Heisenberg-type subalgebras of $\mathbf{U_q(\widehat{sl_2})}$

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Abstract

Using the second Drinfeld formulation of the quantized universal enveloping algebra $U'_q(\widehat{sl_2})$ of the affine Lie algebra $\widehat{sl_2}$ extended by auxiliary elements $D^{\pm 1}$, we introduce the notion and explicitly find elements of a Heisenberg-type subalgebra.

1 Introduction

The purpose of this letter is to introduce a new type of subalgebras which would extend the notion of a Heisenberg subalgebra of a quantized enveloping algebra of an affine Lie algebra. Heisenberg subalgebras of affine Lie algebras and of their q-deformed enveloping algebras are being actively used in various domains of mathematical physics. Most important applications of Heisenberg subalgebras can be found in the field of classical and quantum integrable models and field theories. Vertex operator constructions both for affine Lie algebras [6] and for q-deformations of their universal enveloping algebras [5, 7] are essentially based on Heisenberg subalgebras. Given a quantized universal enveloping algebra $U_q(\widehat{g})$ of an affine Kac-Moody Lie algebra $\widehat{g}$, it is rather important to be able to find generators of a Heisenberg subalgebra associated to a chosen grading of $U_q(\widehat{g})$. It is not always a trivial task to extract particular Heisenberg subalgebra even in the case of $U_q(\widehat{sl_2})$. For instance, one can easily recognize elements of the homogeneous Heisenberg subalgebra among generators of $U_q(\widehat{sl_2})$ in the second Drinfeld realization [5], while it is not obvious how to extract the principal Heisenberg subalgebra.

The letter is organized as follows. First, we recall the definition of the quantized universal enveloping algebra $U'_q(\widehat{sl_2})$ in the second Drinfeld realization. Our general idea is to endow (similarly to [1, 2]) the algebra $U'_q(\widehat{sl_n})$

with appropriate elements $D^{\pm 1}$, and to use properties of extended enveloping algebra generators to find explicit expressions for the elements of corresponding *Heisenberg-type subalgebra*. Then we prove their commutation relations and make comments on possible generalizations.

2 Second Drinfeld realization of $\mathbf{U_q(\widehat{sl_2})}$

Let us recall the second Drinfeld realization [3, 4] of the quantized universal enveloping algebra $U_q'(\widehat{sl_2})$. It is generated by the elements $\{x_k^{\pm},\ k \in \mathbb{Z}; a_n,\ n \in \{\mathbb{Z}\backslash 0\}; \gamma^{\pm\frac{1}{2}}, K\}$, subject to the commutation relations

$$\begin{gathered}
[K, a_k] = 0, \quad K\, x_k^{\pm}\, K^{-1} = q^{\pm 2}\, x_k^{\pm}, \\
[a_k, a_l] = \delta_{k,-l}\, \tfrac{[2k]}{k}\, \tfrac{\gamma^k - \gamma^{-k}}{q - q^{-1}}, \\
[a_n, x_k^{\pm}] = \pm \tfrac{[2n]}{n}\, \gamma^{\mp \frac{|n|}{2}}\, x_{n+k}^{\pm}, \\
[x_n^+, x_k^-] = \tfrac{1}{q - q^{-1}} \left(\gamma^{\frac{1}{2}(n-k)}\, \psi_{k+n} - \gamma^{-\frac{1}{2}(n-k)}\, \phi_{k+n} \right), \\
x_{k+1}^{\pm}\, x_l^{\pm} - q^{\pm 2}\, x_l^{\pm}\, x_{k+1}^{\pm} = q^{\pm 2}\, x_k^{\pm}\, x_{l+1}^{\pm} - x_{l+1}^{\pm}\, x_k^{\pm},
\end{gathered} \tag{2.1}$$

where $\gamma^{\pm\frac{1}{2}}$ belong to the center of $U_q'(\widehat{sl_2})$. The elements ϕ_k and ψ_{-k}, $k \in \mathbb{Z}_+$, are related to $a_{\pm k}$ by means of the expressions

$$\begin{aligned}
\sum_{m=0}^{\infty} \psi_m\, z^{-m} &= K\, exp\left((q - q^{-1}) \sum_{k=1}^{+\infty} a_k\, z^{-k} \right), \\
\sum_{m=0}^{\infty} \phi_{-m}\, z^{m} &= K^{-1}\, exp\left(-(q - q^{-1}) \sum_{k=1}^{+\infty} a_{-k}\, z^{k} \right),
\end{aligned} \tag{2.2}$$

i.e., $\psi_m = 0$, $m < 0$, $\phi_m = 0$, $m > 0$. We define the algebra $U_q(\widehat{sl_2})$ as $U_q'(\widehat{sl_2})$ endowed with extra elements $D^{\pm 1}$, which have the properties

$$\begin{aligned}
&D\, D^{-1} = D^{-1}\, D = 1, \quad D\, K = K\, D, \quad D\, \gamma = \gamma\, D, \\
&D\, x\, D^{-1} = q^{\alpha_x}\, x, \quad \alpha_x \in \mathbb{Z},
\end{aligned} \tag{2.3}$$

where $x \in \{a_n, n \in \{\mathbb{Z} \setminus 0\};\ x_n^{\pm}, n \in \mathbb{Z}\}$.

3 Heisenberg-type subalgebras

In this section we introduce elements of a *Heisenberg-type subalgebra* of $U_q(\widehat{sl_2})$ and prove their commutation relations. Consider the elements $E^{\pm}_{\pm n} \in U_q(\widehat{sl_2})$, $n \in \mathbb{Z}_+$:

$$
\begin{aligned}
E^{\pm}_{n} &= \gamma^{\mp(n+\frac{1}{2})}\, K\, x^{+}_{n}\, D^{m} + x^{-}_{n+1}\, D^{\eta}, \\
E^{\pm}_{-n} &= \gamma^{\mp(n+\frac{1}{2})}\, x^{-}_{-n}\, D^{\theta} + K\, x^{+}_{-n-1}\, D^{l},
\end{aligned}
\tag{3.1}
$$

where $m,\ \eta,\ \theta,\ l \in \mathbb{Z}$.

Proposition. *The elements $E^{\pm}_{\pm n}$, $n \in \mathbb{Z}_+$, given by (3.1) with $m = l = \pm 1$, $\eta = \theta = \mp 1$, and elements D (2.3) defined by $\alpha^{+}_{n} = \alpha^{+}_{-k-1} = \alpha^{+} = \alpha$, $\alpha^{-}_{-n} = \alpha^{-}_{k+1} = \alpha^{-} = -\alpha \pm 2$; $n,\ k \in \mathbb{Z}_+$, where $\alpha \in \mathbb{Z}$ is an arbitrary constant, satisfy commutation relations*

$$
\begin{gathered}
\left[E^{+}_{n}, E^{+}_{-k}\right] = \delta_{n,k}\, c^{\pm}_{n}, \quad n \geq k\ ; \qquad \left[E^{-}_{n}, E^{-}_{-k}\right] = 0, \quad n < k\ ; \\
\left[E^{-}_{-n}, E^{-}_{-k}\right] = 0, \quad \left[E^{+}_{n}, E^{+}_{k}\right] = 0, \quad n, k \in \mathbb{Z}_+,
\end{gathered}
\tag{3.2}
$$

where $c^{\pm}_{n} \equiv \frac{q^{-\alpha\pm 2}}{q-q^{-1}}(\gamma^{n+1} - \gamma^{-3n-1})$, is a central element in $U_q(\widehat{sl_2})$.

We call a subalgebra of $U_q(\widehat{sl_2})$ satisfying commutation relations (3.2) a *Heisenberg-type subalgebra*.

Proof. The proof consists in direct calculation of commutation relations (3.2). Elements D and $m,\ \eta,\ \theta,\ l$ are chosen in such a way that the powers of q and γ be equal in all parts of commutators in (3.2) correspondingly. Indeed, consider the commutator

$$
\begin{aligned}
\left[E^{+}_{n}, E^{+}_{-k}\right] &= \gamma^{-(n+k+1)} \left(q^{m\,\alpha^{-}_{-k}}\, K\, x^{+}_{n}\, x^{-}_{-k} - q^{2+\theta\,\alpha^{+}_{n}}\, K x^{-}_{-k}\, x^{+}_{n}\right) D^{m+\theta} \\
&+ \left(q^{2+\eta\,\alpha^{+}_{-k-1}}\, K\, x^{-}_{n+1}\, x^{+}_{-k-1} - q^{l\,\alpha^{-}_{n+1}}\, K\, x^{+}_{-k-1}\, x^{-}_{n+1}\right) D^{\eta+l} \\
&+ \gamma^{-(n+\frac{1}{2})} \left(q^{m\,\alpha^{+}_{-k-1}}\, K\, x^{+}_{n}\, K\, x^{+}_{-k-1} - q^{l\,\alpha^{+}_{n}}\, K\, x^{+}_{-k-1}\, K\, x^{+}_{n}\right) D^{m+l} \\
&+ \gamma^{-(k+\frac{1}{2})} \left(q^{\eta\,\alpha^{-}_{-k}}\, K\, x^{-}_{n+1}\, K\, x^{-}_{-k} - q^{\theta\,\alpha^{-}_{n+1}}\, K\, x^{-}_{-k}\, K\, x^{-}_{n+1}\right) D^{\eta+\theta}.
\end{aligned}
\tag{3.3}
$$

When $m = l = \pm 1$, $\eta = \theta = \mp 1$, and, correspondingly, $\alpha^+_n = \alpha^+_{-k-1} = \alpha$, $\alpha^-_{-n} = \alpha^-_{k+1} = -\alpha \pm 2$; n, $k \in \mathbb{Z}_+$, then

$$\begin{aligned}\left[E^+_n, E^+_{-k}\right] &= \gamma^{-(n+k+1)}\, q^{-\alpha\pm 2}\, K \left[x^+_n, x^-_{-k}\right] - q^{-\alpha\pm 2}\, K\, \left[x^+_{-k-1}, x^-_{n+1}\right] \\ &= \tfrac{q^{-\alpha\pm 2}}{q-q^{-1}}\, K\, \left(\gamma^{\frac{1}{2}(n+k+2)} - \gamma^{-\frac{1}{2}(3n+3k+2)}\right)\, \phi_{n-k}.\end{aligned} \tag{3.4}$$

If $n > k$, then according to (2.2), $\phi_{n-k} \equiv 0$, and $\left[E^+_n, E^+_{-k}\right] = 0$; when $n = k$, then taking into account that $\phi_0 = K^{-1}$, we find $\left[E^+_n, E^+_{-n}\right] = c^{\pm}_n$. Similar calculations lead to the rest of commutation relations in (3.2).

4 Conclusions

In the second Drinfeld realization of $U_q(\widehat{sl_2})$ endowed with extra elements $D^{\pm 1}$ we have defined the *Heisenberg-type subalgebra*, explicitly constructed its elements, and proved their commutation relations. Properties of a *Heisenberg-type subalgebra* are similar to a Heisenberg subalgebra properties, and are very useful in construction of special types of vertex operators in $U_q(\widehat{sl_2})$, which, in particular, have their applications in the soliton theory of nonlinear integrable partial differential equations [7]. More advanced examples of *Heisenberg-type subalgebras* associated to various elements $D^{\pm 1}$ and sets of powers m, η, θ, l in $U_q(\widehat{sl_n})$ in Drinfeld-Jimbo and the second Drinfeld realizations as well as corresponding vertex operators will be presented in a forthcoming paper [8].

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