On Zariski's pairs of mth canonical discriminant curves

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Abstract

In this note we give examples of Zariski's pairs $B_{1,m}$, $B_{2,m}$ ($m \in \mathbb{N}$ and $m \geq 5$) of plane cuspidal curves such that
(i) $B_{i,m}$ is the discriminant curve of a generic morphism $f_{i,m} : S_i \to \mathbb{P}^2$, $i = 1, 2$,
(ii) S_1 and S_2 are homeomorphic surfaces of general type,
(iii) $f_{i,m}$ is given by linear three-dimensional subsystem of the mth canonical class of S_i.

Introduction

By definition, two plane curves B_1, $B_2 \subset \mathbb{P}^2$ over $\mathbb{C}$ are called a Zariski pair if they have the same degree and homeomorphic tubular neighborhoods in $\mathbb{P}^2$, but the pairs $(\mathbb{P}^2, B_1)$ and $(\mathbb{P}^2, B_2)$ are not homeomorphic. A set of plane curves $(B_1, ..., B_k)$ is called *Zariski's k-tuple* if for each $i \neq j$ the pair (B_i, B_j) is a Zariski one.

Let $B \subset \mathbb{P}^2$ be an irreducible plane curve with ordinary cusps and nodes, as its only singularities. Denote by $2d$ the degree of B, and let g be the genus of its desingularization, $c = \#\{\text{cusps of} B\}$, and $n = \#\{\text{nodes of} B\}$. The curve B is called *the discriminant curve* of a generic morphism if there exists a finite morphism $f : S \to \mathbb{P}^2$, $\deg f \geq 3$, satisfying the following conditions:

(i) S is a non-singular irreducible projective surface;
(ii) f is unramified over $\mathbb{P}^2 \setminus B$;
(iii) $f^*(B) = 2R + C$, where R is irreducible and non-singular, and C is reduced;
(iv) $f_{|R} : R \to B$ coincides with the normalization of B.

Such a morphism f is called *generic*.

Two generic morphisms (S_1, f_1), (S_2, f_2) with the same discriminant curve B are said to be equivalent if there exists an isomorphism $\varphi : S_1 \to S_2$ such that $f_1 = f_2 \circ \varphi$. The following assertion is known as Chisini's Conjecture.

Conjecture 1 *Let B be the discriminant curve of a generic morphism $f : S \to \mathbb{P}^2$ of degree $\deg f \geq 5$. Then f is uniquely determined by B.*

*Partly supported by RFFI (No. 96-01-00614) and INTAS (No. 96-0713).

In [5], Chisini's conjecture was proved for discriminant curves of almost all generic morphisms. In particular, Chisini's conjecture holds for the discriminant curves of the generic morphisms $f_m : S \to \mathbb{P}^2$ of the surfaces S of general type, where f_m is given by a linear three-dimensional subsystem of the mth canonical class of S, $m \in \mathbb{N}$. The discriminant curve B of a generic morphism $f_m : S \to \mathbb{P}^2$, given by a three-dimensional subsystem of the mth canonical class of S, will be called *the mth canonical discriminant curve.*

Let $(\mathbb{P}^2, B_1)$ and $(\mathbb{P}^2, B_2)$ be two diffeomorphic (resp. homeomorphic) pairs. One can show that if B_1 is the discriminant curve of a generic morphism (S_1, f_1), then B_2 is also the discriminant curve of some generic morphism (S_2, f_2). Moreover, if (S_1, f_1) is uniquely determined by B_1, then the same is true for (S_2, f_2) and S_1 and S_2 are diffeomorphic (resp. homeomorphic). It is a natural question to ask ([5]):

Question. *Let $S_1, S_2 \subset \mathbb{P}^r$ be two homeomorphic (Case C)(resp. diffeomorphic (Case D)) surfaces of general type embedded by the mth canonical class, and let B_i be the mth canonical discriminant curve of a generic projection of S_i onto $\mathbb{P}^2$. Are pairs $(\mathbb{P}^2, B_1)$ and $(\mathbb{P}^2, B_2)$ homeomorphic (resp. diffeomorphic)?*

In case C, the answer to this Question is negative, as the following theorem shows.

Theorem 1 *For each positive integer k there exists a countable set of Zariski's k-tuples $(B_{1,m}, ..., B_{k,m})$, $m \in \mathbb{N}$, $m \geq 5$, of mth canonical discriminant curves $B_{i,m}$ such that the corresponding generic morphisms $f_{i,m} : S_i \to \mathbb{P}^2$, $i = 1, ..., k$, are the morphisms of pairwise homeomorphic surfaces of general type S_i.*

The proof of this Theorem is based on technique developed in [5]. It also uses the examples constructed of homeomorphic (but not diffeomorphic) surfaces of general type with ample canonical class constructed by Catanese in [2] and [3].

The Question in the case D remains open.

This paper was written during my stay at the Max-Planck-Institut für Mathematik in Bonn. It is a pleasure to thank the Institut for its hospitality and financial support.

1 Proof of Theorem 1.

In [2] and [3], Catanese investigated smooth simple bidouble coverings $\varphi : S \to Q = \mathbb{P}^1 \times \mathbb{P}^1$ of type $(a, b), (m, n)$; thus, φ is a finite $(\mathbb{Z}/2)^2$ Galois covering branched along two generic curves of respective bidegrees $(2a, 2b)$, $(2m, 2n)$. Let a, b, m, n be integers satisfying the following conditions:

$$\begin{array}{ll} a > 2n\,, n \geq 3, & m > 2b\,, b \geq 3\,; \\ a \equiv n\,(mod\,2), & b \equiv m\,(mod\,2). \end{array}$$

Put

$$u = (n + a - 2),\ v = (m + b - 2),\ w = (a - n),\ z = (m - b). \tag{1}$$

By [2], for a bidouble covering $\varphi : S \to Q = \mathbb{P}^1 \times \mathbb{P}^1$ of type $(a, b), (m, n)$, S is a simply connected surface of general type whose canonical class is

$$K_S = \varphi^*(\mathcal{O}_Q(u, v)) \tag{2}$$

and

$$K_S^2 = 8uv,\ \chi = \chi(\mathcal{O}_S) = \frac{3}{2}uv + (u+v) + 2 - \frac{1}{2}wz. \tag{3}$$

For u, v, w, z satisfying (1), the canonical class K_S is 2-divisible in $H^2(S, \mathbb{Z})$ and the index

$$r = r(S) = \max\{s \in \mathbb{N} \mid (1/s)K_S \in H^2(S, \mathbb{Z})\}$$

is equal to the greatest common divisor (u, v).

Since K_S is 2-divisible, the intersection form on $H_2(S, \mathbb{Z})$ is even. Therefore, by Freedman's theorem ([4]) two such bidouble coverings S_1 and S_2 of Q are homeomorphic if and only if $K_{S_1}^2 = K_{S_2}^2$ and $\chi(\mathcal{O}_{S_1}) = \chi(\mathcal{O}_{S_2})$. In [2] and [3], it has been proved that for each integer k there exists at least one k-tuple $S_1, \dots, S_k$ of bidouble coverings of Q of respective types $(a_i, b_i), (m_i, n_i)$ satisfying the conditions described above and such that

(i) S_i and S_j are homeomorphic for each $1 \le i, j \le k$.

(ii) $r(S_i) \neq r(S_j)$ for $i \neq j$ (and therefore S_i and S_j are not diffeomorphic).

We shall call such k-tuple of bidouble coverings of Q *a Catanese k-tuple.*

Example. Two bidouble coverings S_1 of type $(16, 22), (52, 4)$ and S_2 of type $(28, 10), (28, 10)$ form a Catanese pair with

$$K_S^2 = K_{S_i}^2 = 10368, \qquad \chi = \chi(\mathcal{O}_{S_i}) = 1456$$

and

$$r(S_1) = 18, \qquad r(S_2) = 36.$$

Consider the m-canonical embedding $\Phi_{|mK_{S_i}|} : S_i \to \mathbb{P}^r$, $m \ge 5$ ([1]), and let $f_{m,i} : S_i \to \mathbb{P}^2$ be the restriction to $\Phi_{|mK_{S_i}|}(S_i)$ of a generic projection $pr : \mathbb{P}^r \to \mathbb{P}^2$. Denote by $R_{m,i} \subset S_i$ the ramification curve of the generic projection $f_{m,i}$, and by $B_{m,i} \subset \mathbb{P}^2$ the discriminant curve of $f_{m,i}$.

Proposition 1 *Let $f_i : S_i \to \mathbb{P}^2$, $i = 1, 2$, be two generic morphisms with respective discriminant curves B_i. Assume that there exist a homeomorphism of pairs $\psi : (\mathbb{P}^2, B_1) \to (\mathbb{P}^2, B_2)$. Then there exists a commutative diagram*

$$\begin{tikzcd} S_1 \arrow[r, "\Psi"] \arrow[d, "f_1"'] & S_2 \arrow[d, "f_2"] \\ \mathbb{P}^2 \arrow[r, "\psi"'] & \mathbb{P}^2 \end{tikzcd}$$

such that

(i) Ψ is also a homeomorphism;

(ii) $\Psi(R_1) = R_2$, where R_i is the branch curve of f_i.

Proof coincides with the proof of proposition 8 in [5].

Let a generic morphism $f_m : S \to \mathbb{P}^2$ be given by a three-dimensional linear subsystem of the mth canonical class on S. Then, by the adjunction formula, the ramification curve $R_m \subset S$ of f_m is equivalent to $(3m+1)K_S$. Therefore, if we have two generic morphisms $f_{i,m} : S_i \to \mathbb{P}^2$ given by three-dimensional linear subsystems of the mth canonical classes and such that the corresponding pairs $(\mathbb{P}^2, B_{1,m})$ and $(\mathbb{P}^2, B_{2,m})$ are homeomorphic, then by proposition 1 there exists a homeomorphism $\Psi : S_1 \to S_2$ such that $\Psi(R_{1,m}) = R_{2,m}$. In this case, we should have $r(S_1) = r(S_2)$, since Ψ induces an isomorphism $\Psi_* : H_2(S_1, \mathbb{Z}) \to H_2(S_2, \mathbb{Z})$ such that $\Psi_*(R_{1,m}) = R_{2,m}$. But it is impossible, since, by definition, $r(S_1) \neq r(S_2)$ for the Catanese pair S_1, S_2. Theorem 1 is proved.

Example. (Continuation). Applying Theorem 1 for m-canonical discriminant curves of generic morphisms of surfaces S_1 and S_2 in the example considered above, we obtain Zariski's pairs $(B_{1,m}, B_{2,m})$, $m \geq 5$, of plane cuspidal curves. Applying the computation in the proof of theorem 3 in [5], one can see that the m-canonical discriminant curve $B_{i,m}$ has degree

$$\deg B_{i,m} = 10368m(3m+1),$$

genus

$$g = 5184(3m+2)(3m+1) + 1,$$

and

$$c = 10368(12m^2 + 9m) - 13632$$

ordinary cusps.

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