therefore regularizing.

3.8 Definition. Let $\mathcal{P}$ be the set of all pseudodifferential operators with the transmission property of order zero on Ω, supported by $Y \times [-1,1]$;
$\mathcal{P}_+ = \{r^+ P e^+ : P \in \mathcal{P}\}$;
$\mathcal{G}$ is the set of all singular Green operators of order and type zero on X, supported by $Y \times (0,1]$;
$\mathcal{Q}$ is the set of all pseudodifferential operators of order zero on Ω, supported by $X\backslash\{Y \times (0,\frac{1}{2})\}$;
$\mathcal{R}$ is the set of all regularizing operators on X.

In this notation
$$\mathcal{A} = \mathcal{P}_+ + \mathcal{G} + \mathcal{Q} + \mathcal{R},$$
also topologically, as a (non-direct) sum of Fréchet spaces:
If $\{p_k : k \in \mathbb{N}\}$, $\{g_k : k \in \mathbb{N}\}$, $\{q_k : k \in \mathbb{N}\}$, and $\{r_k : k \in \mathbb{N}\}$ are defining systems of semi-norms for the topologies of $\mathcal{P}, \mathcal{G}, \mathcal{Q}$ and $\mathcal{R}$, respectively, then a semi-norm system for the topology of $\mathcal{A}$ is given by
$$\begin{aligned} a_k(A) &= \inf\{p_k(P) + g_k(G) + q_k(Q) + r_k(R) : \\ &\qquad P \in \mathcal{P}, G \in \mathcal{G}, Q \in \mathcal{Q}, R \in \mathcal{R}, A = P_+ + G + Q + R\} \end{aligned}$$
Moreover, multiplication in $\mathcal{A}$ – understood according to 3.7(1) – respects this decomposition.

3.9 Proposition. *In order to establish the submultiplicativity of the algebra $\mathcal{A}$, it is sufficient to show the following:*
There are increasing semi-norm systems $\{p_k : k \in \mathbb{N}\}$ for the topology of $\mathcal{P}$, $\{g_k : k \in \mathbb{N}\}$ for that of $\mathcal{G}$, $\{q_k : k \in \mathbb{N}\}$ for that of $\mathcal{Q}$, and $\{r_k : k \in \mathbb{N}\}$ for that of $\mathcal{R}$, there are constants $C_k \geq 0$, $k \in \mathbb{N}$, and there is an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that

(i)	$p_k(PP') \leq C_k\, p_k(P)p_k(P')$	(ii)	$p_k((PQ')_b) \leq C_k\, p_k(P)q_k(Q')$
(iii)	$p_k((QP')_b) \leq C_k\, q_k(Q)p_k(P')$	(iv)	$g_k(L(P,P')) \leq C_k\, p_{f(k)}(P)p_{f(k)}(P')$
(v)	$g_k(P_+G') \leq C_k\, p_{f(k)}(P)g_k(G')$	(vi)	$g_k(GP'_+) \leq C_k\, p_{f(k)}(P')g_k(G)$
(vii)	$g_k(GG') \leq C_k\, g_k(G)g_k(G')$	(viii)	$q_k((PQ')_i) \leq C_k\, p_k(P)q_k(Q')$
(ix)	$q_k((QP')_i) \leq C_k\, p_k(P')q_k(Q)$	(x)	$q_k(QQ') \leq C_k q_k(Q)q_k(Q')$
(xi)	$r_k((PQ')_r) \leq C_k\, p_{f(k)}(P)q_{f(k)}(Q')$	(xii)	$r_k((QP')_r) \leq C_k\, p_{f(k)}(P')q_{f(k)}(Q)$
(xiii)	$r_k(GQ') \leq C_k\, g_{f(k)}(G)q_{f(k)}(Q')$	(xiv)	$r_k(QG') \leq C_k\, g_{f(k)}(G')q_{f(k)}(Q)$

(xv)	$r_k(P_+R') \leq C_k\, p_{f(k)}(P) r_k(R')$	(xvi)	$r_k\left(RP'_+\right) \leq C_k\, p_{f(k)}(P') r_k(R)$
(xvii)	$r_k(QR') \leq C_k\, q_{f(k)}(Q) r_k(R')$	(xviii)	$r_k(RQ') \leq C_k\, q_{f(k)}(Q') r_k(R)$
(xix)	$r_k(GR') \leq C_k\, g_{f(k)}(G) r_k(R')$	(xx)	$r_k(RG') \leq C_k\, g_{f(k)}(G') r_k(R)$
(xxi)	$r_k(RR') \leq C_k\, r_k(R) r_k(R')$.		

Proof. By playing with the semi-norm system and operations like replacing the systems $\{\cdot_k : k \in \mathbf{N}\}$ by $\left\{\cdot_{f(k)} : k \in \mathbf{N}\right\}$ one finally obtains (i) through (xxi) with $f(k) = k$. Given $\varepsilon > 0$, and A, $A' \in \mathcal{A}$, we can then find P, P', G, G', Q, Q', R, R' such that

$$A = P_+ + G + Q + R, \quad A' = P'_+ + G' + Q' + R',$$

and

$$\begin{aligned} a_k(A) &\geq p_k(P) + g_k(G) + q_k(Q) + r_k(R) - \varepsilon, \\ a_k(A') &\geq p_k(P') + g_k(G') + q_k(Q') + r_k(R') - \varepsilon. \end{aligned}$$

Then

$$\begin{aligned} a_k(AA') \leq\; & p_k(PP') + p_k\left((PQ')_b\right) + p_k\left((QP')_b\right) + \\ & + g_k\left(L(P,P')_b\right) + g_k\left(P_+G'\right) + g_k\left(GP'_+\right) + g_k\left(GG'\right) \\ & + q_k\left((PQ')_i\right) + g_k\left((QP')_i\right) + q_k\left(QQ'\right) \\ & + r_k\left((PQ')_r\right) + r_k\left((QP')_r\right) + r_k\left(GQ'\right) \\ & + r_k(QG') + r_k(P_+R') + r_k(RP'_+) \\ & + r_k(QR') + r_k(RQ') + r_k(GR') \\ & + r_k(RG') + r_k(RR') \\ \leq\; & 3\,C_k\left(p_k(P) + g_k(G) + q_k(Q) + r_k(R)\right) \\ & \quad \left(p_k(P') + g_k(G') + q_k(Q') + r_k(R')\right) \\ \leq\; & 3\,C_k\left(a_k(A) + \varepsilon\right)\left(a_k(A') + \varepsilon\right) \end{aligned}$$

which gives the assertion, since $\mathcal{A}$ is unital.

The following theorem is proven in Coifman & Meyer [7], Théorème 15:

3.10 Theorem. *Let $Q : C^\infty(\Omega) \to \mathcal{D}'(\Omega)$ be a continuous operator. Then the following is equivalent*

(i) *Q is a pseudodifferential operator of order zero.*

(ii) *Given $s \in \mathbf{R}$ and k smooth vector fields $v_1, \ldots, v_k$ on Ω, the iterated commutator $\operatorname{ad} v_1 \ldots \operatorname{ad} v_k(Q)$ has a bounded extension*

$$\operatorname{ad} v_1 \ldots \operatorname{ad} v_k(Q) : H^s(\Omega) \to H^s(\Omega). \tag{1}$$

It is sufficient to check the conditions in (ii) for finitely many vector fields supported in finitely many coordinate patches. We no longer need the conditions on the commutators with multipliers; they are implicitely contained in those with vector fields.

Together with 1.10, 3.10 gives the following characterization

3.11 Theorem. *Let* $P : C^\infty(\Omega) \to \mathcal{D}'(\Omega)$ *be a continuous operator supported by* $Y \times [-1,1]$. *Then the following are equivalent*

(i) *P is a pseudodifferential operator of order zero with the transmission property*

(ii) *P has the following properties*

(ii.a) *Given* $s \in \mathbf{R}$ *and k smooth vector fields* $v_1, \ldots, v_k$ *on* Ω, *the iterated commutator* $\operatorname{ad} v_1 \ldots \operatorname{ad} v_k(P)$ *has a bounded extension*

$$\operatorname{ad} v_1 \ldots \operatorname{ad} v_k(P) : H^s(\Omega) \to H^s(\Omega). \tag{1}$$

(ii.b) *Given* $s, \sigma \in \mathbf{R}$ *and a* $k+m-$*tuple* $(V_1, \ldots, V_{k+m})$ *consisting of k smooth vector fields and m smooth functions on Y, all extended to* $Y \times \left(-\frac{3}{2}, \frac{3}{2}\right)$, *the iterated commutator* $\operatorname{ad} V_1 \ldots \operatorname{ad} V_{k+m}(P_+)$ *has a bounded extension*

$$\operatorname{ad} V_1 \ldots \operatorname{ad} V_{k+m}(P_+) : \mathcal{W}^s(Y, H^\sigma_{\{0\}}(\mathbf{R}_+)) \to \mathcal{W}^{s+m}(Y, H^{\sigma+m}_{\{0\}}(\mathbf{R}_+)). \tag{2}$$

(ii.c) *The conditions in* (ii.b) *also hold for the formal adjoint* ${P_+}^* = {P^*}_+$ *of* P_+.

The extension is simply obtained by choosing $\varphi \in C_0^\infty([0, 1/3))$ with $\varphi \equiv 1$ near zero and letting $V_j^{ext}(y, s) = \varphi(s) V_j(y)$.
Similarly, we may rewrite theorem 1.9 in the following way.

3.12 Theorem. *Let* $G : C^\infty(\overline{X}) \to \mathcal{D}'(X)$ *be a continuous operator supported by* $Y \times [-1,1]$. *Then the following are equivalent*

(i) *G is a singular Green operator of order and type zero.*

(ii) *Given* $s \in \mathbf{R}$ *and a* $k+m-$*tuple* $(V_1, \ldots, V_{k+m})$ *consisting of k smooth vector fields and m smooth functions on Y, all extended to* $Y \times \left(-\frac{3}{2}, \frac{3}{2}\right)$, *the iterated commutator* $\operatorname{ad} V_1 \ldots \operatorname{ad} V_{k+m}(G)$ *has a bounded extension*

$$\operatorname{ad} V_1 \ldots \operatorname{ad} V_{k+m}(G) : \mathcal{W}^s(Y, \mathcal{S}'(\mathbf{R}_+)) \to \mathcal{W}^{s+m}(Y, \mathcal{S}(\mathbf{R}_+)). \tag{1}$$

3.13 Remark. Condition 3.11(ii.a) guarantees that P is a pseudodifferential operator, while (ii.b) and (ii.c) imply the transmission property, cf. 3.10. A regularizing operator always has the transmission property, and so does an operator whose symbol vanishes near the boundary.
Given two smooth functions, φ, ψ, supported in an arbitrarily small neighborhood of the boundary and equal to one near the boundary, the operator P will have the transmission property iff $\varphi P \psi$ has it.

3.14 Remark. We already know from definition 3.3 that an operator R is regularizing on X if and only if, for every choice of $s, t \in \mathbf{R}$ it has a bounded extension

$$R : H^s_0(X) \to H^t(X).$$

3.15 Corollary. Theorems 3.11, 3.12, 3.10 and remark 3.14 allow us to introduce particular defining semi-norms for the topology of $\mathcal{P}$, $\mathcal{G}$, $\mathcal{Q}$, and $\mathcal{R}$.
For all the commutators we may restrict ourselves to finite sets of functions and vector fields and to $s, \sigma \in \mathbf{Z}$. Choose one such set for 3.11(ii.a) and 3.10(ii), and one for 3.11(ii.b, c) and 3.12(ii). Denote the finitely many vector fields for 3.11(ii.a) and 3.10(ii) by $v_1, \dots, v_N$, respectively, the functions and vector fields for 3.11(ii.b), (ii.c) and 3.12(ii) by $f_1, \dots, f_N$ and $v'_1, \dots, v'_N$. Let

$$v = (v_1, \dots, v_N), \quad v' = (v'_1, \dots, v'_N), \quad f = (f_1, \dots, f_N)$$

Then fix $\phi, \psi, \theta \in C_0^\infty(Y \times (-\frac{1}{4}, \frac{1}{4}))$ such that ϕ, ψ and θ are identically 1 in a (small) neighborhood of the boundary Y and $\phi\psi = \phi, \phi\theta = \theta$. Define, similarly as in proposition 2.9,

$$\begin{aligned} p_k^{[1]}(P) &= \max_{\substack{|\beta| \le k \\ -k \le s \le k}} \left\{ \left\| \mathrm{ad}^\beta(v)(P) \right\|_{\mathcal{L}(H^s(\Omega))}, \left\| \mathrm{ad}^\beta(v)(P^*) \right\|_{\mathcal{L}(H^s(\Omega))} \right\}, \\ p_k^{[2]}(P) &= \max_{\substack{|\alpha|+|\beta| \le k \\ -k \le s, \sigma \le k - |\alpha|}} \left\| \mathrm{ad}^\alpha(f) \mathrm{ad}^\beta(v')(P_+) \right\|_{\mathcal{L}(\mathcal{W}^s(Y, H^\sigma_{\{0\}}(\mathbf{R}_+)), \mathcal{W}^{s+|\alpha|}(Y, H^{\sigma+|\alpha|}_{\{0\}}(\mathbf{R}_+)))}, \\ p_k^{[3]}(P) &= p_k^{[2]}(P^*). \end{aligned}$$

Since the commutators will in general not commute, $\mathrm{ad}^\alpha(\cdot)\mathrm{ad}^\beta(\cdot)$ stands for an application in any order; the maximum is taken over all permutations.
From $p_k^{[1]}, p_k^{[2]}$, and $p_k^{[3]}$ we shall now construct a submultiplicative norm. For arbitrary $P_1, P_2 \in \mathcal{P}$ we have

$$\begin{aligned} p_k^{[2]}(\phi P_1 P_2 \psi) &\le p_k^{[2]}(\phi P_1 \psi \phi P_2 \psi) + p_k^{[2]}(\phi P_1 (1-\theta)(1-\phi) P_2 \psi) \\ &\le c_k\, p_k^{[2]}(\phi P_1 \psi)\, p_k^{[2]}(\phi P_2 \psi) + c_k\, p_k^{[2]}(\phi P_1 (1-\theta))\, p_k^{[2]}((1-\phi) P_2 \psi). \end{aligned}$$

For fixed ϕ, ψ, and θ we may consider the Fréchet space $\mathcal{F}$ of all operators F with symbol in $S^0_{1,0}$, such that $\varphi_1 F \varphi_2 = 0$ for all smooth functions φ_1, φ_2 supported in a fixed neighborhood of the boundary Y. On $\mathcal{F}$, the system $\{p_\nu^{[1]} : \nu \in \mathbf{N}\}$ of norms defined on the 'double' Ω of X induces the same complete topology as the system $\{p_\nu^{[1]}, p_\nu^{[2]} : \nu \in \mathbf{N}\}$, since the operators in $\mathcal{F}$ obviously have the transmission property.
Therefore, $p_k^{[2]}(\phi P_1(1-\theta))$ can be estimated by $d_k\, p_{\mu(k)}^{[1]}(P_1)$ for a suitable constant d_k and $\mu(k) \in \mathbf{N}$. The same argument applies to $p_k^{[2]}((1-\phi)P_2\psi)$ and the terms induced by considering $p_k^{[3]}(\phi P_1 P_2 \psi)$; we may use the same $\mu(k)$ in all cases. Letting

$$p_k(P) = \max\left\{ p_k^{[2]}(\phi P \psi), p_k^{[3]}(\phi P^* \psi), p_{\mu(k)}^{[1]}(P) \right\} \tag{1}$$

we get $p_k(P_1 P_2) \le C_k p_k(P_1) p_k(P_2)$.
For $\mathcal{G}$ use

$$g_k(G) = \max_{\substack{|\alpha|+|\beta| \le k \\ -k \le s \le k - |\alpha|}} \left\| \mathrm{ad}^\alpha(f) \mathrm{ad}^\beta(v')(G) \right\|_{\mathcal{L}(\mathcal{W}^s(Y, H^{-k}_{\{0\}}(\mathbf{R}_+)), \mathcal{W}^{s+|\alpha|}(Y, H^k_{\{0\}}(\mathbf{R}_+)))}.$$

For $\mathcal{Q}$ use the norms

$$q_k(Q) = p_{\mu(k)}^{[1]}(Q)$$

with the function $\mu(k)$ in (1); for $\mathcal{R}$ use

$$r_k(R) = \|R\|_{\mathcal{L}(H_0^{-k}(X), H^k(X))}.$$

3.16 Conclusion. By construction, cf. proposition 2.9 and 2.10, the above semi-norms will have properties (i), (vii), (x), and (xxi) of proposition 3.9. We obtain relations (iv), (xi), (xii), (xiii) and (xiv) by the final argument in proposition 2.11.
Now for relation (ii). The operator $(PQ')_b$ is supported by $Y \times [\frac{1}{3}, 1)$, and so is its adjoint. Since the functions ϕ and ψ have their support in $Y \times [-\frac{1}{4}, \frac{1}{4}]$, all semi-norms for $p_k^{[2]}((PQ')_b)$ and $p_k^{[3]}((PQ')_b)$ will vanish. Relation (iii) is obtained in the same way. The compatibility of the norms q_k and p_k together with Leibniz' rule shows that relations (viii) and (ix) hold; Leibniz' rule moreover yields (v) and (vi).
For (xv) through (xx) use the identities in 1.8 (c), (d). By proposition 3.9, this concludes the proof of theorem 3.1.

4 Some Remarks on the Case of the Full Algebra

We have now considered the case where the algebra consists of the terms that usually form the upper left corner in the matrices of Boutet de Monvels calculus. In general, we will have to deal with matrices

$$A = \begin{bmatrix} P_+ + G & K \\ T & S \end{bmatrix} \tag{2}$$

acting on the Hilbert space $H = L^2(X, E) \oplus H^{-\frac{1}{2}}(Y, F)$. As before, X is a manifold with boundary $\partial X = Y$; X is either compact or $\mathbf{R}^n_+$. E and F are smooth vector bundles.
T here is a trace operator of order and type zero, K is a potential operator of order -1 and S is a pseudodifferential operator with symbol in $S^0_{1,0}$.

There are now two ways to proceed. The first and rather straightforward way is to argue by analogy. One starts with the case $X = \mathbf{R}^n_+$ and characterizations of the potential and trace operators by commutators:

4.1 Theorem. *Let $K : \mathcal{S}(\mathbf{R}^{n-1}) \to \mathcal{S}'(\mathbf{R}^n_+)$ and $T : \mathcal{S}(\mathbf{R}^n_+) \to \mathcal{S}'(\mathbf{R}^{n-1})$ be continuous operators. Then K is a potential operator of order -1 and T is a trace operator of order and type zero, if and only if for all multi-indices $\alpha, \beta \in \mathbf{N}^{n-1}$ and all $s \in \mathbf{Z}$ there are continuous extensions*

$$\mathrm{ad}^\alpha(-ix')\mathrm{ad}^\beta(D_{x'})K : H^{s-\frac{1}{2}}(\mathbf{R}^{n-1}) \to \mathcal{W}^s(\mathbf{R}^{n-1}, \mathcal{S}(\mathbf{R}_+))$$

and

$$\mathrm{ad}^\alpha(-ix')\mathrm{ad}^\beta(D_{x'})T : \mathcal{W}^s(\mathbf{R}^{n-1}, \mathcal{S}'(\mathbf{R}_+)) \to H^{s-\frac{1}{2}}(\mathbf{R}^{n-1})$$

This is proven in the same way as theorem 1.9. One then employs a similar construction as before, relying on the decomposition of the algebra and the construction in [15].

The second and more elegant way uses the following theorem.

4.2 Theorem. *There is an operator Λ of order and type zero in Boutet de Monvel's calculus such that* (i) $\Lambda : L^2(X, E) \to L^2(X, E) \oplus H^{-\frac{1}{2}}(Y, F)$ *is an isomorphism.*
(ii) $\Lambda^{-1} : L^2(X, E) \oplus H^{-\frac{1}{2}}(Y, F) \to L^2(X, E)$ *also is an operator of order and type zero in Boutet de Monvel's calculus.*

For a proof cf. Rempel&Schulze [28], [29], Grubb [17].
Suppose now we are given a submultiplicative system of semi-norms $\{a_k : k \in \mathbf{N}\}$ for the topology of the 'upper left corner' algebra. We then define

$$b_k(A) = a_k(\Lambda^{-1} A \Lambda), \quad k \in \mathbf{N}.$$

Clearly, $\{b_k : k \in \mathbf{N}\}$ is a submultiplicative system of semi-norms on the full algebra $\mathcal{A}$. It also is defining for the topology of $\mathcal{A}$:
Suppose $\{p_\ell : \ell \in \mathbf{N}\}$ is any defining system of semi-norms in $\mathcal{A}$. Then the continuity of Λ implies that $b_k(A) \le C_k p_\ell(A)$ for a suitable semi-norm $p_\ell, \ell = \ell(k), C_k \ge 0$. Vice versa, $p_\ell(B) = p_\ell(\Lambda(\Lambda^{-1}B\Lambda)\Lambda^{-1}) \le C_\ell a_k(\Lambda^{-1}B\Lambda) = C_\ell b_k(B)$, where $k = k(\ell)$.
Therefore the systems $\{b_k : k \in \mathbf{N}\}$ and $\{p_\ell : \ell \in \mathbf{N}\}$ are equivalent, and we obtain the assertion.

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