Barth map of the moduli space of stable rank-2 vector bundles on P^2

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BARTH MAP OF THE MODULI SPACE OF STABLE RANK-2 VECTOR BUNDLES ON P^2

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Introduction

Let $M_n := M_{P^2}(2,0,n)$ be the moduli space of rank-2 stable vector bundles on P^2 with $C_1 = 0$, $c_2 = n$, $\dim M_n = 4n-3$, $n \geq 2$, $\bar{M}_n := \overline{M_{P^2}(2,0,n)}$ its Gieseker-Maruyama compactification, i.e. the space of (S-equivalence classas of) semistable rank-2 sheaves on P^2 with $c_1 = 0$, $c_2 = n$, $\bar{M}_n^s$ the subset of stable sheaves in $\bar{M}_n$, $\partial\bar{M}_n := \bar{M}_n \setminus M_n = \{[\mathcal{E}] \in \bar{M}_n | \mathcal{E}$ is not locally free, i.e. $\mathcal{E}^{\check{}\check{}} \neq \mathcal{E}$, i.e. $l(\mathcal{E}^{\check{}\check{}}/\mathcal{E} \geq 1\}$ the subset of non-locally free sheaves, where $\mathrm{codim}_{\bar{M}_n}\partial\bar{M}_n = 1$, and let $\mathcal{M}_n := \{[\mathcal{E}] \in \bar{M}_n | l(\mathcal{E}^{\check{}\check{}}/\mathcal{E}) \leq 1\}$ be the "good" part of $\bar{M}_n$ (this is a dense open subset in $\bar{M}_n$ for $n \geq 3$).

Remark 0.1. $\mathcal{M}_n \subset \bar{M}_n^s$ for $n \geq 3$.

Next, let $D := \mathcal{M}_n \cap \partial\bar{M}_n = \{[\mathcal{E}] \in \mathcal{M}_n | l(\mathcal{E}^{\check{}\check{}}/\mathcal{E}) = 1\}$ be the "good" part of $\partial\bar{M}_n$, so that $\mathrm{codim}_{\mathcal{M}_n} D = 1$, $\mathrm{codim}_{\bar{M}_n}(\bar{M}_n \setminus \mathcal{M}_n) = 2$, $\mathrm{codim}_{\partial\bar{M}_n}(\partial\bar{M}_n \setminus D) = 1$ for $n \geq 3$. We will mostly deal with $\mathcal{M}_n$ and D. Next, for $[\mathcal{E}] \in D$ denote $x = x(\mathcal{E}) := \mathrm{Supp}(\mathcal{E}^{\check{}\check{}}/\mathcal{E})$; we have

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}^{\check{}\check{}} \xrightarrow{\varepsilon} k(x) \longrightarrow 0. \tag{1}$$

Next, the projection $\pi_n : D \longrightarrow M_{n-1} \times P^2 : [\mathcal{E}] \longmapsto ([\mathcal{E}^{\check{}\check{}}], x(\mathcal{E}))$ is a P^1-fibration: in fact, for any pair $([\mathcal{E}_0], x) \in M_{n-1} \times P^2$ we have by (1)

$$\pi_n^{-1}([\mathcal{E}_0], x) = P(\mathrm{Hom}(\mathcal{E}_0, k(x))) \simeq P(\mathcal{E}_0^{\check{}}|x) \simeq P^1. \tag{2}$$

Next, for any $[\mathcal{E}] \in \mathcal{M}_n$ there is defined a curve $C_n(\mathcal{E}) := \{l \in \check{P}^2 | \ \mathcal{E}|l \not\simeq 2\mathcal{O}_{P^1}\}$ in $\check{P}^2$ of jumping lines of $\mathcal{E}$. This curve has a natural structure of a divisor of degree n in $\check{P}^2$ [B],[B1]. Hence we consider $C_n(\mathcal{E})$ as a point of the projective space $P^{N_n} := |\mathcal{O}_{\check{P}^2}(n)|$, $N_n = n(n+3)/2$.

Remark 0.2. If $[\mathcal{E}] \in D$, then from (1) it clearly follows that

$$C_n(\mathcal{E}) = C_{n-1}(\mathcal{E}^{\check{}\check{}}) \cup \check{x}(\mathcal{E}), \tag{3}$$

where $\check{x} := \{l \in \check{P}^2 | x \in l\}$ is a line in $\check{P}^2$ dual to the point $x \in P^2$.

In this paper we consider the map $f_n : \bar{M}_n \longrightarrow P^{N_n} : [\mathcal{E}] \longmapsto C_n(\mathcal{E})$, called the *Barth map* after W.Barth [B]. This map is well known to be a morphism – see, e.g., [M2, Part II, Prop. 1.9], which is generically finite by [B]. Denote $\mathcal{C}_n = f_n(\bar{M}_n)$. The following results are due to J.Le Potier [L],[L*],[L1],[L2],[L3]:

Theorem 0.3. *(Le Potier* [L2]*):* $f_n|M_n$ *is a quasifinite map onto its image,* $n \geq 2$. *Hence* $\dim \mathcal{C}_n = \dim \mathcal{M}_n = 4n-3$.

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Theorem 0.4. *(Le Potier* [L],[L*],[L1],[L3]*):* $f_4 : \bar{M}_4 \longrightarrow C_4$ *is a birational map* [1] *and* $\deg_{P^{14}} C_4 = 54$. [2]

Remark 0.5. It is well known (see, e.g., [ELS]) that $\deg_{P^N} \bar{C}_n = q_{4n-3}/\deg f_n$, where q_{4n-3} are the Donaldson invariants of CP^2, $q_{4n-3} = \int_{\bar{M}_n} f_n^*(c_1(\mathcal{O}_{P^{N_n}}(1))^{4n-3}$. By now the values of q_{4n-3} are known at least for $n \leq 10$: $q_5 = 1$ since f_2 is birational onto $|\mathcal{O}_{\check{P}^2}(2)|$ (see [B]), $q_9 = 3$ (Maruyama [M1]), $q_{13} = 54$ (Le Potier [L],[L1], Tyurin and Tikhomirov [T], Li and Qin [LQ]), $q_{17} = 2540$ and $q_{21} = 233208$ (Ellingsrud, Le Potier and Strømme [ELS]), $q_{25} = 35225553$, $q_{29} = 8365418914$, $q_{33} = 2780195996868$, $q_{37} = 1253555847090600$ (Göttsche, using the method of Ellingsrud and Göttsche [EG]).

The aim of this paper is to prove the following

Theorem 0.6. $f_n : \bar{M}_n \longrightarrow C_n$ *is birational for any* $n \geq 4$.

From this theorem and remark 0.5 follows

Corollary 0.7.

$$\deg_{P^{20}} C_5 = 2540,$$

where C_5 *is the variety of Darboux quintics in* P^{20} *(see* [B, Prop.5], [D]*),*

$$\deg_{P^{27}} C_6 = 233208, \quad \deg_{P^{35}} C_7 = 35225553, \quad \deg_{P^{44}} C_8 = 8365418914,$$

$$\deg_{P^{54}} C_9 = 2780195996868, \quad \deg_{P^{65}} C_{10} = 1253555847090600.$$

1. Outline of the proof of main result

Our proof is inductive, beginning from $n = 4$ (due to theorem 0.4 of Le Potier), and is based on three geometric observations concerning the behaviour of the sets $\mathcal{M}_n$ and $D = \mathcal{M}_n \cap \partial\bar{M}_n$ under f_n. Denote $Z_n := f_n(D)$.

First observation (Maruyama, Hulek, Strømme 1983; see, e.g., [M3, Question 0.2]), a direct corollary of (3):

the map $f_n|D$ factors through the map π_n in the diagram

$$\begin{array}{ccc} [\mathcal{E}] & \in & D \\ \downarrow & & \downarrow \pi_n \\ ([\mathcal{E}^{\check{}}], x(\mathcal{E})) & \in & M_{n-1} \times P^2 \\ \downarrow & & \downarrow \psi_n \quad (f_n) \\ C_{n-1}(\mathcal{E}^{\check{}}) \cup \check{x}(\mathcal{E}) & & \\ \| & & \\ f_{n-1}([\mathcal{E}^{\check{}}]) \cup \check{x}(\mathcal{E}) & \in & Z_n \end{array} \qquad (4)$$

---

[1] Note here that though $f_4|M_4$ is quasifinite and birational, it is not bijective: e.g., over the Humbert desmic quartics its fiber consists of at least 6 points - see [B, remark after Prop.6] and [Ba, p.367].

[2] The number $\deg_{P^{14}} \bar{C}_4 = 54$ interpreted as the degree of the hypersurface W of Lüroth duartics in $P^{14} = |\mathcal{O}_{\check{P}^2}(4)|$ was already known to F.Morley [Mo], whose result actually states that the degree of W is a factor of 54.

Hence, since π_n is a P^1-fibration, $\operatorname{codim}_{\mathcal{C}_n} Z_n \leq 2$. In fact, as it is easily seen, $\operatorname{codim}_{\mathcal{C}_n} Z_n = 2, \quad n \geq 3$. For our further purposes it will be enough to see that

$$\operatorname{codim}_{\mathcal{C}_n} Z_n = 2, \quad n \geq 5. \tag{5}$$

For this, remark that $\mathcal{C}_n^* = \{C_n \in \mathcal{C}_n | C_n \text{ is smooth}\}$ is a dense open subset in $\mathcal{C}_n, \quad n \geq 2$, – see [B, 5.4], so that $M_n^* = f_n^{-1}(\mathcal{C}_n^*)$ is also dense open in M_n; respectively, $Z_n^* = \psi_n(M_{n-1}^* \times P^2)$ is a dense open subset of $Z_n, \ n \geq 3$. Now by theorem 0.4 and the induction step f_{n-1} is birational for $n \geq 5$. Hence, clearly in view of diagram (4) $\psi_n | M_{n-1}^* \times P^2 : M_{n-1}^* \times P^2 \longrightarrow Z_n^*$ is a birational morphism, i.e. there exists a dense open subset Z_n^{**} in Z_n^* such that $\psi_n | \psi_n^{-1}(Z_n^{**})$ is an isomorphism:

$$\psi_n : \psi_n^{-1}(Z_n^{**}) \xrightarrow{\sim} Z_n^{**}; \tag{6}$$

whence (5) follows.

Second observation. There exists a dense open subset Z_n' in Z_n^{**} (hence in Z_n)

$$Z_n' \overset{open}{\hookrightarrow} Z_n \tag{7}$$

such that

$$f_n^{-1}(Z_n') \overset{sets}{=} \pi_n^{-1}\psi_n^{-1}(Z_n'). \tag{8}$$

In other words, there are no locally free sheaves in $\mathcal{M}_n$ mapping by f_n to a general point of Z_n. In fact, consider the set $L_n = \{[\mathcal{E}] \in M_n | C_n(\mathcal{E}) \text{ contains a line}\}$. As it is shown by S.A.Strømme [S, Theorem 3.7(viii)], $\operatorname{codim}_{M_n} L_n \geq n-1$ (we recall the proof in appeddix B below), hence

$$\operatorname{codim}_{M_n} L_n \geq 3, \quad n \geq 4. \tag{9}$$

Now remark that

$$C_n(\mathcal{E}) \notin Z_n^*, \quad [\mathcal{E}] \in \partial \bar{M}_n \setminus D. \tag{10}$$

Hence $f_n^{-1}(Z_n^*) \setminus \pi_n^{-1}\psi_n^{-1}(Z_n^*) = f_n^{-1}(Z_n^*) \cap M_n \subset L_n$, so that (8) follows from (5) and (9). To show (10), take $[\mathcal{E}] \in \partial \bar{M}_n \setminus D$, so that by definition $l := l(\mathcal{E}^{\vee\vee}/\mathcal{E}) \geq 2$, i.e. $\mathcal{E}^{\vee\vee}/\mathcal{E} = \bigoplus_{i=1}^{k} \mathcal{A}_i$, $\operatorname{Supp}(\mathcal{A}_i) = x_i, \ i = 1, ..., k$, where $x_1, ..., x_k$ are distinct points and

$$l = \sum_{i=1}^{k} l(\mathcal{A}_i) \geq 2. \tag{11}$$

Consider the graph of incidence $\Gamma_{0,1} \subset P^2 \times \check{P}^2$ with natural projections $P^2 \overset{q_1}{\leftarrow} \Gamma_{0,1} \overset{q_2}{\rightarrow} \check{P}^2$. Then by [B] the curve $C_n(\mathcal{E})$ as a divisor in P^2 is given by the ideal sheaf $\mathcal{I}_{C_n(\mathcal{E}),\check{P}^2} = \mathcal{F}itt^0(R^1 q_{2*} q_1^* \mathcal{E}(-1))$. Thus applying the functor $R^i q_{2*} q_1^*$ to the exact triple $0 \to \mathcal{E} \to \mathcal{E}^{\vee\vee} \to \bigoplus_{i=1}^{k} \mathcal{A}_i \to 0$, we obtain $C_n(\mathcal{E}) = C_{n-l}(\mathcal{E}^{\vee\vee}) + \sum_{i=1}^{k} l(\mathcal{A}_i)\check{x}_i$ in $Div(\check{P}^2)$, where $\check{x}_i$ are the lines in $\check{P}^2$ dual to the points $x_i \in P^2$. Hence in view of (11) the curve $C_n(\mathcal{E})$ doesn't contain a smooth component of degree $n-1$, wherefrom (10) follows.

It follows now from (7) and (8) that we can explore the map f_n over Z_n (and eventually show that f_n is birational) only via studying f_n around D. For this, we use

Third observation: Let $y = ([\mathcal{E}_0], x) \in M_{n-1} \times P^2$ and $P_y^1 := \pi_n^{-1}(y)$. Then

$$\mathcal{O}_{\mathcal{M}_n}(D)|_{P_y^1} \simeq \mathcal{O}_{P_y^1}(-2) \tag{12}$$

(here D is understood as a smooth Cartier divisor in $\mathcal{M}_n$). To see this, take a point $z = [\mathcal{E}] \in P^1_y$ considered as a triple of data $z = ([\mathcal{E}_0], x, \mathcal{E}_0 \overset{\varepsilon_z}{\twoheadrightarrow} k(x))$, where $\mathcal{E}_0 = \mathcal{E}^{\sim}$ so that z defines a triple (1):

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_0 \xrightarrow{\varepsilon_z} k(x) \longrightarrow 0. \tag{13}$$

Then one has natural maps $T_{[\mathcal{E}]}\mathcal{M}_n = \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \overset{\varphi_1}{\twoheadrightarrow} \mathrm{Ext}^2(k(x),\mathcal{E}) \overset{\varphi_2}{\twoheadleftarrow} \mathrm{Ext}^1(k(x),k(x))$.

Lemma 1.1. *The tangent space to the divisor D at any point $[\mathcal{E}] \in D$ is given by formula $T_{[\mathcal{E}]}D = \varphi_1^{-1}(im\ \varphi_2)$. Hence the normal space $N_D\mathcal{M}_n|_{[\mathcal{E}]}$ is isomorphic to* $\ker(\varepsilon^\sharp : \mathrm{Ext}^2(k(x),\mathcal{E}_0) \twoheadrightarrow \mathrm{Ext}^2(k(x),k(x))$, *where $\varepsilon^\sharp$ is induced by the map $\mathcal{E}_0 \overset{\varepsilon_z}{\twoheadrightarrow} k(x)$.*

This lemma is a matter of standard diagram chasing. We give its proof in the appendix A. Now relativize the triple (13) over $P^1_y \simeq P^1$: $0 \longrightarrow \mathbf{E} \longrightarrow \mathcal{E}_0 \boxtimes \mathcal{O}_{P^1_y} \longrightarrow k(x) \boxtimes \mathcal{O}_{P^1_y}(1) \longrightarrow 0$. Applying to this triple the relative $\mathcal{E}xt^2_{p_2}(k(x)\boxtimes\mathcal{O}_{P^1_y}(1),\cdot)$-functor, where $p_2 : P^2 \times P^1_y \longrightarrow P^1_y$ is the projection, and using the above lemma and the base-change, we obtain the following formula for the restriction of the normal bundle $\mathcal{N}_D\mathcal{M}_n$ onto P^1_y:

$$\mathcal{N}_D\mathcal{M}_n|P^1_y \simeq \ker(\varepsilon^\sharp : \mathcal{E}xt^2_{p_2}(k(x)\boxtimes\mathcal{O}_{P^1_y}(1), \mathcal{E}_0\boxtimes\mathcal{O}_{P^1_y}) \twoheadrightarrow \mathcal{E}xt^2_{p_2}(k(x)\boxtimes\mathcal{O}_{P^1_y}(1), k(x)\boxtimes\mathcal{O}_{P^1_y}(1))$$

Here one easily checks that $\mathcal{E}xt^2_{p_2}(k(x)\boxtimes\mathcal{O}_{P^1_y}(1), \mathcal{E}_0\boxtimes\mathcal{O}_{P^1_y}) \simeq 2\mathcal{O}_{P^1_y}(-1)$ and $\mathcal{E}xt^2_{p_2}(k(x)\boxtimes\mathcal{O}_{P^1_y}(1), k(x)\boxtimes\mathcal{O}_{P^1_y}(1)) \simeq \mathcal{O}_{P^1_y}$. Hence (12) follows.

Now remark that $f_n|M^*_n : M^*_n \to C^*_n$, $n \geq 2$, is clearly an unramified quasifinite morphism (see, e.g., [L1, theorem 4.10]). Also, denoting $D^* := \pi_n^{-1}(M^*_{n-1} \times P^2)$, we have by (4) that $f_n|D^* = \psi_n \cdot \pi_n$, where $\pi_n : D^* \to M^*_{n-1} \times P^2$ is a P^1-fibration and $\psi_n|(M^*_{n-1} \times P^2 \simeq (f_{n-1}|M^*_{n-1}) \times 1_{P^2}$, $n \geq 3$, is an unramified quasifinite morphism. Hence, if we consider the Stein factorization of the map f_n:

$$f_n : \mathcal{M}_n \xrightarrow{\tilde{f}_n} \tilde{C}_n \xrightarrow{\nu_n} C_n,$$

where $\tilde{f}_n$ is birational with connected fibers and ν_n is quasifinite, then $\tilde{f}_n|D^* = \pi_n$ and we obtain a commutative diagram:

$$\begin{array}{ccc} D^* & \hookrightarrow & \mathcal{M}_n \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \tilde{f}_n} \\ M^*_{n-1} \times P^2 & \hookrightarrow & \tilde{C}_n \\ \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \nu_n} \\ Z^*_n & \hookrightarrow & C_n \end{array} \qquad (f_n : \mathcal{M}_n \to C_n) \tag{14}$$

so that $\psi_n = \nu_n|M^*_{n-1} \times P^2$. Since $\tilde{f}_n$ is birational, to prove the birationality of f_n it is enough to show that ν_n is birational. In view of (6), (7) and (8) for any point $y \in \psi_n^{-1}(Z'_n) = \nu_n^{-1}(Z'_n)$ the fiber $\nu_n^{-1}(\nu_n(y))$ set-theoretically consists of this point y. Now theorem 0.6 will follow if we find a point $y \in \nu_n^{-1}(Z'_n)$ such that [3] ν_n is unramified at y, i.e.

$$\ker(d\nu_n|_y : T_y\tilde{C}_n \longrightarrow T_{\nu_n(y)}C_n) = 0. \tag{15}$$

[3]Here and below for a given scheme $\mathcal{X}$ and any point $x \in \mathcal{X}$ we denote by $T_x\mathcal{X}$ the Zariski tangent space to $\mathcal{X}$ at x.

(In fact, the condition (15) means that the sheaf of relative differentials $\Omega_{\tilde{C}_n/C_n}$ vanishes at this point y, hence, since $\tilde{C}_n$ is irreducible, $\Omega_{\tilde{C}_n/C_n}$ vanishes at a general point of $\tilde{C}_n$ (i.e. it is a torsion $\mathcal{O}_{\tilde{C}_n}$-sheaf); thus ν_n is generically an immersion along $\nu_n^{-1}(Z_n)$ and, by the above, a bijective map at a point $y \in \nu_n^{-1}(Z_n')$; hence it is birational.)

To prove the equality (15) we first remark that the third observation above together with a standard argument from birational geometry (see the proof of lemma 3.6 below, in particular, the equality 100) shows that, at this point y, the variety C_n has an ordinary quadratic cDV singularity, namely, a rational singularity which is analytically isomorphic to a direct product $\mathbf{A}^{4n-5} \times S$ of an affine $(4n-5)$-space and of a surface S with a Du Val singularity of type A_1. (In other words, $\hat{\mathcal{O}}_{y,\tilde{C}_n} \simeq \mathbf{C}[[x_1, ..., x_{4n-5}]] \otimes \mathbf{C}[[x, y, z]]/(xy - z^2)$.) On the other hand, specifying the point y in such a way that $y = ([\mathcal{E}^{\check{}}], x(\mathcal{E}))$, where $\mathcal{E}^{\check{}\check{}}$ is a special Hulsbergen bundle, we obtain an effective description of the tangent space of C_n at the point $w = \nu_n(y)$. This is done in the foregoing sections 2 and 3. Our method consists of constructing (by means of the above Hulsbergen bundle) a smooth quasiprojective surface S in $\mathcal{M}_n$ intersecting D transversally along the fibre $\pi_n^{-1}(y)$ and mapping via f_n onto a surface which, roughly speaking, stands (locally in analytic sense around the point w) for the image under ν_n of the fibre $\{0\} \times S$ of the above direct product $\mathbf{A}^{4n-5} \times S$. Finally, comparing the obtained descriptions of the tangent spaces $T_y\tilde{C}_n$ and T_wC_n leads to the proof of (15) (see subsection 3.6).

2. Construction of a smooth quasiprojective surface S in $\mathcal{M}_n$ with special properties with respect to D

In this section we construct a quasiprojective smooth surface S such that:

(i) S contains a projective line $\mathbf{l}$ such that

$$\mathcal{O}_S(\mathbf{l})|\mathbf{l} \simeq \mathcal{O}_{\mathbf{l}}(-2); \tag{16}$$

(ii) there exists a morphism $j : S \to \mathcal{M}_n$ such that j is an embedding around $\mathbf{l}$ such that

$$j(S \setminus \mathbf{l}) \subset M_n \tag{17}$$

and,

(iii) moreover,

$$j(S) \cap D = j(\mathbf{l}) = P^1_y \tag{18}$$

is a transversal intersection of D and $j(S)$ along a certain fiber P^1_y of the projection $\pi_n : D \to M_{n-1} \times P^2$ over a point $y = ([\mathcal{E}_0], x) \in M_{n-1} \times P^2$, where $[\mathcal{E}_0] \in M_{n-1}$ is a (class of a) certain Hulsbergen bundle (see (19) below).

For this we need to introduce some preliminary constructions and notation. Let Q be a fixed smooth conic in P^2, $x_0 \in P^2 \setminus Q$ a fixed point, $G = G(1, S^nQ)$ the Grassmannian of lines in the projective space $S^nQ \simeq P^n$, so that any point $g \in G$ is naturally understood as a 1-dimensional linear series $g = g^1_n$ of degree n on Q. Equivalently, we will interpret g as a two-dimensional subspace (which we denote below by V_g) in the $(n+1)$-dimensional vector space $H^0(\mathcal{O}_Q(Z))$, where Z is any divisor ofthe linear series g, $l(Z) = n$. The space V_g thus defines the composition

$$e(g) : V_g \otimes \mathcal{O}_{P^2} \overset{\otimes \mathcal{O}_Q}{\twoheadrightarrow} V_g \otimes \mathcal{O}_Q \hookrightarrow H^0(\mathcal{O}_Q(Z)) \otimes \mathcal{O}_Q \overset{ev}{\twoheadrightarrow} \mathcal{O}_Q(Z),$$

and the surjectivity of $e(g)$ is equivalent to saying that g has no fixed points.

Remark 2.1. Clearly, $G^* = \{g \in G \mid g \text{ has no fixed points}\}$ is a dense open subset of G, and for any $g \in G^*$ the sheaf

$$\mathcal{E}_0(g) := \ker e(g) \otimes \mathcal{O}_{P^2}(1). \tag{19}$$

is a stable vectorbundle with $c_2 = n-1$, i.e. $[\mathcal{E}_0(g)] \in M_{n-1}$; this vector bundle $\mathcal{E}_0(g)$ is called a *Hulsbergen bundle* (see [B, §5]). We thus have a well-defined morphism $\rho : G^* \to M_{n-1} : g \mapsto [\mathcal{E}_0(g)]$.

Besides, one has a natural identification

$$H^0(\mathcal{E}_0(g)(1)) \simeq V_g \simeq V_g^{\check{}}.$$

Thus, denoting $P_g^1 = P(V_g)$, we get a canonical map $can : \mathcal{O}_{P_g^1} \to H^0(\mathcal{E}_0(g)(1)) \otimes \mathcal{O}_{P_g^1}(1)$, hence a composition:

$$s : \mathcal{O}_{P^2 \times P_g^1} \xrightarrow{id \boxtimes can} H^0(\mathcal{E}_0(g)(1)) \otimes \mathcal{O}_{P^2} \boxtimes \mathcal{O}_{P_g^1}(1) \xrightarrow{ev \boxtimes id} \mathcal{E}_0(g)(1) \boxtimes \mathcal{O}_{P_g^1}(1).$$

Let $Q_0 = (s)_0$, so that $\operatorname{coker} s = \mathcal{I}_{Q_0,X}(2,2)$, where we use standard notation $\mathcal{O}_{P^2 \times P_g^1}(m,n) = \mathcal{O}_{P^2}(m) \boxtimes \mathcal{O}_{P_g^1}(n)$, $m,n \in \mathbf{Z}$. Next, let $l_0 = \{x_0\} \times P_g^1$, with usual notation $\mathcal{O}_{l_0}(k)$, $k \in \mathbf{Z}$, for invertible sheaves on l_0, and $\tilde{Q} = Q_0 \cup l_0$ a disjoint union (remark that $x_0 \notin Q$). Then the exact triples

$$0 \to \mathcal{I}_{\tilde{Q},P^2 \times P_g^1}(2,2) \to \mathcal{I}_{Q_0,P^2 \times P_g^1}(2,2) \xrightarrow{\varepsilon} \mathcal{O}_{l_0}(2) \to 0,$$

$$0 \to \mathcal{O}_{P^2 \times P_g^1} \xrightarrow{s} \mathcal{E}_0(g)(1) \boxtimes \mathcal{O}_{P_g^1}(1) \xrightarrow{\tilde{s}} \mathcal{I}_{Q_0,P^2 \times P_g^1}(2,2) \to 0 \tag{20}$$

fit in the diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{P^2 \times P_g^1} & \longrightarrow & \mathcal{E}(1,1) & \longrightarrow & \mathcal{I}_{\tilde{Q},P^2 \times P_g^1}(2,2) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{P^2 \times P_g^1} & \xrightarrow{s} & \mathcal{E}_0(g)(1) \boxtimes \mathcal{O}_{P_g^1}(1) & \xrightarrow{\tilde{s}} & \mathcal{I}_{Q_0,P^2 \times P_g^1}(2,2) & \longrightarrow & 0 \\
 & & & & \downarrow \varepsilon \cdot \tilde{s} & & \downarrow \varepsilon & & \\
 & & & & \mathcal{O}_{l_0}(2) & = & \mathcal{O}_{l_0}(2) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{21}$$

Remark 2.2. Note that:

a) from this diagram it follows that the sheaf $\mathcal{E} = \ker(\varepsilon \cdot \tilde{s}) \otimes \mathcal{O}_{P^2 \times P_g^1}(-1,-1)$ satisfies the conditions

$$[\mathcal{E}|P^2 \times \{z\}] \in D, \quad z \in P_g^1, \tag{22}$$

$$Sing(\mathcal{E}|P^2 \times \{z\}) = x_0, \quad z \in P_g^1; \tag{23}$$

b) since Q is the zero-scheme of the section $\wedge^2(ev) : \mathcal{O}_{P^2} \simeq \wedge^2 V_g \otimes \mathcal{O}_{P^2} = \wedge^2(\mathcal{E}_0(g)(1)) = \mathcal{O}_{P^2}(2)$ and $x_0 \notin Q$, it follows that the map

$$V_g \simeq H^0(\mathcal{E}_0(g)(1)) \xrightarrow{ev \otimes k(x_0)} \mathcal{E}_0(g)(1) \otimes k(x_0) : s \mapsto s(x_0) \tag{24}$$

is an isomorphism. Thus, denoting

$$y = ([\mathcal{E}_0(g)], x_0), \tag{25}$$

we get a natural identification:

$$P^1_g = P(V_g) \xrightarrow{\simeq} P(\mathcal{E}_0(g)(1)\otimes k(x_0)) \xrightarrow{\simeq} P(\mathcal{E}_0\check{}(g)(1)|x_0) \simeq \pi_n^{-1}(y) =: P^1_y \subset D; \qquad (26)$$

c) one quickly checks that

$$\mathcal{E}xt^1(\mathcal{I}_{Q_0,P^2\times P^1_g}(2,2),\mathcal{O}_{P^2\times P^1_g}) \simeq \mathcal{E}xt^2(\mathcal{O}_{Q_0}(2,2),\mathcal{O}_{P^2\times P^1_g}) \simeq \mathcal{O}_{Q_0}, \qquad (27)$$

hence there is an isomorphism

$$Ext^1(\mathcal{I}_{Q_0,P^2\times P^1_g}(2,2),\mathcal{O}_{P^2\times P^1_g}) \simeq H^0\mathcal{E}xt^2(\mathcal{O}_{Q_0}(2,2),\mathcal{O}_{P^2\times P^1_g}) \simeq H^0\mathcal{O}_{Q_0}, \qquad (28)$$

under which the unit $1 \in H^0\mathcal{O}_{P^1_g}$ corresponds to the element

$$\xi \in Ext^1(\mathcal{I}_{Q_0,P^2\times P^1_g}(2,2),\mathcal{O}_{P^2\times P^1_g}) \qquad (29)$$

defining the extension (20).

Now we proceed to the construction of the surface S with the prescribed properties (i)–(iii). First, similar to (27) one has:

$$\mathcal{E}xt^1(\mathcal{I}_{Q_0\cup l_0,P^2\times P^1_g}(2,2),\mathcal{O}_{P^2\times P^1_g}) \simeq \mathcal{E}xt^2(\mathcal{O}_{Q_0}(2,2),\mathcal{O}_{P^2\times P^1_g}) \simeq \mathcal{O}_{Q_0}\oplus\mathcal{O}_{l_0}(-2). \qquad (30)$$

Next, by construction the curve Q_0 is a divisor of type $(n,1)$ in $Q\times P^1_g$ (under the identification $Q\simeq P^1$), hence it satisfies the triple $0\to\mathcal{O}_{Q\times P^1}(-n,-1)\to\mathcal{O}_{Q\times P^1}\to\mathcal{O}_{Q_0}\to 0$. Applying to this triple the functor R^ip_{2*}, where $p_2 : P^2\times P^1_g\to P^1_g$ is the projection, we obtain $0\to\mathcal{O}_{P^1_g}\to p_{2*}\mathcal{O}_{Q_0}\to(n-1)\mathcal{O}_{P^1_g}(-1)\to 0$, i.e.

$$p_{2*}\mathcal{O}_{Q_0}\simeq\mathcal{O}_{P^1_g}\oplus(n-1)\mathcal{O}_{P^1_g}(-1). \qquad (31)$$

Besides, since $p_2|l_0 : l_0\to P^1_g$ is an isomorphism, it follows that $p_{2*}\mathcal{O}_{l_0}\simeq\mathcal{O}_{P^1_g}(-2)$. Hence by (30) and (31) we have

$$p_{2*}\mathcal{E}xt^1(\mathcal{I}_{Q_0\cup l_0,P^2\times P^1_g}(2,2),\mathcal{O}_{P^2\times P^1_g}) \simeq \mathcal{O}_{P^1_g}\oplus\mathcal{O}_{P^1_g}(-2)\oplus(n-1)\mathcal{O}_{P^1_g}(-1), \qquad (32)$$

and also $\mathcal{E}xt^i(\mathcal{I}_{Q_0\cup l_0,P^2\times P^1_g}(2,2),\mathcal{O}_{P^2\times P^1_g}) = 0,\ i\neq 1$, respectively, $R^ip_{2*}\mathcal{E}xt^1(\mathcal{I}_{Q_0\cup l_0,P^2\times P^1_g}(2,2),\mathcal{O}_{P^2\times P^1_g}) = 0,\ i\neq 1$. Hence the spectral sequence of local-to-relative $\mathcal{E}xt$–sheaves implies:

$$\mathcal{F} := \mathcal{E}xt^1_{p_2}(\mathcal{I}_{Q_0\cup l_0,P^2\times P^1_g}(2,2),\mathcal{O}_{P^2\times P^1_g}) = \mathcal{O}_{P^1_g}\oplus\mathcal{O}_{P^1_g}(-2)\oplus(n-1)\mathcal{O}_{P^1_g}(-1). \qquad (33)$$

Consider the variety $\mathbf{P}(\mathcal{F}\check{}) := Proj(Sym_{\mathcal{O}_{P^1_g}}\mathcal{F}\check{})$ with its natural projection $p : \mathbf{P}(\mathcal{F}\check{})\to P^1_g$ and let $\mathbf{p} : P^2\times\mathbf{P}(\mathcal{F}\check{})\to P^2\times P^1_g$ and $\mathbf{p_2} : P^2\times\mathbf{P}(\mathcal{F}\check{})\to\mathbf{P}(\mathcal{F}\check{})$ be the induced projections. Similar to (32) one sees that $\mathcal{E}xt^i_{p_2}(\mathcal{I}_{Q_0\cup l_0,P^2\times P^1_g}(2,2),\mathcal{O}_{P^2\times P^1_g}) = 0,\ i\neq 1$. Hence the spectral sequence of global-to-relative $\mathcal{E}xt$'- together with the base change gives:

$$H^0(p^*\mathcal{F}\otimes\mathcal{O}_{\mathbf{P}(\mathcal{F}\check{})}(1)) = H^0(\mathcal{E}xt^1_{\mathbf{p_2}}(\mathbf{p}^*\mathcal{I}_{Q_0\cup l_0,P^2\times P^1_g}(2,2),\mathcal{O}_{P^2}\boxtimes\mathcal{O}_{\mathbf{P}(\mathcal{F}\check{})}(1))) =$$

$$Ext^1(\mathbf{p}^*\mathcal{I}_{Q_0\cup l_0,P^2\times P^1_g}(2,2),\mathcal{O}_{P^2}\boxtimes\mathcal{O}_{\mathbf{P}(\mathcal{F}\check{})}(1)). \qquad (34)$$

Thus the canonical (evaluation) morphism $ev_{\mathcal{F}} : \mathcal{O}_{\mathbf{P}(\mathcal{F}\check{})}\to p^*\mathcal{F}\otimes\mathcal{O}_{\mathbf{P}(\mathcal{F}\check{})}(1)$ considered as the element

$$ev_{\mathcal{F}}\in Ext^1(\mathbf{p}^*\mathcal{I}_{Q_0\cup l_0,P^2\times P^1_g}(2,2),\mathcal{O}_{P^2}\boxtimes\mathcal{O}_{\mathbf{P}(\mathcal{F}\check{})}(1)) \qquad (35)$$

defines the extension:

$$0\to\mathcal{O}_{P^2}\boxtimes\mathcal{O}_{\mathbf{P}(\mathcal{F}\check{})}(1)\to\tilde{\mathbf{E}}(1)\to\mathbf{p}^*\mathcal{I}_{Q_0\cup l_0,P^2\times P^1_g}(2,2)\to 0. \qquad (36)$$

Now according to (32) we have a surjection $\tau : \mathcal{F}^{\check{}} \twoheadrightarrow \mathcal{O}_{P^1_g} \oplus \mathcal{O}_{P^1_g}(2)$ which implies an embedding

$$\bar{S} := \mathbf{P}(\mathcal{O}_{P^1_g} \oplus \mathcal{O}_{P^1_g}(2)) \overset{t}{\hookrightarrow} \mathbf{P}(\mathcal{F}^{\check{}}). \tag{37}$$

Let $\mathbf{t} := 1 \times t : P^2 \times \bar{S} \hookrightarrow P^2 \times \mathbf{P}(\mathcal{F}^{\check{}})$, respectively, $r := p \cdot t : \bar{S} \to P^1_g$ and $\mathbf{r} := \mathbf{p} \cdot \mathbf{t} : P^2 \times \bar{S} \to P^2 \times P^1_g$ be the induced projections. The natural (evaluation) morphism $ev_{\bar{S}} : \mathcal{O}_{\bar{S}} \to r^*(\mathcal{O}_{P^1_g} \oplus \mathcal{O}_{P^1_g}(-2)) \otimes \mathcal{O}_{\bar{S}/P^1_g}(1)$ clearly fits in the diagram:

$$\begin{array}{ccc} r^*(\mathcal{O}_{P^1_g} \oplus \mathcal{O}_{P^1_g}(-2)) \otimes \mathcal{O}_{\bar{S}/P^1_g}(1) & \overset{r^*\tau^{\check{}}\otimes id}{\hookrightarrow} & r^*\mathcal{F}^{\check{}} \otimes \mathcal{O}_{\bar{S}/P^1_g}(1) \\ \uparrow ev_{\bar{S}} & & \| \\ \mathcal{O}_{\bar{S}} & \overset{t^*ev_{\mathcal{F}}}{\to} & r^*(\mathcal{F}^{\check{}} \otimes \mathcal{O}_{\mathbf{P}(\mathcal{F}^{\check{}})}(1)) \end{array} \tag{38}$$

Now similar to (35) the morphism $ev_{\bar{S}}$ can be considered as an element

$$ev_{\bar{S}} \in Ext^1(\mathbf{r}^*\mathcal{I}_{Q_0 \cup l_0, P^2 \times P^1_g}(2,2), \mathcal{O}_{P^2} \boxtimes \mathcal{O}_{\bar{S}/P^1_g}(1)) \tag{39}$$

defining the extension

$$0 \to \mathcal{O}_{P^2} \boxtimes \mathcal{O}_{\bar{S}/P^1_g}(1) \to \mathbf{E}_{\bar{S}}(1) \to \mathbf{r}^*\mathcal{I}_{Q_0 \cup l_0, P^2 \times P^1_g}(2,2) \to 0 \tag{40}$$

which is obtained from (36) by applying the functor $\mathbf{t}^*$.

Now let $s_1 : \mathcal{O}_{\bar{S}} \to \mathcal{O}_{\bar{S}/P^1_g}(1) \otimes r^*\mathcal{O}_{P^1_g}(-2))$ be the canonical morphism such that

$$\mathbf{l} := (s_1)_0 \underset{\simeq}{\overset{r}{\longrightarrow}} P^1_g \tag{41}$$

is a unique (-2)-curve on $\bar{S}$. Next, fix a general section $s \in H^0(\mathcal{O}_{\bar{S}/P^1_g}(1))$ such that $(s)_0$ is a smooth section of the projection $r : \bar{S} \to P^1_g$ disjoint to s_1. Then the evaluation morphism $ev_{\bar{S}}$ can be written as

$$ev_{\bar{S}} : r^*(\mathcal{O}_{P^1_g} \oplus \mathcal{O}_{P^1_g}(2)) \overset{(s,s_1(2))}{\to} \mathcal{O}_{\bar{S}/P^1_g}(1). \tag{42}$$

Now let

$$S := \bar{S} \smallsetminus (s)_0, \quad . \tag{43}$$

By the above, $\mathbf{l} \subset S$ is a (-2)-curve on S, i.e. it satisfies (16) (the condition (i) from the beginning of this section). We are going to show that S is a desired surface, i.e. it satisfies the rest two conditions (ii) and (iii) above.

For this, fix any point $y_0 \in P^1_g$ $U := P^1_g \smallsetminus y_0 \simeq \mathbf{A}^1$ and let

$$S^* = r^{-1}(U) \cap S = \bar{S} \smallsetminus (r^{-1}(y_0) \cup (s)_0). \tag{44}$$

Clearly

$$S^* \simeq \mathbf{A}^2 \tag{45}$$

with affine coordinates (z,t) in $\mathbf{A}^2$, where z is a standard coordinate in $U \simeq \mathbf{A}^1$ and t is defined as the image of 1 under the map of sections $H^0(\mathcal{O}_{\bar{S}}) \to H^0(\mathcal{O}_{S^*})$ defined by the morphism

$$\mathcal{O}_{\bar{S}} \underset{\simeq}{\overset{res}{\longrightarrow}} \mathcal{O}_{S^*} \underset{\simeq}{\overset{\cdot(2r^{-1}(y_0))}{\longrightarrow}} r^*\mathcal{O}_{P^1_g}(2)|S^* \underset{\simeq}{\overset{s_1|S^*}{\longrightarrow}} \mathcal{O}_{\bar{S}/P^1_g}(1)|S^* \underset{\simeq}{\overset{(s|S^*)^{-1}}{\longrightarrow}} \mathcal{O}_{S^*}. \tag{46}$$

In these coordinates clearly

$$\mathbf{l}^* := \mathbf{l} \cap S^* = \{t = 0\}. \tag{47}$$

Now restricting the extension (40) onto S^* and denoting $r_0 := r|S^* : S^* \to U$, $\mathbf{r_0} := 1 \times r_0$, $Q^* := Q_0 \cap U$, $l^* := l_0 \cap U$, we get the $\mathcal{O}_{P^2 \times S^*}$-extension:

$$0 \to \mathcal{O}_{P^2 \times S^*} \to \mathbf{E}_{S^*}(1) \to \mathbf{r_0}^* \mathcal{I}_{Q_0 \cup l_0, P^2 \times P^1_g}(2,2) \to 0 \tag{48}$$

given by the element $\tilde{\xi} \in Ext^1(\mathbf{r_0}^* \mathcal{I}_{Q_0 \cup l_0, P^2 \times P^1_g}(2,2), \mathcal{O}_{P^2 \times S^*})$ which in view of (29) and the definition of t corresponds to the element

$$(1,t) \in H^0(\mathcal{O}_{S^*}) \oplus H^0(\mathcal{O}_{S^*}) \tag{49}$$

under the isomorphism

$$Ext^1(\mathbf{r_0}^* \mathcal{I}_{Q_0 \cup l_0, P^2 \times P^1_g}(2,2), \mathcal{O}_{P^2 \times S^*}) \simeq Ext^1(\mathbf{r_0}^* \mathcal{I}_{Q_0, P^2 \times P^1_g}(2,2), \mathcal{O}_{P^2 \times S^*}) \oplus$$

$$\oplus Ext^1(\mathbf{r_0}^* \mathcal{I}_{l_0, P^2 \times P^1_g}(2,2), \mathcal{O}_{P^2 \times S^*}) \simeq H^0(r_0^* \mathcal{O}_{Q^*}) \oplus H^0(r_0^* \mathcal{O}_{l^*}(-2)) \simeq H^0(\mathcal{O}_{S^*}) \oplus H^0(\mathcal{O}_{S^*}). \tag{50}$$

Now (47), (48) and (49) clearly imply that

a) $[\mathbf{E}_{S^*}|P^2 \times \{(z,t)\}] \in \mathcal{M}_n$ for any $(z,t) \in S^*$, i.e. we obtain a morphism

$$j : S^* \to \mathcal{M}_n : (z,t) \to [\mathbf{E}_{S^*}|P^2 \times \{(z,t)\}]; \tag{51}$$

b) by construction, this map extends to the morphism

$$j : S \to \mathcal{M}_n \tag{52}$$

satisfying (17) and (18),

c) the restriction of (48) onto $P^2 \times \mathbf{l}^* \simeq P^2 \times U \subset P^2 \times P^1_g$ coincides with the restriction of the triple (20) onto $P^2 \times U$.

We have only to show the transversality of the intersection of $j(S)$ with D along $\mathbf{l}$. For this, take any point $z \in U \subset P^1_g$ (i.e., equivalently, the point $(z,0) \in \mathbf{l}^*$ (we use here (47) and the identification (41)) and denote

$$h_z := r_0^{-1}(z) = \{(z,t) | t \in \mathbf{A}^1\}.$$

Then an easy computation (using (49)) of the differential $d(j|h_z)$ at the point $(z,0)$ shows that this differential is nondegenerate and its image V_z is transversal to the space $T_{(z,0)}D$, so that $j(h_z)$ intersects D transversally at $(z,0)$. This together with a)-c) above shows that S satisfies the above conditions (ii) and (iii). Besides, we have in the above notations:

$$T_z \mathcal{M}_n = T_z D \oplus V_z, \quad z \in P^1_y. \tag{53}$$

Convention on notations: since our surface S (respectively, its affine open part S^*) depends on the choice of a pair $(g, x_0) \in G(1, S^n Q) \times (P^2 \setminus C)$, we will below specify sometimes its notation as S_{g,x_0} (respectively, S^*_{g,x_0}).

3. Description of the map $f_n : S \to C_n$

In this section we study the image of the surface $S = S_{g,x_0}$ under the Barth map f_n. For this introduce some more notations. Let $\check{x}$ be a line in $\check{P}^2$ corresponding to an arbitrary point $x \in P^2$, with a picked equation $\{L_x = 0\}$, $L_x \in H^0 \mathcal{O}_{\check{P}^2}(1)$; $\sum_{i=1}^n x_i(z) \in Div\ Q$ a divisor of degree n on Q corresponding to a given point $z \in g$, where a linear series $g \in G(1, S^n Q)$ is understood here as a line in $S^n Q$; respectively, $D^n(z) = \cup_{i=1}^n \check{x}_i(z) \in |\mathcal{O}_{\check{P}^2}(n)|$ a reducible curve with the equation

$$\Psi_z^n := \prod_{i=1}^n L_{x_i(z)} = 0, \quad \Psi_z^n \in H^0(\mathcal{O}_{\check{P}^2}(n)), \tag{54}$$

corresponding to the above point $z \in g$;

$$E(g) = \{D_n(z) \in |\mathcal{O}_{\check{P}^2}(n)| \mid z \in g\} \tag{55}$$

an irreducible conic in the projective space $|\mathcal{O}_{\check{P}^2}(n)|$ corresponding to a given point $g \in G(1, S^nQ)$; $\{L_{x_0} = 0\}$ fixed equation of the line $\check{x}_0$ in $\check{P}^2$.

Now for any point $(z,t) \in S^*_{g,x_0}$ understood via the map j from (51) as (a class of) a sheaf from $\mathcal{M}_n$ consider the corresponding curve of jumping lines $C^n(z,t) = f_n(z,t)$. Repeating now the argument from the proof of theorem 4 of [B] – see theorem 6.2 from appendix C below, we obtain the following equation of the curve $C^n(z,t)$ in P^2:

$$C^n(z,t) = \{c_0 L_{x_1(z)} \cdots L_{x_n(z)} + L_{x_0}(\sum_{i=1}^{n} c_i L_{x_1(z)} \cdots \check{L}_{x_i(z)} \cdots L_{x_n(z)} = 0\},\ c_i \in k. \tag{56}$$

Next, consider the so called *Poncelet curve*

$$\mathcal{P}onc^{n-1}(g) := \{\sum_{i=1}^{n} c_i L_{x_1(z)} \cdots \check{L}_{x_i(z)} \cdots L_{x_n(z)} = 0\} \in |\mathcal{O}_{\check{P}^2}(n-1)|. \tag{57}$$

According to Barth [B, Theorem 4] (see theorem 6.2 below) the curve $\mathcal{P}onc^{n-1}(g)$ doesn't depend on the choice of the point $z \in U$ and

$$\mathcal{P}onc^{n-1}(g) = f_{n-1}([\mathcal{E}_0(g)]), \tag{58}$$

where $\mathcal{E}_0(g) = \ker e(g) \otimes \mathcal{O}_{P^2}(1)$ (see the definition (19)). Moreover, for any fixed $z \in U$ the precise statement of theorem 4 of [B] (see the assertion i) of theorem 6.2 below) together with (48)– (49) shows that the coefficient c_0 in (57) equals

$$c_0 = \lambda_z t, \tag{59}$$

where $\lambda_z \neq 0$ depends on the choice of scalar factors in the forms $L_{x_i(z)}$, $i = 1, ..., n$. In particular, for $t = 0$ we obtain:

$$w := C^n(z,0) = \check{x}_0 \cup \mathcal{P}onc^{n-1}(g). \tag{60}$$

Remark 3.1. Let $P^{N_n *} := \{C \in P^{N_n} | C \textit{ is smooth}\}$. By [B, sec. 5],$\mathcal{P}onc^{n-1}(g) \in P^{N_{n-1}*}$ for general $g \in G(1, S^nQ)$. In other words,

$$G^{**} := \{g \in G^* | \mathcal{P}onc^{n-1}(g) \in P^{N_{n-1}*}\}$$

is a dense open subset of $G(1, S^nQ)$ such that

$$(\rho \times 1)(G^{**} \times (P^2 \smallsetminus Q)) \subset M^*_{n-1} \times P^2, \tag{61}$$

where ρ is the morphism defined in remark 2.1.

Now fixing for $\mathcal{P}onc^{n-1}(g)$ any equation, say, $\{\Phi^{n-1}_g = 0\}$, $\Phi^{n-1}_g \in H^0(\mathcal{O}_{\check{P}^2}(n-1))$, and choosing appropriately the scalar factor of the form Ψ^n_z from (54), in view of (59) we can rewrite the equation (56) of the curve $C^n(z,t)$ in the form

$$C^n(z,t) = \{t\Psi^n_z + L_{x_0}\Phi^{n-1}_g = 0\},\ \ (z,t) \in S^*_{g,x_0}. \tag{62}$$

This shows that $f_n(h_z) = \{C^n(z,t) | t \in \mathbf{A}^1\}$ (recall that $h_z = \{(z,t) | t \in \mathbf{A}^1\}$). Hence we see that

$$R = R_{g,x_0} := f_n(S_{g,x_0})$$

is an open part of the quadric cone in $|\mathcal{O}_{\check{P}^2}(n)|$ ruled by lines joining w to the points of the conic $E(g)$. By construction these lines are images under f_n of lines h_z, $z \in P^1_y$, of the ruling of S. Thus in view of (16) and (18) we obtain

Lemma 3.2. *For a general point* $(g,x_0)\in G(1,S^nQ)\times(P^2\setminus Q)$

i) the surface $R_{g,x_0}=f_n(S_{g,x_0})$ *is an open subset of a quadric cone in the projective space* $|\mathcal{O}_{\check{P}^2}(n)|$, *and the morphism* $f_n: S_{g,x_0}\to R_{g,x_0}$ *is a contraction of a (-2)-curve* $P^1_y\simeq 1$ *on* S_{g,x_0}, *where* $y=([\mathcal{E}_0(g)],x_0)$.

ii) Moreover, for $w=\nu_n(y)$,

$$T_wR_{g,x_0}=\mathrm{Span}(\bigcup_{z\in P^1_y}(df_n|_z)(V_z))\simeq k^3,\quad \textit{where } V_z=T_zh_z,\quad z\in P^1_y. \tag{63}$$

Now return to diagram (14) and remark that by construction we have a diagram

$$\begin{array}{ccc} M^*_{n-1}\times P^2 & \xrightarrow{\psi_n} & P^{N_n} \\ {\scriptstyle f_{n-1}\times 1}\downarrow & & \uparrow{\scriptstyle \mu} \\ C^*_{n-1}\times P^2 & \hookrightarrow & P^{N_{n-1}*}\times P^2 \end{array} \tag{64}$$

Since $\pi_n: D^*\to Z^*_n$ is a P^1-fibration, it follows that for any $y\in M^*_{n-1}\times P^2$

$$\mathrm{Span}(\bigcup_{z\in P^1_y}(d\pi_n|_z)(T_zD^*))=T_y(M^*_{n-1}\times P^2).$$

Hence from the diagrams (14) and (96), since $\psi_n=\nu_n|M^*_{n-1}\times P^2$ and $\mu|P^{N_{n-1}*}\times P^2$ and $f_{n-1}\times 1|M^*_{n-1}\times P^2$ are unramified, we have

$$(df_n|_z)(T_zD^*))=(d(\nu_n\cdot\pi_n)|_z)(T_zD^*))=\mathrm{Span}(\bigcup_{z\in P^1_y}(d\nu_n|_y)(d\pi_n|_z)(T_zD^*))=$$

$$=(d\nu_n|_y)(T_y(M^*_{n-1}\times P^2))=(d(\nu_n|M^*_{n-1}\times P^2)|_y)(T_y(M^*_{n-1}\times P^2))=(d\psi_n|_y)(T_y(M^*_{n-1}\times P^2))=$$

$$(d\mu|_{w_0})(d(f_{n-1}\times 1)|_y)(T_y(M^*_{n-1}\times P^2))\simeq T_y(M^*_{n-1}\times P^2)\simeq k^{4n-5}. \tag{65}$$

Next, returning to (60)-(62) we get [4]:

$$\mathcal{P}T_wR_{g,x_0}=\mathrm{Span}(w,E(g))=\mathrm{Span}(R_{g,x_0})\simeq P^3. \tag{66}$$

Since $E(g)$is a conic, $\mathrm{Span}(w,E(g))$ is a projective 3-subspace in $P^N=|H^0(\mathcal{O}_{\check{P}^2}(n))|$ for any linear series $g\in G(1,S^nQ)$.

Consider the morphism

$$\mu: P^{N_{n-1}}\times P^2\simeq|\mathcal{O}_{\check{P}^2}(n-1)|\times|\mathcal{O}_{\check{P}^2}(1)|\to P^{N_n}:(C,x)\mapsto C\cup\check{x}$$

and let $B_n:=im(\mu)$, respectively, $B^*_n:=\mu(P^{N_{n-1}*}\times P^2)$. Evidently, $\mu: P^{N_{n-1}*}\times P^2\to B^*_n$ is an isomorphism (hence $\mu: P^{N_{n-1}}\times P^2\to B_n$ is birational). Moreover, from the definition of μ it follows that for any $w=\mu(C,x)\in B^*_n$ we have

$$T_w\mu(P^{N_{n-1}}\times\{x\})\cap T_w\mu(\{C\}\times P^2)=\{0\}, \tag{67}$$

hence

$$T_wB^*_n=T_w\mu(P^{N_{n-1}}\times\{x\})\oplus T_w\mu(\{C\}\times P^2)\simeq k^{(n^2+n+2)/2}, \tag{68}$$

respectively,

$$\mathcal{P}(C,x):=\mathcal{P}T_wB^*_n=\mathrm{Span}(\mu(P^{N_{n-1}}\times\{x\}),\mu(\{C\}\times P^2)). \tag{69}$$

[4]Here and everywhere below for a given subscheme $\mathcal{X}$ of a projective space $P^N:=|H^0(\mathcal{O}_{\check{P}^2}(n))|$ and a closed point $x\in\mathcal{X}$ we denote by $\mathcal{P}T_x\mathcal{X}$ the projective subspace of P^N passing through x and uniquely determined by the condition that $T_x\mathcal{P}T_x\mathcal{X}=T_x\mathcal{X}$, where $T_x\mathcal{X}$ is the Zariski tangent space to $\mathcal{X}$ at x.

Next, let

$$\mathcal{U}_n := \{(C,x) \in P^{N_{n-1}} \times P^2 \mid T_{\mu(C,x)}\mu(P^{N_{n-1}} \times \{x\}) \cap T_{\mu(C,x)}\mu(\{C\} \times P^2) = \{0\} \}. \tag{70}$$

Since by (67) $P^{N_{n-1}*} \times P^2 \subset \mathcal{U}_n$, it follows that $\mathcal{U}_n$ is a *dense open* subset in $P^{N_{n-1}*} \times P^2$. (Openness of $\mathcal{U}_n$ follows from the openness of the condition $T_{\mu(C,x)}\mu(P^{N_{n-1}} \times \{x\}) \cap T_{\mu(C,x)}\mu(\{C\} \times P^2) = \{0\}$ (provided that the spaces under intersection have fixed dimensions: in fact, $\dim T_{\mu(C,x)}\mu(P^{N_{n-1}} \times \{x\}) = N_{n-1} = (n^2+n-2)/2$, $\dim T_{\mu(C,x)}\mu(\{C\} \times P^2) = 2$).

Besides, since clearly for any $(C,x) \in P^{N_{n-1}} \times P^2$

$$im(d\mu|(C,x) : T_{\mu(C,x)}(P^{N_{n-1}} \times P^2) \to T_{\mu(C,x)}B_n) =$$

$$= \mathrm{Span}(T_{\mu(C,x)}\mu(P^{N_{n-1}} \times \{x\}), T_{\mu(C,x)}\mu(\{C\} \times P^2)), \tag{71}$$

we can extend the definition of $\mathcal{P}(C,x)$ in (69) to $\mathcal{U}_n$:

$$\mathcal{P}(C,x) := \mathrm{Span}(\mu(P^{N_{n-1}} \times \{x\}), \mu(\{C\} \times P^2)), \quad (C,x) \in \mathcal{U}_n. \tag{72}$$

Thus

$$im(d\mu|(C,x)) = T_{\mu(C,x)}\mathcal{P}(C,x) \simeq k^{(n^2+n+2)/2}, \quad \ker(d\mu|(C,x)) = 0, \quad (C,x) \in \mathcal{U}_n. \tag{73}$$

Note that since $\mathcal{U}_n$ is open in $P^{N_{n-1}} \times P^2$, the set

$$\mathcal{V}_n := \{(g,x_0) \in G(1,S^nQ) \times (P^2 \setminus Q) | \mathcal{P}onc^{n-1}(g) \in \mathcal{U}_n$$

is an *open* subset in $G(1,S^nQ) \times (P^2 \setminus Q)$.

Lemma 3.3. *$\mathcal{V}_n$ is a dense open subset in $G(1,S^nQ) \times (P^2 \setminus Q)$ and for a general point $(g,x_0) \in \mathcal{V}_n$*

$$\mathrm{Span}(w, E(g)) \cap \mathcal{P}(\mathcal{P}onc^{n-1}(g), x_0) = \{w\}, \tag{74}$$

where $w = \mu(\mathcal{P}onc^{n-1}(g), x_0)$.

Proof. Since $\mathcal{V}_n$ is open in $G(1,S^nQ) \times (P^2 \setminus Q)$, we need only to pick a point $(g,x_0) \in \mathcal{V}_n$. For this, fix a point $x_1 \in P^2 \setminus Q$, $x_1 \neq x_0$. Then one has a pair of distinct points $\{a_1,a_2\} = \check{x}_1 \cap \check{Q}$ on $\check{Q}$, where $\check{Q} \subset \check{P}^2$ is the conic dual to Q. This pair $\{a_1,a_2\}$ defines uniquely the involution $i : \check{x}_1 \to \check{x}_1$ of which a_1 and a_2 are the fixed points. Now for any line $v \in \check{x}_1$ denote $\{u_1(v),u_2(v)\} = v \cap Q$, $\{u_3(v),u_4(v)\} = i(v) \cap Q$. Then $g^1_4(x_1) := \{\sum_{i=1}^4 u_i(v) | v \in \check{x}_1\}$ is clearly a linear series of degree 4 on Q without fixed points, hence fixing $n-4$ points $x_2,...,x_{n-3} \in Q$ we get a linear series $g = g^1_4(x_1) + \sum_{i=2}^{n-3} x_i$ as a point of $G(1,S^nQ)$. This together with the definitions (54) and (55) implies that any curve $C \in \mathrm{Span}(E(g))$ contains the union of lines $\check{x}_2,...,\check{x}_{n-3}$, i.e.

$$C = \{F^n = 0\} \in \mathrm{Span}(E(g)) \quad \Longrightarrow \quad F^n = F^4 L_{x_2} \cdots L_{x_{n-3}}, \quad F^4 \in H^0(\mathcal{O}_{\check{P}^2}(4)). \tag{75}$$

Besides, the series g defines a Poncelet curve $\mathcal{P}onc^{n-1}(g) \in |\mathcal{O}_{\check{P}^2}(n-1)|$ which is clearly decomposable and contains the lines $\check{x}_1, \check{x}_2, ..., \check{x}_{n-3}$ as components:

$$\mathcal{P}onc^{n-1}(g) = \check{x}_1 \cup \check{x}_2 \cup ... \cup \check{x}_{n-3} \cup C^2_g, \tag{76}$$

where

$$C^2_g = \bigcup_{v \in \check{x}_1} (\check{u}_1(v) \cup \check{u}_2(v)) \cap (\check{u}_3(v) \cup \check{u}_4(v)) \tag{77}$$

is a *smooth* conic in $\check{P}^2$ with an equation, say, $\Phi^2_g = 0$. Then

$$\mathcal{P}onc^{n-1}(g) = \{\Phi^{n-1}_g = 0\}, \quad where \quad \Phi^{n-1}_g = \Phi^2_g L_{x_1} L_{x_2} \cdots L_{x_{n-3}}. \tag{78}$$

Now remark that by the definition (76) all the components of the curve $w = \mathcal{P}onc^{n-1}(g) \cup \check{x}_0$ are distinct and reduced, hence clearly

$$\mu^{-1}(w) = \{w_0, ..., w_{n-3}\} \tag{79}$$

is a finite set of $n-2$ distinct points

$$\mu^{-1}(w) = \{w_0, ..., w_{n-3}\}, w_i = (C_g^2 \cup \bigcup_{j \neq i} \check{x}_j, x_i), \qquad i = 0, ..., n-3, \tag{80}$$

such that μ is an immersion at each of these points w_i, i.e. the differential $d\mu | w_i$ is injective. This means that $\mu^{-1}(w) \subset \mathcal{U}_n$. In particular, taking $i = 0$, we have

$$w_0 = (\mathcal{P}onc^{n-1}(g), x_0) \in \mathcal{U}_n. \tag{81}$$

Now to prove (74) it is clearly enough to prove that

$$\mathrm{Span}(E(g)) \cap \mathcal{P}(\mathcal{P}onc^{n-1}(g), x_0) = \emptyset. \tag{82}$$

Assume the contrary, i.e. that there exists a point $C \in \mathrm{Span}(E(g)) \cap \mathcal{P}(\mathcal{P}onc^{n-1}(g), x_0)$, this point C as a curve of degree n in $\check{P}^2$ being given by an equation, say, $F^n = 0$, where $F^n \in H^0(\mathcal{O}_{\check{P}^2}(n))$. Since $w = \mathcal{P}onc^{n-1}(g) \cup \check{x}_0$, one clearly has in view of (78):

$$\mathcal{P}(\mathcal{P}onc^{n-1}(g), x_0) = P(\{L\Phi_g^2 L_{x_1} L_{x_2} \cdots L_{x_{n-3}} + L_{x_0}\Phi^{n-1} \mid \Phi^{n-1} \in H^0(\mathcal{O}_{\check{P}^2}(n-1)),\ L \in H^0(\mathcal{O}_{\check{P}^2}(1))\}). \tag{83}$$

Hence $F^n = L\Phi_g^2 L_{x_1} L_{x_2} \cdots L_{x_{n-3}} + L_{x_0}\Phi^{n-1}$ for some $\Phi^{n-1} \in H^0(\mathcal{O}_{\check{P}^2}(n-1))$, $L \in H^0(\mathcal{O}_{\check{P}^2}(1))$. Now, since $C \in \mathcal{P}(\mathcal{P}onc^{n-1}(g), x_0)$, it follows from (75) that $F^n = F^4 L_{x_1} \cdot \cdots L_{x_{n-3}}$ for some $F^4 \in H^0\mathcal{O}_{\check{P}^2}(4)$. Hence

$$F^4 = L\Phi_g^2 L_{x_1} + L_{x_0}\Phi^3 \tag{84}$$

for some $L \in H^0\mathcal{O}_{\check{P}^2}(1)$, $\Phi^3 \in H^0\mathcal{O}_{\check{P}^2}(3)$. Next, remark that the intersection of the subspaces in $P^{14} = |\mathcal{O}_{\check{P}^2}(4)$:

$$P(\{L\Phi_g^2 L_{x_1} | L \in H^0\mathcal{O}_{\check{P}^2}(1)\} \cap P(\{L_{x_0}\Phi^3 | \Phi^3 \in H^0\mathcal{O}_{\check{P}^2}(3)\} \tag{85}$$

is clearly a unique point $\{CL_{x_0}\Phi_g^2 L_{x_1}\}$, hence

$$P^{11}(g, x_0) := \mathrm{Span}(P(\{L\Phi_g^2 L_{x_1} | L \in H^0\mathcal{O}_{\check{P}^2}(1)\}), P(\{L_{x_0}\Phi^3 | \Phi^3 \in H^0\mathcal{O}_{\check{P}^2}(3)\})) = = P(\{L\Phi_g^2 L_{x_1} + L_{x_0}\Phi^3 \ | L \in H^0\mathcal{O}_{\check{P}^2}(1),\ \ \Phi^3 \in H^0\mathcal{O}_{\check{P}^2}(3)\})) \tag{86}$$

is a 11-dimensional subspace in P^{14}. Remark that the condition (84) above can be rewritten now as

$$C^4 := \{F^4 = 0\} \in P^{11}(g, x_0). \tag{87}$$

Now pick the points $x_0, x_1 \in P^2 \setminus Q$ and choose affine coordinates y, z in $\check{P}^2$ so that

$$\check{Q} = \{2z - y^2 = 0\}, \quad \check{x}_0 = \{z - y - 1 = 0\}, \quad \check{x}_1 = \{y = 0\}. \tag{88}$$

In this coordinates the involution $i : \check{x}_1 \to \check{x}_1$ is given by $(z, 0) \mapsto (-z, 0)$, hence one easily computes the equation of the conic C_g^2 from (77):

$$\Phi_g^2 = z - y^2; \tag{89}$$

respectively, the condition (76) can be rewritten in terms of the quartic C^4 from (87) as:

$$C \in \mathrm{Span}(E(g)) \iff C^4 \in \mathrm{Span}(E'(g)), \tag{90}$$

where $E'(g)$ is a conic in P^{14} described as:

$$E'(g) = \{\lambda[(z^2 + a)^2 - a(2z - y^2)^2] \mid a \in \mathbf{C} \cup \{\infty\},\ \lambda \in \mathbf{C}\}. \tag{91}$$

The condition (87) here precisely means that

$$P^{11}(g,x_0)\cap \mathrm{Span}(E'(g)) \neq \emptyset. \tag{92}$$

Now consider the (rational) restriction map

$$r : P^{14} - - \to P^4 = P(H^0\mathcal{O}_{\check{x}_0}(4)) : C^4 \mapsto C^4 \cap \check{x}_0$$

By (99) we may consider y as an affine coordinate on the line $\check{x}_0$. Now taking $a = 0,\ 1,\ \infty$ in (91) and putting there $z = y+1$, we obtain 3 linearly independent polynomials

$$f_1 = (y+1)^4,\quad f_2 = y^3 + y^2,\quad f_3 = 1 \quad \in \quad H^0\mathcal{O}_{\check{x}_0}(4). \tag{93}$$

This means that $r|\mathrm{Span}(E'(g))$ is an embedding such that

$$r(\mathrm{Span}(E'(g))) = \mathrm{Span}(\sum_{i=1}^{3} \alpha_i f_i | \alpha_i \in \mathbf{C}).$$

Respectively, (86)), (99)) and (89)) imply that $r(P^{11}(g,x_0))$ is a projective line in P^4:

$$P^1(g,x_0) := r(P^{11}(g,x_0)) = \mathrm{Span}(\sum_{i=4}^{5} \alpha_i f_i | \alpha_i \in \mathbf{C})$$

spanned by the polynomials

$$f_4 = y^3 - y^2 - y,\quad f_5 = y^3 - 2y - 1. \tag{94}$$

Now since $r|\mathrm{Span}(E'(g))$ is an embedding, (92) implies that $r(\mathrm{Span}(E'(g)))\cap P^1(g,x_0) \neq \emptyset$. On the other hand, one checks immediately that the polynomials $f_1, ..., f_5$ in (93) and (94) are linearly independent. Hence, a contradiction. □

Now remark that in view of (66) the condition (74) can be rewritten as:

$$\mathcal{P}T_w R_{g,x_0} \cap \mathcal{P}(C,x_0) = \{w\},\quad w = \mu(C,x_0),\ C = \mathcal{P}onc^{n-1}(g),\ (g,x_0) \in \mathcal{V}_n,$$

or, equivalently, as

$$\mathcal{P}T_w R_{g,x_0} \cap im(d\mu|(C,x_0)) = \{0\},\quad w = \mu(C,x_0),\ C = \mathcal{P}onc^{n-1}(g),\ (g,x_0) \in \mathcal{V}_n.$$

Thus in view of (73) lemma 3.3 implies

Corollary 3.4.

$$\begin{aligned} \mathcal{V}_n^* := \{(g,x_0) \in \mathcal{V}_n \mid T_w R_{g,x_0} \cap im(d\mu|(C,x_0)) = \ker(d\mu|(C,x_0)) = \{0\}, \\ w = \mu(C,x_0),\ C = \mathcal{P}onc^{n-1}(g),\ (g,x_0) \in \mathcal{V}_n\} = \\ = \{(g,x_0) \in \mathcal{V}_n \mid T_w R_{g,x_0} + im(d\mu|(C,x_0)) = T_w R_{g,x_0} \oplus im(d\mu|(C,x_0)), \\ \ker(d\mu|(C,x_0)) = \{0\},\quad w = \mu(C,x_0),\ C = \mathcal{P}onc^{n-1}(g),\ (g,x_0) \in \mathcal{V}_n\} \end{aligned} \tag{95}$$

is a dense open subset of $\mathcal{V}_n$. Hence also in view of remark 3.1

$$\mathcal{V}_n^{**} := \mathcal{V}_n^* \cap (G^{**} \times (P^2 \setminus Q))$$

is dense open in $\mathcal{V}_n$.

Now return to diagram (14) and remark that by construction we have a diagram

$$\begin{array}{ccc} M_{n-1}^* \times P^2 & \xrightarrow{\psi_n} & P^{N_n} \\ {\scriptstyle f_{n-1}\times 1}\downarrow & & \uparrow{\scriptstyle \mu} \\ C_{n-1}^* \times P^2 & \hookrightarrow & P^{N_{n-1}*} \times P^2 \end{array} \tag{96}$$

Here $\mu|P^{N_{n-1}*} \times P^2$ and $f_{n-1} \times 1|M^*_{n-1} \times P^2$ are unramified, hence $\psi_n = \nu_n|M^*_{n-1} \times P^2$ is unramified as well.

Since $\pi_n : D^* \to Z^*_n$ is a P^1-fibration, it follows that for any $y \in M^*_{n-1} \times P^2$

$$\mathrm{Span}(\bigcup_{z\in P^1_y} (d\pi_n|_z)(T_zD^*)) = T_y(M^*_{n-1} \times P^2).$$

Hence from the diagrams (14) and (96), since ψ_n is unramified, we have

$$\mathrm{Span}(\bigcup_{z\in P^1_y} (df_n|_z)(T_zD^*)) = \mathrm{Span}(\bigcup_{z\in P^1_y} (d(\nu_n{\cdot}\pi_n)|_z)(T_zD^*)) = \mathrm{Span}(\bigcup_{z\in P^1_y} (d\nu_n|_y)(d\pi_n|_z)(T_zD^*))$$
$$= (d\nu_n|_y)(T_y(M^*_{n-1} \times P^2)) = (d(\nu_n|M^*_{n-1} \times P^2)|_y)(T_y(M^*_{n-1} \times P^2)) =$$
$$(d\psi_n|_y)(T_y(M^*_{n-1} \times P^2)) \simeq T_y(M^*_{n-1} \times P^2) \simeq k^{4n-5}. \qquad (97)$$

Thus taking $y = (\rho \times 1)(g, x_0) \in M^*_{n-1} \times P^2$ [5] for $(g, x_0) \in \mathcal{V}^{**}_n$ and using (53) and lemma 3.2, we get:

$$\mathrm{Span}(\bigcup_{z\in P^1_y} (df_n|_z)(T_z\mathcal{M}_n)) = \mathrm{Span}(\bigcup_{z\in P^1_y} (df_n|_z)(T_zD^*)) + \mathrm{Span}(\bigcup_{z\in P^1_y} (df_n|_z)(V_z)) =$$
$$= (d\psi_n|_y)(T_y(M^*_{n-1} \times P^2)) + T_wR_{g,x_0}, \qquad (98)$$

where $w = \psi_n(y)$. Since by the above $(d\psi_n|_y)(T_y(M^*_{n-1} \times P^2)) \subset im(d\mu|w_0)$ for $w_0 = (f_{n-1} \times 1)(y)$, (95), (98) and lemma 3.2 imply

Corollary 3.5. *For* $(g, x_0) \in \mathcal{V}^{**}_n$, $y = (\rho \times 1)(g, x_0) \in M^*_{n-1} \times P^2$ *and* $w = \psi_n(y)$ *we have:*

$$\mathrm{Span}(\bigcup_{z\in P^1_y} (df_n|_z)(T_z\mathcal{M}_n)) = (d\psi_n|_y)(T_y(M^*_{n-1} \times P^2)) \oplus T_wR_{g,x_0} \simeq k^{4n-5} \oplus k^3 \simeq k^{4n-2}.$$

Now prove the following

Lemma 3.6. *In conditions of the above corollary,*

i) $\dim T_y\tilde{\mathcal{C}}_n = 4n - 2$*;*

ii) $T_y\tilde{\mathcal{C}}_n = \mathrm{Span}(\bigcup_{z\in P^1_y} (d\tilde{f}_n|_z)(T_z\mathcal{M}_n))$.

Proof. Consider the diagram (14) and denote shortly $\mathcal{Z} := \tilde{f}_n(D^*) = M^*_{n-1} \times P^2$, $\mathrm{codim}_{\tilde{\mathcal{C}}_n} \mathcal{Z} = 2$, so that $y \in \mathcal{Z}$. By the choice of the point y we have:

$$P^1_y := \tilde{f}^{-1}_n(y) \simeq P^1, \quad \mathcal{O}_{\mathcal{M}_n}(D)|P^1_y \simeq \mathcal{O}_{P^1}(-2),$$

hence $\omega_{\mathcal{M}_n}|P^1_y \simeq \mathcal{O}_{P^1}$. Hence, since $\tilde{\mathcal{C}}_n$ is normal, by theorem of Grauert-Riemenschneider [GR] $R^i\tilde{f}_{n*}\mathcal{O}_{\mathcal{M}_n} = 0$, $i \geq 1$, i.e. y is a rational (and also canonical) singularity of $\tilde{\mathcal{C}}_n$, so that, by [KKMS, chap.1,§3], the local ring $\mathcal{O}_{\tilde{\mathcal{C}}_n,y}$ is Cohen-Macaulay. Now let $\tilde{\mathcal{C}}_n \hookrightarrow P^M$ be any projective embedding and $H_1, ..., H_{4n-5}$ be general hyperplanes in P^M through the point y, such that, by Bertini's theorem, there exists a neighbourhood $U \subset \tilde{\mathcal{C}}_n$ of the point y with the following properties:

1) $S = L \cap U$ is an irreducible surface, smooth outside y, where $L := H_1 \cap ... \cap H_{4n-5}$, and the local ring $\mathcal{O}_{S,y}$ is also Cohen-Macaulay; hence S is normal by Serre criterion;

[5] Recall that ρ is defined in remark 2.1; see also remark 3.1.

2) L intersects $\mathcal{Z}$ transversally at y, i.e. $S \cap \mathcal{Z} = y$ is scheme-theoretically a reduced point (here and below we consider $\mathcal{Z}$ as a reduced irreducible variety; recall that it is birationally isomorphic to $M_{n-1} \times P^2$); in particular,

$$T_y\mathcal{Z} \cap T_yS = \{0\}. \tag{99}$$

Now prove that $\tilde{S} = \tilde{f}_n^{-1}(S)$ is a smooth surface. Since $P^1_y \subset \tilde{S}$ and by construction $\tilde{f}_n|(\tilde{S} \setminus P^1_y) : \tilde{S} \setminus P^1_y \to S \setminus y$ is an isomorphism, it follows that $\tilde{S} \setminus P^1_y$ is smooth. Now show that $\tilde{S}$ is smooth along P^1_y, hence it is smooth. In fact, if there exists a point $z \in P^1_y$ such that $z \in Sing\ \tilde{S}$, then since D is smooth at z, we have $\dim T_z(\tilde{S} \cap D) \geq 2$. Hence there exists a vector $0 \neq \tau \in T_z(\tilde{S} \cap D)$ such that $\tau \notin T_zP^1_y$; hence $\tau' = d\tilde{f}_n(\tau) \neq 0$. Considering τ' as a scheme $Spec\ k[t]/(t^2)$, we see that $\tau' \in f(D) \cap f(\tilde{S}) = \mathcal{Z} \cap S = y$, where by the property 2) above y is a reduced point, a contradiction.

Now as $\tilde{S}$ is smooth, S normal and $\tilde{f}_n|\tilde{S} : \tilde{S} \to S$ is a contraction of P^1_y to the point y, where $\mathcal{O}_{\tilde{S}}(P^1_y)|P^1_y \simeq \mathcal{O}_{\mathcal{M}_n}(D)|P^1_y \simeq \mathcal{O}_{P^1}(-2)$, i.e. P^1_y is a (-2)-curve on $\tilde{S}$, we obtain by [A, Cor.6] that y is a Du Val singularity of the type A_1 on S. Hence, since DuVal singularities have no moduli, a standard argument shows

(see, e.g.,[R, Cor.1.14]) [6] that $\mathcal{C}_n$ is analytically, around y, isomorphic to $S \times \mathcal{Z}$, i.e., more precisely,

$$\hat{\mathcal{O}}_{y,\tilde{\mathcal{C}}_n} \simeq \mathbf{C}[[x_1, ..., x_{4n-5}]] \otimes \mathbf{C}[[x,y,z]]/(xy - z^2). \tag{100}$$

Hence, in particular, $T_y\tilde{\mathcal{C}}_n = T_y\mathcal{Z} \oplus T_yS = T_ySing\ \tilde{\mathcal{C}}_n \oplus T_yS \simeq k^{4n-2}$, i.e. we obtain the statement i) of lemma. Whence by (100) the statement ii) follows. □

3.7. **Proof of (15).** Using lemma 3.6 and corollaries 3.5 and 3.4 we have $(d\nu_n|_y)(T_y\tilde{\mathcal{C}}_n) = (d\nu_n|_y)(\mathrm{Span}(\bigcup_{z\in P^1_y}(d\tilde{f}_n|_z)(T_z\mathcal{M}_n)) = \mathrm{Span}(\bigcup_{z\in P^1_y}(d\nu_n|_y)(d\tilde{f}_n|_z)(T_z\mathcal{M}_n)) = \mathrm{Span}(\cup_{z\in P^1_y}(df_n|_z)(T_zM_n)) = k^{4n-2} \simeq T_y\tilde{\mathcal{C}}_n$. Hence $\ker(d\nu_n|y) = 0$, q.e.d.

4. Appendix A: proof of lemma 1.1

First, for any point $[\mathcal{E}] \in D$, where $\mathcal{E}$ satisfies (13), and any tangent vector $\tau \in T_{[\mathcal{E}]}\mathcal{M}_n$, $\tau = Spec(k[t]/(t^2))$, we get a $\mathcal{O}_{P^2\times\tau}$-sheaf E such that $E|P^2 \times [\mathcal{E}] \simeq \mathcal{E}$. Now the condition that $\tau \in T_{[\mathcal{E}]}D$ precisely means that E fits into the $\mathcal{O}_{P^2\times\tau}$-triples

$$0 \to \mathcal{E} \to E \to \mathcal{E} \to 0, \quad 0 \to E \to E_0 \to \varkappa \to 0, \tag{101}$$

where E_0 is a locally free rank-2 $\mathcal{O}_{P^2\times\tau}$-sheaf and $\varkappa$ is an artinian $\mathcal{O}_{P^2\times\tau}$-sheaf of length 2, and these triples fit in the diagram:

$$\begin{array}{rccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \uparrow & & \uparrow & & \uparrow & \\
\gamma: & 0 \to & k(x) & \to & \varkappa & \to & k(x) & \to 0 \\
 & & \uparrow & & \uparrow & & \uparrow & \\
\beta: & 0 \to & \mathcal{E}_0 & \to & E_0 & \to & \mathcal{E}_0 & \to 0 \\
 & & \uparrow & & \uparrow & & \uparrow & \\
\alpha: & 0 \to & \mathcal{E} & \to & E & \to & \mathcal{E} & \to 0 \\
 & & \uparrow & & \uparrow & & \uparrow & \\
 & & 0 & & 0 & & 0 &
\end{array} \tag{102}$$

[6] In [R] the case of dimension 3 is treated, and the case of any dimension is taken similarly.

where the left and the right columns coincide with (13). We may consider (102) as a $\mathcal{O}_{P^2}$-diagram via applying to it the functor p_{1*}, where $p_1 : P^2 \times \tau \longrightarrow P^2$ is the projection. Thus the horizontal extensions α, β and γ of this diagram can be treated as elements of the groups $\mathrm{Ext}^1_{P^2}(\mathcal{E},\mathcal{E})$, $\mathrm{Ext}^1_{P^2}(\mathcal{E}_0,\mathcal{E}_0)$ and $\mathrm{Ext}^1_{P^2}(k(x),k(x))$ respectively. Now by (101) these groups satisfy the diagram:

$$
\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \uparrow \\
0 \longrightarrow & \mathrm{Ext}^1(k(x),k(x)) & \xrightarrow{\varphi_2} & \mathrm{Ext}^2(k(x),\mathcal{E}) & \longrightarrow & \mathrm{Ext}^2(k(x),\mathcal{E}_0) \\
 & & & \uparrow \varphi_1 & & \uparrow \\
 & & & \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) & \xrightarrow{\varphi_3} & \mathrm{Ext}^1(\mathcal{E},\mathcal{E}_0) \\
 & & & & & \uparrow \varphi_4 \\
 & & & & & \mathrm{Ext}^1(\mathcal{E}_0,\mathcal{E}_0) \\
 & & & & & \uparrow \\
 & & & & & 0
\end{array}
\tag{103}
$$

(for simplicity here and below we omit the subscript $_{P^2}$ in notations of Ext-groups). In terms of this diagram the condition that the triple (13) and the first triple (101) extend to the diagram (102) (i.e. that $\tau \in T_{[\mathcal{E}]}D$) can be written as: $\varphi_3(\alpha) \in im\ \varphi_4$. Now the diagram (103) shows that this condition is eqivalent to the condition $\varphi_1(\alpha) \in im\ \varphi_2$, which gives the first statement of lemma. Now the last statement of lemma immediately follows from the diagram outcoming from (103):

$$
\begin{array}{ccccccccc}
 & 0 & & 0 & & & & \\
 & \uparrow & & \uparrow & & & & \\
0 \longrightarrow & \mathrm{Ext}^1(k(x),k(x)) & \xrightarrow{\varphi_2} & \mathrm{Ext}^2(k(x),\mathcal{E}) & \longrightarrow & \mathrm{Ext}^2(k(x),\mathcal{E}_0) & \xrightarrow{\varepsilon^!:} & \mathrm{Ext}^2(k(x),k(x) \\
 & \uparrow & & \uparrow \varphi_1 & & \uparrow & & \\
0 \longrightarrow & T_{[\mathcal{E}]}D & \longrightarrow & \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) & \longrightarrow & N_D\mathcal{M}_n|_{[\mathcal{E}]} & \longrightarrow 0 & \\
 & \uparrow & & \uparrow & & \uparrow & & \\
 & \mathrm{Ext}^1(\mathcal{E}_0,\mathcal{E}) & = & \mathrm{Ext}^1(\mathcal{E}_0,\mathcal{E}) & & 0 & & \\
 & \uparrow & & \uparrow & & & & \\
 & 0 & & 0 & & & &
\end{array}
\tag{104}
$$

Remark 4.1. Alternative proof of the last diagram is given in [T, Prop. 1.5.1].

5. Appendix B: proof of (9)

In this section for convenience of the reader we recall the proof of the following result of S.A.Strømme leading to (9):

$$
\mathrm{codim}_{M_n} L_n \geq n-1, \tag{105}
$$

where $L_n = \{[\mathcal{E}] \in M_n | C_n(\mathcal{E}) \text{ contains a line}\}$. We quote [S, section 3, in particular, theorem 3.7(viii)] here. Fix a closed point y of P^2 and let $p : F \to P^2$ be the blowing up of P^2 at y, with natural projection $q : F \to P^1$. Let $\delta \in Pic(P^1)$ be the positive generator and $\tau = p^*(c_1(\mathcal{O}_{P^2}(1)))$, so that the class of the exceptional divisor $R = p^{-1}(y)$ is $\tau - \delta$. For any bundle $\mathcal{E} \in M_n$, put $\tilde{\mathcal{E}} = p^*\mathcal{E}$. Then $\tilde{\mathcal{E}}|R \simeq 2\mathcal{O}_R$. It is well known that, conversely, if D is a 2-bundle on F such that $D|R \simeq 2\mathcal{O}_R$, then $\mathcal{E} = p_*D$ is a bundle and the natural map $\tilde{\mathcal{E}} \to D$ is an isomorphism. So we are reduced to the classification of rank-2 bundles $\tilde{\mathcal{E}}$ on F such that (i) $\tilde{\mathcal{E}}|R \simeq 2\mathcal{O}_R$, (ii) $c_1(\tilde{\mathcal{E}}) = 0$, $c_2(\tilde{\mathcal{E}}) = n\tau^2$, (iii) $h^0(\tilde{\mathcal{E}}) = 0$.

First put $k = rank(R^1q_*(\tilde{\mathcal{E}}(-\tau))$; then the restriction of E to a general fiber of q is of the form $\mathcal{O}(k) \oplus \mathcal{O}(-k)$. Next, clearly $L_n = \bigcup_{y \in P^2} L_n(y)$, where $L_n(y) = \{[\mathcal{E}] \in M_n | C_n(\mathcal{E})$ contains a line $\check{y}$ in $\check{P}^2$ dual to the point $y \in P^2\}$, and the condition $\mathcal{E} \in L_n(y)$ clearly means that $k > 0$.

Now evidently $q_*(\tilde{\mathcal{E}}(-k\tau)) \simeq \mathcal{O}_{P^1}(-i\delta)$ for some $i \geq 0$, and we call the pair (i, k) the type of $\mathcal{E}$, respectively denote $L_{(i,k)}(y) = \{\mathcal{E} \in L_n(y) | \mathcal{E}$ has the type $(i, k)\}$. Now one quickly sees that $i > k$. (In fact, if $i \leq k$, then $\tilde{\mathcal{E}}(i\delta - k\tau) \subseteq \tilde{\mathcal{E}}(k(\delta - \tau)) \subseteq \tilde{\mathcal{E}}$, contradicting to the fact that $h^0(\tilde{\mathcal{E}}) = 0$.) Any non-zero section of $\tilde{\mathcal{E}}(i\delta - k\tau)$ induces a short exact sequence

$$0 \to \mathcal{O}(k\tau - i\delta) \to \tilde{\mathcal{E}} \to \mathcal{I}_Y(i\delta - k\tau) \to 0,$$

where Y is a finite subscheme of F of length $c_2(\tilde{\mathcal{E}}(i\delta - k\tau) = n - k(2i - k) \geq 0$. Hence this number is non-negative. Conversely, let (i, k) be given, satisfying the conditions

$$k \geq 0, \quad i - k > 0, \quad n - k(2i - k) \geq 0, \tag{*}$$

and let $Y \subseteq F$ be a group of $n - k(2i - k)$ general points, we construct $\tilde{\mathcal{E}}$ as a general extension as above. It is easily veryfied that $E = p_*\tilde{\mathcal{E}}$ is a bundle of type (i, k). Note that the association $\mathcal{E} \mapsto Y$ induces a dominating morphism $L_{(i,k)}(y) \to H_{(i,k)}$, where $H_{(i,k)}$ is the open part of the Hilbert scheme of F parametrizing locally complete intersection subschemes of the finite length $n - k(2i - k)$. Hence $\dim H_{(i,k)} = 2n - 2k(2i - k)$. Furthermore, all the fibers of of this morphism are open subsets of a projective space of constant dimension, say, d, and, in fact, there is a locally free sheaf $\mathcal{E}xt$ of rank $rank(\mathcal{E}xt) = n - k(2i - k) + (2k + 1)(2i - 1 - k)$ on $H_{(i,k)}$ (so that $d = rank(\mathcal{E}xt) - 1$) and an open embedding $L_{(i,k)}(y) \to \mathbf{P}(\mathcal{E}xt)$ over $H_{(i,k)}$. Hence $\dim L_{(i,k)}(y) = \dim H_{(i,k)} + d = 3n - 3k(2i - k) + (2k + 1)(2i - k - 1) - 1$ and $codim_{M_n} L_{(i,k)}(y) = n + (2i - k)(k - 1) + 2k - 1 \geq n + 1$; moreover, $codim_{M_n} L_{(i,k)}(y) \geq n + 2$ if $k \geq 2$ (we use (*)). Since clearly $L_n = \bigcup_{y \in P^2} L_n(y) = \bigcup_{y \in P^2} (\bigcup_{i \geq k > 0} L_{(i,k)}(y))$, (105) follows.

6. Appendix C: Barth's results on Hulsbergen bundles

Here we recall the results of Barth on Hulsbergen bundles from [B, 5.1-3].

Consider an N-tuple of distinct points $x_1, ..., x_N \in P^2$. The $\binom{N}{2}$ pairs x_i, x_j among these points determine lines $L_{ij} \subset P^2$ not necessary all different. Denote by ν_{ij} the number of points x_k on L_{ij}. The dual configuration in $\check{P}^2$ consists of a complete N-side with sides X_i dual to x_i and vertices l_{ij} dual to L_{ij}. In each vertex l_{ij} there intersect ν_{ij} sides X_k.

Lemma 6.1. *Let $\mathcal{I} \subset \mathcal{O}_{\check{P}^2}$ be the ideal sheaf of functions vanishing at each vertice l_{ij} at least of order $\nu_{ij} - 1$. Then*

$$h^0(\mathcal{I}(N - 1)) = N. \tag{106}$$

Proof. Take a line $X \subset \check{P}^2$ not through any l_{ij}, then $\mathcal{I}|X \simeq \mathcal{O}_{P^1}$, and there is the exact sequence

$$0 \to \Gamma(\mathcal{I}(N - 2)) \to \Gamma(\mathcal{I}(N - 1)) \to \Gamma(\mathcal{I}(N - 1)|X) = \Gamma(\mathcal{O}_{P^1}(N - 1)).$$

Every $g \in \Gamma(\mathcal{I}(N - 2))$ vanishes on every line X_i at least of order $N - 1$, hence identically. So $h^0(\mathcal{I}(N - 2)) = 0$ and the exact sequence shows $h^0(\mathcal{I}(N - 1)) \leq N$.

Let $v_1, ..., v_N \in \Gamma(\mathcal{O}_{\check{P}^2}(1))$ be the equations for $X_1, ..., X_N$ and put $f_k := \prod_{i \neq k} v_i \in \Gamma(\mathcal{O}_{\check{P}^2}(N-1))$. These f_k are sections in $\Gamma(\mathcal{I}(N-1))$ and they are linearly independent: if $\sum c_k f_k = 0$, $c_k \in \mathbf{C}$, then restricting to X_i one obtains $c_i = 0$. So $h^0(\mathcal{I}(N-1)) \geq N$ too, and $f_1, ..., f_N$ form a basis of this space. □

Now following W.Hulsbergen, W.Barth considers vector bundles $\mathcal{E} \in M_n$ such that $\mathcal{E}(1)$ admits a a section s with N ordinary zeroes precisely at $x_1, ..., x_N$. Every such $\mathcal{E}$ is obtained by an extension

$$0 \to \mathcal{O}_{P^2} \xrightarrow{s} \mathcal{E}(1) \to \mathcal{I}(2) \to 0, \tag{107}$$

with $\mathcal{I} \subset \mathcal{O}_{P^2}$ the ideal sheaf of $x_1, ..., x_N$. Conversely, such extensions are classified by elements in the vector space

$$Ext^1_{\mathcal{O}_{P^2}}(\mathcal{I}(2), \mathcal{O}_{P^2}) \simeq \bigoplus_i \mathcal{O}_{x_i}(1) \tag{108}$$

of dimension N. Such an extension defines a locally free sheaf $\mathcal{E}(1)$ iff all its components in the direct sum decomposition (108) are nonzero.

Let $L \subset P$ be a line through some zero x_i of s, then it is easily seen that

$$\mathcal{E}|L \simeq \mathcal{O}_L(\nu - 1) \oplus \mathcal{O}_L(1 - \nu)$$

with $\nu \geq 1$ the number of points x_i on L. From [B1, Theorem 2 i) and ii)] it follows immediately that the curve of jumping lines $C_n(\mathcal{E}) \subset \check{P}^2$ belongs to the linear system described by equations in $\Gamma(\mathcal{I}(N-1))$. In other words, $C_n(\mathcal{E})$ is circumscribed about the complete N-side in $\check{P}^2$ with sides $X_1, ..., X_N$. Hulsbergen's main result is a converse of this statement:

Theorem 6.2. *There is an isomorphism*

$$\sigma : Ext^1_{\mathcal{O}_{P^2}}(\mathcal{I}(2), \mathcal{O}_{P^2}) \to \Gamma(\mathcal{I}(N-1)) \tag{109}$$

with these twoproperties:

i) if $\mathcal{E}$ is a locally free sheaf defined by an extension ε, then $\sigma(\varepsilon)$ is an equation for the curve $C_n(\mathcal{E})$;

ii) an extension ε defines a sheaf $\mathcal{E}$ locally free at x_i iff $\sigma(\varepsilon) = \sum c_k f_k$ with $c_i \neq 0$.

Proof. Let $F \subset P^2 \times \check{P}^2$ be the flag manifold, p, q its projections onto $P^2, \check{P}^2$, and $\mathcal{O}_F(k,l) = p^*\mathcal{O}_{P^2}(k) \otimes q^*\mathcal{O}_{\check{P}^2}(l)$. Then $q_*p^*\mathcal{I}$ is the ideal sheaf of $X_1 \cup ... \cup X_N$. Pick an isomorphism $h : \mathcal{O}_{\check{P}^2} \to q_*((p^*\mathcal{I})(0,N))$.

Definition of $\sigma : Ext^1_{\mathcal{O}_{P^2}}(\mathcal{I}(2), \mathcal{O}_{P^2}) \to \Gamma(\mathcal{I}(N-1))$. Via canonical isomorphisms

$$Ext^1_{\mathcal{O}_{P^2}}(\mathcal{I}(2), \mathcal{O}_{P^2}) \xrightarrow{p^*} Ext^1_{\mathcal{O}_F}(p^*(\mathcal{I}(2)), \mathcal{O}_F) \to Ext^1_{\mathcal{O}_F}((p^*\mathcal{I})(0,N), \mathcal{O}_F(-2,N))$$

to ε there corresponds an extension on F

$$0 \to \mathcal{O}_F(-2,N) \to (p^*\mathcal{E})(-1,N) \to (p^*\mathcal{I})(0,N) \to 0 \tag{110}$$

and an exact sequence

$$\begin{array}{ccccc} 0 \longrightarrow \Gamma((p^*\mathcal{I})(0,N)) & \longrightarrow & H^1(\mathcal{O}_F(-2,N)) & \longrightarrow & H^1((p^*\mathcal{E})(-1,N)) \\ \uparrow q^*h & & \uparrow \wr & & \\ \Gamma(\mathcal{O}_{\check{P}^2}) & \xrightarrow{\sigma(\varepsilon)} & \Gamma(\mathcal{O}_{\check{P}^2}(N-1)) & & \end{array} \tag{111}$$

defining σ.

Proof of i). The direct image sequence of (110) under q is

$$
\begin{array}{ccccccc}
0 \longrightarrow & q_*p^*\mathcal{I})(N) & \longrightarrow & R^1q_*(\mathcal{O}_F(-2,N)) & \longrightarrow & R^1q_*((p^*\mathcal{E}(-1))(N) \to (R^1q_*p^*\mathcal{I})(N) \\
& \| & & \| & & \\
& \mathcal{O}_{\check{P}2} & \xrightarrow{\sigma(\varepsilon)} & \mathcal{O}_{\check{P}2}(N-1). & &
\end{array}
\tag{112}
$$

The support of $R^1q_*p^*\mathcal{E}(-1)$ is the curve $C_n(\mathcal{E})$. The support of $R^1q_*p^*\mathcal{I})(N)$ is the discrete set $\{l_{ij}\}$. So $\sigma(\varepsilon) = 0$ is an equation for $C_n(\mathcal{E})$, even with multiplicities [B1, Theorem 2].

Proof that σ is an isomorphism onto $\Gamma(\mathcal{I}(N-1))$: Since $C_n(\mathcal{E})$ has its equation in $\Gamma(\mathcal{I}(N-1))$ and since

$$
\dim \Gamma(\mathcal{I}(N-1)) = N = \dim Ext^1_{\mathcal{O}_{P2}}(\mathcal{I}(2), \mathcal{O}_{P^2}), \tag{113}
$$

one only has to show that σ is injective. To do this, assume that $\sigma(\varepsilon)$ vanishes. The section h in $(p^*\mathcal{I})(0,N)$ then lifts to a section h' in $(p^*\mathcal{E})(-1,N)$. If $\mathcal{E}$ would be locally free at x_i, then $\mathcal{E}|L \simeq 2\mathcal{O}_L$ for the general line through x_i, hence for almost all lines $L \subset P$. One would obtain the contradiction $h' = 0$. This shows that the extension ε defining $\mathcal{E}$ must be trivial at each x_i, i.e., $\varepsilon = 0$.

Proof of ii). If ε is non-trivial at x_i, then $\sigma(\varepsilon)$ cannot vanish on X_i, because every bundle $\mathcal{E}|L$, $x_i \in L$, would then be a limit of bundles $\mathcal{O}_L(k) \oplus \mathcal{O}_L(-k)$, $k > 0$. So the hyperplane $\{\sum c_k f_k,\ c_i = 0\} \subset \Gamma(\mathcal{I}(N-1))$ under σ^{-1} corresponds to extensions ε which are trivial at x_i. □

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