The transition constant for arithmetic hyperbolic reflection groups

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Abstract

The transition constant was introduced in our 1981 paper and denoted as $N(14)$. It is equal to the maximal degree of the ground fields of V-arithmetic connected edge graphs with 4 vertices and of the minimality 14. This constant is fundamental since if the degree of the ground field of an arithmetic hyperbolic reflection group is greater than $N(14)$, then the field comes from special plane reflection groups. In [14], we claimed its upper bound 56. Using similar but more difficult considerations, here we show that the upper bound is 25.

As applications, using this result and our methods, we show that the degree of ground fields of arithmetic hyperbolic reflection groups in dimensions at least 6 has the upper bound 25; in dimensions 3, 4, 5 it has the upper bound 44. This significantly improves our results in [14, 15, 16]. Additionally using recent results by Belolipetsky and Maclachlan, the last upper bound can be improved to 35.

In Appendix, we give a review and corrections to Section 1 of our papers [10] which is important for our methods.

1 Introduction

The transition constant was introduced in our paper [10] and was denoted in [10] as $N(14)$. It is equal to the maximal degree of the ground fields of V-arithmetic connected edge graphs with 4 vertices and of the minimality 14. We showed in [10] that the number of these fields is finite, and $N(14)$ is a finite effective constant.

It was shown in [10] that the degree of ground fields of arithmetic hyperbolic reflection groups in dimensions at least 10 has the upper bound $N(14)$. In [14], we had shown that the degree of ground fields of arithmetic hyperbolic reflection groups in dimensions at least 6 is bounded by the maximum from $N(14)$ and 11 (here 11 is the upper bound for the degree of ground fields of plain arithmetic hyperbolic reflection groups with quadrangle fundamental polygon of the minimality 14). In [13], we showed that the degree of ground fields of

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arithmetic hyperbolic reflection groups is bounded by the maximum of $N(14)$ and the degree of ground fields of arithmetic hyperbolic reflection groups in dimensions two and three. In general, this constant is fundamental since if the degree of the ground field of an arithmetic hyperbolic reflection group is greater than $N(14)$, then the field comes from special plane reflection groups. See [10], [13] — [16].

In [14], we claimed that $N(14) \le 56$. Here we improve this bound. Using similar but more difficult considerations, we show that $N(14) \le 25$.

As applications (see Section 5), using our methods, we show that the degree of ground fields of arithmetic hyperbolic reflection groups in dimensions $n \ge 6$ has the upper bound 25; in dimensions $n = 3, 4, 5$, it has the upper bound 44. Remark that this also gives another proof of finiteness in dimension $n = 3$ which is different from the first proof by I. Agol [1]. These significantly improve results of our recent papers [14]—[16].

In [2], Belolipetsky obtained the upper bound 35 for $n = 3$. In [8], Maclachlan obtained the upper bound 11 for $n = 2$. Using these results and our results from [13], we can improve the upper bound 44 in dimensions $n = 4, 5$. We show that the upper bound is 35 in these dimensions.

We hope that our results will be important for further classification. See an example in Theorem 23.

In Appendix, we review and correct results of Section 1 of our paper [10] which are very important for our methods.

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2 Reminding of some basic facts about hyperbolic fundamental polyhedra

Here we remind some basic definitions and results about fundamental chambers (always for discrete reflection groups) in hyperbolic spaces and their Gram matrices. See [21], [25] and [9], [10].

We work with Klein model of a hyperbolic space $\mathcal{L}$ associated to a hyperbolic form Φ over the field of real numbers $\mathbb{R}$ with signature $(1, n)$, where $n = \dim \mathcal{L}$. Let $V = \{x \in \Phi | x^2 > 0\}$ be the cone determined by Φ, and let V^+ be one of the two halves of this cone. Then $\mathcal{L} = \mathcal{L}(\Phi) = V^+/\mathbb{R}^+$ is the set of rays in V^+; we let $[x]$ denote the element of $\mathcal{L}$ determined by the ray $\mathbb{R}^+ x$ where $x \in V^+$ and $\mathbb{R}^+$ is the set of all positive real numbers. The hyperbolic distance is given by the formula

$$\rho([x], [y]) = (x \cdot y)/\sqrt{x^2 y^2},\ \ [x], [y] \in \mathcal{L},$$

then the curvature of $\mathcal{L}$ is equal to -1.

Every half-space $\mathcal{H}^+$ in $\mathcal{L}$ determines and is determined by the orthogonal element $e \in \Phi$ with square $e^2 = -2$:

$$\mathcal{H}^+ = \mathcal{H}_e^+ = \{[x] \in \mathcal{L} | x \cdot e \geq 0\}.$$

It is bounded by the hyperplane

$$\mathcal{H}^+ = \mathcal{H}_e^+ = \{[x] \in \mathcal{L} | x \cdot e = 0\}$$

orthogonal to e. If two half-spaces $\mathcal{H}_{e_1}^+$, $\mathcal{H}_{e_2}^+$ where $e_1^2 = e_2^2 = -2$ have a common non-empty open subset in $\mathcal{L}$, then $\mathcal{H}_{e_1} \cap \mathcal{H}_{e_2}$ is an angle of the value ϕ where $2\cos\phi = e_1 \cdot e_2$ if $-2 < e_1 \cdot e_2 \leq 2$, and the distance between hyperplanes $\mathcal{H}_{e_1}$ and $\mathcal{H}_{e_2}$ is equal to ρ where $2\,\mathrm{ch}\,\rho = e_1 \cdot e_2$ if $e_1 \cdot e_2 > 2$.

A convex polyhedron $\mathcal{M}$ in $\mathcal{L}$ is intersection of a finite number of half-spaces $\mathcal{H}_e^+$, $e \in P(\mathcal{M})$, where $P(\mathcal{M})$ are all the vectors with square -2 which are orthogonal to the faces (of the codimension one) of $\mathcal{M}$ and are directed outward. The matrix

$$A = (a_{ij}) = (e_i \cdot e_j),\ e_i, e_j \in P(\mathcal{M}), \tag{1}$$

is the Gram matrix $\Gamma(\mathcal{M}) = \Gamma(P(\mathcal{M}))$ of $\mathcal{M}$. It determines $\mathcal{M}$ uniquely up to motions of $\mathcal{L}$. If $\mathcal{M}$ is sufficiently general, then $P(\mathcal{M})$ generates Φ, and the form Φ is

$$\Phi = \sum_{e_i, e_j \in P(\mathcal{M})} a_{ij} X_i Y_j \ \bmod \text{Kernel}, \tag{2}$$

and $P(\mathcal{M})$ naturally identifies with a subset of Φ and defines $\mathcal{M}$.

The polyhedron $\mathcal{M}$ is a fundamental chamber of a discrete reflection group W in $\mathcal{L}$ if and only if $a_{ij} \geq 0$ and $a_{ij} = 2\cos\frac{\pi}{m_{ij}}$ where $m_{ij} \geq 2$ is an integer if $a_{ij} < 2$ for all $i \neq j$. Symmetric real matrices A satisfying these conditions and having all their diagonal elements equal to -2 are called *fundamental* (then the set $P(\mathcal{M})$ formally corresponds to indices of the matrix A). As usual, further we identify fundamental matrices with fundamental graphs Γ. Their vertices correspond to $P(\mathcal{M})$. Two different vertices $e_i \neq e_j \in P(\mathcal{M})$ are connected by the thin edge of the integer weight $m_{ij} \geq 3$ if $0 < a_{ij} = 2\cos\frac{\pi}{m_{ij}} < 2$, by the thick edge if $a_{ij} = 2$, and by the broken edge of the weight a_{ij} if $a_{ij} > 2$. In particular, the vertices e_i and e_j are disjoint if and only if $e_i \cdot e_j = a_{ij} = 2\cos\frac{\pi}{2} = 0$. Equivalently, e_i and e_j are perpendicular (or orthogonal). See some examples of such graphs in Figures 1 — 8 below.

For a real $t > 0$, we say that a fundamental matrix $A = (a_{ij})$ (and the corresponding fundamental chamber $\mathcal{M}$) *has minimality* t if $a_{ij} < t$ for all a_{ij}. Here we follow [9], [10]. Further, the minimality $t = 14$ will be especially important.

It is known that fundamental domains of arithmetic hyperbolic groups must have finite volume. Let us assume that it is valid for a fundamental chamber $\mathcal{M}$ of a hyperbolic discrete reflection group. As Vinberg had shown [21], in order for $\mathcal{M}$ to be a fundamental chamber of an arithmetic reflection group W in $\mathcal{L}$, it is necessary and sufficient that all of the cyclic products

$$b_{i_1\ldots i_m} = a_{i_1 i_2} \cdot a_{i_2 i_3} \cdots a_{i_{m-1} i_m} \cdot a_{i_m i_1} \tag{3}$$

be algebraic integers, that the field $\widetilde{\mathbb{K}} = \mathbb{Q}(\{a_{ij}\})$ be totally real, and that, for any embedding $\widetilde{\mathbb{K}} \to \mathbb{R}$ not the identity over the *ground field* $\mathbb{K} = \mathbb{Q}(\{b_{i_1\ldots i_m}\})$ generated by all of the cyclic products (3), the form (2) be negative definite.

Fundamental real matrices $A = (a_{ij})$, $a_{ij} = e_i \cdot e_j$, e_i, $e_j \in P(\mathcal{M})$ (or the corresponding graphs), with a hyperbolic form Φ in (2) and satisfying these Vinberg's conditions will be further called *V-arithmetic* (here we don't require that the corresponding hyperbolic polyhedron $\mathcal{M}$ has finite volume). It is well-known (and easy to see; see arguments in Sect. 4.3) that a subset $P \subset P(\mathcal{M})$ also defines a V-arithmetic matrix $(e_i \cdot e_j)$, $e_i, e_j \in P$, with the same ground field $\mathbb{K}$ if the subset P is hyperbolic, i. e. the corresponding to P form (2) is hyperbolic.

3 V-arithmetic edge polyhedra

A fundamental chamber $\mathcal{M}$ (and the corresponding Gram matrix A or a graph) is called *edge chamber (matrix, graph)* if all hyperplanes $\mathcal{H}_e$, $e \in P(\mathcal{M})$, contain one of two distinct vertices v_1 and v_2 of the 1-dimensional edge v_1v_2 of $\mathcal{M}$. Assume that both vertices v_1 and v_2 are finite (further we always consider this case). Further we call this edge chambers *finite*. Assume that $\dim \mathcal{L} = n$. Then $P(\mathcal{M})$ consists of $n+1$ elements: e_1, e_2 and $n-1$ elements $P(\mathcal{M}) - \{e_1, e_2\}$. Here $P(\mathcal{M}) - \{e_1, e_2\}$ corresponds to hyperplanes which contain the edge v_1v_2 of $\mathcal{M}$. The e_1 corresponds to the hyperplane which contains v_1 and does not contain v_2. The e_2 corresponds to the hyperplane which contains v_2 and does not contain v_1. Then the set $P(\mathcal{M})$ is hyperbolic (it has hyperbolic Gram matrix), but its subsets $P(\mathcal{M}) - \{e_1\}$ and $P(\mathcal{M}) - \{e_2\}$ are negative definite (they have negative definite Gram matrix) and define Coxeter graphs. Only the element $u = e_1 \cdot e_2$ of the Gram matrix of $\mathcal{M}$ can be greater than 2. Thus, $\mathcal{M}$ will have the minimality t if and only if $u = e_1 \cdot e_2 < t$.

From considerations above, the Gram graph $\Gamma(P(\mathcal{M}))$ of an edge chamber has only one hyperbolic connected component $P(\mathcal{M})^{hyp}$ (containing e_1 and e_2) and several negative definite connected components. Gram matrix $\Gamma(P(\mathcal{M})^{hyp})$ evidently also corresponds to an edge chamber of the dimension $\#P(\mathcal{M})^{hyp} - 1$. If $\mathcal{M}$ is V-arithmetic, the ground field $\mathbb{K}$ of $\mathcal{M}$ is the same as for the hyperbolic connected component $\Gamma(P(\mathcal{M})^{hyp})$

The following result had been proved in [10].

Theorem 1. *([10, Theorem 2.3.1]) Given $t > 0$, there exists an effective constant $N(t)$ such that every V-arithmetic edge chamber of the minimality t with ground field $\mathbb{K}$ of degree greater that $N(t)$ over $\mathbb{Q}$ has the hyperbolic connected component of its Gram graph which has less than 4 elements.*

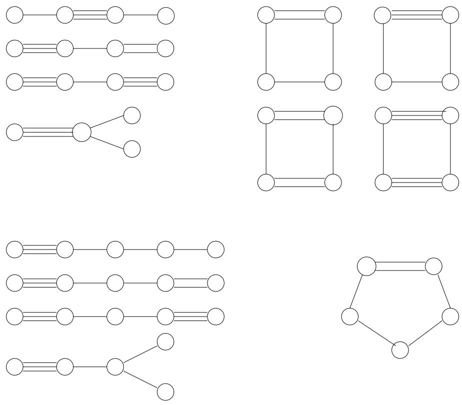

Figure 1: All arithmetic Lannér graphs with at least 4 vertices

Considerations in [10] (and also [9]) also show that the set of possible ground fields $\mathbb{K}$ of hyperbolic connected components with at least 4 vertices of V-arithmetic edge chambers of minimality t is finite. Even the set of Gram graphs $\Gamma(P(\mathcal{M})^{hyp})$ of minimality t with fixed ≥ 4 number of vertices is finite. Taking this under consideration, here we want to formulate and prove more efficient variant of this theorem. We restrict by the minimality $t = 14$ to get an exact estimate for the constant $N(14)$, but the same finiteness results are valid for any $t > 0$.

Following [14], below we formulate a more efficient definition of the constants $N(t)$, $t > 0$, and $N(14)$.

3.1 Arithmetic Lannér graphs with ≥ 4 elements

We remind that Lannér graphs are Gram graphs of bounded fundamental hyperbolic simplexes. They are characterized as hyperbolic fundamental graphs such that any their proper subgraph is a Coxeter graph. They were classified by Lannér [6]. In Figure 1 we give all arithmetic Lannér graphs with at least 4 vertices (only one Lannér graph with ≥ 4 vertices is not arithmetic). As usual, we replace a thin edge of the weight k by $k - 2$-edges for a small k. Ground fields of Lannér graphs with ≥ 4 vertices give three fields:

$$\mathcal{F}L^4 = \{\mathbb{Q},\, \mathbb{Q}(\sqrt{2}),\ \mathbb{Q}(\sqrt{5})\}. \tag{4}$$

See [23] for details.

3.2 Arithmetic triangle graphs

Triangle graphs are Gram graphs of bounded fundamental triangles on hyperbolic plane (we don't consider non-bounded triangles). Equivalently, they are

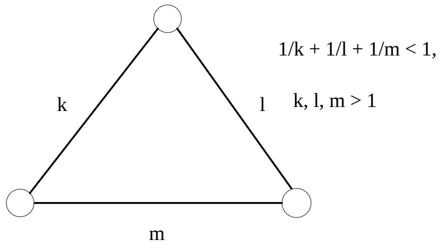

Figure 2: Triangle graphs

Lannér graphs with 3 vertices. They are given in Figure 2 where $2 \le k, l, m$ and

$$\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1.$$

Arithmetic triangles were enumerated by Takeuchi [18]. All bounded arithmetic triangles are given by the following triplets (k, l, m):

$$(2,3,7\ -\ 12),\ (2,3,14),\ (2,3,16),\ (2,3,18),\ (2,3,24),\ (2,3,30),\ (2,4,5\ -\ 8),$$
$$(2,4,10),\ (2,4,12),\ (2,4,18),\ (2,5,5),\ (2,5,6),\ (2,5,8),\ (2,5,10),\ (2,5,20),$$
$$(2,5,30),\ (2,6,6),\ (2,6,8),\ (2,6,12),\ (2,7,7),\ (2,7,14),\ (2,8,8),\ (2,8,16),$$
$$(2,9,18),\ (2,10,10),\ (2,12,12),\ (2,12,24),\ (2,15,30),\ (2,18,18),\ (3,3,4\ -\ 9),$$
$$(3,3,12),\ (3,3,15),\ (3,4,4),\ (3,4,6),\ (3,4,12),\ (3,5,5),\ (3,6,6),\ (3,6,18),$$
$$(3,8,8),\ (3,8,24),\ (3,10,30),\ (3,12,12),\ (4,4,4\ -6),\ (4,4,9),\ (4,5,5),\ (4,6,6),$$
$$(4,8,8),\ (4,16,16),\ (5,5,5),\ (5,5,10),\ (5,5,15),\ (5,10,10),\ (6,6,6),\ (6,12,12),$$
$$(6,24,24),\ (7,7,7),\ (8,8,8),\ (9,9,9),\ (9,18,18),\ (12,12,12),\ (15,15,15).$$

Their ground fields were found by Takeuchi [19]. They give the set of fields

$$\begin{array}{l} \mathcal{F}T = \{\mathbb{Q}\} \cup \{\mathbb{Q}(\sqrt{a}) \mid a = 2,\ 3,\ 5,\ 6\} \cup \{\mathbb{Q}(\sqrt{2}, \sqrt{3}),\ \mathbb{Q}(\sqrt{2}, \sqrt{5})\} \cup \\ \cup \{\mathbb{Q}(\cos\frac{2\pi}{b}) \mid b = 7,\ 9,\ 11,\ 15,\ 16,\ 20\}. \end{array} \tag{5}$$

3.3 V-arithmetic connected finite edge graphs with 4 vertices for $2 < u < 14$

Using classification of Coxeter graphs, it is easy to draw all possible pictures of connected finite edge graphs $\Gamma^{(4)}$ with 4 vertices and $u = e_1 \cdot e_2 > 2$. They correspond to all 3-dimensional finite fundamental edge polyhedra with connected Gram graph and $u > 2$. They are given in Figure 3 and give five types of graphs $\Gamma = \Gamma_i^{(4)}$, $i = 1, 2, 3, 4, 5$. All possible natural parameters s, k, r, $p \ge 2$ for these graphs can be easily enumerated by the condition that $\Gamma - \{e_1\}$, $\Gamma - \{e_2\}$ are Coxeter graphs. They will be given in Sec. 4 below.

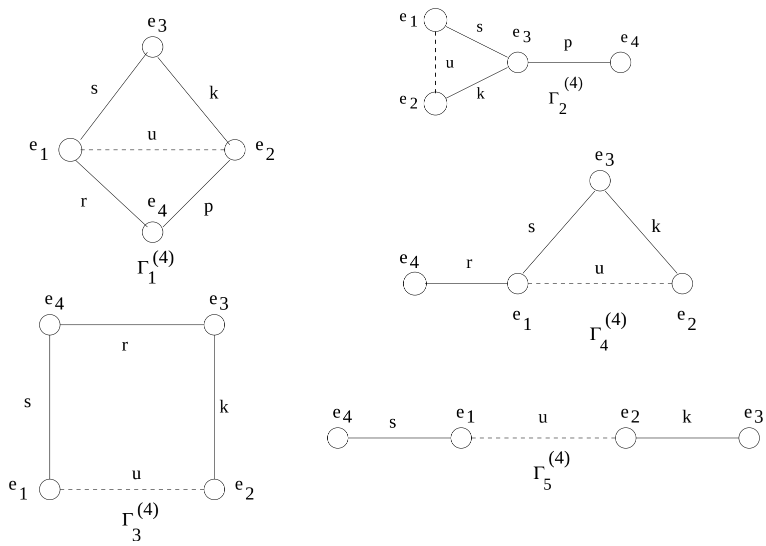

Figure 3: Five graphs $\Gamma_i^{(4)}$, $i = 1, 2, 3, 4, 5$.

Definition 2. *For $i = 1, 2, 3, 4, 5$ and $t > 0$ we denote by $\Gamma_i^{(4)}(t)$ the set of all V-arithmetic connected finite edge graphs with 4 vertices $\Gamma_i^{(4)}$ of the minimality t, i. e. for $2 < u < t$, and by*

$$\mathcal{F}\Gamma_i^{(4)}(t)$$

the set of all their ground fields.

All V-arithmetic graphs $\Gamma_i^{(4)}$ for $2 < u < t$ give particular cases of graphs of V-arithmetic edge polyhedra with hyperbolic connected component having 4 vertices and minimality t. Thus, by Theorem 1, degree (over $\mathbb{Q}$) of fields from $\mathcal{F}\Gamma_i^{(4)}(t)$ is bounded by the effective constant $N(t)$. It follows that the sets of V-arithmetic graphs $\Gamma_i^{(4)}(t)$ and fields $\mathcal{F}\Gamma_i^{(4)}(t)$ are also finite.

Vice verse, Theorem 2 can be deduced from finiteness of the sets of fields above because of the following easy statement.

Proposition 3. *The ground field of any V-arithmetic edge chamber of the minimality $t > 0$ with the hyperbolic connected component of its Gram graph having at least 4 vertices belongs to one of the finite sets of fields $\mathcal{F}L^4$, $\mathcal{F}T$ and $\mathcal{F}\Gamma_i^{(4)}(t)$, $1 \le i \le 5$, introduced above.*

In particular, Theorem 1 is equivalent to finiteness of the sets of fields $\mathcal{F}\Gamma_i^{(4)}(t)$, $i = 1, 2, 3, 4, 5$.

Proof. See [14]. □

Degree of fields from $\mathcal{F}L^4$ is bounded by 2, and degree of fields from $\mathcal{F}T$ is bounded by 5.

For arithmetic hyperbolic reflection groups, the minimality $t = 14$ is especially important. Using the same methods as for the proof of Theorem 1 in [10], we can prove the following effective upper bounds which improve the upper bounds which we claimed in [14] (e.g., in [14] we claimed that $N(14) \leq 56$).

Theorem 4. *The degree of fields from* $\mathcal{F}\Gamma_1^{(4)}(14)$ *is bounded* ($\leq$) *by* 23.

The degree of fields from $\mathcal{F}\Gamma_2^{(4)}(14)$ *is bounded by* 23.

The degree of fields from $\mathcal{F}\Gamma_3^{(4)}(14)$ *is bounded by* 25.

The degree of fields from $\mathcal{F}\Gamma_4^{(4)}(14)$ *is bounded by* 25.

The degree of fields from $\mathcal{F}\Gamma_5^{(4)}(14)$ *is bounded by* 25.

Thus, the constant $N(14)$ *of Theorem 1 can be taken to be* $N(14) = 25$.

Proof. The proof will be given in the next Section 4. □

4 Ground fields of V-arithmetic connected finite edge graphs with four vertices of the minimality 14.

Here we shall obtain explicit upper bounds of degrees of fields from the finite sets $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \leq i \leq 5$ (see Definition 2), and prove Theorem 4. Moreover, our considerations will deliver important information about these sets of fields.

4.1 Some results on hyperbolic numbers

Like for the proof of Theorem 1 from [10], we use the following general results from [10, Section 1] (see Section 6 which also contains some corrections).

Theorem 5. *([10, Theorem 1.2.1]) Let* $\mathbb{F}$ *be a totally real algebraic number field, and let each embedding* $\sigma : \mathbb{F} \to \mathbb{R}$ *corresponds to an interval* $[a_\sigma, b_\sigma]$ *in* $\mathbb{R}$ *where*

$$\prod_\sigma \frac{b_\sigma - a_\sigma}{4} < 1.$$

In addition, let the natural number m *and the intervals* $[s_1, t_1], \dots, [s_m, t_m]$ *in* $\mathbb{R}$ *be fixed. Then there exists a constant* $N(s_i, t_i)$ *such that, if* α *is a totally real algebraic integer and if the following inequalities hold for the embeddings* $\tau : \mathbb{F}(\alpha) \to \mathbb{R}$*:*

$$s_i \leq \tau(\alpha) \leq t_i \quad for \quad \tau = \tau_1, \dots, \tau_m,$$

$$a_{\tau|\mathbb{F}} \leq \tau(\alpha) \leq b_{\tau|\mathbb{F}} \quad for \quad \tau \neq \tau_1, \dots, \tau_m,$$

then

$$[\mathbb{F}(\alpha) : \mathbb{F}] \leq N(s_i, t_i).$$

Theorem 6. *([10, Theorem 1.2.2]) Under the conditions of Theorem 5, $N(s_i, t_i)$ can be taken to be $N(s_i, t_i) = N$, where N is the least natural number solution of the inequality*

$$N \ln (1/R) - M \ln (2N + 2) - \ln B \geq \ln S. \tag{6}$$

Here

$$M = [\mathbb{F} : \mathbb{Q}], \quad B = \sqrt{|\text{discr } \mathbb{F}|}; \tag{7}$$

$$R = \sqrt{\prod_{\sigma} \frac{b_\sigma - a_\sigma}{4}}, \quad S = \prod_{i=1}^{m} \frac{2er_i}{b_{\sigma_i} - a_{\sigma_i}} \tag{8}$$

where

$$\sigma_i = \tau_i|\mathbb{F}, \quad r_i = \max\{|t_i - a_{\sigma_i}|, |b_{\sigma_i} - s_i|\}. \tag{9}$$

We note that the proof of Theorems 5 and 6 uses a variant of Fekete's Theorem (1923) about existence of non-zero integer polynomials of bounded degree which differ only slightly from zero on appropriate intervals. See [10, Theorem 1.1.1] (see its corrections in Section 6, Theorems 24, 25).

Unfortunately, Theorems 5 and 6 usually give a poor upper bound for the degree. They should mainly be considered as existence theorems. Usually we shall use explicit polynomials to bound the degree in similar cases.

We use the following statement. Similar arguments one should use to prove Theorems 5 and 6, see Section 6.

Lemma 7. *Let $\mathbb{F}$ be a totally real algebraic number field, and let each embedding $\sigma : \mathbb{F} \to \mathbb{R}$ corresponds to an interval $[a_\sigma, b_\sigma]$ in $\mathbb{R}$. In addition, let the natural number m and the intervals $[s_1, t_1], \dots, [s_m, t_m]$ in $\mathbb{R}$ be fixed.*

Let $P(x)$ be a non-zero polynomial over the ring of integers of $\mathbb{F}$, and

$$\delta(\sigma) = \max_{x \in [a_\sigma, b_\sigma]} |P^\sigma(x)| \quad \text{for} \quad \sigma : \mathbb{F} \to \mathbb{R},$$

and

$$a_i = \max_{x \in [s_i, t_i]} |P^{(\sigma_i|\mathbb{F})}(x)|.$$

Suppose that $\prod_\sigma \delta(\sigma) < 1$.

Let α be a totally real algebraic integer such that the following inequalities hold for all embeddings $\tau : \mathbb{F}(\alpha) \to \mathbb{R}$:

$$s_i \leq \tau(\alpha) \leq t_i \quad for \quad \tau = \tau_1, \dots, \tau_m,$$

$$a_{\tau|\mathbb{F}} \leq \tau(\alpha) \leq b_{\tau|\mathbb{F}} \quad for \quad \tau \neq \tau_1, \dots, \tau_m,$$

Then we have the following bound for $[\mathbb{F}(\alpha) : \mathbb{F}]$:

$$1 \leq [\mathbb{F}(\alpha) : \mathbb{F}] \leq \frac{\ln \prod_i a_i - \ln \prod_i \delta(\tau_i|\mathbb{F})}{-\ln \prod_\sigma \delta(\sigma)}$$

if $P(\alpha) \neq 0$. Note that α does not exist if the right hand side is < 1.

Proof. Since $P(\alpha)\neq 0$, we have the inequalities for the norm

$$1 \le |N_{\mathbb{F}(\alpha)/\mathbb{Q}}(P(\alpha))| =$$

$$|\prod_{\tau}\tau(P(\alpha))| = \prod_{\tau}|P^{\tau}(\tau(\alpha))| = \prod_{\tau\neq\tau_i}P^{\tau}(\tau(\alpha))\prod_{i=1}^{m}|P^{\tau_i}(\tau_i(\alpha))|$$

$$\le \prod_{\tau\neq\tau_i}\max_{[a_{\tau|\mathbb{F}},b_{\tau|\mathbb{F}}]}|P^{\tau}(x)|\prod_{i=1}^{m}\max_{[s_i,t_i]}|P^{\sigma_i}(x)|$$

$$\le \prod_{\tau}\max_{[a_{\tau|\mathbb{F}},b_{\tau|\mathbb{F}}]}|P^{\tau}(x)|\prod_{i=1}^{m}\max_{[s_i,t_i]}|P^{\sigma_i}(x)|/\prod_{i=1}^{i=m}\max_{[a_{\tau_i|\mathbb{F}},b_{\tau_i|\mathbb{F}}]}|P^{\tau_i}(x)| =$$

$$\le \left(\prod_{\sigma}\delta(\sigma)\right)^{[\mathbb{F}(\alpha):\mathbb{F}]}\prod_{i=1}^{m}a_i/\prod_{i=1}^{i=m}\delta(\tau_i).$$

Taking logarithm from both sides and using that $\prod_{\sigma}\delta(\sigma)<1$, we obtain the statement.

This is similar to the proof of Theorems 5 and 6 in Section 6. □

4.2 Ground fields of some V-arithmetic connected edge graphs with 3 vertices with the given minimality.

In spite of we are mainly interested in connected edge graphs with four vertices, some connected edge graphs with three vertices will be very important.

All connected edge graphs with three vertices are given in Figure 4. They are $\Gamma_1^{(3)}$ where $s,k\ge 3$ and $\Gamma_2^{(3)}$ where $d\ge 3$. We denote by $\mathcal{F}\Gamma_1^{(3)}(t)$ and $\mathcal{F}\Gamma_2^{(3)}(t)$ their ground fields for the minimality $t>2$. It means that $2<\sigma^{(+)}(u)<t$.

4.2.1 Ground fields of $\Gamma_2^{(3)}(t)$

First let us consider the graphs $\Gamma_2^{(3)}(t)$. The corresponding Gram matrix is

$$\begin{pmatrix} -2 & u & 0\\ u & -2 & 2\cos\frac{\pi}{d}\\ 0 & 2\cos\frac{\pi}{d} & -2\end{pmatrix} \qquad (10)$$

where $d\ge 3$ is an integer, u is a totally real algebraic integer. The ground field $\mathbb{K}=\mathbb{Q}(u^2,\cos^2(\pi/d))$. The determinant $d(u)$ of the Gram matrix is given by the equality

$$\frac{d(u)}{2}=u^2-4\sin^2(\pi/d).$$

It follows that $\Gamma_2^{(3)}$ is V-arithmetic of the minimality $t>2$, if and only if for $\alpha=u^2$ we have

$$0<\sigma(\alpha)<\sigma(4\sin^2\frac{\pi}{d}) \qquad (11)$$

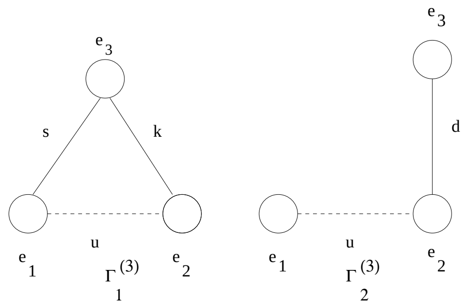

Figure 4: Connected edge graphs with 3 vertices

for all $\sigma : \mathbb{K} \to \mathbb{R}$ which are different from $\sigma^{(+)}$, and

$$4 < \sigma^{(+)}(\alpha) < t^2. \tag{12}$$

It follows that $\mathbb{F}_d = \mathbb{Q}(\cos^2(\pi/d)) \subset \mathbb{K} = \mathbb{Q}(\alpha)$. We will be especially interested in $t = 14$ and $t = 16$. Let us estimate the degree $n = [\mathbb{K} : \mathbb{Q}]$.

Assume that $d = 3$. Then α satisfies

$$0 < \sigma(\alpha) < 3, \qquad 4 < \sigma^{(+)}(\alpha) < t^2.$$

We take the polynomial

$$P(x) = x^3(x-1)^4(x-2)^4(x-3)^3(x^2-3x+1)^3. \tag{13}$$

of the degree 20. The maximum of $|P(x)|$ on the interval $[0, 3]$ is equal to $\delta = 884736/9765625 = 0.0905969664$. Using $P(x)$, by Lemma 7, we get the upper bound

$$[\mathbb{K} : \mathbb{Q}] \leq 1 + \frac{\ln(P(t^2))}{(-\ln(\delta))}.$$

For $t = 14$, we get $[\mathbb{K} : \mathbb{Q}] \leq 44$, and for $t = 16$, we get $[\mathbb{K} : \mathbb{Q}] \leq 47$. (Theorems 5 and 6 give more poor results: 76 for $t = 14$ and 78 for $t = 16$.)

Assume that $d = 4$. Then α satisfies the inequalities

$$0 < \sigma(\alpha) < 2, \qquad 4 < \sigma^{(+)}(\alpha) < t^2.$$

We take the polynomial

$$P(x) = x(x-1)^2(x-2). \tag{14}$$

of the degree 4. The maximum of $|P(x)|$ on the interval $[0, 2]$ is equal to $\delta = 1/4$. Using $P(x)$, by Lemma 7, we obtain the bound

$$[\mathbb{K} : \mathbb{Q}] \leq 1 + \frac{\ln(P(t^2))}{(-\ln(\delta))}.$$

For $t = 14$ and $t = 16$, we get $[\mathbb{K} : \mathbb{Q}] \leq 16$. (Theorems 5 and 6 give more poor bound: 31.)

Assume that $d = 5$. We take the polynomial

$$P(x) = x(x-1)\left(x + 1 - 4\sin^2\frac{\pi}{5}\right)\left(x - 4\sin^2\frac{\pi}{5}\right). \tag{15}$$

of the degree 4 over the ring of integers of $\mathbb{F}_5$. The maximum of $|P(x)|$ on the interval $[0, 4\sin^2(\pi/5)]$ is equal to $\delta_1 = 0.04559....$ For its conjugate polynomial

$$P^\sigma(x) = x(x-1)\left(x + 1 - 4\sin^2\frac{2\pi}{5}\right)\left(x - 4\sin^2\frac{2\pi}{5}\right),$$

the maximum of $|P^\sigma(x)|$ on the conjugate interval $[0, 4\sin^2(2\pi/5)]$ is equal to $\delta_2 = 2.1419....$ Evidently, $P(\alpha)$ is not zero. By Lemma 7, we get the bound

$$[\mathbb{K} : \mathbb{F}_5] \leq \frac{\ln P(t^2) - \ln \delta_1}{-\ln(\delta_1 \cdot \delta_2)}.$$

For $t = 14$ and $t = 16$, we get $[\mathbb{K} : \mathbb{F}_5] \leq 10$, and $[\mathbb{K} : \mathbb{Q}] \leq 20$. (Theorems 5 and 6 give the bounds $[\mathbb{K} : \mathbb{F}_5] \leq 27$ for $t = 14$, and $[\mathbb{K} : \mathbb{F}_5] \leq 28$ for $t = 16$ only.)

Assume that $d \geq 6$. We take the polynomial

$$P(x) = x\left(x - 4\sin^2\frac{\pi}{d}\right). \tag{16}$$

over the ring of integers of $\mathbb{F}_d$. For $\sigma : \mathbb{F}_d \to \mathbb{R}$, the maximum of $|P^\sigma(x))|$ on the interval $[0, \sigma(4\sin^2(\pi/d))]$ is equal to $(4\sigma(\sin^4(\pi/d))$. Obviously, $P(\alpha) \neq 0$. Then we have (in fact, we repeat the proof of Lemma 7 for this case):

$$1 \leq |N_{\mathbb{K}/\mathbb{Q}}(P(\alpha))| < N_{\mathbb{F}_d/\mathbb{Q}}(4\sin^4(\pi/d))^{[\mathbb{K}:\mathbb{F}_d]}\frac{|P(t^2)|}{4\sin^4(\pi/d)} =$$

$$= \left(\frac{\gamma(d)^2}{4^{[\mathbb{F}_d:\mathbb{Q}]}}\right)^{[\mathbb{K}:\mathbb{F}_d]}\frac{|P(t^2)|}{4\sin^4(\pi/d)}.$$

where for $d \geq 3$ we have

$$N_{\mathbb{F}_d/\mathbb{Q}}(4\sin^2(\pi/d)) = \gamma(d) = \begin{cases} p & \text{if } d = p^t > 2 \text{ where } p \text{ is prime,} \\ 1 & \text{otherwise,} \end{cases} \tag{17}$$

and $[\mathbb{F}_d : \mathbb{Q}] = \varphi(d)/2$ where $\varphi(d)$ is the Euler function.

Denoting $m = [\mathbb{K} : \mathbb{F}_d]$, we obtain the inequality

$$m(\varphi(d)\ln 2 - 2\ln\gamma(d)) < \ln t^2 + \ln(t^2 - 4\sin^2(\pi/d)) - \ln(4\sin^4(\pi/d)).$$

It is easy to see that $\varphi(d)\ln 2 - 2\ln\gamma(d) > 0$ for $d \geq 6$. Thus, we obtain

$$1 \leq m < \frac{\ln t^2 + \ln(t^2 - 4\sin^2(\pi/d)) - \ln(4\sin^4(\pi/d))}{\varphi(d)\ln 2 - 2\ln\gamma(d)} \tag{18}$$

Respectively $[\mathbb{K}:\mathbb{Q}] \le m\varphi(d)/2$ where m satisfies (18).

We use the trivial bound: $\varphi(d) \ge \sqrt{d-2}$. Then for $t = 14$, we obtain that only the following $d \ge 6$ are possible, and we get the corresponding estimates for $[\mathbb{K}:\mathbb{Q}]$:

$d = 6$, then $[\mathbb{K}:\mathbb{Q}] \le 8$; $d = 7$, then $[\mathbb{K}:\mathbb{Q}] \le 138$; $d = 8$, then $[\mathbb{K}:\mathbb{Q}] \le 18$; $d = 9$, then $[\mathbb{K}:\mathbb{Q}] \le 18$; $d = 10$, then $[\mathbb{K}:\mathbb{Q}] \le 10$; $d = 11$, then $[\mathbb{K}:\mathbb{Q}] \le 30$; $d = 12$, then $[\mathbb{K}:\mathbb{Q}] \le 10$; $d = 13$, then $[\mathbb{K}:\mathbb{Q}] \le 24$; $d = 14$, then $[\mathbb{K}:\mathbb{Q}] \le 9$; $d = 15$, then $[\mathbb{K}:\mathbb{Q}] \le 8$; $d = 16$, then $[\mathbb{K}:\mathbb{Q}] \le 8$; $d = 17$, then $[\mathbb{K}:\mathbb{Q}] \le 16$; $d = 18$, then $[\mathbb{K}:\mathbb{Q}] \le 9$; $d = 19$, then $[\mathbb{K}:\mathbb{Q}] \le 18$; $d = 20$, then $[\mathbb{K}:\mathbb{Q}] \le 8$; $d = 21$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 22$, then $[\mathbb{K}:\mathbb{Q}] \le 10$; $d = 23$, then $\mathbb{K} = \mathbb{F}_{23}$; $d = 24$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 25$, then $\mathbb{K} = \mathbb{F}_{25}$; $d = 26$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 27$, then $\mathbb{K} = \mathbb{F}_{27}$; $d = 28$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 29$, then $\mathbb{K} = \mathbb{F}_{29}$; $d = 30$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 31$, then $\mathbb{K} = \mathbb{F}_{31}$; $d = 32$, then $\mathbb{K} = \mathbb{F}_{32}$; $d = 33$, then $\mathbb{K} = \mathbb{F}_{33}$; $d = 34$, then $\mathbb{K} = \mathbb{F}_{34}$; $d = 35$, then $\mathbb{K} = \mathbb{F}_{35}$; $d = 36$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 37$, then $\mathbb{K} = \mathbb{F}_{37}$; $d = 38$, then $\mathbb{K} = \mathbb{F}_{38}$; $d = 39$, then $\mathbb{K} = \mathbb{F}_{39}$; $d = 40$, then $\mathbb{K} = \mathbb{F}_{40}$; $d = 42$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 44$, then $\mathbb{K} = \mathbb{F}_{44}$; $d = 45$, then $\mathbb{K} = \mathbb{F}_{45}$; $d = 46$, then $\mathbb{K} = \mathbb{F}_{46}$; $d = 48$, then $\mathbb{K} = \mathbb{F}_{48}$; $d = 50$, then $\mathbb{K} = \mathbb{F}_{50}$; $d = 52$, then $\mathbb{K} = \mathbb{F}_{52}$; $d = 54$, then $\mathbb{K} = \mathbb{F}_{54}$; $d = 56$, then $\mathbb{K} = \mathbb{F}_{56}$; $d = 58$, then $\mathbb{K} = \mathbb{F}_{58}$; $d = 60$, then $\mathbb{K} = \mathbb{F}_{60}$; $d = 62$, then $\mathbb{K} = \mathbb{F}_{62}$; $d = 66$, then $\mathbb{K} = \mathbb{F}_{66}$; $d = 70$, then $\mathbb{K} = \mathbb{F}_{70}$; $d = 72$, then $\mathbb{K} = \mathbb{F}_{72}$; $d = 78$, then $\mathbb{K} = \mathbb{F}_{78}$; $d = 80$, then $\mathbb{K} = \mathbb{F}_{80}$; $d = 84$, then $\mathbb{K} = \mathbb{F}_{84}$; $d = 90$, then $\mathbb{K} = \mathbb{F}_{90}$; $d = 96$, then $\mathbb{K} = \mathbb{F}_{96}$; $d = 102$, then $\mathbb{K} = \mathbb{F}_{102}$; $d = 120$, then $\mathbb{K} = \mathbb{F}_{120}$;

For $t = 16^2$, we obtain that only the following $d \ge 6$ are possible, and we get the corresponding estimates for $[\mathbb{K}:\mathbb{Q}]$:
$d = 6$, then $[\mathbb{K}:\mathbb{Q}] \le 8$; $d = 7$, then $[\mathbb{K}:\mathbb{Q}] \le 144$; $d = 8$, then $[\mathbb{K}:\mathbb{Q}] \le 18$; $d = 9$, then $[\mathbb{K}:\mathbb{Q}] \le 21$; $d = 10$, then $[\mathbb{K}:\mathbb{Q}] \le 10$; $d = 11$, then $[\mathbb{K}:\mathbb{Q}] \le 30$; $d = 12$, then $[\mathbb{K}:\mathbb{Q}] \le 10$; $d = 13$, then $[\mathbb{K}:\mathbb{Q}] \le 24$; $d = 14$, then $[\mathbb{K}:\mathbb{Q}] \le 9$; $d = 15$, then $[\mathbb{K}:\mathbb{Q}] \le 8$; $d = 16$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 17$, then $[\mathbb{K}:\mathbb{Q}] \le 24$; $d = 18$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 19$, then $[\mathbb{K}:\mathbb{Q}] \le 18$; $d = 20$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 21$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 22$, then $[\mathbb{K}:\mathbb{Q}] \le 10$; $d = 23$, then $\mathbb{K} = \mathbb{F}_{23}$; $d = 24$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 25$, then $\mathbb{K} = \mathbb{F}_{25}$; $d = 26$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 27$, then $\mathbb{K} = \mathbb{F}_{27}$; $d = 28$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 29$, then $\mathbb{K} = \mathbb{F}_{29}$; $d = 30$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 31$, then $\mathbb{K} = \mathbb{F}_{31}$; $d = 32$, then $\mathbb{K} = \mathbb{F}_{32}$; $d = 33$, then $\mathbb{K} = \mathbb{F}_{33}$; $d = 34$, then $\mathbb{K} = \mathbb{F}_{34}$; $d = 35$, then $\mathbb{K} = \mathbb{F}_{35}$; $d = 36$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 37$, then $\mathbb{K} = \mathbb{F}_{37}$; $d = 38$, then $\mathbb{K} = \mathbb{F}_{38}$; $d = 39$, then $\mathbb{K} = \mathbb{F}_{39}$; $d = 40$, then $\mathbb{K} = \mathbb{F}_{40}$; $d = 42$, then $[\mathbb{K}:\mathbb{Q}] \le 12$; $d = 44$, then $\mathbb{K} = \mathbb{F}_{44}$; $d = 45$, then $\mathbb{K} = \mathbb{F}_{45}$; $d = 46$, then $\mathbb{K} = \mathbb{F}_{46}$; $d = 48$, then $\mathbb{K} = \mathbb{F}_{48}$; $d = 50$, then $\mathbb{K} = \mathbb{F}_{50}$; $d = 52$, then $\mathbb{K} = \mathbb{F}_{52}$; $d = 54$, then $\mathbb{K} = \mathbb{F}_{54}$; $d = 56$, then $\mathbb{K} = \mathbb{F}_{56}$; $d = 58$, then $\mathbb{K} = \mathbb{F}_{58}$; $d = 60$, then $\mathbb{K} = \mathbb{F}_{60}$; $d = 62$, then $\mathbb{K} = \mathbb{F}_{62}$; $d = 64$, then $\mathbb{K} = \mathbb{F}_{64}$; $d = 66$, then $\mathbb{K} = \mathbb{F}_{66}$; $d = 68$, then $\mathbb{K} = \mathbb{F}_{68}$; $d = 70$, then $\mathbb{K} = \mathbb{F}_{70}$; $d = 72$, then $\mathbb{K} = \mathbb{F}_{72}$; $d = 78$, then $\mathbb{K} = \mathbb{F}_{78}$; $d = 80$, then $\mathbb{K} = \mathbb{F}_{80}$; $d = 84$, then $\mathbb{K} = \mathbb{F}_{84}$; $d = 90$, then $\mathbb{K} = \mathbb{F}_{90}$; $d = 96$, then $\mathbb{K} = \mathbb{F}_{96}$; $d = 102$, then $\mathbb{K} = \mathbb{F}_{102}$; $d = 120$, then $\mathbb{K} = \mathbb{F}_{120}$;

To improve these results, let us apply a similar polynomial to the one we used for $d = 5$. For $d \geq 5$, we consider the integral polynomial

$$P(x) = x(x-1)(x+1-4\sin^2\frac{\pi}{d})\left(x-4\sin^2\frac{\pi}{d}\right) \tag{19}$$

over the ring of integers of $\mathbb{F}_d$. For $\sigma : \mathbb{F}_d \to \mathbb{R}$ we denote by $\delta(\sigma)$ the maximum of $|P^\sigma(x)|$ on the interval $[0, \sigma(4\sin^2(\pi/d)]$ (by calculating zeros of the derivative). Remark that all $\varphi(d)/2$ embeddings σ are defined by $\sigma(4\sin^2(\pi/d)) = 4\sin^2(k\pi/d)$ where $1 \leq k \leq d/2$ and $(k, d) = 1$, and $\sigma^{(+)}$ corresponds to $k = 1$.

Since evidently $P(\alpha) \neq 0$, by Lemma 7, we get

$$1 \leq m \leq \frac{\ln P^{\sigma^{(+)}}(t^2) - \ln \delta(\sigma^{(+)})}{-\ln\left(\prod_\sigma \delta(\sigma)\right)} \tag{20}$$

if $\prod_\sigma \delta(\sigma) < 1$. It usually gives a better upper bound for m than (18), but it is harder to calculate. Calculating (20) for $t = 14^2$, $t = 16^2$ and for d from the lists above, for some d we get a better upper bound for m. We formulate the final result in the theorems below.

Theorem 8. *For $d \geq 3$, V-arithmetic connected edge graphs $\Gamma_2^{(3)}(14)$ (see Figure 4) of the minimality* 14 *(equivalently, totally real algebraic integers $\alpha = u^2$ over $\mathbb{F}_d = \mathbb{Q}(\cos^2(\pi/d))$ satisfying the conditions* (11) *and* (12) *above for $t = 14$) are possible only for d which are shown below. Moreover, we have the shown below upper bounds for the degree $m = [\mathbb{K} : \mathbb{F}_d]$ of the ground field $\mathbb{K} = \mathbb{Q}(\alpha) \supset \mathbb{F}_d$: $d = 3$, $m \leq 44$; $d = 4$, $m \leq 16$; $d = 5$, $m \leq 10$; $d = 6$, $m \leq 8$; $d = 7$, $m \leq 9$; $d = 8$, $m \leq 8$; $d = 9$, $m \leq 5$; $d = 10$, $m \leq 5$; $d = 11$, $m \leq 4$; $d = 12$, $m \leq 5$; $d = 13$, $m \leq 3$; $d = 14$, $m \leq 3$; $d = 15$, $m \leq 2$; $d = 16$, $m \leq 3$; $d = 17$, $m \leq 2$, $d = 18$, $m \leq 3$; $d = 20$, $m \leq 2$, $d = 21$, $m \leq 2$; $d = 22$, $m \leq 2$; $d = 24$, $m \leq 3$; $d = 26$, $m \leq 2$; $d = 28$, $m \leq 2$; $d = 30$, $m \leq 3$; $d = 36$, $m \leq 2$; $d = 42$, $m \leq 2$; for all remaining $d = 19$, 23, 25, 27, 29, 31 $-$ 35, 38 $-$ 40, 44, 45, 46, 48, 50, 52, 54, 56, 58, 60, 66, 70, 72, 78, 84, 90, 102, 120 we have $m = 1$ and $\mathbb{K} = \mathbb{F}_d$.*

In particular, $[\mathbb{K} : \mathbb{Q}] \leq 44$ for $d = 3$; $[\mathbb{K} : \mathbb{Q}] \leq 27$ for $d = 7$; $[\mathbb{K} : \mathbb{Q}] \leq 20$ for $d = 5$, 11; $[\mathbb{K} : \mathbb{Q}] \leq 18$ for $d = 13$; $[\mathbb{K} : \mathbb{Q}] \leq 16$ for all remaining d.

Theorem 9. *For $d \geq 3$, V-arithmetic connected edge graphs $\Gamma_2^{(3)}(14)$ (see Figure 4) of the minimality* 16 *(equivalently, totally real algebraic integers $\alpha = u^2$ over $\mathbb{F}_d = \mathbb{Q}(\cos^2(\pi/d))$ satisfying the conditions* (11) *and* (12) *above for $t = 16$) are possible only for d which are shown below. Moreover, we have the shown below upper bounds for the degree $m = [\mathbb{K} : \mathbb{F}_d]$ of the ground field $\mathbb{K} = \mathbb{Q}(\alpha) \supset \mathbb{F}_d$:*

Then only the following d are possible, and we have the following upper bounds for $m = [\mathbb{K} : \mathbb{F}_d]$: $d = 3$, $m \leq 47$; $d = 4$, $m \leq 16$; $d = 5$, $m \leq 10$; $d = 6$, $m \leq 8$; $d = 7$, $m \leq 10$; $d = 8$, $m \leq 9$; $d = 9$, $m \leq 5$; $d = 10$, $m \leq 5$; $d = 11$, $m \leq 4$; $d = 12$, $m \leq 5$; $d = 13$, $m \leq 3$; $d = 14$, $m \leq 3$; $d = 15$, $m \leq 2$; $d = 16$, $m \leq 3$; $d = 17$, $m \leq 2$, $d = 18$, $m \leq 4$; $d = 19$, $m \leq 2$; $d = 20$, $m \leq 3$, $d = 21$, $m \leq 2$; $d = 22$, $m \leq 2$; $d = 24$, $m \leq 3$; $d = 26$, $m \leq 2$; $d = 28$, $m \leq 2$; $d = 30$, $m \leq 3$; $d = 36$, $m \leq 2$; $d = 42$, $m \leq 2$; for all remaining $d = 23$, 25, 27, 29, 31 $-$ 35, 38 $-$ 40, 44, *45, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 70, 72, 78, 80, 84, 90, 96, 102, 120 we have* $m = 1$ *and* $\mathbb{K} = \mathbb{F}_d$.

In particular, $[\mathbb{K} : \mathbb{Q}] \le 47$ *for* $d = 3$*;* $[\mathbb{K} : \mathbb{Q}] \le 30$ *for* $d = 7$*;* $[\mathbb{K} : \mathbb{Q}] \le 20$ *for* $d = 5,\ 11$*;* $[\mathbb{K} : \mathbb{Q}] \le 18$ *for* $d = 8,\ 13,\ 19$*;* $[\mathbb{K} : \mathbb{Q}] \le 16$ *for all remaining* d.

By Theorem 8, we obtain

Theorem 10. *The set of fields* $\mathcal{F}\Gamma_2^{(3)}(14)$ *is finite and their degree over* $\mathbb{Q}$ *has the upper bound* 44*. See more exact information about these fields depending on the parameter* d *(see Figure 4) in Theorem 8.*

4.2.2 Ground fields of some $\Gamma_1^{(3)}(t)$

Here we consider ground fields $\mathcal{F}\Gamma_1^{(3)}(t)$ of V-arithmetic graphs $\Gamma_1^{(3)}(t)$ for parameters $3 \le s \le k \le 5$. See Figure 4.

The corresponding Gram matrix is

$$\begin{pmatrix} -2 & u & 2\cos\frac{\pi}{s} \\ u & -2 & 2\cos\frac{\pi}{k} \\ 2\cos\frac{\pi}{s} & 2\cos\frac{\pi}{k} & -2 \end{pmatrix} \qquad (21)$$

where $s, k \ge 3$ are integers, and u is a totally real algebraic integer. The ground field is

$$\mathbb{K} = \mathbb{Q}\left(u^2, \cos^2(\pi/s), \cos^2(\pi/k), u\cos(\pi/s)\cos(\pi/k)\right).$$

The determinant $d(u)$ of the Gram matrix is given by the equality

$$\frac{d(u)}{2} = u^2 + 4\cos\frac{\pi}{s}\cos\frac{\pi}{k}u + 4\cos^2\frac{\pi}{s} + 4\cos^2\frac{\pi}{k} - 4.$$

It follows that $\Gamma_1^{(3)}$ is V-arithmetic of the minimality $t > 2$, if and only if we have

$$0 < -\widetilde{\sigma}(2\cos\frac{\pi}{s}\cos\frac{\pi}{k}) - \sqrt{\sigma(4\sin^2\frac{\pi}{s}\sin^2\frac{\pi}{k})} < \widetilde{\sigma}(u) <$$

$$< -\widetilde{\sigma}(2\cos\frac{\pi}{s}\cos\frac{\pi}{k}) + \sqrt{\sigma(4\sin^2\frac{\pi}{s}\sin^2\frac{\pi}{k})} \qquad (22)$$

for all $\sigma : \mathbb{K} \to \mathbb{R}$ which are different from $\sigma^{(+)}$, and

$$4 < \sigma^{(+)}(u^2) < t^2. \qquad (23)$$

where $\widetilde{\sigma}$ extends $\sigma : \mathbb{K} = \mathbb{Q}(u^2) \to \mathbb{R}$ to $\mathbb{Q}(u) = \mathbb{K}(\cos(\pi/s)\cos(\pi/k))$.

Case $s = k = 3$**.** In this case, $\mathbb{K} = \mathbb{Q}(u)$ and

$$-2 < \sigma(u) < 1, \quad 2 < \sigma^{(+)}(u) < t.$$

We take the polynomial

$$P(x) = (x+2)^3(x+1)^4x^4(x-1)^3(x^2+x-1)^2 \qquad (24)$$

which is similar to (13), and we similarly get

$$[\mathbb{K}:\mathbb{Q}] \leq 1 + \frac{\ln(P(t))}{(-\ln(\delta))}$$

where $\delta = 0.0905969664$. For $t = 14$, we get $[\mathbb{K}:\mathbb{Q}] \leq 23$. (Theorems 5 and 6 give only the upper bound 57.)

Case $s = 3$**,** $k = 4$**.** Then $\mathbb{Q}(\sqrt{2}) \subset \mathbb{K}(u)$ and $\mathbb{K} = \mathbb{Q}(u^2)$. We have $\sqrt{2} \in \mathbb{K}$ if and only if $u \in \mathbb{K}$. In this case,

$$d(u)/2 = u^2 + \sqrt{2}u - 1.$$

We have

$$\widetilde{\sigma}(-\frac{1}{\sqrt{2}}) - \sqrt{\frac{3}{2}} < \widetilde{\sigma}(u) < \widetilde{\sigma}(-\frac{1}{\sqrt{2}}) + \sqrt{\frac{3}{2}},$$
$$4 < \sigma^{(+)}(u^2) < t^2.$$

We take the polynomial

$$P(x) = x^4(x+\sqrt{2})^4(x^2+\sqrt{2}x-1) \tag{25}$$

over the ring of integers of $\mathbb{Q}(\sqrt{2})$. The maximum of $P(x)$ on the interval $[-1/\sqrt{2} - \sqrt{3/2}, -1/\sqrt{2} + \sqrt{3/2}]$ is equal to $\delta_1 = 0.09375....$ The maximum of the conjugate polynomial $P^g(x) = x^4(x-\sqrt{2})^4(x^2-\sqrt{2}x-1)$ on the interval $[1/\sqrt{2} - \sqrt{3/2}, 1/\sqrt{2} + \sqrt{3/2}]$ is equal to $\delta_2 = 0.09375....$ By Lemma 7, we get

$$[\mathbb{K}:\mathbb{Q}(\sqrt{2})] < \frac{\ln P(t) - \ln \delta_1}{-\ln(\delta_1\delta_2)}$$

if $\sqrt{2} \in \mathbb{K}$, and

$$[\mathbb{K}(u):\mathbb{Q}(\sqrt{2})] < \frac{\ln((P(t)P^g(-t)) - \ln \delta_1\delta_2}{-log(\delta_1\delta_2)}$$

if $\sqrt{2}$ does not belong to $\mathbb{K}$.

For $t = 14$, in both cases we get $[\mathbb{K}:\mathbb{Q}] \leq 12$.

Case $s = k = 4$**.** Then $u \in \mathbb{K}$ and $\mathbb{K} = \mathbb{Q}(u)$. We have

$$-2 < \sigma(u) < 0, \quad 2 < \sigma^{(+)}(u) < 14.$$

We take

$$P(x) = (x+2)(x+1)^2x \tag{26}$$

which is similar to the polynomial (14). By Lemma 7,

$$[\mathbb{K}:\mathbb{Q}] \leq 1 + \frac{\ln P(t)}{-\ln(1/4)}.$$

For $t = 14$, we obtain $[\mathbb{K} : \mathbb{Q}] \le 8$.

Case $s = 3$**,** $k = 5$**.** Then $u \in \mathbb{K}$ and $\mathbb{F}_5 = \mathbb{Q}(\cos^2(\pi/5)) \subset \mathbb{K} = \mathbb{Q}(u)$. We have

$$\frac{d(u)}{2} = u^2 + 2\cos\frac{\pi}{5}u + 4\cos^2\frac{\pi}{5} - 3\ .$$

We have

$$-\sigma(\cos\frac{\pi}{5}) - \sqrt{3}\,\sigma(\sin\frac{\pi}{5}) < \sigma(u) < -\sigma(\cos\frac{\pi}{5}) + \sqrt{3}\,\sigma(\sin\frac{\pi}{5}),\ \ 2 < \sigma^{(+)}(u) < 14. \tag{27}$$

We take the polynomial

$$P(x) = x(x+1)(x+3-4\sin^2(\pi/5))(x+2-4\sin^2(\pi/5))(x^2+2\cos\frac{\pi}{5}x+4\cos^2\frac{\pi}{5}-3) \tag{28}$$

over the ring of integers of $\mathbb{F}_5$. The maxima $\delta(\sigma)$ of $|P^\sigma(x)|$ on the corresponding intervals (27) are equal to $\delta_1 = 0.0690098...$ if $\sigma(\sin(\pi/5)) = \sin(\pi/5)$, and $\delta_2 = 1.2383316...$ if $\sigma(\sin(\pi/5)) = \sin(3\pi/5)$. By Lemma 7,

$$[\mathbb{K} : \mathbb{F}_5] < \frac{\ln P(t) - \ln\delta_1}{-\ln(\delta_1\delta_2)}.$$

For $t = 14$, we get $[\mathbb{K} : \mathbb{F}_5] \le 7$ and $[\mathbb{K} : \mathbb{Q}] \le 14$.

Case $s = 4$**,** $k = 5$**.** (This case is the most difficult.)
Then $\mathbb{F}_5 = \mathbb{Q}(\cos^2(\pi/5)) \subset \mathbb{K} = \mathbb{Q}(u^2)$, and $u \in \mathbb{K}$ if and only if $\sqrt{2} \in \mathbb{K}$. We have

$$\frac{d(u)}{2} = u^2 + 2\sqrt{2}\cos\frac{\pi}{5}\,u + 4\cos^2\frac{\pi}{5} - 2 =$$
$$u^2 + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{10}}{2}\right)u + \left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right),$$

and

$$0 < a(\widetilde{\sigma}) = -\widetilde{\sigma}(\sqrt{2}\cos\frac{\pi}{5}) - \sqrt{\sigma(2\sin^2\frac{\pi}{5})} < \widetilde{\sigma}(u) <$$
$$< -\widetilde{\sigma}(\sqrt{2}\cos\frac{\pi}{5}) + \sqrt{\sigma(2\sin^2\frac{\pi}{5})} = b(\widetilde{\sigma}) \tag{29}$$

for all $\sigma : \mathbb{K} \to \mathbb{R}$ which are different from $\sigma^{(+)}$, and

$$4 < \sigma^{(+)}(u^2) < t^2. \tag{30}$$

We take the polynomial $P(x)$ of the degree 8,

$$P(x) = \left(x^2 + \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{10}}{2}\right)x - \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\times$$

$$\left(4x^3+(3\sqrt{2}+3\sqrt{10})x^2+(6+4\sqrt{5})x+\frac{5\sqrt{2}}{2}+\frac{\sqrt{10}}{2}\right)\times$$
$$\left(4x^3+(3\sqrt{2}+3\sqrt{10})x^2+(7+3\sqrt{5})x+\frac{3\sqrt{2}}{2}+\frac{\sqrt{10}}{2}\right) \tag{31}$$

over the ring of integers of $\mathbb{F}_{4,5}=\mathbb{Q}(\sqrt{2},\cos^2(\pi/5))=\mathbb{Q}(\sqrt{2},\sqrt{5})$.

All roots of the polynomials $P^{\widetilde{\sigma}}(x)$ belong to the corresponding intervals $[a(\widetilde{\sigma}),\ b(\widetilde{\sigma})]$. The maxima $\delta(\widetilde{\sigma})$ of $|P^{\widetilde{\sigma}}(x)|$ on the corresponding intervals (29) are as follows. The maxima of $|P^{\widetilde{\sigma}}(x)|$ on the interval

$$[a(\widetilde{\sigma}),\ b(\widetilde{\sigma})]=[-1.97537668..,\ -0.31286893...]$$

is equal to $\delta_1=0.045593135...$ if $\widetilde{\sigma}(\sqrt{2})=\sqrt{2}$ and $\widetilde{\sigma}(\sqrt{5})=\sqrt{5}$; on the interval

$$[a(\widetilde{\sigma}),\ b(\widetilde{\sigma})]=[0.31286893...,1.97537668...]$$

it is equal to $\delta_2=0.045593135...$ if $\widetilde{\sigma}(\sqrt{2})=-\sqrt{2}$ and $\widetilde{\sigma}(\sqrt{5})=\sqrt{5}$; on the interval

$$[a(\widetilde{\sigma}),\ b(\widetilde{\sigma})]=[-0.90798099...,\ 1.78201304...]$$

it is equal to $\delta_3=2.14190686...$ if $\widetilde{\sigma}(\sqrt{2})=\sqrt{2}$ and $\widetilde{\sigma}(\sqrt{5})=-\sqrt{5}$; on the interval

$$[a(\widetilde{\sigma}),\ b(\widetilde{\sigma})]=[-1.78201304...,\ 0.90798099...]$$

it is equal to $\delta_4=2.14190686...$ if $\widetilde{\sigma}(\sqrt{2})=-\sqrt{2}$ and $\widetilde{\sigma}(\sqrt{5})=-\sqrt{5}$.

By Lemma 7,

$$[\mathbb{K}:\mathbb{F}_{4,5}]\le\frac{\ln P(t)-\ln\delta_1}{-\ln(\delta_1\delta_2\delta_3\delta_4)}$$

if $\sqrt{2}\in\mathbb{K}$, and

$$[\mathbb{K}(u):\mathbb{F}_{4,5}]\le\frac{\ln(P(t)P^{\tau}(-t))-\ln(\delta_1\delta_2)}{-\ln(\delta_1\delta_2\delta_3\delta_4)}$$

if $\sqrt{2}\notin\mathbb{K}$ where $\tau(\sqrt{2})=-\sqrt{2}$ and $\tau(\sqrt{5})=\sqrt{5}$. For $t=14$, in the first case, we get $[\mathbb{K}:\mathbb{F}_{4,5}]\le 5$ and $[\mathbb{K}:\mathbb{Q}]\le 20$; for the second case, we get $[\mathbb{K}(u):\mathbb{F}_{4,5}]\le 11$ and $[\mathbb{K}(u):\mathbb{Q}]\le 44$. Then $[\mathbb{K}:\mathbb{Q}]\le 22$. Thus, in both cases, we get $[\mathbb{K}:\mathbb{Q}]\le 22$. (Theorems 5 and 6 give $[\mathbb{K}:\mathbb{Q}]\le 72$ only.)

Case $s=k=5$. Then $\mathbb{F}_5=\mathbb{Q}(\cos^2(\pi/5))\subset\mathbb{K}=\mathbb{Q}(u)$, and we have

$$-2<\sigma(u)<\sigma(4\sin^2\frac{\pi}{5})-2,\quad 2<\sigma^{(+)}(u)<14. \tag{32}$$

We take the polynomial

$$P(x)=(x+2)(x+1)(x+3-4\sin^2\frac{\pi}{5})(x+2-4\sin^2\frac{\pi}{5}) \tag{33}$$

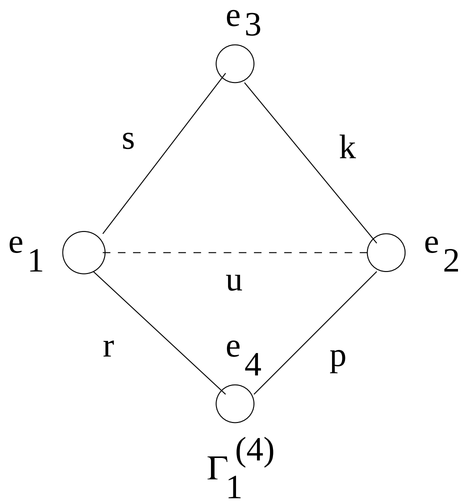

Figure 5: The graph $\Gamma_1^{(4)}$

which is similar to (15). The maxima $\delta(\sigma)$ of $|P^\sigma(x)|$ on the corresponding intervals (32) are equal to $\delta_1 = 0.0455931...$ if $\sigma(\sin^2(\pi/5)) = \sin^2(\pi/5)$; $\delta_2 = 2.1419068...$ if $\sigma(\sin^2(\pi/5)) = \sin^2(2\pi/5)$.

By Lemma 7,

$$[\mathbb{K} : \mathbb{F}_5] \le \frac{\ln P(t) - \ln \delta_1}{-\ln(\delta_1 \delta_2)}.$$

For $t = 14$, we get $[\mathbb{K} : \mathbb{F}_5] \le 6$ and $[\mathbb{K} : \mathbb{Q}] \le 12$.

Thus, finally we get the following statement.

Theorem 11. *We have the following upper bounds for the degrees of ground fields* $\mathbb{K} \in \mathcal{F}\Gamma_1^{(3)}(14)$ *that is for V-arithmetic edge graphs* $\Gamma_1^{(3)}$ *of the minimality* 14 *(i.e.,* $2 < u < 14$*) with parameters* $3 \le s \le k \le 5$ *(see Figure 4):*
$[\mathbb{K} : \mathbb{Q}] \le 23$ *for* $s = k = 3$*;*
$[\mathbb{K} : \mathbb{Q}] \le 12$ *for* $s = 3,\ k = 4$*;*
$[\mathbb{K} : \mathbb{Q}] \le 8$ *for* $s = k = 4$*;*
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K} : \mathbb{F}_5] \le 7$ *for* $s = 3,\ k = 5$*;*
for $s = 4,\ k = 5$ *we have* $\mathbb{F}_{4,5} = \mathbb{Q}(\sqrt{2}, \sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K} : \mathbb{F}_{4,5}] \le 5$ *if* $\sqrt{2} \in \mathbb{K}$*, and* $\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K} : \mathbb{F}_5] \le 11$ *if* $\sqrt{2} \notin \mathbb{K}$*;*
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K} : \mathbb{F}_5] \le 6$ *for* $s = k = 5$*.*

In particular, $[\mathbb{K} : \mathbb{Q}] \le 23$ *for all fields* $\mathbb{K} \in \mathcal{F}\Gamma_1^{(3)}(14)$ *with parameters* $3 \le s \le k \le 5$*.*

4.3 Fields from $\mathcal{F}\Gamma_1^{(4)}(14)$

For $\Gamma_1^{(4)}(14)$ (see Figure 5) we assume that integers $s, k, r, p \ge 3$. Subgraphs $\Gamma_1^{(4)} - \{e_1\}$ and $\Gamma_1^{(4)} - \{e_2\}$ must be Coxeter graphs. It follows that we must consider only (up to obvious symmetries) the following cases: either $s = k = 3$ and $5 \geq r \geq p \geq 3$, or $s = p = 3$ and $5 \geq r \geq k \geq 4$; the totally real algebraic integer u satisfies the inequality $2 < u < 14$.

Since the graph $\Gamma_1^{(4)}(14)$ with the parameters s, k, r, p contains two subgraphs $\Gamma_1^{(3)}(14)$ with the parameters s, k and r, p, we obtain that any field $\mathbb{K} \in \mathcal{F}\Gamma_1^{(4)}(14)$ with the parameters s, k, r, p belongs to $\mathcal{F}\Gamma_1^{(3)}(14)$ with the parameters s, k and to $\mathcal{F}\Gamma_1^{(3)}(14)$ with the parameters p, r. Thus, the degree of fields $\mathbb{K} \in \mathcal{F}\Gamma_1^{(4)}(14)$ with the parameters s, k, r, p is less than minimum of degrees of fields from $\mathcal{F}\Gamma_1^{(3)}(14)$ with the parameters s, k and fields from $\mathcal{F}\Gamma_1^{(3)}(14)$ with the parameters p, r. Thus, applying Theorem 11 we obtain an "easy" result.

Proposition 12. *We have the following upper bounds for the degrees of ground fields $\mathbb{K} \in \mathcal{F}\Gamma_1^{(4)}(14)$ that is for V-arithmetic edge graphs $\Gamma_1^{(4)}$ of the minimality 14 (i.e., $2 < u < 14$) depending on their parameters s, k, p, r (see Figure 5):*
$[\mathbb{K} : \mathbb{Q}] \leq 23$ *for* $s = k = p = r = 3$*;*
$[\mathbb{K} : \mathbb{Q}] \leq 12$ *for* $s = k = 3$, $r = 4$, $p = 3$*;*
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K} : \mathbb{F}_5] \leq 7$ *for* $s = k = 3$, $r = 5$, $p = 3$*;*
$[\mathbb{K} : \mathbb{Q}] \leq 8$ *for* $s = k = 3$, $r = p = 4$*;*
$\mathbb{F}_{4,5} = \mathbb{Q}(\sqrt{2}, \sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K} : \mathbb{F}_{4,5}] \leq 5$ *if* $s = k = 3$, $r = 5$, $p = 4$*;*
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K} : \mathbb{F}_5] \leq 6$ *for for* $s = k = 3$, $r = p = 5$.
$[\mathbb{K} : \mathbb{Q}] \leq 12$ *for* $s = p = 3$, $r = k = 4$*;*
$\mathbb{F}_{4,5} = \mathbb{Q}(\sqrt{2}, \sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K} : \mathbb{F}_{4,5}] \leq 3$ *for* $s = p = 3$, $r = 5$, $k = 4$*;*
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K} : \mathbb{F}_5] \leq 6$ *for* $s = p = 3$, $r = k = 5$.

In particular, $[\mathbb{K} : \mathbb{Q}] \leq 23$ *for all fields* $\mathbb{K} \in \mathcal{F}\Gamma_1^{(4)}(14)$.

By considering the graphs $\Gamma_1^{(4)}$ directly, one can significantly improve Proposition 12. We hope to consider this in further variants of the paper and further publications.

4.4 Fields from $\mathcal{F}\Gamma_2^{(4)}(14)$

For $\Gamma_2^{(4)}(14)$ (see Figure 6), $s, k, p \geq 3$ are natural numbers and $2 < u < 14$ is a totally real algebraic integer. Moreover, we have only the following possibilities: $3 \leq s \leq k \leq 5$, $p = 3$; $s = k = 3$, $p = 4, 5$.

Since the graph $\Gamma_2^{(4)}(14)$ with the parameters s, k, p contains the subgraph $\Gamma_1^{(3)}(14)$ with the parameters s, k, any field $\mathbb{K} \in \mathcal{F}\Gamma_2^{(4)}(14)$ with the parameters s, k, p belongs to $\mathcal{F}\Gamma_1^{(3)}(14)$ with the parameters s, k. Thus, the degree of fields $\mathbb{K} \in \mathcal{F}\Gamma_2^{(4)}(14)$ with the parameters s, k, p is less than the degrees of fields from $\mathcal{F}\Gamma_1^{(3)}(14)$ with the parameters s, k. Thus, applying Theorem 11 we obtain an "easy" result.

Proposition 13. *We have the following upper bounds for the degrees of ground fields $\mathbb{K} \in \mathcal{F}\Gamma_2^{(4)}(14)$ that is for V-arithmetic edge graphs $\Gamma_2^{(4)}$ of the minimality 14 (i.e., $2 < u < 14$) depending on their parameters s, k, p (see Figure 6):*

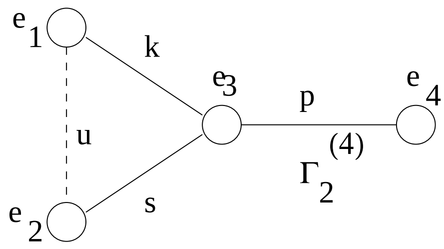

Figure 6: The graph $\Gamma_2^{(4)}$

$[\mathbb{K}:\mathbb{Q}] \leq 23$ *for* $s = k = 3,\ p = 3, 4, 5$*;*
$[\mathbb{K}:\mathbb{Q}] \leq 12$ *for* $s = 3,\ k = 4,\ p = 3$*;*
$[\mathbb{K}:\mathbb{Q}] \leq 8$ *for* $s = k = 4,\ p = 3$*;*
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K}:\mathbb{F}_5] \leq 7$ *for* $s = 3,\ k = 5,\ p = 3$*;*
for $s = 4,\ k = 5,\ p = 3$ *we have* $\mathbb{F}_{4,5} = \mathbb{Q}(\sqrt{2}, \sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K}:\mathbb{F}_{4,5}] \leq 5$ *if* $\sqrt{2} \in \mathbb{K}$*, and* $\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K}:\mathbb{F}_5] \leq 11$ *if* $\sqrt{2} \notin \mathbb{K}$*;*
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K}:\mathbb{F}_5] \leq 6$ *for* $s = k = 5,\ p = 3$.

In particular, $[\mathbb{K}:\mathbb{Q}] \leq 23$ *for all fields* $\mathbb{K} \in \mathcal{F}\Gamma_2^{(4)}(14)$*.*

By considering the graphs $\Gamma_2^{(4)}$ directly, one can significantly improve Proposition 13. We hope to consider this in further variants of the paper and further publications.

4.5 Fields from $\mathcal{F}\Gamma_3^{(4)}(14)$

For $\Gamma_3^{(4)}(14)$ (see Figure 7), $s \geq 2$, $k, r \geq 3$ are natural numbers, and $2 < u < 14$ is a totally real algebraic integer. Moreover, we have only the following possibilities: $s = 2$, $k = 3$, $r = 3$, 4, 5; $s = 2$, $k = 4$, 5, $r = 3$; $3 \leq s \leq k \leq 5$, $r = 3$; $s = k = 3$, $r = 4$, 5.

The ground field $\mathbb{K} = \mathbb{Q}(u^2)$ contains cyclic products

$$\cos^2\frac{\pi}{s},\ \cos^2\frac{\pi}{k},\ \cos^2\frac{\pi}{r},\ u^2,\ u\cos\frac{\pi}{s}\cos\frac{\pi}{k}\cos\frac{\pi}{r}\,.$$

The determinant $d(u)$ of the Gram matrix is determined by the equality

$$-\frac{d(u)}{4} = \sin^2\frac{\pi}{r}\,u^2 + 2\cos\frac{\pi}{s}\cos\frac{\pi}{k}\cos\frac{\pi}{r}\,u + 4\cos^2\frac{\pi}{r} - 4\sin^2\frac{\pi}{s}\sin^2\frac{\pi}{k}. \quad (34)$$

Let

$$D = 4\cos^2\frac{\pi}{s}\cos^2\frac{\pi}{k}\cos^2\frac{\pi}{r} + 16\sin^2\frac{\pi}{s}\sin^2\frac{\pi}{k}\sin^2\frac{\pi}{r} - 16\sin^2\frac{\pi}{r}\cos^2\frac{\pi}{r}$$

be the discriminant of this quadratic polynomial of the variable u. The graph $\Gamma_3^{(4)}(14)$ is V-arithmetic if and only if for $\tau : \mathbb{K}(u) \to \mathbb{R}$ which is different from

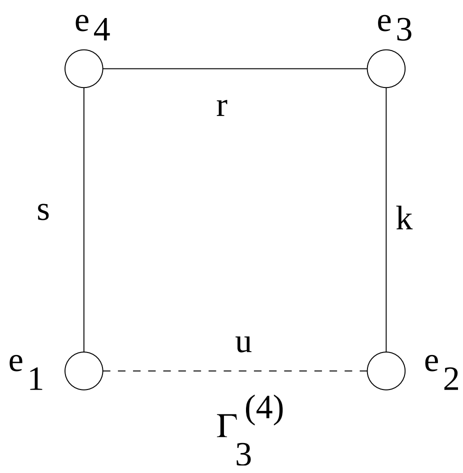

Figure 7: The graph $\Gamma_3^{(4)}$

$\sigma^{(+)}$ on $\mathbb{K}$, one has

$$\frac{-2\widetilde{\tau}\left(\cos\frac{\pi}{s}\cos\frac{\pi}{k}\cos\frac{\pi}{r}\right)-\sqrt{\tau(D)}}{2\tau(\sin^2\frac{\pi}{r})} < \tau(u) <$$

$$< \frac{-2\widetilde{\tau}\left(\cos\frac{\pi}{s}\cos\frac{\pi}{k}\cos\frac{\pi}{r}\right)+\sqrt{\tau(D)}}{2\tau(\sin^2\frac{\pi}{r})} \qquad (35)$$

where $\widetilde{\tau}$ extends τ.

Since the graph $\Gamma_3^{(4)}(14)$ with the parameters s, k, r contains subgraphs $\Gamma_2^{(3)}(14)$ with the parameters s, if $s \geq 3$, and $3 \leq k \leq 5$, we obtain that any field $\mathbb{K} \in \mathcal{F}\Gamma_3^{(4)}(14)$ with the parameters s, k, r belongs to $\mathcal{F}\Gamma_2^{(3)}(14)$ with the parameter s, if $s \geq 3$, and to $\mathcal{F}\Gamma_2^{(3)}(14)$ with the parameter $3 \leq k \leq 5$. Thus, the degree of fields $\mathbb{K} \in \mathcal{F}\Gamma_3^{(4)}(14)$ with the parameters s, k, r is less than minimum of degrees of fields from $\mathcal{F}\Gamma_2^{(3)}(14)$ with the parameter s, if $s \geq 3$, and fields from $\mathcal{F}\Gamma_2^{(3)}(14)$ with the parameters $3 \leq k \leq 5$. Thus, applying Theorem 8 we obtain an "easy" result.

Proposition 14. *We have the following upper bounds for the degrees of ground fields* $\mathbb{K} \in \mathcal{F}\Gamma_3^{(4)}(14)$ *that is for V-arithmetic edge graphs* $\Gamma_3^{(4)}$ *of the minimality* 14 *(i.e.,* $2 < u < 14$*) depending on their parameters* s, k, r *(see Figure 7):*
$[\mathbb{K}:\mathbb{Q}] \leq 44$ *for* $s = 2,\ k = 3,\ r = 3, 4, 5$*;*
$[\mathbb{K}:\mathbb{Q}] \leq 16$ *for* $s = 2,\ k = 4,\ r = 3$*;*
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K}:\mathbb{F}_5] \leq 10$ *for* $s = 2,\ k = 5,\ r = 3$*;*
$[\mathbb{K}:\mathbb{Q}] \leq 44$ *for* $s = k = 3,\ r = 3, 4, 5$*;*
$[\mathbb{K}:\mathbb{Q}] \leq 16$ *for* $s = 3,\ k = 4,\ r = 3$*;*
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K}:\mathbb{F}_5] \leq 10$ *for* $s = 3,\ k = 5,\ r = 3$*;*

$[\mathbb{K}:\mathbb{Q}] \leq 16$ *for* $s = k = 4$, $r = 3$*;*
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K}:\mathbb{F}_5] \leq 8$ *for* $s = 4$, $k = 5$, $r = 3$*;*
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K}:\mathbb{F}_5] \leq 10$ *for* $s = k = 5$, $r = 3$*;*

Let us improve the poor bounds for $s = 2$, $k = 3$ and $s = k = 3$.

For $s = 2$, we have for $\alpha = u^2$ and $\sigma : \mathbb{K} \to \mathbb{R}$

$$0 < \sigma(\alpha) < \sigma\left(D/(4\sin^4(\pi/r))\right)$$

if $\sigma \neq \sigma^{(+)}$.

For $s = 2$, $k = r = 3$,

$$0 < \sigma(\alpha) < \frac{8}{3}, \quad 4 < \sigma^{(+)}(\alpha) < 14^2.$$

We take the polynomial

$$P(x) = x(x-1)^2(x-2)^2(x^2-3x+1)^2. \tag{36}$$

The maximum of $|P(x)|$ on $[0, 8/3]$ is $\delta = 0.148148....$ By Lemma 7,

$$[\mathbb{K}:\mathbb{Q}] \leq 1 + \frac{\ln|P(14^2)|}{-\ln\delta} \leq 25.9.$$

Thus, $[\mathbb{K}:\mathbb{Q}] \leq 25$.

For $s = 2$, $k = 3$, $r = 4$,

$$0 < \sigma(\alpha) < 2, \quad 4 < \sigma^{(+)}(\alpha) < 14^2.$$

We take the polynomial

$$P(x) = x(x-1)^2(x-2). \tag{37}$$

The maximum of $|P(x)|$ on the interval $[0, 2]$ is equal to $\delta = 1/4$. By Lemma 7,

$$[\mathbb{K}:\mathbb{Q}] \leq 1 + \frac{\ln P(14^2)}{-\ln\delta} < 16.3.$$

Thus, $[\mathbb{K}:\mathbb{Q}] \leq 16$.

For $s = 2$, $k = 3$, $r = 5$,

$$0 < \sigma(\alpha) < 4 - (16/5)\sigma(\sin^2(2\pi/5)), \quad 4 < \sigma^{(+)}(\alpha) < 14^2.$$

We take the polynomial

$$P(x) = x(x-1)(x+1-4\sin^2(\pi/5)) \tag{38}$$

over the ring of integers of the field $\mathbb{F}_5 = \mathbb{Q}(\cos^2(\pi/5))$. The maximum of $|P(x)|$ on $[0, 4-(16/5)\sin^2(2\pi/5)]$ is equal to $\delta_1 = 0.0844582....$ The maximum of $|P^g(x)|$ on the interval $[0, 4 - (16/5)\sin^2(\pi/5)]$ where $g : \mathbb{F}_5 \to \mathbb{R}$ is not identity, is equal to $\delta_2 = 1.51554175....$ By Lemma 7,

$$[\mathbb{K} : \mathbb{F}_5] \le \frac{\ln P(14^2) - \ln \delta_1}{-\ln(\delta_1\delta_2)} \le 8.91.$$

Thus, $[\mathbb{K} : \mathbb{F}_5] \le 8$.

Let $s = k = r = 3$. Then

$$0 < \sigma(u^2) < x_1^2 = 2.15612498..., \quad 4 < \sigma^{(+)}(u^2) < 14^2$$

where x_1 is the left root of the polynomial $3x^2 + x - 5$ defined by (34). We take the polynomial

$$P(x) = x(x-1)(x-2). \tag{39}$$

The maximum of $|P(x)|$ on the interval $[0, 2.15612498...]$ is equal to $\delta = 0.3849001....$ By Lemma 7,

$$[\mathbb{K} : \mathbb{Q}] \le 1 + \frac{\ln P(14^2)}{-\ln \delta} \le 17.2.$$

Thus, $[\mathbb{K} : \mathbb{Q}] \le 17$.

Let $s = k = 3$, $r = 4$. Then

$$0 < \sigma(u^2) < x_1^2 = 1.30901699..., \quad 4 < \sigma^{(+)}(u^2) < 14^2,$$

where x_1 is the left root of the polynomial $2x^2 + \sqrt{2}x - 1$ defined by (34). We take the polynomial

$$P(x) = x(x-1)^2. \tag{40}$$

The maximum of $|P(x)|$ on the interval $[0, 1.30901699...]$ is equal to $\delta = 0.14814....$ By Lemma 7,

$$[\mathbb{K} : \mathbb{Q}] \le 1 + \frac{\ln P(14^2)}{-\ln \delta} \le 8.3.$$

Thus, $[\mathbb{K} : \mathbb{Q}] \le 8$.

Let $s = k = 3$, $r = 5$. Then $\mathbb{F}_5 = \mathbb{Q}(\cos^2(\pi/5)) \subset \mathbb{K} = \mathbb{Q}(u)$. In this case, $\sigma(D) < 0$ if $\sigma(\sin^2(\pi/5)) = \sin^2(\pi/5)$. It follows that $\mathbb{K} = \mathbb{F}_5$.

Thus, finally, we get the following result.

Theorem 15. *We have the following upper bounds for the degrees of ground fields $\mathbb{K} \in \mathcal{F}\Gamma_3^{(4)}(14)$ that is for V-arithmetic edge graphs $\Gamma_3^{(4)}$ of the minimality 14 (i.e., $2 < u < 14$) depending on their parameters s, k, r (see Figure 7):*
$[\mathbb{K} : \mathbb{Q}] \le 25$ *for* $s = 2$, $k = r = 3$;
$[\mathbb{K} : \mathbb{Q}] \le 16$ *for* $s = 2$, $k = 3$, $r = 4$;
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K} : \mathbb{F}_5] \le 8$ *for* $s = 2$, $k = 3$, $r = 5$;
$[\mathbb{K} : \mathbb{Q}] \le 16$ *for* $s = 2$, $k = 4$, $r = 3$;

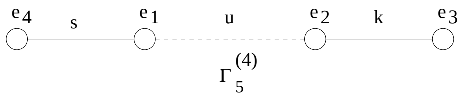

Figure 8: The graph $\Gamma_5^{(4)}$

$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K} : \mathbb{F}_5] \le 10$ *for* $s = 2,\ k = 5,\ r = 3$;
$[\mathbb{K} : \mathbb{Q}] \le 17$ *for* $s = k = r = 3$;
$[\mathbb{K} : \mathbb{Q}] \le 8$ *for* $s = k = 3,\ r = 4$;
$\mathbb{K} = \mathbb{F}_5 = \mathbb{Q}(\sqrt{5})$ *if* $s = k = 3,\ r = 5$.
$[\mathbb{K} : \mathbb{Q}] \le 16$ *for* $s = 3,\ k = 4,\ r = 3$;
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K} : \mathbb{F}_5] \le 10$ *for* $s = 3,\ k = 5,\ r = 3$;
$[\mathbb{K} : \mathbb{Q}] \le 16$ *for* $s = k = 4,\ r = 3$;
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K} : \mathbb{F}_5] \le 8$ *for* $s = 4,\ k = 5,\ r = 3$;
$\mathbb{F}_5 = \mathbb{Q}(\sqrt{5}) \subset \mathbb{K}$ *and* $[\mathbb{K} : \mathbb{F}_5] \le 10$ *for* $s = k = 5,\ r = 3$;

In particular, $[\mathbb{K} : \mathbb{Q}] \le 25$ *for all fields* $\mathbb{K} \in \mathcal{F}\Gamma_3^{(4)}(14)$.

By considering the graphs $\Gamma_2^{(4)}$ directly in other cases, one can significantly improve Theorem 14. We hope to consider this in further variants of the paper and further publications.

4.6 Fields from $\mathcal{F}\Gamma_5^{(4)}(14)$

For $\Gamma_5^{(4)}(14)$ (see Figure 8), $3 \le s \le k$ are natural numbers and $2 < u < 14$ is a totally real algebraic integer.

The ground field $\mathbb{K} = \mathbb{Q}(u^2)$ contains cyclic products

$$\cos^2 \frac{\pi}{k},\ \cos^2 \frac{\pi}{s},\ u^2\ .$$

Thus, $\mathbb{F}_{s,k} = \mathbb{Q}(\cos^2(\pi/s), \cos^2(\pi/k)) \subset \mathbb{K} = \mathbb{Q}(u^2)$. The determinant $d(u)$ of the Gram matrix is determined by the equality

$$-\frac{d(u)}{4} = u^2 - 4\sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{s}.$$

The $\Gamma_5^{(4)}$ is V-arithmetic if and only if for $\alpha = u^2$ we have

$$0 < \sigma(\alpha) < \sigma(4\,\sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{s}) < 4$$

for any $\sigma : \mathbb{K} \to \mathbb{R}$ which is different from identity $\sigma^{(+)}$. For $\sigma^{(+)}$, we have

$$4 < \sigma^{(+)}(\alpha) < 14^2.$$

The graph $\Gamma_5^{(4)}(14)$ with parameters s, k, contains two subgraphs $\Gamma_2^{(3)}(14)$, with parameters $d = s$ and $d = k$. Thus the field $\mathbb{K} \in \mathcal{F}\Gamma_5^{(4)}(14)$ with parameters

$3 \le s \le k$ belongs to $\mathcal{F}\Gamma_2^{(3)}(14)$ with the parameter s and to $\mathcal{F}\Gamma_2^{(3)}(14)$ with the parameter k. By Theorem 8, the degree $[\mathbb{K} : \mathbb{Q}] \le 20$ if one of s and k does not belong to $\{3, 7\}$. Let us assume that both s and k belong to $\{3, 7\}$.

Let $s = k = 3$. Then $0 < \sigma(\alpha) < 9/4$. We take the polynomial

$$P(x) = x(x-1)^2(x-2)^2(x^2-3x+1). \tag{41}$$

The maximum of $|P(x)|$ on the interval $[0, 9/4]$ is equal to $\delta = 0.21570956...$. By Lemma 7

$$[\mathbb{K} : \mathbb{Q}] \le 1 + \frac{\ln P(14^2)}{-\ln \delta} \le 25.06$$

and $[\mathbb{K} : \mathbb{Q}] \le 25$.

Let $s = 3$, $k = 7$. In this case,

$$0 < \sigma(\alpha) < 3\sigma(\sin^2(\pi/7))$$

for $\sigma : \mathbb{F}_7 = \mathbb{Q}(\cos^2(\pi/7)) \to \mathbb{R}$. We take the polynomial

$$P(x) = x^2(x-1)^2(x-2)(x^2-3x+1)(x+1-4\sin^2(\pi/7))(x-4\sin^2(\pi/7)) \tag{42}$$

which can be considered as an appropriate combination of the polynomials (13) for $d = 3$ and (19) for $d = 7$. For $\sigma = \sigma_k : \mathbb{F}_7 \to \mathbb{R}$ where $\sigma_k(\sin^2(\pi/7)) = \sin^2(k\pi/7)$, $k = 1, 2, 3$, we obtain that the maximum of $|P^\sigma(x)|$ on the interval $[0, 3\sigma_k(\sin^2(\pi/7))]$ is equal to $\delta_1 = 0.0050709048...$, $\delta_2 = 0.110711589...$, $\delta_3 = 1.23817314...$ respectively. By Lemma 7,

$$[\mathbb{K} : \mathbb{F}_7] \le \frac{\ln(P(14^2)) - \ln(\delta_1)}{-\ln(\delta_1\delta_2\delta_3)} \le 7.3. \tag{43}$$

Thus $[\mathbb{K} : \mathbb{F}_7] \le 7$ and $[\mathbb{K} : \mathbb{Q}] \le 21$.

Case $s = k = 7$. In this case, $0 < \sigma(\alpha) < \sigma(4\sin^4(\pi/7))$. We take the polynomial

$$P(x) = x(x-1)(x+1-4\sin^2(\pi/7))(x-4\sin^2(\pi/7)) \tag{44}$$

which is the same as we used in (19) for $d = 7$. For $\sigma = \sigma_k$, $k = 1, 2, 3$, in notations above, $|P^\sigma(x)|$ on the interval $[0, \sigma(4\sin^4(\pi/7))]$ has maximum $\delta_1 = 0.0289099468...$, $\delta_2 = 0.52203650...$, $\delta_3 = 2.93338845...$. Like above, we get $[\mathbb{K} : \mathbb{F}_7] \le 7$ and $[\mathbb{K} : \mathbb{Q}] \le 21$.

Thus, we obtain the following result.

Theorem 16. *We have the following upper bounds for the degrees of ground fields $\mathbb{K} \in \mathcal{F}\Gamma_5^{(4)}(14)$ that is for V-arithmetic edge graphs $\Gamma_5^{(4)}$ of the minimality 14 (i.e., $2 < u < 14$) depending on their parameters s, k (see Figure 8):*
$[\mathbb{K} : \mathbb{Q}] \le 25$ *for* $s = k = 3$*;*
$[\mathbb{K} : \mathbb{Q}] \le 21$ *for* $s = 3$, $k = 7$*;*
$[\mathbb{K} : \mathbb{Q}] \le 21$ *for* $s = k = 7$
$[\mathbb{K} : \mathbb{Q}] \le 20$ *for all other* $3 \le s \le k$*.*

In particular, $[\mathbb{K} : \mathbb{Q}] \le 25$ *for all fields* $\mathbb{K} \in \mathcal{F}\Gamma_5^{(4)}(14)$*.*

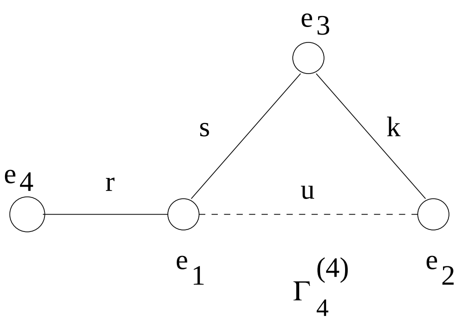

Figure 9: The graph $\Gamma_4^{(4)}$

This Theorem can be significantly improved using our results and methods from [14]—[16]. We hope to do that in further variants of this paper and further publications.

4.7 Fields from $\mathcal{F}\Gamma_4^{(4)}(14)$

For $\Gamma_4^{(4)}(14)$ (see Figure 9), $k \geq 2$, $s, r \geq 3$ are natural numbers and $2 < u < 14$ is a totally real algebraic integer. Moreover, we have only the following possibilities: $s = 3$, $r = 3, 4, 5$; $s = 4, 5$, $r = 3$.

The ground field $\mathbb{K} = \mathbb{Q}(u^2)$ contains cyclic products

$$\cos^2 \frac{\pi}{s},\ \cos^2 \frac{\pi}{r},\ \cos^2 \frac{\pi}{k},\ u^2,\ u \cos \frac{\pi}{s} \cos \frac{\pi}{k}\,.$$

This case can be considered as a specialization of the graph $\Gamma_1^{(4)}$ when we take $p = 2$. The determinant $d(u)$ of the Gram matrix is determined by the equality

$$\frac{-d(u)}{4} = u^2 + 4 \cos \frac{\pi}{s} \cos \frac{\pi}{k}\, u + 4\, \cos^2 \frac{\pi}{s} - 4 \sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{r}.$$

We obtain

$$\sigma(u^2 + 4 \cos \frac{\pi}{s} \cos \frac{\pi}{k}\, u + 4\, \cos^2 \frac{\pi}{s} - 4 \sin^2 \frac{\pi}{k} \sin^2 \frac{\pi}{r}) < 0 \tag{45}$$

for $\sigma \neq \sigma^{(+)}$, and

$$4 < \sigma^{(+)}(u^2) < 14^2. \tag{46}$$

We then have $\mathbb{K} = \mathbb{Q}(u^2)$.

Assume that $k = 2$. Then we obtain

$$0 < \sigma(u^2) < 4 - \sigma(4 \cos^2 \frac{\pi}{s}) - \sigma(4 \cos^2 \frac{\pi}{r}).$$

If $s = r = 3$, we obtain $0 < \sigma(u^2) < 2$. Applying (like above) the polynomial (14), $P(x) = x(x-1)^2(x-2)$, we obtain $[\mathbb{K} : \mathbb{Q}] \leq 16$.

If $(s,r) = (3,4)$, $(4,3)$, we obtain $0 < \sigma(u^2) < 1$. Applying the polynomial $P(x) = x(x-1)$, we get $[\mathbb{K}:\mathbb{Q}] \leq 8$.

If $(s,r) = (3,5)$ or $(5,3)$, we obtain $0 < \sigma(u^2) < \sigma(4\sin^2(\pi/5)) - 1$. Applying the polynomial $P(x) = x(x-1)(x+1-4\sin^2(\pi/5))$, we obtain $[\mathbb{K}:\mathbb{F}_5] \leq 5$ and $[\mathbb{K}:\mathbb{Q}] \leq 10$.

Assume that $k \geq 3$. We have that

$$\widetilde{u} = u^2 + 4\cos\frac{\pi}{s}\cos\frac{\pi}{k}\,u + 4\cos^2\frac{\pi}{s} \tag{47}$$

is a totally positive algebraic integer (since minimum of this quadratic polynomial of u is equal to $4\cos^2\frac{\pi}{s}\sin^2\frac{\pi}{k}$) which belongs to $\mathbb{K}$, and $\Gamma_4^{(4)}$ is V-arithmetic if and only if

$$0 < \sigma(4\cos^2\frac{\pi}{s}\sin^2\frac{\pi}{k}) \leq \sigma(\widetilde{u}) < \sigma(4\,\sin^2\frac{\pi}{r}\sin^2\frac{\pi}{k}) < 4 \tag{48}$$

for any $\sigma : \mathbb{K} \to \mathbb{R}$ which is different from identity $\sigma^{(+)}$. For $\sigma^{(+)}$, we have

$$4 < 4+8\cos\frac{\pi}{s}\cos\frac{\pi}{k}+4\cos^2\frac{\pi}{s} < \sigma^{(+)}(\widetilde{u}) < 14^2+56\cos\frac{\pi}{s}\cos\frac{\pi}{k}+4\cos^2\frac{\pi}{s} < 16^2. \tag{49}$$

It follows that $\mathbb{K} = \mathbb{Q}(\widetilde{u})$.

Further we use only the inequalities

$$0 < \sigma(\widetilde{u}) < \sigma(4\,\sin^2\frac{\pi}{r}\sin^2\frac{\pi}{k}) < 4$$

and

$$4 < \sigma^{(+)}(\widetilde{u}) < 16^2.$$

Then this case is similar to the previous case of $\Gamma_5^{(4)}(14)$ if we replace 14^2 by 16^2 (or use (49) if necessary), and take $s = \min(r,k)$. Instead of Theorem 8 we should use Theorem 9.

Here similar bad cases are as follows:

Case $r = k = 3$. Then $[\mathbb{K}:\mathbb{Q}] \leq 25$ (here we use $14^2+56\cos\frac{\pi}{s}\cos\frac{\pi}{k}+4\cos^2\frac{\pi}{s}$ instead of 14^2).

Case $r = 3$, $k = 7$. Then $[\mathbb{K}:\mathbb{Q}] \leq 21$

Thus, we obtain the following.

Theorem 17. *We have the following upper bounds for the degrees of ground fields $\mathbb{K} \in \mathcal{F}\Gamma_4^{(4)}(14)$ that is for V-arithmetic edge graphs $\Gamma_4^{(4)}$ of the minimality 14 (i.e., $2 < u < 14$) depending on their parameters k, s, r (see Figure 8):*
$[\mathbb{K}:\mathbb{Q}] \leq 16$ *for* $k = 2$, $s = r = 3$;
$[\mathbb{K}:\mathbb{Q}] \leq 8$ *for* $k = 2$, $(s,r) = (3,4)$, $(4,3)$;
$[\mathbb{K}:\mathbb{Q}] \leq 10$ *for* $k = 2$, $(s,r) = (3,5)$, $(5,3)$;
$[\mathbb{K}:\mathbb{Q}] \leq 25$ *for* $r = k = 3$;
$[\mathbb{K}:\mathbb{Q}] \leq 21$ *for* $r = 3$, $k = 7$;
$[\mathbb{K}:\mathbb{Q}] \leq 20$ *for all remaining* k, s, r.

In particular, $[\mathbb{K}:\mathbb{Q}] \leq 25$ *for all fields* $\mathbb{K} \in \mathcal{F}\Gamma_4^{(4)}(14)$.

This Theorem can be significantly improved using our results and methods from [14]—[16]. We hope to demonstrate that in further variants of this paper and further publications.

This finishes the proof of Theorem 4.

5 Applications

Using results of [9]—[16], we obtain the following applications to arithmetic hyperbolic reflection groups.

Theorem 18. *In dimensions $n \ge 10$, the ground field of any arithmetic hyperbolic reflection group belongs to one of finite sets of fields $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \le i \le 5$. In particular, its degree has the upper bound 25.*

Proof. In [9], [10] (see more details in [14]) we showed that the field belongs to $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \le i \le 5$. By Theorem 4 of this paper, than the degree has the upper bound 25. (See more details in the proof of the next Theorem.) □

Theorem 19. *In dimensions $n \ge 6$, the ground field of any arithmetic hyperbolic reflection group belongs to one of finite sets of fields $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \le i \le 5$, and $\mathcal{F}\Gamma_{2,4}(14)$. In particular, its degree has the upper bound 25.*

Proof. This is similar to [14, Theorem 9] where we claimed this theorem for 56.

In [14], we showed that the field belongs to one of sets $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \le i \le 5$, and $\mathcal{F}\Gamma_{2,4}(14)$. It is well-known that the degree of fields from $\mathcal{F}L^4$ has the upper bound 2. By Takeuchi [18] (see also [17]—[20]), the degree of fields from $\mathcal{F}T$ has the upper bound 5. In [14], we showed that the degree of fields from $\mathcal{F}\Gamma_{2,4}(14)$ has the upper bound 11. By Theorem 4 of this paper, the degree of fields from $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \le i \le 5$, has the upper bound 25.

It follows the statement. □

Theorem 20. *In dimensions $n = 4, 5$, the ground field of any arithmetic hyperbolic reflection group belongs to one of finite sets of fields $\mathcal{F}\Gamma_1^{(6)}(14)$, $\mathcal{F}\Gamma_2^{(6)}(14)$, $\mathcal{F}\Gamma_3^{(6)}(14)$, $\mathcal{F}\Gamma_1^{(7)}(14)$, $\mathcal{F}\Gamma_2^{(7)}(14)$, and $\mathcal{F}\Gamma_{2,5}(14)$ if it is different from a field from the sets $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \le i \le 5$, $\mathcal{F}\Gamma_{2,4}(14)$ in Theorem 19 above.*

In particular, its degree has the upper bound 44.

Proof. This is similar to [15, Theorem 2.6] where we claimed the upper bound 138.

In [15], we showed that if a field is different from a field from $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \le i \le 5$, $\mathcal{F}\Gamma_{2,4}(14)$, then it belongs to $\mathcal{F}\Gamma_1^{(6)}(14)$, $\mathcal{F}\Gamma_2^{(6)}(14)$, $\mathcal{F}\Gamma_3^{(6)}(14)$, $\mathcal{F}\Gamma_1^{(7)}(14)$, $\mathcal{F}\Gamma_2^{(7)}(14)$, and $\mathcal{F}\Gamma_{2,5}(14)$. In [15] we showed that the degree of fields from $\mathcal{F}\Gamma_{2,5}(14)$ has the upper bound 12.

Assume that a field belongs to $\mathcal{F}\Gamma_1^{(6)}(14)$, $\mathcal{F}\Gamma_2^{(6)}(14)$, $\mathcal{F}\Gamma_3^{(6)}(14)$, $\mathcal{F}\Gamma_1^{(7)}(14)$, $\mathcal{F}\Gamma_2^{(7)}(14)$. All diagrams $\Gamma_1^{(6)}(14)$, $\Gamma_2^{(6)}(14)$, $\Gamma_3^{(6)}(14)$, $\Gamma_1^{(7)}(14)$, $\Gamma_2^{(7)}(14)$ have a subdiagram $\Gamma_2^{(3)}(14)$. By Theorem 10, then the degree of the field has the upper bound 44.

It follows the statement. $\square$

Theorem 21. *In dimension $n = 3$, the ground field of any arithmetic hyperbolic reflection group belongs to one of finite sets of fields $\mathcal{F}\Gamma_6^{(4)}(14)$, $\mathcal{F}\Gamma_1^{(5)}(14)$, $\mathcal{F}\Gamma_4^{(6)}(14)$, and $\mathcal{F}\Gamma_2(14)$ if it is different from a field from the sets $\mathcal{F}\Gamma_1^{(6)}(14)$, $\mathcal{F}\Gamma_2^{(6)}(14)$, $\mathcal{F}\Gamma_3^{(6)}(14)$, $\mathcal{F}\Gamma_1^{(7)}(14)$, $\mathcal{F}\Gamma_2^{(7)}(14)$, $\mathcal{F}\Gamma_{2,5}(14)$ and $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \le i \le 5$, $\mathcal{F}\Gamma_{2,4}(14)$ in Theorems 19 and 20 above.*

In particular, its degree has the upper bound 44.

Proof. This is similar to [16, Theorem 3.8] where we claimed the upper bound 909.

In [16], we showed that if a field is different from a field from $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \le i \le 5$, $\mathcal{F}\Gamma_{2,4}(14)$, and $\mathcal{F}\Gamma_1^{(6)}(14)$, $\mathcal{F}\Gamma_2^{(6)}(14)$, $\mathcal{F}\Gamma_3^{(6)}(14)$, $\mathcal{F}\Gamma_1^{(7)}(14)$, $\mathcal{F}\Gamma_2^{(7)}(14)$, $\mathcal{F}\Gamma_{2,5}(14)$, then it belongs to one of sets $\mathcal{F}\Gamma_6^{(4)}(14)$, $\mathcal{F}\Gamma_1^{(5)}(14)$, $\mathcal{F}\Gamma_4^{(6)}(14)$, and $\mathcal{F}\Gamma_2(14)$.

Assume that a field belongs to one of sets $\mathcal{F}\Gamma_6^{(4)}(14)$, $\mathcal{F}\Gamma_1^{(5)}(14)$, $\mathcal{F}\Gamma_4^{(6)}(14)$, and $\mathcal{F}\Gamma_2(14)$.

The set $\mathcal{F}\Gamma_2(14)$ consists of ground fields of arithmetic hyperbolic reflection groups in dimension 2 with the fundamental polygon of the minimality 14. We showed in [14] using results of [7] that the degree of ground fields of arithmetic hyperbolic reflection groups of dimension 2 has the upper bound 44.

All diagrams $\Gamma_6^{(4)}(14)$, $\Gamma_1^{(5)}(14)$, $\Gamma_4^{(6)}(14)$ have a subdiagram $\Gamma_2^{(3)}(14)$. By Theorem 10, then the degree of the field has the upper bound 44.

It follows the statement. $\square$

We remark that Theorem 21 also gives another proof of finiteness in dimension 3 which is different from the proof by I. Agol [1].

In [2], Belolipetsky showed (using results of [1]) that the degree of ground fields of arithmetic hyperbolic reflection groups in dimension 3 has the upper bound 35. Using this result and results of [14], we can improve the upper bound above in dimensions 4 and 5.

Theorem 22. *In dimensions $n = 4$, 5, the ground field of an arithmetic hyperbolic reflection group belongs to one of sets $\mathcal{F}L^4$, $\mathcal{F}T$, $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \le i \le 5$, or it is equal to the ground field of an arithmetic hyperbolic reflection group in dimension 2 or 3.*

In particular, its degree has the upper bound 35.

Proof. In [13] we proved the first statement of the theorem.

The degree of fields from $\mathcal{F}L^4$, $\mathcal{F}T$ has the upper bound 5. By Theorem 4 of this paper, the degree of fields from $\mathcal{F}\Gamma_i^{(4)}(14)$, $1 \le i \le 5$, has the upper bound 25.

By [2], the degree of fields in dimension 3 has the upper bound 35.
By [8], the degree of fields in dimension 2 has the upper bound 11.
It follows the statement. □

Theorems 10 and 8 can be also considered as a small step to classification.

Theorem 23. *Assume that a narrow face of the minimality* 14 *of the fundamental chamber (it does exist by [9]) of an arithmetic hyperbolic reflection group contains an edge with the hyperbolic connected component of its graph containing a subgraph* $\Gamma_2^{(3)}$ *with the parameter* d *(see Figure 4).*

Then $d \leq 120$*, and the ground field is* $\mathbb{F}_d = \mathbb{Q}(\cos^2(\pi/d))$ *if* $d \geq 44$ *(see more exact results in Theorem 8).*

6 Appendix: Hyperbolic numbers (the review of [10, Sec. 1])

Here we review our results in [10, Sec.1] and correct some arithmetic mistakes (Theorems 1.1.1 and 1.2.2 in [10] which are similar to Theorems 25 and 27 here). This mistakes are unessential for results of [10].

6.1 Fekete's theorem

The following important theorem, to which this section is devoted, was obtained by Fekete [5]. Although Fekete considered (as we know) the case of $\mathbb{Q}$ his method of proof can be immediately carried over to totally real algebraic number fields.

Theorem 24. *(M. Fekete). Suppose that* $\mathbb{F}$ *is a totally real algebraic number field, and to every embedding* $\sigma : \mathbb{F} \to \mathbb{R}$ *there corresponds an interval* $[a_\sigma, b_\sigma]$ *in* $\mathbb{R}$ *and the real number* $\lambda_\sigma > 0$*. Suppose that* $\prod_\sigma \lambda_\sigma = 1$*. Then for every nonnegative integer* n *there exists a nonzero polynomial* $P_n(T) \in \mathbb{O}[T]$ *of degree no greater than* n *over the ring of integers* $\mathbb{O}$ *of* $\mathbb{F}$ *such that the following inequality holds for each* σ*:*

$$\max_{x\in[a_\sigma,b_\sigma]} |P_n^\sigma(x)| \leq \lambda_\sigma |\operatorname{discr} \mathbb{F}|^{1/(2[\mathbb{F}:\mathbb{Q}])} 2^{n/(n+1)}(n+1) \left(\prod_\sigma \frac{b_\sigma - a_\sigma}{4}\right)^{n/(2[\mathbb{F}:\mathbb{Q}])}. \tag{50}$$

Proof. Suppose that $N = [\mathbb{F} : \mathbb{Q}]$ and that $\gamma_1, \dots \gamma_N$ is the basis for $\mathbb{O}$ over $\mathbb{Z}$. Suppose we are given a nonzero polynomial

$$P_n(T) = \sum_{i=0}^{n} \sum_{j=1}^{N} \alpha_{ij}\gamma_j T^i \in \mathbb{O}[T]$$

of degree no greater than n, where the $\alpha_{ij} \in \mathbb{Z}$ are not all zero. For every $\sigma : \mathbb{F} \to \mathbb{R}$ we consider the real functions $P_n^\sigma(x)$ on the interval $[a_\sigma, b_\sigma]$.

We make the change of variables

$$x = x(z) = \frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z.$$

If z runs through $[0, \pi]$, then x runs through $[a_\sigma, b_\sigma]$. We also set $Q_n^\sigma(z) = P_n^\sigma(x(z))$.

Since $Q_n^\sigma(z)$ is an even trigonometric polynomial, it follows that

$$Q_n^\sigma(z) = \sum_{k=0}^{n} A_{k\sigma} \cos kz, \tag{51}$$

where

$$A_{k\sigma} = \frac{1}{\pi} \int_{-\pi}^{\pi} P_n^\sigma \left(\frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right) \cos kz \, dz$$

$$= \sum_{i=0}^{n} \sum_{j=1}^{N} \left(\frac{1}{\pi} \int_{-\pi}^{\pi} \gamma_j^\sigma \left(\frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right)^i \cos kz \, dz \right) \alpha_{ij} \,,$$

if $k \geq 1$, and

$$A_{0\sigma} = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_n^\sigma \left(\frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right) dz$$

$$= \sum_{i=0}^{n} \sum_{j=1}^{N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \gamma_j^\sigma \left(\frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right)^i dz \right) \alpha_{ij} \,.$$

Thus,

$$A_{k\sigma} = \sum_{i=0}^{n} \sum_{j=1}^{N} c_{k\sigma ij} \alpha_{ij} \tag{52}$$

are linear functions of the α_{ij}, where

$$c_{k\sigma ij} = \gamma_j^\sigma \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right)^i \cos kz \, dz \,,$$

if $k \geq 1$, and

$$c_{0\sigma ij} = \gamma_j^\sigma \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{b_\sigma + a_\sigma}{2} + \frac{b_\sigma - a_\sigma}{2} \cos z \right)^i dz \,.$$

We note that, because of these formulas, $c_{k\sigma ij} = 0$ for $i < k$, and

$$c_{k\sigma kj} = \gamma_j^\sigma \cdot 2 \left(\frac{b_\sigma - a_\sigma}{4} \right)^k \,, \quad \text{if } k \geq 1$$

($c_{0\sigma 0j} = \gamma_j^\sigma$). Hence, if we order the indices $k\sigma$ and ij lexicographically, we find that the matrix of the linear forms (52) is an upper block-triangular matrix with the shown above $N \times N$ matrices $(c_{0\sigma 0j})$ and $(c_{k\sigma kj})$, $1 \le k \le n$, on the diagonal. It follows that its determinant is equal to

$$\Delta = \det(\gamma_j^\sigma)^{n+1} \cdot 2^{Nn} \left(\prod_\sigma \frac{b_\sigma - a_\sigma}{4} \right)^{n(n+1)/2}.$$

Since $\prod_\sigma \lambda_\sigma^{n+1} = (\prod_\sigma \lambda_\sigma)^{n+1} = 1$, according to Minkowski's theorem on linear forms (see, for example, [3], [4]), there exist $\alpha_{ij} \in \mathbb{Z}$, not all zero, such that $|A_{k\sigma}| \le \lambda_\sigma |\Delta|^{1/N(n+1)}$, and hence, by (51),

$$\max_z |Q_n^\sigma(z)| \le \lambda_\sigma \cdot (n+1)|\Delta|^{1/N(n+1)}.$$

Taking into account that $\det(\gamma_j^\sigma)^2 = \operatorname{discr} \mathbb{F}$, we obtain the proof of the theorem. □

Taking $\lambda_\sigma = 1$, we get a particular statement which we later use.

Theorem 25. *(M. Fekete). Suppose that $\mathbb{F}$ is a totally real algebraic number field, and to every embedding $\sigma : \mathbb{F} \to \mathbb{R}$ there corresponds an interval $[a_\sigma, b_\sigma]$ in $\mathbb{R}$. Then for every nonnegative integer n there exists a nonzero polynomial $P_n(T) \in \mathbb{O}[T]$ of degree no greater than n over the ring of integers $\mathbb{O}$ of $\mathbb{F}$ such that the following inequality holds for each σ:*

$$\max_{x \in [a_\sigma, b_\sigma]} |P_n^\sigma(x)| \le |\operatorname{discr} \mathbb{F}|^{1/(2[\mathbb{F}:\mathbb{Q}])} 2^{n/(n+1)} (n+1) \left(\prod_\sigma \frac{b_\sigma - a_\sigma}{4} \right)^{n/(2[\mathbb{F}:\mathbb{Q}])}. \quad (53)$$

6.2 Hyperbolic numbers

The totally real algebraic integers $\{\alpha\}$ which we consider here are very similar to Pisot-Vijayaraghavan numbers [4], although the later are not totally real.

Theorem 26. *Let $\mathbb{F}$ be a totally real algebraic number field, and let each embedding $\sigma : \mathbb{F} \to \mathbb{R}$ corresponds to an interval $[a_\sigma, b_\sigma]$ in $\mathbb{R}$, where*

$$\prod_\sigma \frac{b_\sigma - a_\sigma}{4} < 1.$$

In addition, let the natural number m and the intervals $[s_1, t_1], \ldots, [s_m, t_m]$ in $\mathbb{R}$ be fixed.

Then there exists a constant $N(s_i, t_i)$ such that, if α is a totally real algebraic integer and if the following inequalities hold for the embeddings $\tau : \mathbb{F}(\alpha) \to \mathbb{R}$:

$$s_i \le \tau(\alpha) \le t_i, \text{ for } \tau = \tau_1, \ldots, \tau_m,$$

$$a_{\tau|\mathbb{F}} \le \tau(\alpha) \le b_{\tau|\mathbb{F}} \text{ for } \tau \ne \tau_1, \ldots \tau_m,$$

then

$$[\mathbb{F}(\alpha):\mathbb{F}] \le N(s_i, t_i).$$

Theorem 27. *Under the conditions of Theorem 26, $N(s_i, t_i)$ can be taken to be $N(s_i, t_i) = N(S)$, where $N(S)$ is the least natural number solution of the inequality*

$$nln(1/R) - M\ln(2n+2) - \ln B \ge \ln S. \tag{54}$$

Here

$$M = [\mathbb{F}:\mathbb{Q}], \quad R = \sqrt{\prod_{\sigma} \frac{b_\sigma - a_\sigma}{4}}, \tag{55}$$

$$B = \sqrt{|\operatorname{discr}\mathbb{F}|}, \quad S = \prod_{i=1}^{m} \left(2er_i(b_{\sigma_i} - a_{\sigma_i})^{-1}\right), \tag{56}$$

where $\sigma_i = \tau_i|\mathbb{F}$ and $r_i = \max\{|t_i - a_{\sigma_i}|,\ |b_{\sigma_i} - s_i|\}$.

Asymptotically,

$$N(s_i, t_i) \sim \frac{\ln S}{\ln(1/R)}.$$

Proof. We use the following statement.

Lemma 28. *Suppose that $Q_n(T) \in \mathbb{R}[T]$ is a non-zero polynomial over $\mathbb{R}$ of degree no greater than $n > 0$, $a < b$ and $M_0 = \max_{[a,b]}\{|Q_n(x)|\}$. Then for $x \ge b$*

$$|Q_n(x)| \le \frac{M_0(x-a)^n n^n}{((b-a)/2)^n\, n!} <$$

$$\frac{M_0(x-a)^n e^n}{((b-a)/2)^n\sqrt{2\pi n}} < \frac{M_0(x-a)^n e^n}{((b-a)/2)^n}.$$

Proof. Let $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. Then we have the Lagrange interpolation formula

$$Q_n(x) = \sum_{i=0}^{n} Q_n(\alpha_i)F_i(x)$$

where

$$F_i(x) = \frac{(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{i-1})(x-\alpha_{i+1})\cdots(x-\alpha_n)}{(\alpha_i-\alpha_0)(\alpha_i-\alpha_1)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_i-\alpha_n)}.$$

Taking $\alpha_i = a + i(b-a)/n$, $0 \le i \le n$, we obtain for $x \ge b$ that

$$|Q_n(x)| \le \frac{M_0(x-a)^n}{((b-a)/n)^n}\sum_{i=0}^{n}\frac{1}{i!(n-i)!} = \frac{M_0(x-a)^n 2^n}{((b-a)/n)^n n!}.$$

By Stirling formula, $n! = \sqrt{2\pi n}(n/e)^n e^{\lambda_n}$ where $0 < \lambda_n < 1/(12n)$. Thus, $n^n/n! < e^n/\sqrt{2\pi n} < e^n$. It follows the statement. □

We continue the proof of theorems.

For given n we consider the polynomial $P_n(T) \in \mathbb{O}[T]$ whose existence is ensured by Fekete's theorem 25. Setting $N = [\mathbb{F}(\alpha) : \mathbb{F}]$ and $M = [\mathbb{F} : \mathbb{Q}]$, we use Fekete's theorem and the lemma to conclude that

$$|\prod_\tau \tau(P_n(\alpha))| = \prod_\tau |P_n^\tau(\tau(\alpha))| = \prod_{\tau\neq\tau_i} P_n^\tau(\tau(\alpha)) \prod_{i=1}^m |P_n^{\tau_i}(\tau_i(\alpha))|$$

$$\leq \prod_{\tau\neq\tau_i} \max_{[a_{\tau|\mathbb{F}}, b_{\tau|\mathbb{F}}]} |P_n^\tau(x)| \prod_{i=1}^m \max_{[s_i,t_i]} |P_n^{\sigma_i}(x)|$$

$$\leq \Big(|\operatorname{discr} \mathbb{F}|^{1/(2M)} \cdot 2 \cdot (n+1) R^{n/M}\Big)^{NM} \prod_{i=1}^m \frac{r_i^n e^n}{((b_{\sigma_i} - a_{\sigma_i})/2)^n}$$

$$= R^{nN} B^N \cdot S^n \cdot (2n+2)^{MN}.$$

Since $R < 1$, there exists n_0 large enough so that

$$R^{n_0} \cdot B \cdot (2n_0 + 2)^M \leq \frac{1}{S}. \tag{57}$$

Then if $N > n_0$, we find that

$$R^{n_0 N} \cdot B^N \cdot S^{n_0} (2n_0 + 2)^{MN} \leq S^{n_0 - N} < 1,$$

since $S > 1$. From this and the above chain of inequalities we have

$$|\prod_\tau \tau(P_{n_0}(\alpha))| < 1.$$

But

$$\prod_\tau \tau(P_{n_0}(\alpha)) = N_{\mathbb{F}(\alpha)/\mathbb{Q}}(P_{n_0}(\alpha)) \in \mathbb{Z},$$

and hence $P_{n_0}(\alpha) = 0$. Consequently, $N \leq n_0$, and we have obtained a contradiction. We have thereby proved that $N \leq n_0$, where n_0 is a natural number solution of (57). The inequality (57) is obviously equivalent to

$$n_0 \ln(1/R) - M \ln(2n_0 + 2) - \ln B \geq \ln S,$$

and this completes the proof of the theorems. □

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