ARITHMETICS OF CONSERVATIVE POLYNOMIALS AND YET ANOTHER ACTION OF Gal$(\bar{\mathbb{Q}}/\mathbb{Q})$ ON PLANE TREES

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1. Introduction

In this paper we introduce a "dynamical" action D of the absolute Galois group $\Gamma = \mathrm{Gal}\,(\bar{\mathbb{Q}}/\mathbb{Q})$ on bicolored plane trees which is distinct from the "geometric" action G provided by the Grothendieck theory of "Dessins d'enfants". Like the action G, D is defined as an action on certain equivalence classes of complex polynomials which are in one-to-one correspondence with classes of isotopical equivalence of bicolored plane trees. However, instead of Shabat polynomials, i.e., of polynomials with only two critical values, to define D we use conservative polynomials, i.e., polynomials with all critical points fixed.

Conservative polynomials were introduced by S. Smale [16] in connection with his "mean value conjecture" which states that for any non-constant polynomial $p(z)$ and $z \in \mathbb{C}$, $p'(z) \neq 0$, there exists a critical point $\zeta \in \mathbb{C}$ of $p(z)$ such that

$$\left| \frac{p(\zeta) - p(z)}{\zeta - z} \right| \leq |p'(z)| \, .$$

Smale's conjecture has been verified for $\deg p(z) \leq 4$ and in some other special cases (see [14], [17], [18]) but the general case still remains open.

The value of

$$S_\zeta = \left| \frac{p(\zeta) - p(z)}{\zeta - z} \right| \left| \frac{1}{p'(z)} \right|$$

is unchanged if we replace $p(z)$ by the polynomial $\tilde{p}(z) = (A \circ p \circ A^{-1})(z)$, and z, ζ by the points $\tilde{z} = A(z), \tilde{\zeta} = A(\zeta)$, where $A(z)$ is a degree one polynomial. Moreover, S_ζ is also unchanged if we replace $p(z)$ by $B(p(z))$, where $B(z)$ is a degree one polynomial. Therefore, in the mean value conjecture one can suppose without loss of generality that $z = 0$ and $p(0) = 0$. Then the question is to estimate

$$\min \left\{ \left| \frac{p(\zeta)}{\zeta p'(0)}, \right| \; : \; p'(\zeta) = 0 \right\}.$$

Now, if $p(z)$ is a conservative polynomial, then all $p(\zeta)/\zeta$ are equal to 1 and Smale's conjecture reduces to the inequality $|p'(0)| \geq 1$.

Motivated by Smale's conjecture A. Kostrikin proposed in [5] several conjectures concerning conservative polynomials. In particular, basing on numerical experiments A. Kostrikin conjectured that, under some normalization conditions, the number of degree d conservative polynomials is finite and is equal to C_{2d-2}^{d-1}. This conjecture was proved by D. Tischler in the paper [17]. Moreover, in this paper a one-to-one correspondence between equivalence classes of conservative polynomials as holomorphic dynamical systems on $\mathbb{C}$ and equivalence classes of embeddings of bicolored trees into the plane was established.

From the finiteness of equivalence classes of conservative polynomials of a given degree one can deduce that in each equivalence class there exist polynomials with algebraic coefficients. This fact makes possible to define the action D similarly to the action G. Indeed, Γ acts on conservative polynomials over $\bar{\mathbb{Q}}$ coefficientwise and this action descends to an action on their equivalence classes. Therefore, by Tischler's theorem we obtain an action on bicolored plane trees.

The action D seems to be very different from the action G. For instance, while the lists of valencies of "black" and "white" vertices of a tree are Galois invariant with respect to the action G, for the action D only the list of valencies of "white" vertices is Galois invariant. In particular, for the action D the natural "combinatorial orbit" containing a given tree λ is much longer than the corresponding orbit for the action G. Furthermore, unlike the Grothendieck correspondence the change of coloring of a tree leads to a complete alteration of the corresponding polynomials. Finally, it is not clear whether there exist any natural "morphisms" compatible with the action D. In this connection note that many results of the "Dessins d'enfants" theory (for instance, the Belyi theorem) make use of compositions of corresponding functions; on the other hand, we show below that all conservative polynomials not equivalent to z^n for composite n are indecomposable.

Note that conservative polynomials are simplest examples of postcritically finite polynomials that is of polynomials for which orbits of critical points under iterations are finite. The corresponding dynamical systems also can be classified in certain combinatorial terms by means of so-called Hubbard trees but this general construction is more complicated (see [8], [9]). Note also that for any Shabat polynomial $P(z)$ and a linear polynomial $B(z)$ that sends critical values of $P(z)$ to some critical points of $P(z)$ the composition $B \circ P(z)$ is a postcritically finite polynomial. Therefore, the Shabat polynomials and the action G also can be investigated in the context of dynamical systems theory (see paper [7] for such an approach).

This paper is organized as follows. In the first part we review basic definitions and results concerning Shabat polynomials and the action G. In the second part we discuss the corresponding topics for conservative polynomials and the action D. Our exposition is rather informal; the main goal is to define the action D and provide a number of examples. In particular, we describe all the trees the combinatorial orbit of which with respect to the action D consists of a single tree and calculate the corresponding polynomials. Besides, we provide two examples of combinatorial orbits containing two elements; in the first of them the Galois orbit coincides with the combinatorial orbit and in the second one the combinatorial orbit splits into two Galois orbits.

2. Shabat polynomials and the action G

2.1. **Correspondence between plane trees and Shabat polynomials.** A complex polynomial $P(z)$ is called *a Shabat polynomial* if $P(z)$ has only two finite critical values p_1, p_2 that is if the equality $P'(\zeta) = 0$, $\zeta \in \mathbb{C}$, implies that $P(\zeta) = p_1$ or $P(\zeta) = p_2$. Two Shabat polynomials $P(z), p_1, p_2$ and $Q(z), q_1, q_2$ are called *equivalent* if there exist complex polynomials $A(z)$, $B(z)$ of degree one such that $Q = B \circ P \circ A$ and $q_i = B(p_i)$, $i = 1, 2$. *A plane tree* λ is a tree embedded into the plane. *A bicolored plane tree* is a plane tree vertices of which are colored in two colors in such a way that any edge connects vertices of different colors. Two

Figure 1.

bicolored plane trees λ and $\tilde{\lambda}$ are called *equivalent* if $\tilde{\lambda} = h(\lambda)$, where $h : \mathbb{C} \to \mathbb{C}$ is a homeomorphism preserving the orientation of $\mathbb{C}$ and colors of vertices of λ and $\tilde{\lambda}$.

For a degree n Shabat polynomial $P(z)$, p_1, p_2 define a bicolored plane n-edged graph λ_P as a preimage of the segment $[p_1, p_2]$ under the map $P : \mathbb{C} \to \mathbb{C}$. By definition, white (resp. black) vertices of λ_P are preimages of p_1 (resp. p_2) and edges of λ_P are preimages of $[p_1, p_2]$. Clearly, the valency of a vertex v of λ_P coincides with the multiplicity of the point v with respect to the map $P : \mathbb{C} \to \mathbb{C}$. It is not difficult to show that the graph λ_P is actually a tree. Furthermore, the following theorem is true (see e.g. [13]).

Theorem 2.1. *The map $P \to \lambda_P$ descends to a bijection between equivalence classes of degree n Shabat polynomials and equivalence classes of bicolored plane trees with n edges.*

This theorem is a very particular case of the correspondence between "dessins" and "Belyi pairs" provided by the Grothendieck theory of "Dessins d'enfants" (see [11],[12],[6]).

The simplest example of a Shabat polynomial is the polynomial $P(z) = z^n$ with the unique critical value $p_1 = 0$. For this "degenerate" case we can take as p_2 any complex number distinct from zero. The tree λ_P is *a star* (see Fig. 1). An other simple example of a Shabat polynomial is the n-th Chebyshev polynomial $T_n(z)$, $T_n(\cos\varphi) = \cos(n\varphi)$. For $P(z) = T_n(z)$ the tree λ_P is *a chain* (see Fig. 1).

Let us explain now how starting from a tree λ to find a Shabat polynomial $P(z)$ from the corresponding equivalence class. Let $\alpha = \alpha_1, \alpha_2, ..., \alpha_p$ (resp. $\beta = \beta_1, \beta_2, ..., \beta_q$) be sequences of valencies of white (resp. black) vertices of λ in decreasing order; in this case we say that λ is of the "type"

$$< \alpha \,|\, \beta > = < \alpha_1, \alpha_2, ..., \alpha_p \,;\, \beta_1, \beta_2, ..., \beta_q >$$

with respect to the action G. Denote by $a_1, a_2, ..., a_p$ (resp. $b_1, b_2, ..., b_q$) coordinates of white (resp. black) vertices of λ and by c the leading coefficient of $P(z)$. Let us fix p_1, p_2. Then we have the following system to define $P(z)$:

$$(1) \qquad \begin{cases} P(z) - p_1 = c(z - a_1)^{\alpha_1}(z - a_2)^{\alpha_2} ... (z - a_p)^{\alpha_p}, \\ P(z) - p_2 = c(z - b_1)^{\beta_1}(z - b_2)^{\beta_2} ... (z - b_q)^{\beta_q}, \end{cases}$$

with the additional condition that $a_1, a_2, ..., a_p$, $b_1, b_2, ..., b_q$ are mutually distinct.

Note that any other tree $\tilde{\lambda}$ of the type $< \alpha \,|\, \beta >$ also leads to system (1) and vice versa any solution of (1) corresponds to a tree of the type $< \alpha \,|\, \beta >$. The set of all trees of the type $< \alpha \,|\, \beta >$ is called *a combinatorial orbit* of λ with respect to the action G. Note also that system (1) provides n algebraic equations in $n+2$ unknowns $a_1, a_2, ..., a_p$, $b_1, b_2, ..., b_q, c$. The inequality between the number of equations and the number of unknowns corresponds to the possibility to perform a linear change of variable z. For instance, we can fix coordinates of two distinguished vertices of λ. Such a normalization can be very useful for practical calculations.

2.2. **Galois group action. Fields of definition and fields of modules.** In each equivalence class of Shabat polynomials there exist polynomials with algebraic coefficients and the group $\Gamma = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on the set of Shabat polynomials over $\bar{\mathbb{Q}}$ in a natural way: if $P(z), p_1, p_2$ is a Shabat polynomial with algebraic coefficients and $\sigma \in \Gamma$ then $P^\sigma(z), p_1^\sigma, p_2^\sigma$ again is a Shabat polynomial. Since $(B \circ P \circ A)^\sigma = B^\sigma \circ P^\sigma \circ A^\sigma$, this action descends to an action on equivalence classes and hence by theorem 2.1 we get an action G of the group Γ on plane bicolored trees.

Theorem 2.2 (Lenstra, Schneps [10]). *The action G is faithful.*

Since the action G obviously preserves the type of λ and a number of trees of any given type is finite, the index $[\Gamma : \mathrm{St}\,\lambda]$, where $\mathrm{St}\,\lambda \subset \Gamma$ denotes the stabilizer of λ, is finite. The fixed field $k_\lambda \subset \bar{\mathbb{Q}}$ of $\mathrm{St}\,\lambda$ is called *the field of modules* of λ. So, k_λ is a number field whose degree over $\mathbb{Q}$ is equal to the length of the orbit containing λ.

For a tree λ any field K_λ for which there exists a Shabat polynomial $P(z)$ from the corresponding to λ equivalence class such that $P(z) \in K_\lambda[z]$ is called *the field of definition* of λ. It is easy to show that every field of definition contains the field of modules. The converse problem (in a more general setting) was investigated in [1], [2] where the following theorem was proved.

Theorem 2.3 ([1], [2]). *For any bicolored plane tree λ there exists a Shabat polynomial $P(z)$ from the corresponding equivalence class such that $P(z) \in k_\lambda[z]$.*

Note that the similar statement for arbitrary dessins and corresponding Belyi functions fails to be true (see [1], [2]).

To illustrate the notions introduced above we calculate, following [13], the Shabat polynomials corresponding to the combinatorial orbit $< 2, 2, 1, 1, 1 \,|\, 3, 2, 2 >$ consisting of two trees shown on Fig. 2.

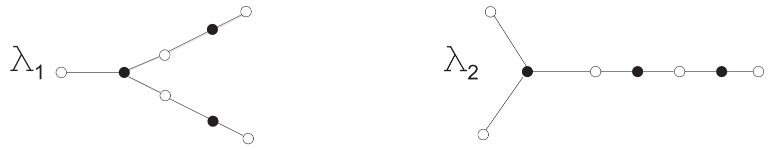

Figure 2.

We place the black vertex of valency 3 at the point $z = 0$ and choose the position of two black vertices of valency 2 so that the sum of the corresponding complex numbers equals 2. Assuming that $p_1 = 0$ and that $P(z)$ is monic we conclude that

$$P(z) = z^3(z^2 - 2z + a)^2,$$

where $a \in \mathbb{C}$ is a parameter to define. Moreover, since

$$P'(z) = z^2(z^2 - 2z + a)(7z^2 - 10z + 3a)$$

and the roots μ_1, μ_2 of $Q(z) = 7z^2 - 10z + 3a$ must be distinct, the inequality $21a - 25 \neq 0$ holds.

Furthermore, since

$$P(z) = S(z)Q(z) + Az + B,$$

where $S(z)$ is a polynomial and

$$A = -\frac{16}{7^6}(21a - 25)(49a^2 - 476a + 400), \qquad B = -\frac{196}{7^6}a(28a - 25)(7a - 10),$$

the condition that $P(\mu_1) = P(\mu_2)$ reduces to the condition that $A = 0$. Therefore, taking into account that $21a - 25 \neq 0$, we find that

$$a = \frac{1}{7}(34 \pm 6\sqrt{21}).$$

One of these values corresponds to the tree λ_1 and the other one to the tree λ_2. Therefore, in this example the combinatorial orbit coincides with the Galois orbit and the corresponding field of modules is $\mathbb{Q}(\sqrt{21})$.

Note that combinatorial orbits do not always coincide with the corresponding Galois orbits. For instance, the combinatorial orbit $< 4, 1, 1; 2, 2, 1, 1 >$ shown on Fig. 3 *splits* into two Galois orbits; in other words, the field of modules of both trees is $\mathbb{Q}$. This is a corollary of the fact that the group of symmetries of a tree is a Galois invariant; since λ_1 has a symmetry of order 2 while λ_2 has no any symmetries they can not be in the same Galois orbit (see [13]).

Figure 3.

More generally, define *an edge rotation group* $ER(\lambda)$ of λ as a group of permutations of edges of λ generated by two permutations ρ_w, ρ_b, where ρ_w (resp. ρ_b) cyclically permutes edges of λ in the counter-clockwise order around white (resp. black) vertices. Clearly, $ER(\lambda)$ can be identified with the monodromy group of any Shabat polynomial from the corresponding equivalence class.

Theorem 2.4 ([3],[4])**.** *The group* $ER(\lambda)$ *is invariant with respect to the action* G.

The invariant $ER(\lambda)$ is very powerful and absorbs many other combinatorial Galois invariants. Nevertheless, $ER(\lambda)$ does not distinguish *all* Galois orbits. For an example of such a situation see the paper [19] where an other very interesting Galois invariant is introduced.

3. Conservative polynomials and the action D

3.1. **Correspondence between plane trees and conservative polynomials.** A complex polynomial $C(z)$ is called *conservative* if all its critical points are fixed that is if the equality $C'(\zeta) = 0$, $\zeta \in \mathbb{C}$, implies that $C(\zeta) = \zeta$. Two conservative polynomials $\tilde{C}(z)$, $C(z)$ are called *equivalent* if there exists a complex polynomial $A(z)$ of degree one such that $\tilde{C} = A^{-1} \circ C \circ A$. Note that this equivalence relation is more restrictive than the equivalence relation for Shabat polynomials.

For a conservative polynomial $C(z)$ define a bicolored plane graph λ_P as follows. Let ζ_i, $1 \leq i \leq p$, be all finite critical points of $C(z)$. Then the immediate attractive basin B_{ζ_i} of each ζ_i, $1 \leq i \leq p$, is a disk. Furthermore, if d_i is the multiplicity of $C(z)$ at the point ζ_i, then there is an analytic conjugation of $C(z)$ on B_{ζ_i} to $z \to z^{d_i}$ on the unit disk D such that conjugating maps $\varphi_{\zeta_i} : \mathrm{D} \to B_{\zeta_i}$, $1 \leq i \leq p$, extend continuously to the closed unit disk $\bar{\mathrm{D}}$ (see [17]). Let S_i, $1 \leq i \leq p$, be an union of $d_i - 1$ radial segments which are forward invariant under the map $z \to z^{d_i}$ on $\bar{\mathbb{D}}$. Define the bicolored graph λ_C as an union of all images S_i, $1 \le i \le p$, under corresponding conjugating maps φ_{ζ_i} , i.e., $\lambda_C = \cup_{i=1}^{p} \varphi_{\zeta_i}(S_i)$. By definition, the white vertices of λ_C are images of zero (attractive fixed points of $C(z)$) and black vertices of λ_C are images of end-points of S_i, $1 \le i \le p$ (repelling fixed points of $C(z)$). Clearly, the valency of a white vertex v of λ_C coincides with the multiplicity of the point v with respect to the map $P' : \mathbb{C} \to \mathbb{C}$. Note also that by construction the graph λ_C is a forward invariant. It turns out that λ_C is actually a tree. Moreover, the following theorem proved by D. Tischler is true.

Theorem 3.1 ([17]). *The map $C \to \lambda_C$ descends to a bijection between equivalence classes of degree n conservative polynomials and equivalence classes of bicolored plane trees with $n-1$ edges.*

The simplest example of a conservative polynomial is again the polynomial $P(z) = z^n$ with the unique critical point $\zeta = 0$. Clearly, the corresponding tree is an $n-1$-edged star. The following non-trivial example (cf. [16],[5]) is the polynomial $f_n(z) = z^n + (n/n-1)z$, $n \ge 2$. Since all the zeroes of $f_n'(z)$ are simple, all the white vertices of the corresponding tree λ are of valency 1. Therefore, λ is also an $n-1$-edged star but *with the bicoloring changed.* On the right side of Fig. 4 the dynamics of $f_6(z)$ are shown: the points for which the limits of iterations have the same value are painted by the same color[1].

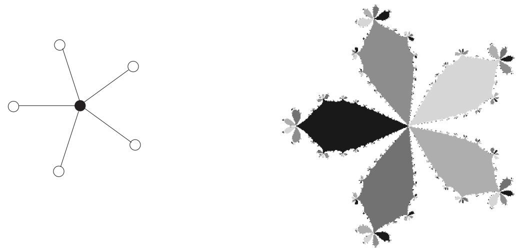

Figure 4.

Starting from a tree λ we can find a conservative polynomial $C(z)$ from the corresponding equivalence class as follows. Let $\alpha = \alpha_1, \alpha_2, ..., \alpha_p$ be the sequence of valencies of white vertices of λ in decreasing order; in this case we say that λ is of the "type" $< \alpha >$ with respect to the action D. We can suppose that $C(z)$ satisfies $C'(0) = 0$ or in other words that one of white vertices of λ is located at the point zero. We can suppose also that $C(z)$ is monic. Under these two normalization assumptions the position of λ in the plane is defined up to a change $z \to \varepsilon z$, $\varepsilon^{n-1} = 1$, and we obtain the following system to determine $C(z)$:

$$(2) \qquad \begin{cases} C'(z) = nz^{\alpha_1}(z-a_1)^{\alpha_2} ... \, (z-a_{p-1})^{\alpha_p}, \\ C(a_j) = a_j, \; 1 \le j \le p-1, \\ C(0) = 0, \end{cases}$$

[1]To prepare dynamical pictures for this paper we used C. McMullen's programs available on http://www.math.harvard.edu/ ctm/programs.html

with the additional condition that the numbers $0, a_1, a_2, ..., a_{p-1}$ are mutually distinct.

Of course, the normalization above is not unique possible. In general, like the case of Shabat polynomials the choice of the most appropriate normalization depends on combinatorics of the tree λ.

3.2. **Galois group action. Fields of definition and fields of modules.** In each equivalence class of conservative polynomials there exist polynomials with algebraic coefficients. Indeed, it follows from the Tischler theorem that system (2) has only a finite number of solutions. Therefore, since equations of (2) have rational coefficients, all these solutions are algebraic. Furthermore, the group Γ acts on the set of the conservative polynomials with algebraic coefficients in a natural way: if $C(z)$ is a conservative polynomial with algebraic coefficients and $\sigma \in \Gamma$ then $C^{\sigma}(z)$ again is a conservative polynomial. Since $(A^{-1} \circ P \circ A)^{\sigma} = (A^{\sigma})^{-1} \circ P^{\sigma} \circ A^{\sigma}$, this action descends to an action on equivalence classes. Hence, by the Tischler theorem, we obtain an action D of Γ on bicolored plane trees. Finally, it is easy to observe (see Remark 3.3 below) that this action is distinct from the action G. Let us summarize these facts in the form of a theorem.

Theorem 3.2. *D is a well-defined action of the group* Gal $(\bar{\mathbb{Q}}/\mathbb{Q})$ *on bicolored plane trees distinct from the action G.*

Define *the dynamical field of modules* k_{λ}^{D} of a plane bicolored tree λ as the fixed field of the stabilizer of λ with respect to the action D. Again, by simple combinatorial reasons k_{λ}^{D} is a number field whose degree over $\mathbb{Q}$ is equal to the length of the orbit containing λ. Define *the dynamical field of definition* of λ as any field K_{λ}^{D} for which there exists a conservative polynomial $C(z)$ from the corresponding to λ equivalence class such that $C(z) \in K_{\lambda}^{D}[z]$. These definitions agree with the corresponding definitions concerning rational functions considered as dynamical systems on $\mathbb{P}^1$ (see [15]). Like Belyi functions these dynamical systems in general are not necessarily can be defined over its fields of modules. Nevertheless, in certain cases (for instance, if the corresponding equivalence class contains a polynomial) we can find a representative defined over its field of modules ([15]). In particular, the following result similar to theorem 2.3 holds.

Theorem 3.3 ([15])**.** *For any bicolored plane tree λ there exists a conservative polynomial $C(z)$ from the corresponding equivalence class such that $C(z) \in k_{\lambda}^{D}[z]$.*

By theorem 2.2 each number field appears as the field of modules of some tree with respect to the action G. It is interesting to establish whether the similar statement is true for the action D.

Problem 3.1. *Does the action D is faithful ?*

An other interesting question concerning the action D is the following one:

Problem 3.2. *What are combinatorial invariants of the action D ?*

In connection with problem 3.1 note that the proof of theorem 2.2 relies on *the Belyi lemma* which states that for any polynomial $A(z)$ defined over a number field there exists a polynomial $S(z)$ with rational coefficients such that the composition $R(z) = S(A(z))$ is a Shabat polynomial. On the other hand, for the conservative polynomials the following proposition holds.

Proposition 3.1. *All conservative polynomials not equivalent to z^n for composite n are indecomposable.*

Proof. Indeed, suppose that $C(z) = C_1(C_2(z))$ with $\deg C_1(z), \deg C_2(z) > 1$. Let $\zeta \in \mathbb{C}$ be a critical point of the polynomial $C_1(z)$. The chain rule implies that any point $\mu \in \mathbb{C}$ such that $C_2(\mu) = \zeta$ is a critical point of $C(z)$. If $C_2(z)$ is not equal to $A(z-\mu)^l + \zeta$ for some $\mu \in \mathbb{C}$ and integer $l \geq 2$, then there exist $\mu_1, \mu_2 \in \mathbb{C}$, $\mu_1 \neq \mu_2$, such that $C_2(\mu_1) = C_2(\mu_2) = \zeta$. For these points we have $C'(\mu_1) = C'(\mu_2) = 0$ while $C(\mu_1) = C(\mu_2) = C_1(\zeta)$. Since $\mu_1 \neq \mu_2$ this contradicts to the condition that μ_1, μ_2 are fixed points of $C(z)$. Therefore, $C_2(z) = A(z-\mu)^l + \zeta$. Furthermore, if $C_1(z)$ has a critical point $\tilde{\zeta} \in \mathbb{C}$ distinct from ζ then it follows from the form of $C_2(z)$ that there exist mutually distinct $\tilde{\mu}_1, \tilde{\mu}_2, \dots \tilde{\mu}_l \in \mathbb{C}$ such that $C_2(\tilde{\mu}_i) = \tilde{\zeta}$, $1 \leq i \leq l$, and as above this leads to a contradiction. Therefore, ζ is a unique finite critical point of $C_1(z)$. It follows that $C_1(z)$ is equal to $\tilde{A}(z-\zeta)^{\tilde{l}} + \tilde{\zeta}$ for some $\tilde{\zeta} \in \mathbb{C}$ and integer $\tilde{l} \geq 2$, and hence $C(z) = C_1(C_2(z))$ is equivalent to z^n for composite n.

Remark 3.1. Essentially we proved that any polynomial $C(z)$ such that the preimage of any critical value of $C(z)$ contains a unique critical point is either indecomposable or equal to $C(z) = A(z-\zeta)^l + \mu$ for some $a, \zeta, \mu \in \mathbb{C}$ and integer composite l.

3.3. **Examples of Galois orbits.** We start from a very simple example. Consider the tree $\lambda_{r,s}$ shown on Fig. 5.

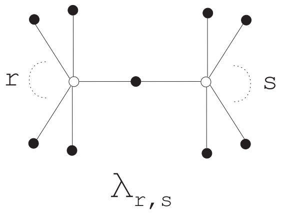

Figure 5.

Clearly, we can place white vertices to the points $0, 1$. Then

$$C'(z) = cz^r(1-z)^s,$$

where $c \in \mathbb{C}$ is a parameter to define. Integrating and taking into account that $C(0) = 0$, we conclude that

$$C(z) = cz^{r+1}\ {}_2F_1(-s, r+1, 2+r, z).$$

Since

$$C(1) = c\frac{r!s!}{(r+s+1)!},$$

the condition $C(1) = 1$ implies that

$$c = \frac{(r+s+1)!}{r!s!}.$$

Note that conservative polynomials corresponding to the trees $\lambda_{r,s}$ together with conservative polynomials corresponding to the stars with the white center are only conservative polynomials which are also Shabat polynomials. Note also that the

Figure 6.

equality $k_{\lambda_{r,s}} = \mathbb{Q}$ can be observed without any calculations since the combinatorial orbit $< r, s >$ with respect to the action D contains only the tree $\lambda_{r,s}$. The proposition below shows that such a phenomenon is rather exceptional.

Proposition 3.2. *Suppose that a tree λ is a unique tree in its combinatorial orbit with respect to the action D. Then either λ is a star with the black center, or λ is a star with the white center, or $\lambda = \lambda_{r,s}$.*

Proof. Let $< \alpha_1, \alpha_2, ..., \alpha_p ; \beta_1, \beta_2, ..., \beta_q >$ be the type of λ with respect to the action G. If $q = 1$ then λ is a star with the black center so we will suppose that $q > 1$. Show first that if $\beta_2 > 1$ then the combinatorial orbit of λ with respect to D contains more than one tree. Indeed, let v_1, v_2 be the black vertices of valencies β_1, β_2 respectively and let f be a path connecting v_1 and v_2. Consider the following operation: cut off any *branch* b of λ growing from the vertex v_2 (that is any maximal subgraph of λ having a vertex v_2 as a vertex of valency 1) such that b does not contain f, and glue b to the vertex v_1 (see Fig. 6). Since $\beta_2 > 1$ we always can perform such an operation and though, generally, there are many ways for doing it, in any case the obtained tree $\bar{\lambda}$ can not be isotopically equivalent to λ since the maximal valency of a black vertex of $\bar{\lambda}$ is greater than the corresponding valency of λ. On the other hand, the type of $\bar{\lambda}$ with respect to the action D remains the same. Hence, we can suppose that $\beta_2 = 1$.

Furthermore, if $\beta_1 > 2$, then cutting off any branch b of λ growing from v_1 such that b does not contain f and gluing b to the vertex v_2, we obtain again a tree $\bar{\lambda}$ which is not isotopically equivalent to λ since the sets $\{\beta_1, 1, 1, ..., 1\}$ and $\{\beta_1 - 1, 2, 1, ..., 1\}$ can not coincide due to the condition $\beta_1 > 2$. Therefore, either $\beta_1 = 2$ and then $\lambda = \lambda_{r,s}$, or $\beta_1 = 1$ and then λ is a star with the white center.

Remark 3.2. The corresponding list of the trees whose combinatorial orbit with respect to the action G contains a single tree was obtained by N. Adrianov. Clearly, this list is longer; it contains six infinite series and one “sporadic” tree (see [13]).

Consider now the combinatorial orbit $< 3, 1, 1 >$ consisting of two trees shown on Fig. 7.

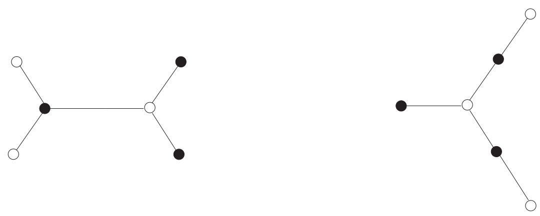

Figure 7.

Place the white vertex of valency 3 at zero. Then

$$C'(z) = az^3(z^2 + dz + b) = az^3(z - \mu_1)(z - \mu_2), \qquad a, b, d, \mu_1, \mu_2 \in \mathbb{C},$$

where the numbers $0, \mu_1, \mu_2$ are mutually distinct. Moreover, $d \neq 0$ since otherwise $C(z) = \tilde{C}(z^2)$ with $\tilde{C}(z) \in \mathbb{C}[z]$ in contradiction with proposition 3.1. Hence, we can suppose that $d = 1$. Then, since $C(0) = 0$,

$$C(z) = a(z^6/6 + z^5/5 + bz^4/4).$$

Furthermore,

$$C(z) - z = A(z)(z^2 + z + b) + B(z), \tag{3}$$

where

$$\begin{aligned} A(z) &= az^4/6 + az^3/30 + (ab/12 - a/30)z^2 + (-7ab/60 + a/30)z \\ &\quad - ab^2/12 + 3ab/20 - a/30, \\ B(z) &= (a/30 - 1 + ab^2/5 - 11ab/60)z + ab^3/12 - 3ab^2/20 + ab/30. \end{aligned}$$

In view of (3) the condition that μ_1, μ_2 are fixed points of $C(z)$ reduces to the condition that μ_1, μ_2 are roots of $B(z)$. Since $\mu_1 \neq \mu_2$ and $\deg B(z) = 1$ this fact implies that $B(z) \equiv 0$ or equivalently that

$$\begin{cases} a/30 - 1 + ab^2/5 - 11ab/60 = 0, \\ \quad ab^3/12 - 3ab^2/20 + ab/30 = 0. \end{cases}$$

Solving this system we conclude that either

$$a = 30, \qquad b = 0,$$

or

$$a = -5055/8 \pm 795\sqrt{41}/8, \qquad b = 9/10 \pm \sqrt{41}/10.$$

All these solutions correspond conservative polynomials but in the first case the condition that $0, \mu_1, \mu_2$ are distinct is violated. Geometrically it means that one of the white vertices of valency 1 merges with the white vertex of valency 3. It follows that the first solution actually corresponds to the tree $\lambda_{4,1}$ and therefore to the combinatorial orbit $< 3, 1, 1 >$ corresponds the pair of algebraically conjugated solutions. So, in this example the combinatorial orbit coincides with the Galois orbit and the corresponding field of modules is $\mathbb{Q}(\sqrt{41})$.

Remark 3.3. Note the difference between the actions G and D provided by this example. Indeed, both trees shown on Fig. 7 are unique trees in their combinatorial orbits with respect to the action G and therefore the field of modules with respect to G of each of them is $\mathbb{Q}$.

Finally, consider the combinatorial orbit $< 2, 1, 1 >$. There are two trees of this type; one of them is shown on Fig. 8 and the other one on the Fig. 9. If we place the white vertex of valency 2 at zero and set the sum of the coordinates of two white vertices of valency 1 equals 2, then

$$C'(z) = az^2(z^2 + 2z + b), \qquad C(z) = a(z^5/5 + z^4/2 + bz^3/3)$$

and we conclude as above that the remainder

$$B(z) = (14ab/15 - 4a/5 - 1 - 2ab^2/15)z + 11ab^2/30 - 2ab/5$$

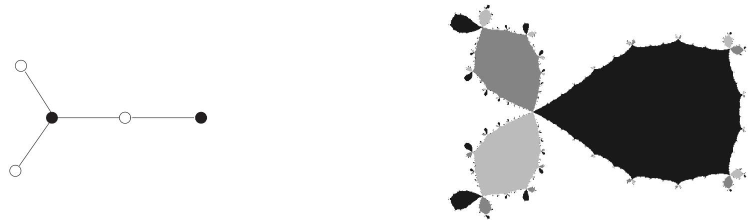

Figure 8.

of $C(z)-z$ divided by z^2+2z+b must be equal to zero. Solving the corresponding system

$$\left\{\begin{aligned} 14ab/15 - 4a/5 - 1 - 2ab^2/15 = 0, \\ 11ab^2/30 - 2ab/5 = 0 \end{aligned}\right. \tag{4}$$

and rejecting the "parasitic" solution

$$a = -5/4, \qquad b = 0$$

for which one of the white vertices of valency 1 merges with the white vertex of valency 2, we find that

$$a = 605/36, \qquad b = 12/11.$$

Therefore, system (4) gives us only *one* polynomial

$$C(z) = 121z^5/36 + 605z^4/72 + 55z^3/9.$$

Making a picture we see that $C(z)$ corresponds to the tree shown on Fig. 8.

Figure 9.

The cause for which system (4) provides no solution corresponding to the tree shown on Fig. 9 consists in the supposition that the sum of the coordinates α_1, α_2 of two white vertices of valency 1 is not equal to zero. In reality for this tree this sum *does* equal to zero. Indeed, setting $\alpha_1 + \alpha_2 = 0$, $\alpha_1\alpha_2 = -1$ we arrive to the equalities

$$C'(z) = az^2(z^2-1), \qquad C(z) = az^5/5 - az^3/3.$$

The conditions $C(1) = 1$, $C(-1) = -1$ reduce to the equality $-2a/15 = 1$, and, therefore,

$$C(z) = -3z^5/2 + 5z^3/2.$$

On the right side of Fig. 9 the dynamics of $C(z)$ are shown. Note that in this example both polynomials have rational coefficients and therefore the combinatorial orbit splits into two Galois orbits .

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