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Abstract

In this paper, we give several necessary conditions for a holomorphic Siegel modular form on $\mathrm{Sp}_2(\mathbb{Z})$ to be obtained as a Borcherds lift. As an application, we show that Siegel Eisenstein series are not Borcherds lifts. We also give a condition satisfied by the weight of a Borcherds lift.

1 Introduction and the main results

1.1 Introduction

The lifting of elliptic modular forms to Siegel modular forms of degree two was, suggested by a conjecture by Saito and Kurokawa, introduced by a pioneering work [[20]] by Maass (for an overview of this theory, see Zagier's article [[26]]). Using theta lifting, Oda [[22]] and Rallis-Schiffmann [[23]], independently, constructed a lifting to automorphic forms on $O(2,m)$ generalizing Maass's lifting as well as Shimura correspondence ([[24]]). Later the *multipicative* analogue of Oda-Rallis-Schiffmann lifting was discovered by Borcherds ([[2, 3]]). In our previous paper [[17]], we showed that Borcherds lifts satisfy certain symmetries. The object of the paper is to study Borcherds lifts on $\Gamma_2 = \mathrm{Sp}_2(\mathbb{Z})$ (namely the case of $O(2,3)$) by using these symmetries.

A Borcherds lift on Γ_2 is a meromorphic automorphic form F on Γ_2 (with a multiplier system of finite order) whose divisor is of the form $\sum_d A(d)H_d$, where d runs over the positive integers congruent to 0 or 1 modulo 4, $A(d) \in \mathbb{Z}$ ($A(d) = 0$ except for a finite number of d) and H_d is the Humbert surface of discriminant d (for the precise definition of H_d, see **2.2**; for Borcherds lifts, we refer to [[2, 3, 5]]; see also [[15, 1]]). Since every Borcherds lift is a quotient of two holomorphic Borcherds lifts, we are mainly concerned with holomorphic Borcherds lifts in this paper.

From the fundamental results of Borcherds ([[2, 3, 4]]), it follows that the weight of F is expressed in terms of $A(d)$ and Cohen numbers ([[7]]). By using this weight formula and certain properties of Cohen numbers, we determine the holomorphic Borcherds lifts of small weights (Proposition 4.13).

To discuss the arbitrary weight case, we employ our previous result on the multiplicative symmetries for Borcherds lifts ([[17]]; see Theorem 3.1). We first characterize the powers of the modular discriminant Δ in terms of multiplicative Hecke operators (Proposition 3.2). We then show that the image of a holomorphic Borcherds lift F on Γ_2 under the Siegel operator is proportional to a power of Δ by using the multiplicative symmetry for F and the above characterization. This immediately implies that the Siegel Eisenstein series are never Borcherds lifts. We also show that a holomorphic Borcherds lift on Γ_2 with trivial character is proportional to $\chi_{10}^a\chi_{35}^b F'$, where χ_{10} and χ_{35} are Borcherds lifts of weight 10 and 35, respectively, $a \in \mathbb{Z}_{\geq 0}, b \in \{0,1\}$ and F' is a Borcherds lift of weight divisible by 12 such that the image of F' under the Witt operator is nonzero (Corollary 1.5).

1.2 Siegel modular forms

To explain our results more precisely, let

$$\Gamma_n := \left\{ \gamma \in \mathrm{GL}_{2n}(\mathbb{Z}) \mid {}^t\gamma \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \gamma = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix} \right\}$$

be the Siegel modular group of degree n and $\mathfrak{H}_n := \{Z \in \mathrm{M}_n(\mathbb{C}) \mid {}^tZ = Z, \mathrm{Im}(Z) > 0\}$ the upper half space of degree n, where 0_n (respectively 1_n) is the zero (respectively identity) matrix of degree n.

Let $M_k(\Gamma_n)$ denote the space of holomorphic automorphic forms of weight k on Γ_n. By definition, $F \in M_k(\Gamma_n)$ is a holomorphic function on $\mathfrak{H}_n$ such that $F(\gamma\langle Z\rangle) = j(\gamma, Z)^k F(Z)$ holds for any $\gamma \in \Gamma_n$ and $Z \in \mathfrak{H}_n$, and that F is holomorphic at each cusp if $n = 1$. Here

$$\gamma\langle Z\rangle := (AZ+B)(CZ+D)^{-1},\ j(\gamma, Z) := \det(CZ+D)$$
$$\left(\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n, Z \in \mathfrak{H}_n\right).$$

In what follows, we are mainly concerned with the case of $n = 2$ and often write (τ_1, z, τ_2) for a point

$$\begin{pmatrix} \tau_1 & z \\ z & \tau_2 \end{pmatrix} \in \mathfrak{H}_2.$$

For $F \in M_k(\Gamma_2)$, we put

$$\Phi(F)(\tau) := \lim_{y \to \infty} F(\tau, 0, iy) \qquad (\tau \in \mathfrak{H}_1),$$
$$\mathcal{W}(F)(\tau_1, \tau_2) := F(\tau_1, 0, \tau_2) \qquad (\tau_1, \tau_2 \in \mathfrak{H}_1).$$

Then $\Phi(F) \in M_k(\Gamma_1)$ and $\mathcal{W}(F) \in \mathrm{Sym}^2(M_k(\Gamma_1))$, where $\mathrm{Sym}^2(M_k(\Gamma_1))$ is the space of functions on $\mathfrak{H} \times \mathfrak{H}$ generated by $f(\tau_1)f'(\tau_2) + f'(\tau_1)f(\tau_2)$ with $f, f' \in M_k(\Gamma_1)$. The operator Φ (respectively $\mathcal{W}$) is called the *Siegel* (respectively *Witt*) operator. Then $S_k(\Gamma_2) = \{F \in M_k(\Gamma_2) \mid \Phi(F) = 0\}$ is the space of cusp forms. Note that $\mathcal{W}(F) \neq 0$ if $\Phi(F) \neq 0$. A Siegel modular form $F \in M_k(\Gamma_2)$ admits the Fourier expansion

$$F(\tau_1, z, \tau_2) = \sum_{n,r,m \in \mathbb{Z}} A_F(n, r, m)\, \mathbf{e}(n\tau_1 + rz + m\tau_2),$$

where we put $\mathbf{e}(z) = \exp(2\pi i z)$ for $z \in \mathbb{C}$. Note that $A_F(n, r, m) = 0$ unless $n, m, 4nm - r^2 \geq 0$.

For $k \geq 4$, we define the Siegel Eisenstein series on Γ_2 of weight k by

$$E_k(Z) := \sum_{\gamma \in \Gamma_2^\infty \backslash \Gamma_2} j(\gamma, Z)^{-k} \qquad (Z \in \mathfrak{H}_2),$$

where

$$\Gamma_2^\infty := \left\{ \begin{pmatrix} A & B \\ 0_2 & {}^tA^{-1} \end{pmatrix} \in \Gamma_2 \right\}.$$

Then $E_k \in M_k(\Gamma_2)$ and $\Phi(E_k) = e_k$, where

$$\begin{aligned} e_k(\tau) &:= \frac{1}{2} \sum_{c,d\in\mathbb{Z},\,(c,d)=1} (c\tau + d)^{-k} \\ &= 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\mathbf{e}(n\tau) \qquad (\tau \in \mathfrak{H}_1) \end{aligned}$$

is the elliptic Eisenstein series of weight k. Here B_k denotes the k-th Bernoulli number and

$$\sigma_{k-1}(n) = \sum_{0<d|n} d^{k-1}.$$

We define the modular discriminant by

$$\begin{aligned} \Delta(\tau) &:= q \prod_{n\geq 1} (1 - q^n)^{24} \\ &= \frac{1}{1728}\Big(e_4^3(\tau) - e_6^2(\tau)\Big) \in S_{12}(\Gamma_1) \qquad (\tau \in \mathfrak{H}_1, q := \mathbf{e}(\tau)). \end{aligned}$$

Due to Igusa ([[18]]), the graded ring $\bigoplus_{k\geq 0} M_k(\Gamma_2)$ is generated by $E_4, E_6, \chi_{10}, \chi_{12}$ and χ_{35}, where

$$\chi_{10} := -43867 \cdot 2^{-12} \cdot 3^{-5} \cdot 5^{-2} \cdot 7^{-1} \cdot 53^{-1}(E_4E_6 - E_{10}) \in S_{10}(\Gamma_2),$$

$$\begin{aligned} \chi_{12} &:= 131 \cdot 593 \cdot 2^{-13} \cdot 3^{-7} \cdot 5^{-3} \cdot 7^{-2} \cdot 337^{-1} \\ &\quad \times \left(3^2 \cdot 7^2\, E_4^3 + 2 \cdot 5^3\, E_6^2 - 691\, E_{12}\right) \in S_{12}(\Gamma_2) \end{aligned}$$

and χ_{35} is a unique element of $S_{35}(\Gamma_2)$ up to constant multiples (for an explicit form of χ_{35}, see **2.3**). Note that we follow Igusa's normalizations of χ_{10} and χ_{12} so that

$$\begin{aligned} A_{\chi_{10}}(1,1,1) &= -1/4, \\ A_{\chi_{12}}(1,1,1) &= 1/12. \end{aligned}$$

We also recall that van der Geer ([[10]]) defined a Siegel modular form

$$G_{24} := \left(\chi_{12} - 2^{-12} \cdot 3^{-6}(E_6^2 + E_4^3)\right)^2 - E_4\left(2 \cdot 3^{-1}\chi_{10} - 2^{-11} \cdot 3^{-6} E_4E_6\right)^2,$$

of weight 24 on Γ_2 whose divisor is the Humbert surface of discriminant 5. It is known that χ_{10}, χ_{35} and G_{24} are Borcherds lifts (see [[13, 14]]), but χ_{12} is not a Borcherds lift (see [[17]]).

In this paper, employing our previous result on the multiplicative symmetries for Borcherds lifts ([[17]]; see Theorem 3.1), we give several necessary conditions for $F \in M_k(\Gamma_2)$ to be a Borcherds lift.

Theorem 1.1. *Assume that $F \in M_k(\Gamma_2)$ is a Borcherds lift. Then $\Phi(F)$ is proportional to a power Δ^r of the modular discriminant Δ with $r \geq 0$.*

As direct consequences of Theorem 1.1, we have

Corollary 1.2. *If $F \in M_k(\Gamma_2)$ is a Borcherds lift with $\Phi(F) \neq 0$, then the weight k is divisible by* 12.

Corollary 1.3. *The Siegel Eisenstein series E_k is not a Borcherds lift.*

We also show the following results.

Theorem 1.4. *Let $F \in M_k(\Gamma_2)$ be a Borcherds lift with $\mathcal{W}(F) \neq 0$.*

(i) The weight k is divisible by 12.

(ii) We have $k > 12$.

Corollary 1.5. *Let $F \in M_k(\Gamma_2)$ be a Borcherds lift. We put $b = 0$ if k is even and $b = 1$ otherwise. Let $a \in \mathbb{Z}_{\geq 0}$ be the nonnegative integer such that the multiplicity of H_1 in the divisor of F is equal to $3a+b$. Then F is proportional to $\chi_{10}^a \chi_{35}^b F'$. Here either $F' = 1$ or $F' \in M_{12c}(\Gamma_2)$ is a Borcherds lift with $\mathcal{W}(F') \neq 0$ (note that $c \geq 2$ in this case). In particular, the weight k of F is of the form $10a + 35b + 12c$ $(a \in \mathbb{Z}_{\geq 0}, b \in \{0,1\}, c \in \mathbb{Z}_{\geq 0}, c \neq 1)$.*

1.3 The organization of the paper

The paper is organized as follows. In Section 2, we recall the definition of Borcherds lifts (**2.4**) attached to weakly holomorphic Jacobi forms of weight 0 and index 1 (**2.1**) after recalling the definition of Humbert surfaces (**2.2**). We also recall the construction of several known Borcherds lifts $\chi_5, \chi_{10}, \chi_{35}$ and G_{24} (**2.5**). The main theorems (Theorem 1.1 and Theorem 1.4 (i)) are proved in Section 3 in the following way:

By our previous results ([[17]]), Borcherds lifts satisfy certain symmetries $(\mathrm{MS})_p$ for any prime number p (see Theorem 3.1). Suppose that $F \in M_k(\Gamma_2)$ satisfies $(\mathrm{MS})_2$. Then $f = \Phi(F)$ satisfies the condition

$$(*)_2 \qquad f(2\tau) f\left(\frac{\tau}{2}\right) f\left(\frac{\tau+1}{2}\right) = \epsilon_2 f(\tau)^3$$

with some $\epsilon_2 \in \mathbb{C}, |\epsilon_2| = 1$ (see Proposition 3.6). The key fact is that, if $f \in M_k(\Gamma_1)$ satisfies $(*)_2$, f is a constant multiple of a power of the modular discriminant Δ (Proposition 3.2 and Remark 3.3). Theorem 1.1 is then immediately deduced. We also show that the image φ of F under the Witt operator satisfies a similar symmetry $(\mathrm{ms})_2$ (Proposition 3.6) and that the weight of such a φ is divisible by 12 if $\varphi \neq 0$ (Proposition 3.5), which completes the proof of Theorem 1.4 (i). In Section 4, we give a weight formula for the Borcherds lifts (Theorem 4.11 (iv)), essentially due to Borcherds, after recalling the definition of Cohen numbers. Using the weight formula together with a lower bound of Cohen numbers (Lemma 4.5), we determine the holomorphic Borcherds lifts of weight less than or equal to 60 (Proposition 4.13), which proves the second part of Theorem 1.4.

1.4 Notation

We let $\mathbb{C}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. For a nonnegative integer n, we write $n = \square$ if n is a square and $n \neq \square$ otherwise. The transpose of a matrix X is denoted by ${}^tX$. We denote by $\mathrm{Sym}_m := \{X \in \mathrm{M}_m \mid {}^tX = X\}$ the space of symmetric matrices of degree m. For $S \in \mathrm{Sym}_m(\mathbb{C})$, we put $S(X,Y) = {}^tXSY$ and $S[X] = {}^tXSX$ for $X, Y \in \mathbb{C}^m$.

For a condition P, we put $\delta(P) = 1$ if P holds and $\delta(P) = 0$ otherwise.

2 Borcherds lifts

2.1 Jacobi forms

For $k \in \mathbb{Z}$, let $\mathcal{J}_{k,1}$ denote the space of holomorphic functions on $\mathfrak{H} \times \mathbb{C}$ satisfying the following conditions:

(i) For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$ and $(\tau, z) \in \mathfrak{H} \times \mathbb{C}$, we have

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k \mathbf{e}\left(\frac{cz^2}{c\tau + d}\right) \phi(\tau, z).$$

(ii) For $\lambda, \mu \in \mathbb{Z}$, we have

$$\phi(\tau, z + \lambda\tau + \mu) = \mathbf{e}(-\lambda^2\tau - 2\lambda z)\, \phi(\tau, z).$$

(iii) Let $\phi(\tau, z) = \sum_{n,r \in \mathbb{Z}} a_\phi(n, r)\mathbf{e}(n\tau + rz)$ be the Fourier expansion of ϕ. Then $a_\phi(n, r) = 0$ if $4n - r^2$ is sufficiently small.

The Fourier coefficient $a_\phi(n, r)$ depends only on $N = 4n - r^2$ and is often denoted by $a_\phi(N)$. We put $a_\phi(N) = 0$ if $N \equiv 1$ or $2 \pmod 4$. We then have

$$\phi(\tau, z) = \sum_{N \in \mathbb{Z}} a_\phi(N) \sum_{r \in \mathbb{Z},\, r^2 \equiv -N \bmod 4} \mathbf{e}\left(\frac{N + r^2}{4}\tau + rz\right).$$

Let $J_{k,1} := \{\phi \in \mathcal{J}_{k,1} \mid a_\phi(N) = 0 \text{ if } N < 0\}$ be the space of holomorphic Jacobi forms of weight k and index 1, and let $J_{k,1}^{\mathrm{cusp}} := \{\phi \in J_{k,1} \mid a_\phi(0) = 0\}$ be the subspace of cusp forms in $J_{k,1}$.

In what follows, we are mainly concerned with the weight 0 case. For $\phi \in \mathcal{J}_{0,1}$, we call $\{a_\phi(N) \mid N < 0\}$ the *principal part* of ϕ, which determines ϕ since $J_{0,1} = \{0\}$. The vanishing of $J_{0,1}$ is proved as follows. Suppose that ϕ is a nonzero element of $J_{0,1}$. Then $\Delta\phi \in J_{12,1}^{\mathrm{cusp}}$. Since $J_{12,1}^{\mathrm{cusp}}$ is spanned by ϕ' with $a_{\phi'}(3) = 1$ (cf. [[9]]), we have $\Delta\phi = c\,\phi'$ with $c \in \mathbb{C}^\times$ and hence $\phi = c\,\Delta^{-1}\phi'$. This implies that $a_\phi(-1) = c$, which contradicts to the holomorphy of ϕ.

2.2 Humbert surfaces

Let

$$Q := \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & -2 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}.$$

Let $O(Q) := \{g \in GL_5 \mid {}^t gQg = Q\}$ be the orthogonal group of Q. For $Z = (\tau_1, z, \tau_2) \in \mathfrak{H}_2$, put $\widetilde{Z} := {}^t(-\tau_1\tau_2 + z^2, \tau_1, z, \tau_2, 1) \in \mathbb{C}^5$. Note that $Q[\widetilde{Z}] = 0$ and $Q(\widetilde{Z}, \overline{\widetilde{Z}}) = 4\det(\mathrm{Im}(Z)) > 0$. There exists a homomorphism $\iota\colon \mathrm{Sp}_2(\mathbb{R}) \to O(Q, \mathbb{R})$ such that $\widetilde{g\langle Z\rangle} = j(g, Z)^{-1}\iota(g)\widetilde{Z}$ for $g \in \mathrm{Sp}_2(\mathbb{R})$ and $Z \in \mathfrak{H}_2$.

Let

$$L := \mathbb{Z}^5, \ L^* := Q^{-1}L,$$
$$L^*_{\mathrm{prim}} := \{\lambda \in L^* \mid n^{-1}\lambda \notin L^* \text{ for any integer } n > 1\}.$$

For an integer d, let

$$\mathcal{H}_d := \sum_{X \in \mathcal{L}_d} \left\{Z \in \mathfrak{H}_2 \mid Q(X, \widetilde{Z}) = 0\right\}$$

be a divisor in $\mathfrak{H}_2$, where $\mathcal{L}_d := \{X \in L^*_{\mathrm{prim}} \mid Q[X] = -d/2\}$. Note that $\mathcal{H}_d = 0$ unless $d > 0$ and $d \equiv 0$ or $1 \pmod 4$. Since $\mathcal{L}_d$ is $\iota(\Gamma_2)$-invariant, $\mathcal{H}_d$ is Γ_2-invariant. Denote by H_d the image of $\mathcal{H}_d$ in $\Gamma_2\backslash\mathfrak{H}_2$ by the natural projection $\mathfrak{H}_2 \to \Gamma_2\backslash\mathfrak{H}_2$. The divisor H_d of $\Gamma_2\backslash\mathfrak{H}_2$ is called the *Humbert surface* of discriminant d. It is known that H_d is nonzero and irreducible if $d \equiv 0$ or $1 \pmod 4$ (see [[11]], page 212, Theorem 2.4; see also [[12]], Section 3). Note that

$$\mathcal{H}_1 = \bigcup_{\gamma \in \Gamma_2} \gamma\left\{(\tau_1, 0, \tau_2) \mid \tau_1, \tau_2 \in \mathfrak{H}\right\},$$
$$\mathcal{H}_4 = \bigcup_{\gamma \in \Gamma_2} \gamma\left\{(\tau, z, \tau) \mid \tau \in \mathfrak{H}, z \in \mathbb{C}\right\}.$$

2.3 Siegel modular forms with a nontrivial character

Recall that the character group of Γ_2 consists of two elements, the trivial character and the nontrivial one υ given by

$$\upsilon\begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} = 1,$$
$$\upsilon(n(B)) = (-1)^{b_1+b_2+b_3},$$
$$\upsilon(d(A)) = (-1)^{(1+a_1+a_4)(1+a_2+a_3)+a_1a_4},$$

where

$$n(B) := \begin{pmatrix} 1_2 & B \\ 0_2 & 1_2 \end{pmatrix} \qquad \left(B = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Z}) \right),$$
$$d(A) := \begin{pmatrix} A & 0_2 \\ 0_2 & {}^tA^{-1} \end{pmatrix}, \qquad \left(A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \right)$$

(cf. [[21]]). Let $M_k(\Gamma_2, \upsilon)$ be the space of holomorphic functions F on $\mathfrak{H}_2$ satisfying $F(\gamma Z) = \upsilon(\gamma) j(\gamma, Z)^k F(Z)$ for any $\gamma \in \Gamma_2$.

Let

$$\theta_\epsilon(Z) := \sum_{l \in \mathbb{Z}^2} \mathbf{e}\left(\frac{1}{2}\, {}^t\!\left(l + \frac{1}{2}\epsilon'\right) Z \left(l + \frac{1}{2}\epsilon'\right) + \frac{1}{2}\, {}^t\!\left(l + \frac{1}{2}\epsilon'\right) \epsilon'' \right) \qquad (Z \in \mathfrak{H}_2)$$

be the theta constant associated with a characteristic $\epsilon = (\epsilon', \epsilon'') \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$. Then $\theta_\epsilon \neq 0$ if and only if ϵ is even, that is ${}^t\epsilon'\epsilon'' \equiv 0 \pmod 2$. There are 10 even characterstics. Set $\chi_5 := 2^{-8} \prod_{\epsilon \text{ even}} \theta_\epsilon$. It is known that $\chi_5 \in M_5(\Gamma_2, \upsilon)$, $\mathrm{div}(\chi_5) = H_1$ and $\chi_{10} = \chi_5^2$ (see [[19]], IV. 9 for example). Put

$$\begin{aligned} \chi_{35}(\tau_1, z, \tau_2) := {} & \chi_5(\tau_1, z, \tau_2)^{-8} \chi_5(2\tau_1, 2z, 2\tau_2) \\ & \times \prod_{a,b,c \bmod 2} \chi_5\left(\frac{\tau_1 + a}{2}, \frac{z + b}{2}, \frac{\tau_2 + c}{2} \right) \\ & \times \prod_{a \bmod 2} \chi_5\left(\frac{\tau_1 + a}{2}, z, 2\tau_2 \right) \chi_5\left(2\tau_1, z, \frac{\tau_2 + a}{2} \right) \\ & \times \prod_{b \bmod 2} \chi_5\left(2\tau_1, -\tau_1 + z, \frac{\tau_1 - 2z + \tau_2 + b}{2} \right). \end{aligned}$$

Then $\chi_{35} \in S_{35}(\Gamma_2)$ and $\mathrm{div}(\chi_{35}) = H_1 + H_4$. It follows that $\chi_{30} := \chi_5^{-1}\chi_{35} \in M_{30}(\Gamma_2, \upsilon)$ and $\mathrm{div}(\chi_{30}) = H_4$. The following fact is easily proved.

Lemma 2.1. *Let $F \in M_k(\Gamma_2, \upsilon)$. If k is odd, $\chi_5^{-1}F \in M_{k-5}(\Gamma_2)$. If k is even, $\chi_{30}^{-1}F \in M_{k-30}(\Gamma_2)$.*

2.4 Borcherds lifts on Γ_2

As a special case of Borcherds theory ([[2, 3]]; see also [[15]], §2.1), we have the following result:

Theorem 2.2. *Let $\phi \in \mathcal{J}_{0,1}$ and write $a(N)$ for $a_\phi(N)$. Assume that $a(N) \in \mathbb{Z}$ if $N < 0$.*

(i) Set

$$\delta := \sum_{r \in \mathbb{Z}} a(-r^2),$$
$$\rho := \frac{1}{2} \sum_{r \in \mathbb{Z},\, r>0} a(-r^2) r,$$
$$\nu := \frac{1}{4} \sum_{r \in \mathbb{Z}} a(-r^2) r^2$$

and

$$\Lambda = \left\{(m,n,r) \in \mathbb{Z}^3 \mid m > 0\right\} \cup \left\{(0,n,r) \in \mathbb{Z}^3 \mid n > 0\right\} \cup \left\{(0,0,r) \in \mathbb{Z}^3 \mid r > 0\right\}.$$

Then

$$\Psi_\phi(\tau_1, z, \tau_2) := \mathbf{e}\left(\nu\tau_1 - \rho z + \frac{\delta}{24}\tau_2\right) \prod_{(m,r,n) \in \Lambda} (1 - \mathbf{e}(m\tau_1 + rz + n\tau_2))^{a(4mn-r^2)}$$

converges absolutely if $\det(\mathrm{Im}(Z))$ *is sufficiently large, and is meromorphically continued to* $\mathfrak{H}_2$*.*

(ii) The function Ψ_ϕ *is a meromorphic modular form on* Γ_2 *of weight* $k_\phi = a(0)/2$ *and character* υ^α $(\alpha \in \{0,1\})$*.*

(iii) The divisor of Ψ_ϕ *is*

$$\sum_d \left(\sum_{f=1}^{\infty} a(-f^2 d)\right) H_d,$$

where d *runs over the positive integers congruent to* 0 *or* 1 *modulo* 4*.*

The meromorphic modular form Ψ_ϕ is called the *Borcherds lift* of ϕ.

Remark 2.3. As we will see in Remark 4.9, the weight of Ψ_ϕ is always an integer.

In view of Lemma 2.1, we are mainly concerned with Borcherds lifts of trivial character in this paper.

2.5 Examples of Borcherds lifts

For an even integer $k \geq 4$, let

$$e_{k,1}(\tau, z) := \frac{1}{2} \sum_{c,d \in \mathbb{Z},\, (c,d)=1} (c\tau + d)^{-k} \sum_{\lambda \in \mathbb{Z}} \mathbf{e}\left(\lambda^2 \frac{a\tau + b}{c\tau + d} + 2\lambda \frac{z}{c\tau + d} - \frac{cz^2}{c\tau + d}\right)$$

be the Jacobi Eisenstein series of weight k and index 1. Then $e_{k,1} \in J_{k,1}$. Set

$$\begin{aligned}\phi^{(1)}(\tau, z) &:= \frac{e_4(\tau)^2 e_{4,1}(\tau, z) - e_6(\tau) e_{6,1}(\tau, z)}{144\Delta(\tau)} \\ &= \left(\zeta + 10 + \zeta^{-1}\right) + q\left(10\zeta^2 - 64\zeta + 108 - 64\zeta^{-1} + 10\zeta^{-2}\right) + \cdots,\end{aligned}$$

where $q := \mathbf{e}(\tau), \zeta := \mathbf{e}(z)$ and $e_k(\tau)$ is the elliptic Eisenstein series of weight k (cf. **1.2**). It is easily verified that $\phi^{(1)}$ is an element of $\mathcal{J}_{0,1}$ with principal part

$$a(N) = \begin{cases} 1 & \text{if } N = -1, \\ 0 & \text{if } N < -1 \end{cases}$$

and that χ_5 (respectively χ_{10}) is proportional to the Borcherds lift of $\phi^{(1)}$ (respectively $2\phi^{(1)}$).

Let

$$\phi^{(2)}(\tau, z) = q^{-1} + \left(\zeta^2 - \zeta + 60 - \zeta^{-1} + \zeta^{-2}\right) + O(q)$$

be the element of $\mathcal{J}_{0,1}$ with principal part

$$a(N) = \begin{cases} 1 & \text{if } N = -4, \\ -1 & \text{if } N = -1, \\ 0 & \text{if } N < 0, N \neq -4, N \neq -1. \end{cases}$$

Then χ_{30} is proportional to the Borcherds lift of $\phi^{(2)}$.

Similarly we let

$$\phi^{(3)}(\tau, z) = q^{-1}(\zeta + \zeta^{-1}) + 48 + O(q)$$

be the element of $\mathcal{J}_{0,1}$ with principal part

$$a(N) = \begin{cases} 1 & \text{if } N = -5, \\ 0 & \text{if } N < 0, N \neq -5. \end{cases}$$

Then G_{24} is proportional to the Borcherds lift of $\phi^{(3)}$.

3 Proof of the main results

3.1 The multiplicative symmetry

Let $F \in M_k(\Gamma_2)$ and let p be a prime number. Put

$$\begin{aligned}\left(F|\mathcal{T}_p^{\uparrow}\right)(\tau_1, z, \tau_2) &= F(p\tau_1, pz, \tau_2) \prod_{a=0}^{p-1} F\left(\frac{\tau_1 + a}{p}, z, \tau_2\right), \\ \left(F|\mathcal{T}_p^{\downarrow}\right)(\tau_1, z, \tau_2) &= F(\tau_1, pz, p\tau_2) \prod_{a=0}^{p-1} F\left(\tau_1, z, \frac{\tau_2 + a}{p}\right).\end{aligned}$$

We say that F satisfies the *multiplicative symmetry* for p if the condition

$$(\mathrm{MS})_p \qquad F|\mathcal{T}_p^{\uparrow} = \epsilon_p(F)\, F|\mathcal{T}_p^{\downarrow}$$

holds with $\epsilon_p(F) \in \mathbb{C}^1$. The following result is a special case of [[17]].

Theorem 3.1. *Suppose that $F \in M_k(\Gamma_2)$ is a Borcherds lift. Then F satisfies the multiplicative symmetry for any prime number p.*

3.2 A characterization of powers of the modular discriminant

Let k be a nonnegative integer. Denote by $M_k(\Gamma_1)$ (respectively $S_k(\Gamma_1)$) the space of holomorphic automorphic (respectively cusp) forms on $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$ of weight k. Recall that $S_{12}(\Gamma_1) = \mathbb{C} \cdot \Delta$ and that Δ has no zeros in $\mathfrak{H} = \mathfrak{H}_1$.

For $f \in M_k(\Gamma_1)$ and a prime number p, we define the *multiplicative Hecke operator* by

$$(f|\mathcal{T}_p)(\tau) := f(p\tau) \prod_{c=0}^{p-1} f\left(\frac{\tau + c}{p}\right).$$

Then $f|\mathcal{T}_p \in M_{(p+1)k}(\Gamma_1)$. The following elementary fact plays a crucial role in the proof of Theorem 1.1.

Proposition 3.2. *A nonzero element f of $M_k(\Gamma_1)$ satisfies*

$$(*)_p \qquad (f|\mathcal{T}_p)(\tau) = \epsilon_p(f) f(\tau)^{p+1} \qquad (\tau \in \mathfrak{H})$$

for any prime number p with $\epsilon_p(f) \in \mathbb{C}^1$ if and only if f is a constant multiple of Δ^r $(r \in \mathbb{Z}_{\geq 0})$.

Proof. We first show the "if" part. It suffices to prove $(*)_p$ for $f = \Delta$. For a prime number p, we have

$$\frac{(\Delta|\mathcal{T}_p)(\tau)}{\Delta(\tau)^{p+1}} = \mathbf{e}\left(p\tau + \sum_{a=0}^{p-1} \frac{\tau + a}{p} - (p+1)\tau\right) \times \prod_{n\geq 1} \left(\frac{(1 - \mathbf{e}(pn\tau)) \prod_{a=0}^{p-1} \left(1 - \mathbf{e}\left(n\frac{\tau + a}{p}\right)\right)}{(1 - \mathbf{e}(n\tau))^{p+1}} \right)^{24}.$$

Since

$$\prod_{a=0}^{p-1} \left(1 - \mathbf{e}\left(n\frac{\tau + a}{p}\right)\right) = \begin{cases} (1 - \mathbf{e}(p^{-1}n\tau))^p & \text{if } p|n, \\ 1 - \mathbf{e}(n\tau) & \text{if } p \nmid n, \end{cases}$$

we have

$$\begin{aligned} \frac{(\Delta|\mathcal{T}_p)(\tau)}{\Delta(\tau)^{p+1}} &= \mathbf{e}\left(\frac{p-1}{2}\right) \\ &\times \left(\frac{\prod_{n\geq 1}(1 - \mathbf{e}(pn\tau)) \prod_{n\geq 1,\, p|n}(1 - \mathbf{e}(p^{-1}n\tau))^p \prod_{n\geq 1,\, p\nmid n}(1 - \mathbf{e}(n\tau))}{\prod_{n\geq 1}(1 - \mathbf{e}(n\tau))^{p+1}} \right)^{24} \\ &= (-1)^{p-1}, \end{aligned}$$

which proves the claim. This also shows that $\epsilon_p(\Delta^r) = (-1)^{(p-1)r}$. We next show the "only if" part. Suppose that f satisfies $(*)_p$ for $p = 2$. Let $f(\tau) = a_r q^r + O(q^{r+1})$ with $a_r \neq 0$ ($q = \mathbf{e}(\tau)$). We may assume $a_r = 1$ without loss of generality. Then $g := f\Delta^{-r} = 1 + O(q)$ is in $M_{k-12r}(\Gamma)$ and we have

$$g(2\tau)g\left(\frac{\tau}{2}\right)g\left(\frac{\tau+1}{2}\right) = \epsilon\, g(\tau)^3 \tag{3.1}$$

with $\epsilon = (-1)^r \epsilon_2(f) \in \mathbb{C}^1$. Suppose that g has a zero τ_0 in $\mathfrak{H}$. Let $\mathcal{F} = \{\tau \in \mathfrak{H} \mid |\tau| \geq 1, -1/2 \leq \mathrm{Re}(\tau) \leq 1/2\}$. Since $\mathcal{F}$ is the closure of a fundamental domain of $\Gamma\backslash\mathfrak{H}$, we may suppose that $\tau_0 \in \mathcal{F}$. Note that $\mathrm{Im}(2\tau_0) \geq \sqrt{3} > 1$. Since $g(2\tau_0)^3 = \epsilon^{-1} g(4\tau_0)g(\tau_0)g(\tau_0 + 1/2) = 0$, we have $g(2\tau_0) = 0$. Take a point τ_1 in $\mathcal{F}$ such that $\tau_1 - 2\tau_0 \in \mathbb{Z}$. Then $g(\tau_1) = 0$ and $\tau_1 \neq \tau_0$. Repeating this procedure, we see that g has infinitely many zeros in $\mathcal{F}$, a contradiction. Thus g has no zeros in $\mathfrak{H}$. It follows that $g^{-1} \in M_{-k+12r}$. This implies that $k = 12r$ and g is a constant. This completes the proof of the proposition. □

Remark 3.3. The proof of the proposition shows that f is a constant multiple of Δ^r if $f \in M_k(\Gamma_1)$ satisfies $(*)_2$.

3.3 The multiplicative symmetry for $\mathrm{Sym}^2(M_k(\Gamma_1))$

For $\varphi \in \mathrm{Sym}^2(M_k(\Gamma_1))$ and a prime number p, we define the multiplicative Hecke operators by

$$(\varphi|\mathcal{T}_p^{\uparrow})(\tau_1, \tau_2) := \varphi(p\tau_1, \tau_2) \prod_{c=0}^{p-1} \varphi\left(\frac{\tau_1 + c}{p}, \tau_2\right),$$
$$(\varphi|\mathcal{T}_p^{\downarrow})(\tau_1, \tau_2) := \varphi(\tau_1, p\tau_2) \prod_{c=0}^{p-1} \varphi\left(\tau_1, \frac{\tau_2 + c}{p}\right).$$

We say that φ satisfies the *multiplicative symmetry* for p if there exists an $\epsilon_p(\varphi) \in \mathbb{C}^1$ such that

$$\varphi|\mathcal{T}_p^{\uparrow} = \epsilon_p(\varphi)\varphi|\mathcal{T}_p^{\downarrow} \tag{ms$_p$}$$

holds. For $\varphi \in \mathrm{Sym}^2(M_k(\Gamma_1))$, we put

$$\Phi'(\varphi)(\tau) = \lim_{y\to\infty} \varphi(\tau, iy).$$

Then $\Phi'(\varphi) \in M_k(\Gamma_1)$. The following fact is easily verified.

Lemma 3.4. *If $\varphi \in \mathrm{Sym}^2(M_k(\Gamma_1))$ satisfies* $(\mathrm{ms})_2$ *and $f := \Phi'(\varphi)$ is not identically equal to zero, then f satisfies $(*)_2$. In particular, f is a constant multiple of Δ^r and k is divisible by* 12.

Proposition 3.5. *If $\varphi \in \mathrm{Sym}^2(M_k(\Gamma_1)) \setminus \{0\}$ satisfies* $(\mathrm{ms})_2$*, k is divisible by* 12.

Proof. If $\Phi'(\varphi) \neq 0$, the assertion follows from Lemma 3.4. Suppose that $\Phi'(\varphi) = 0$. Since $S_k(\Gamma_1) = \Delta\, M_{k-12}(\Gamma_1)$, there exist a positive integer r and $\psi \in \mathrm{Sym}^2(M_{k-12r}(\Gamma_1))$ such that $\varphi = (\Delta^r \otimes \Delta^r)\psi$ and $\Phi'(\psi) \neq 0$. Since ψ satisfies $(\mathrm{ms})_2$, we have $12|(k - 12r)$ by Lemma 3.4 and hence $12|k$. □

The following is easily verified.

Proposition 3.6. *Suppose that $F \in M_k(\Gamma_2)$ satisfies* $(\mathrm{MS})_p$ *for a prime p. Put $f := \Phi(F)$ and $\varphi := \mathcal{W}(F)$. Then f (respectively φ) satisfies* $(*)_p$ *(respectively* $(\mathrm{ms})_p$*) and $\epsilon_p(F) = \epsilon_p(f) = \epsilon_p(\varphi)$.*

3.4 Proofs of Theorem 1.1 and Theorem 1.4 (i)

Proposition 3.7. *Assume that $F \in M_k(\Gamma_2)$ satisfies* $(\mathrm{MS})_2$ *and that $f := \Phi(F)$ is not identically equal to zero. Then $f = c\,\Delta^r$ $(c \in \mathbb{C}^\times, r \in \mathbb{Z}_{\geq 0})$. In particular, the weight k is divisible by* 12.

Proof. The proposition follows from Proposition 3.6 and Remark 3.3. ☐

Theorem 1.1 is a direct consequence of Theorem 3.1 and Proposition 3.7. Theorem 1.4 (i) also follows from Theorem 3.1, Proposition 3.6 and Proposition 3.5.

4 The weight formula

4.1 Cohen numbers

To express the weight k_ϕ of Ψ_ϕ in terms of the principal part of ϕ, we recall the definitions of Cohen numbers $c(r, N)$ and $C(r, N)$.

Let $L(s, \chi)$ denote the Dirichlet L-function attached to a Dirichlet character χ modulo N. For $r, N \in \mathbb{Z}_{>0}$, define

$$c(r,N) := \begin{cases} (-1)^{[r/2]}(r-1)!\,N^{r-1/2}2^{1-r}\pi^{-r}L(r,\chi_{(-1)^rN}) & \\ \qquad\qquad \text{if } (-1)^rN \equiv 0 \text{ or } 1 \pmod 4, \\ 0 & \\ \qquad\qquad \text{if } (-1)^rN \equiv 2 \text{ or } 3 \pmod 4, \end{cases} \tag{4.1}$$

where $\chi_M(d) := \left(\dfrac{M}{d}\right)$ for $M \in \mathbb{Z}$ with $M \equiv 0$ or $1 \pmod 4$. We put $c(r, 0) := \zeta(1-2r)$ for $r \in \mathbb{Z}_{>0}$. For $r \in \mathbb{Z}_{>0}$ and $N \in \mathbb{Z}$, we define

$$C(r,N) := \begin{cases} \sum_{d^2|N} c(r, d^{-2}N) & \text{if } (-1)^rN \equiv 0 \text{ or } 1 \pmod 4 \text{ and } N > 0, \\ \zeta(1-2r) & \text{if } N = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Note that

$$c(r,N) = \sum_{f>0,\, f^2|N} \mu(f) C(r, f^{-2}N)$$

for $N > 0$, where μ is the Möbius function. We set

$$C^*(N) := -60\,C(2,N),$$
$$c^*(N) := -60\,c(2,N)$$

for $N \geq 0$. If $N > 0$, we have

$$c^*(N) = \sum_{f>0,\, f^2|N} \mu(f) C^*\left(f^{-2}N\right). \tag{4.3}$$

The formula (4.3) implies that

$$c^*(D) = C^*(D) \tag{4.4}$$

for a fundamental discriminant D.

Remark 4.1. We give a table of $c^*(N)$ and $C^*(N)$ for $N \leq 20$ with $N \equiv 0$ or $1 \pmod 4$.

N	0	1	4	5	8	9	12	13	16	17	20
$c^*(N)$	-1/2	5	30	24	60	120	120	120	240	240	240
$C^*(N)$	-1/2	5	35	24	60	125	120	120	275	240	264

The following result is proved in [[7]] (Propositition 4.1).

Proposition 4.2. *For a nonnegative integer N, we have*

$$C^*(N) = 12 \sum_{s \in \mathbb{Z}} \sigma_1\left(\frac{N-s^2}{4}\right) + \begin{cases} 6N & \textit{if } N = \square, \\ 0 & \textit{otherwise,} \end{cases}$$

where

$$\sigma_1(m) = \begin{cases} \sum_{0<d|m} d & \textit{if } m \in \mathbb{Z}_{>0}, \\ -1/24 & \textit{if } m = 0, \\ 0 & \textit{otherwise.} \end{cases} \tag{4.5}$$

In particular, $C^*(N)$ is a positive integer if $N > 0$.

Let N be a positive integer with $N \equiv 0$ or $1 \pmod 4$. Then we have $N = g^2 D$ with $g \in \mathbb{Z}_{>0}$ and D a fundamental discriminant. We set

$$B(N) := g^3 \prod_{p|g} \left(1 - \frac{\chi_D(p)}{p^2}\right).$$

This number is a positive integer divisible by g. The following is easily verified.

Lemma 4.3. *(i) For a fundamental discriminant D, we have $B(D) = 1$.*

(ii) Let N, g, D be as above. Then we have $B(N) \geq g^2$.

Lemma 4.4. *Let N be a positive integer with $N \equiv 0$ or $1 \pmod 4$ and let $N = g^2 D$ with $g \in \mathbb{Z}_{>0}$ and D a fundamental discriminant. Then we have*

$$c^*(N) = B(N)\, C^*(D).$$

In particular $c^*(N)$ is a positive integer.

Proof. By (4.2) and (4.3), we have

$$\begin{aligned} c^*(N) &= B(N)\, \frac{60 D^{3/2}}{2\pi^2} L(2, \chi_D) \\ &= B(N)\, C^*(D), \end{aligned}$$

which proves the lemma. □

Lemma 4.5. *For $N \in \mathbb{Z}_{>0}$, we have*

$$C^*(N) \geq c^*(N) \geq 3N.$$

Proof. Since $c^*(N) \geq 0$ and $C^*(N) = \sum_{f>0,\, f^2 | N} c^*(f^{-2}N)$, we have $C^*(N) \geq c^*(N)$. By Proposition 4.2, we have $C^*(N) \geq 3N$. Then Lemma 4.4 and Lemma 4.3 imply that

$$c^*(N) \geq g^2 C^*(D) \geq 3g^2 D = 3N.$$

□

Lemma 4.6. *If $D > 1$ is a fundamental discriminant with $D \neq 5$ and $D \neq 8$, then $C^*(D)$ is divisible by 120.*

Proof. The lemma immediately follows from Proposition 4.2 and [[8]] Corollary 10.3.9. □

Lemma 4.7. *If $N \geq 9$, $c^*(N)$ is divisible by 120.*

Proof. We may suppose that $N = g^2 D$ with $g \in \mathbb{Z}_{>0}$ and D a fundamental discriminant. By Lemma 4.4, we have $c^*(N) = B(N)\, C^*(D)$. In view of Lemma 4.6, it suffices to consider the case where $D = 1$, $D = 5$ or $D = 8$. Since $C^*(1) = 5, C^*(5) = 24$ and $C^*(8) = 60$, the proof of the lemma is reduced to the following elementary fact, whose proof we omit.

(a) If $D = 1$ and $g \geq 3$, $B(N)$ is divisible by 24.

(b) If $D = 5$ and $g \geq 2$, $B(N)$ is divisible by 5.

(c) If $D = 8$ and $g \geq 2$, $B(N)$ is divisible by 2.

□

4.2 The weight formula for Borcherds lifts

Proposition 4.8. *(i) For each positive integer d with $d \equiv 0$ or $1 \pmod 4$, there uniquely exists an element ϕ_d of $\mathcal{J}_{0,1}$ with principal part*

$$a_{\phi_d}(N) = \begin{cases} 1 & \text{if } N = -d, \\ 0 & \text{if } N < 0 \text{ and } N \neq -d. \end{cases}$$

(ii) For $\phi \in \mathcal{J}_{0,1}$, we have

$$\sum_{N \geq 0} a_\phi(-N) C^*(N) = 0$$

and hence

$$\frac{1}{2} a_\phi(0) = \sum_{N > 0} a_\phi(-N) C^*(N).$$

Proof. This fact is essentially due to Borcherds ([[4]]; see also [[6]], page 1721). For completeness, we give a sketch of the proof. For $k \in \mathbb{Z}$, denote by $J^+_{k,1}$ and $J^{+,\,\mathrm{cusp}}_{k,1}$ the space of skew-holomorphic Jacobi forms of weight k and index 1 and its subspace consisting of cusp forms, respectively (for the definition, see [[25]]). For an odd integer k, we also denote by $M^+_{k/2}(\Gamma_0(4))$ and $S^+_{k/2}(\Gamma_0(4))$ the Kohnen plus space on $\Gamma_0(4)$ of weight $k/2$ and its subspace consisting of cusp forms, respectively. It is known that $J^+_{k,1} \cong M^+_{k-1/2}(\Gamma_0(4))$ and $J^{+,\,\mathrm{cusp}}_{k,1} \cong S_{k-1/2}(\Gamma_0(4))$ if k is odd (see [[25]]; see also [[16]]). By [[4]], the obstruction space for $\mathcal{J}_{0,1}$ is $J^{+,\,\mathrm{cusp}}_{3,1}$. The first assertion of the proposition follows from the fact $J^{+,\,\mathrm{cusp}}_{3,1} \cong S^+_{5/2}(\Gamma_0(4)) \cong S_4(\mathrm{SL}_2(\mathbb{Z})) = \{0\}$. Next observe that $J^+_{3,1} \cong M^+_{5/2}(\Gamma_0(4))$ and that the latter space is one-dimensional and spanned by Cohen's Eisenstein series

$$\sum_{N \geq 0} C(2, N) \mathbf{e}(N\tau).$$

The second assertion follows from this. □

Remark 4.9. Proposition 4.8 implies that, if $\phi \in \mathcal{J}_{0,1}$ satisfies the assumption of Theorem 2.1, the weight k_ϕ of the Borcherds lift Ψ_ϕ is an integer.

Remark 4.10. The Borcherds lift Ψ_ϕ is holomorphic if and only if

$$\sum_{f=1}^{\infty} a_\phi(-f^2 d) \geq 0$$

holds for any positive integer d with $d \equiv 0$ or $1 \pmod 4$.

Theorem 4.11. *(i) For each positive integer d with $d \equiv 0$ or $1 \pmod 4$, there exists an $F_d \in M_{k_d}(\Gamma_2, \upsilon^{\alpha_d})$ with $\alpha_d \in \{0, 1\}$ satisfying $\mathrm{div}(F_d) = H_d$.*

(ii) We have $k_d = c^*(d)$.

(iii) We have $F_1 \in M_5(\Gamma_2, \upsilon), F_4 \in M_{30}(\Gamma_2, \upsilon)$ and $F_d \in M_{k_d}(\Gamma_2)$ if $d > 4$.

(iv) A Borcherds lift $F \in M_k(\Gamma_2, \upsilon^\alpha)$ $(\alpha \in \{0,1\})$ is a constant multiple of $\prod_d F_d^{A(d)}$, where d runs over the positive integers with $d \equiv 0$ or $1 \pmod 4$, and $A(d)$ is a nonnegative integer ($A(d) = 0$ except for a finite number of d) satisfying $A(1) + A(4) \equiv \alpha \pmod 2$. Furthermore we have

$$k = \sum_{d>0} A(d)c^*(d).$$

Proof. Set

$$\psi_d := \sum_{f>0,\, f^2|d} \mu(f)\, \phi_{f^{-2}d}$$

and let F_d be the Borcherds lift of ψ_d. By Theorem 2.2 (iii), we have $\mathrm{div}(F_d) = H_d$ and hence F_d is holomorphic. The weight of F_d is equal to

$$\sum_{f>0,\, f^2|d} \mu(f)C^*(f^{-2}d) = c^*(d)$$

by (4.3). By Lemma 2.1, the character of F_d is equal to 1 (respectively υ) if $d > 4$ (respectively if $d = 1$ or $d = 4$). The remaining assertion of the theorem is easily verified. □

Proposition 4.12. *The weight k_d of F_d is divisible by 24 if and only if $d > 4$ and $d \neq 8$. If $d \geq 9$, k_d is divisible by 120.*

Proof. The proposition is a direct consequence of Theorem 4.11, Lemma 4.7 and Remark 4.1. □

Proposition 4.13. *The Borcherds lifts in $M_k(\Gamma_2)$ with $k \leq 60$ are listed as follows:*

Borcherds lift	*weight*	*divisor*
F_1^{2a} $(1 \leq a \leq 6)$	$10a$	$2aH_1$
$F_1^{2a+1}F_4$ $(1 \leq a \leq 2)$	$10a + 35$	$(2a+1)H_1 + H_4$
$F_1^{2a}F_5$ $(1 \leq a \leq 3)$	$10a + 24$	$2aH_1 + H_5$
F_4^2	60	$2H_4$
F_5^2	48	$2H_5$
F_8	60	H_8

Remark 4.14. The above table shows that every Borcherds lift of weight less than or equal to 60 is a monomial of F_1, F_4, F_5 and F_8. We also see that there is no holomorphic Borcherds lift of weight 12, which proves the second assertion of Theorem 1.4. This also gives another proof of the fact that χ_{12} is not a Borcherds lift, which was proved in [[17]] in a different way.

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