Max-Planck-Institut für Mathematik
Bonn

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by

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Max-Planck-Institut für Mathematik
Preprint Series 2011 (65)

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MPIM 11-65

COUNTING NUMBERS IN MULTIPLICATIVE SETS: LANDAU VERSUS RAMANUJAN

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Abstract. A set S of integers is said to be *multiplicative* if for every pair m and n of coprime integers we have that mn is in S iff both m and n are in S. Both Landau and Ramanujan gave approximations to $S(x)$, the number of $n \le x$ that are in S, for specific choices of S. The asymptotical precision of their respective approaches are being compared and related to Euler-Kronecker constants, a generalization of Euler's constant $\gamma = 0.57721566\ldots$.

This paper claims little originality, its aim is to give a survey on the literature related to this theme with an emphasis on the contributions of the author (and his coauthors).

1. Introduction

To every prime p we associate a set $E(p)$ of positive allowed exponents. Thus $E(p)$ is a subset of $\mathbb{N}$. We consider the set S of integers consisting of 1 and all integers n of the form $n = \prod_i p_i^{e_i}$ with $e_i \in E(p_i)$. Note that this set is *multiplicative*, i.e., if m and n are coprime integers then mn is in S iff both m and n are in S. It is easy to see that in this way we obtain all multiplicative sets of natural numbers. As an example, let us consider the case where $E(p)$ consists of the positive even integers if $p \equiv 3 (\mathrm{mod}\ 4)$ and $E(p) = \mathbb{N}$ for the other primes. The set S_B obtained in this way can be described in another way. By the well-known result that every positive integer can be written as a sum of two squares iff every prime divisor p of n of the form $p \equiv 3 (\mathrm{mod}\ 4)$ occurs to an even exponent, we see that S_B is the set of positive integers that can be written as a sum of two integer squares.

In this note we are interested in the counting function associated to S, $S(x)$, which counts the number of $n \le x$ that are in S. By $\pi_S(x)$ we denote the number of primes $p \le x$ that are in S. We will only consider S with the property that $\pi_S(x)$ can be well-approximated by $\delta\pi(x)$ with $\delta > 0$ real and $\pi(x)$ the prime counting function (thus $\pi(x) = \sum_{p \le x} 1$). Recall that the Prime Number Theorem states that asymptotically $\pi(x) \sim x/\log x$. Gauss as a teenager conjectured that the logarithmic integral, $\mathrm{Li}(x)$, defined as $\int_2^x dt/\log t$ gives a much better approximation to $\pi(x)$. Indeed, it is now known that, for any $r > 0$ we have $\pi(x) = \mathrm{Li}(x) + O(x \log^{-r} x)$. On the other hand, the result that $\pi(x) = x/\log x + O(x \log^{-r} x)$, is false for $r > 2$. In this note two types of approximation of $\pi_S(x)$ by $\delta\pi(x)$ play an important role.

Date: November 10, 2011.

2000 *Mathematics Subject Classification.* 11N37; 11Y60.

We say S satisfies Condtion A if, asymptotically,

$$\pi_S(x) \sim \delta \frac{x}{\log x}. \tag{1}$$

We say that S satisfies Condition B if there are some fixed positive numbers δ and ρ such that asymptotically

$$\pi_S(x) = \delta \mathrm{Li}(x) + O\Big(\frac{x}{\log^{2+\rho} x}\Big). \tag{2}$$

The following result is a special case of a result of Wirsing [36], with a reformulation following Finch et al. [9, p. 2732]. As usual Γ will denote the gamma function. By χ_S we deonte the characteristic function of S, that is we put $\chi_S(n) = 1$ if n is in S and zero otherwise.

Theorem 1. *Let S be a multiplicative set satisfying Condtion A, then*

$$S(x) \sim C_0(S) x \log^{\delta-1} x,$$

where

$$C_0(S) = \frac{1}{\Gamma(\delta)} \lim_{P\to\infty} \prod_{p<P} \Big(1 + \frac{\chi_S(p)}{p} + \frac{\chi_S(p^2)}{p^2} + \dots\Big)\Big(1 - \frac{1}{p}\Big)^{\delta},$$

converges and hence is positive.

In case $S = S_B$ we have $\delta = 1/2$ by Dirichlet's prime number theoorem for arithmetic progressions. Recall that for fixed $r > 0$ this theorem states that

$$\pi(x; d, a) := \sum_{p\le x,\ p\equiv a (\mathrm{mod}\ d)} 1 = \frac{\mathrm{Li}(x)}{\varphi(d)} + O\Big(\frac{x}{\log^r x}\Big).$$

Theorem 1 thus gives that, asymptotically, $S_B(x) \sim C_0(S_B)x/\sqrt{\log x}$, a result derived in 1908 by Edmund Landau. Ramanujan, in his first letter to Hardy (1913), wrote in our notation that

$$S_B(x) = C_0(S_B) \int_2^x \frac{dt}{\sqrt{\log t}} + \theta(x), \tag{3}$$

with $\theta(x)$ very small. In reply to Hardy's question what 'very small' is in this context Ramanujan wrote back $O(\sqrt{x/\log x})$. (For a more detailed account and further references see Moree and Cazaran [20].) Note that by partial integration Ramanujan's claim, if true, implies the result of Landau. This leads us to the following defintion.

Definition 1. *Let S be a multiplicative set such that $\pi_S(x) \sim \delta x/\log x$ for some $\delta > 0$. If for all x sufficiently large*

$$|S(x) - C_0(S)x\log^{\delta-1} x| < |S(x) - C_0(S)\int_2^x \log^{\delta-1} dt|,$$

for every x sufficiently large, we say that the Landau approximation is better than the Ramanujan approximation. If the reverse inequality holds for every x sufficiently large, we say that the Ramanujan approximation is better than the Landau approximation.

We denote the formal Dirichlet series $\sum_{n=1,\ n\in S}^{\infty} n^{-s}$ associated to S by $L_S(s)$. For $\mathrm{Re}(s)>1$ it converges. If

$$\gamma_S := \lim_{s\to 1+0}\Big(\frac{L_S'(s)}{L_S(s)}+\frac{\delta}{s-1}\Big) \tag{4}$$

exists, we say that S has *Euler-Kronecker constant* γ_S. In case S consists of all positive integers we have $L_S(s)=\zeta(s)$ and it is well known that

$$\lim_{s\to 1+0}\Big(\frac{\zeta'(s)}{\zeta(s)}+\frac{1}{s-1}\Big)=\gamma. \tag{5}$$

If the multiplicative set S satisfies condtion B, then it can be shown that γ_S exists. Indeed, we have the following result.

Theorem 2. [19]. *If the multiplicative set S satisfies Condition B, then*

$$S(x)=C_0(S)x\log^{\delta-1}x\Big(1+(1+o(1))\frac{C_1(S)}{\log x}\Big),\qquad as\quad x\to\infty,$$

where $C_1(S)=(1-\delta)(1-\gamma_S)$.

Corollary 1. *Suppose that S is multiplicative and satisfies Condition B. If $\gamma_S<1/2$, then the Ramanujan approximation is asymptotically better than the Landau one. If $\gamma_S>1/2$ it is the other way around.*

The corollary follows on noting that by partial integration we have

$$\int_2^x \log^{\delta-1} dt = x\log^{\delta-1}x\Big(1+\frac{1-\delta}{\log x}+O\Big(\frac{1}{\log^2 x}\Big)\Big). \tag{6}$$

On comparing (6) with Theorem 2 we see Ramanujan's claim (3), if true, implies $\gamma_{S_B}=0$.

A special, but common case, is where the primes in the set S are, with finitely many exceptions, precisely those in a finite union of arithmetic progressions, that is, there exists a modulus d and integers $a_1,\ldots,a_s$ such that for all sufficiently large primes p we have $p\in S$ iff $p\equiv a_i(\mathrm{mod}\ d)$ for some $1\le i\le s$. (Indeed, all examples we consider in this paper belong to this special case.) Under this assumption it can be shown, see Serre [28], that $S(x)$ has an aysmptotic expansion in the sense of Poincaré, that is, for every integer $m\ge 1$ we have

$$S(x)=C_0(S)x\log^{\delta-1}x\Big(1+\frac{C_1(S)}{\log x}+\frac{C_2(S)}{\log^2 x}+\ldots+\frac{C_m(S)}{\log^m x}+O(\frac{1}{\log^{m+1}x})\Big), \tag{7}$$

where the implicit error term may depend on both m and S. In particular $S_B(x)$ has an expansion of the form (7) (see, e.g., Hardy [12, p. 63] for a proof).

2. On the numerical evaluation of γ_S

We discuss various ways of numerically approximating γ_S. A few of these approaches involve a generalization of the von Mangoldt function $\Lambda(n)$ (for more details see Section 2.2 of [20]).

We define $\Lambda_S(n)$ implicitly by

$$-\frac{L'_S(s)}{L_S(s)} = \sum_{n=1}^{\infty} \frac{\Lambda_S(n)}{n^s}. \tag{8}$$

As an example let us compute $\Lambda_S(n)$ in case $S = \mathbb{N}$. Since

$$L_{\mathbb{N}}(s) = \zeta(s) = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1},$$

we obtain $\log \zeta(s) = -\sum_p \log(1 - p^{-s})$ and hence

$$-\frac{L'_S(s)}{L_S(s)} = -\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1}.$$

We infer that $\Lambda_S(n) = \Lambda(n)$, the von Mangoldt function. Recall that

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^e; \\ 0 & \text{otherwise.} \end{cases}$$

In case S is a multiplicative semigroup generated by $q_1, q_2, \ldots \ldots$, we have

$$L_S(s) = \prod_i \Big(1 - \frac{1}{{q_i}^s}\Big)^{-1},$$

and we find

$$\Lambda_S(n) = \begin{cases} \log q_i & \text{if } n = q_i^e; \\ 0 & \text{otherwise.} \end{cases}$$

Note that S_B is a multiplicative semigroup. It is generated by 2, the primes $p \equiv 1 (\text{mod } 4)$ and the squares of the primes $p \equiv 3 (\text{mod } 4)$.

For a more general multiplicative set $\Lambda_S(n)$ can become more difficult in nature as we will now argue. We claim that (8) gives rise to the identity

$$\chi_S(n) \log n = \sum_{d|n} \chi_S(\frac{n}{d}) \Lambda_S(d). \tag{9}$$

In the case $S = \mathbb{N}$, e.g., we obtain $\log n = \sum_{d|n} \Lambda(d)$. In order to derive (9) we use the observation that if $F(s) = \sum f(n)n^{-s}$, $G(s) = \sum g(n)n^{-s}$ and $F(s)G(s) = H(s) = \sum h(n)n^{-s}$ are formal Dirichlet series, then h is the Dirichlet convolution of f and g, that is $h(n) = (f * g)(n) = \sum_{d|n} f(d)g(n/d)$. By an argument similar to the one that led us to the von Mangoldt function, one sees that $\Lambda_S(n) = 0$ in case n is not a prime power. Thus we can rewrite (9) as

$$\chi_S(n) \log n = \sum_{p^j|n} \chi_S(\frac{n}{d}) \Lambda_S(d). \tag{10}$$

By induction one then finds that $\Lambda_S(p^e) = c_S(p^e)\log p$, where $c_S(p) = \chi_S(p)$ and $c_S(p^e)$ is defined recursively for $e > 1$ by

$$c_S(p^e) = e\chi_S(p^e) - \sum_{j=1}^{e-1} c_s(p^j)\chi_S(p^{e-j}).$$

Also a more closed expression for $\Lambda_S(n)$ can be given ([20, Proposition 13]), namely

$$\Lambda_S(n) = e\log p \sum_{m=1}^{e} \frac{(-1)^{m-1}}{m} \sum_{k_1+k_2+\ldots+k_m=e} \chi_S(p^{k_1})\chi_S(p^{k_2})\cdots\chi_S(p^{k_m}),$$

if $n = p^e$ for some $e \geq 1$ and $\Lambda_S(n) = 0$ otherwise, or alternatively $\Lambda_S(n) = We\log p$, where

$$W = \sum_{l_1+2l_2+\ldots+el_e=e} \frac{(-1)^{l_1+\ldots+l_e-1}}{l_1+l_2+\ldots+l_e} \Big(\frac{l_1+l_2+\ldots+l_e}{l_1!l_2!\cdots l_e!}\Big)\chi_S(p)^{l_1}\chi_S(p^2)^{l_2}\cdots\chi_S(p^e)^{l_e},$$

if $n = p^e$ and $\Lambda_S(n) = 0$ otherwise, where the k_i run through the natural numbers and the l_j through the non-negative integers.

Now that we can compute $\Lambda_S(n)$ we are ready for some formulae expressing γ_S in terms of this function.

Theorem 3. *Suppose that S is a multiplicative set satisfying Condition B. Then*

$$\sum_{n\leq x} \frac{\Lambda_S(n)}{n} = \delta\log x - \gamma_S + O(\frac{1}{\log^\rho x}).$$

Moreover, we have

$$\gamma_S = -\delta\gamma + \sum_{n=1}^{\infty} \frac{\delta - \Lambda_S(n)}{n}.$$

In case S furthermore is a semigroup generated by $q_1, q_2, \ldots$, then one has

$$\gamma_S = \lim_{x\to\infty}\Big(\delta\log x - \sum_{q_i\leq x}\frac{\log q_i}{q_i-1}\Big).$$

The second formula given in Theorem 3 easily follows from the first on invoking the classical definition of γ:

$$\gamma = \lim_{x\to\infty}\Big(\sum_{n\leq x}\frac{1}{n} - \log x\Big).$$

Theorem 3 is quite suitable for getting an approximative value of γ_S. The formulae given there, however, do not allow one to compute γ_S with a prescribed numerical precision. For doing that another approach is needed, the idea of which is to relate the generating series $L_S(s)$ to $\zeta(s)$ and then take the logarithmic derivative. We illustrate this in Section 4 by showing how γ_{S_D} (defined in that section) can be computed with high numerical precision.

3. Non-divisibility of multiplicative arithmetic functions

Given a multiplicative arithmetic function f taking only integer values, it is an almost immediate obervation that, with q a prime, the set $S_{f;q} := \{n : q \nmid f(n)\}$ is multiplicative.

3.1. **Non-divisibility of Ramanujan's τ.** In his so-called 'unpublished' manuscript on the partition and tau functions [1, 3], Ramanujan considers the counting function of $S_{\tau;q}$, where $q \in \{3, 5, 7, 23, 691\}$ and τ is the Ramanujan τ-function. Ramanujan's τ-function is defined as the coefficients of the power series in q;

$$\Delta := q \prod_{m=1}^{\infty} (1 - q^m)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

After setting $q = e^{2\pi i z}$, the function $\Delta(z)$ is the unique normalized cusp form of weight 12 for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. It turns out that τ is a multiplicative function and hence the set $S_{\tau;q}$ is multiplicative. Given any such $S_{\tau;q}$, Ramanujan denotes $\chi_{S_{\tau;q}}(n)$ by t_n. He then typically writes: "It is easy to prove by quite elementary methods that $\sum_{k=1}^{n} t_k = o(n)$. It can be shown by transcendental methods that

$$\sum_{k=1}^{n} t_k \sim \frac{Cn}{\log^{\delta_q} n}; \tag{11}$$

and

$$\sum_{k=1}^{n} t_k = C \int_2^n \frac{dx}{\log^{\delta_q} x} + O\Big(\frac{n}{\log^r n}\Big), \tag{12}$$

where r is any positive number'. Ramanujan claims that $\delta_3 = \delta_7 = \delta_{23} = 1/2$, $\delta_5 = 1/4$ and $\delta_{691} = 1/690$. Except for $q = 5$ and $q = 691$ Ramanujan also writes down an Euler product for C. These are correct, except for a minor omission he made in case $q = 23$.

Theorem 4. ([17]). *For $q \in \{3, 5, 7, 23, 691\}$ we have $\gamma_{S_{\tau;q}} \neq 0$ and thus Ramamnujan's claim* (12) *is false for $r > 2$.*

The reader might wonder why this specific small set of q. The answer is that in these cases Ramanujan established easy congruences such as

$$\tau(n) \equiv \sum_{d|n} d^{11} (\text{mod } 691)$$

that allow one to easily describe the non-divisibility of $\tau(n)$ for these q. Serre, see [34], has shown that for every odd prime q a formula of type (11) exists, although no simple congruences as above exist. This result requires quite sophisticated tools, e.g., the theory of l-adic representations. The question that arises is whether $\gamma_{S_{\tau;q}}$ exists for every odd q and if yes, to compute it with enough numerical precision as to determine whether it is zero or not and to be able to tell whether the Landau or the Ramanujan approximation is better.

3.2. **Non-divisibility of Euler's totient function φ.** Spearman and Williams [32] determined the asymptotic behaviour of $S_{\varphi;q}(x)$. Here invariants from the cyclotomic field $\mathbb{Q}(\zeta_q)$ come into play. The mathematical connection with cyclotomic fields is not very direct in [32]. However, this connection can be made and in this way the results of Spearman and Williams can then be rederived in a rather straightforward way, see [10, 19]. Recall that the Extended Riemann Hypothesis (ERH) says that the Riemann Hypothesis holds true for every Dirichlet L-series $L(s,\chi)$.

Theorem 5. ([10]). *For $q \le 67$ we have $1/2 > \gamma_{S_{\varphi;q}} > 0$. For $q > 67$ we have $\gamma_{S_{\varphi;q}} > 1/2$. Furthermore we have $\gamma_{S_{\varphi;q}} = \gamma + O(\log^2 q/\sqrt{q})$, unconditionally with an effective constant, $\gamma_{S_{\varphi;q}} = \gamma + O(q^{\epsilon-1})$, unconditionally with an ineffective constant and $\gamma_{S_{\varphi;q}} = \gamma + O((\log q)(\log\log q)/q)$ if ERH holds true.*

The explicit inequalities in this result were first proved by the author [19], who established them assuming ERH. Note that the result shows that Landau wins over Ramanujan for every prime $q \ge 71$.

Given a number field K, the Euler-Kronecker constant $\mathcal{E}K_K$ of the number field K is defined as

$$\mathcal{E}K_K = \lim_{s\downarrow 1}\Big(\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s-1}\Big),$$

where $\zeta_K(s)$ denotes the Dedekind zeta-function of K. Given a prime $p \ne q$, let f_p the smallest positive integer such that $p^{f_p} \equiv 1 (\mathrm{mod}\ q)$. Put

$$S(q) = \sum_{p\ne q, f_p\ge 2} \frac{\log p}{p^{f_p}-1}.$$

We have

$$\gamma_{S_{\varphi;q}} = \gamma - \frac{(3-q)\log q}{(q-1)^2(q+1)} - S(q) - \frac{\mathcal{EK}_{\mathbb{Q}(\zeta_q)}}{q-1}. \tag{13}$$

(This is a consequence of Theorem 2 and Proposition 2 of Ford et al. [10].)

The Euler-Kronecker constants $\mathcal{E}K_K$ and in particular $\mathcal{EK}_{\mathbb{Q}(\zeta_q)}$ have been well-studied, see e.g. Ford et al. [10], Ihara [14] or Kumar Murty [23] for results and references.

4. Some Euler-Kronecker constants related to binary quadratic forms

Hardy [12, p. 9, p. 63] was under the misapprehension that for S_B Landau's approximation is better. However, he based himself on a computation of his student Gertrude Stanley [33] that turned out to be incorrect. Shanks proved that

$$\gamma_{S_B} = \frac{\gamma}{2} + \frac{1}{2}\frac{L'}{L}(1,\chi_{-4}) - \frac{\log 2}{2} - \sum_{p\equiv 3 (\mathrm{mod}\ 4)} \frac{\log p}{p^2-1}. \tag{14}$$

Various mathematicians independently discovered the result that

$$\frac{L'}{L}(1,\chi_{-4}) = \log\Big(M(1,\sqrt{2})^2 e^{\gamma}/2\Big),$$

where $M(1,\sqrt{2})$ denotes the limiting value of Lagrange's AGM algorithm $a_{n+1} = (a_n+b_n)/2$, $b_{n+1}=\sqrt{a_nb_n}$ with starting values $a_0=1$ and $b_0=\sqrt{2}$. Gauss showed (in his diary) that

$$\frac{1}{M(1,\sqrt{2})} = \frac{2}{\pi}\int_0^1 \frac{dx}{\sqrt{1-x^4}}.$$

The total arclength of the lemniscate $r^2 = \cos(2\theta)$ is given by $2l$, where $L = \pi/M(1,\sqrt{2})$ is the so-called lemniscate constant.

Shanks used these formulae to show that $\gamma_{S_B} = -0.1638973186345\ldots \neq 0$, thus establishing the falsity of Ramanujan's claim (3). Since $\gamma_{S_B} < 1/2$, it follows by Corollary 1 that actually the Ramanujan approximation is better.

A natural question is to determine the primitive binary quadratic forms $f(X,Y) = aX^2+bXY+cY^2$ of negative discriminant for which the integers represented form a multiplicative set. This does not seem to be known. However, in the more restrictive case where we require the multiplicative set to be also a semigroup the answer is known, see Earnest and Fitzgerald [7].

Theorem 6. *The value set of a positive definite integral binary quadratic form forms a semigroup if and only if it is in the principal class, i.e. represents 1, or has order 3 (under Gauss composition).*

In the former case, the set of represented integers is just the set of norms from the order $\mathfrak{O}_D$, which is multiplicative. In the latter case, the smallest example are the forms of discriminant -23, for which the class group is cyclic of order 3: the primes p are partitioned into those of the form $X^2-XY+6Y^2$ and those of the form $2X^2 \pm XY + 3Y^2$.

Although the integers represented by $f(X,Y)$ do not in general form a multiplicative set, the associated set I_f of integers represented by f, always satisfies the same type of asymptotic, namely we have

$$I_f(x) \sim C_f \frac{x}{\sqrt{\log x}}.$$

This result is due to Paul Bernays [2], of fame in logic, who did his PhD thesis with Landau. Since his work was not published in a mathematical journal it got forgotten and later partially rediscovered by mathematicians such as James and Pall. For a brief description of the proof approach of Bernays see Brink et al. [4].

We like to point out that in general the estimate

$$I_f(x) = C_f \frac{x}{\sqrt{\log x}}\Big(1 + (1+o(1))\frac{C_f'}{\log x}\Big)$$

does not hold. For example, for $f(X,Y) = X^2+14Y^2$, see Shanks and Schmid [31].

Bernays did not compute C_f, this was only done much later and required the combined effort of various mathematicians. The author and Osburn [21] combined these results to show that of all the two dimensional lattices of covolume 1, the hexagonal lattice has the fewest distances. Earlier Conway and Sloane [6] had identified the lattices with fewest distances in dimensions 3 to 8, also relying on the work of many other mathematicians.

In the special case where $f = X^2 + nY^2$, a remark in a paper of Shanks seemed to suggest that he thought C_f would be maximal in case $n = 2$. However, the maximum does not occur for $n = 2$, see Brink et al. [4].

In estimating $I_f(x)$, the first step is to count $B_D(x)$. Given a discriminant $D \le -3$ we let $B_D(x)$ count the number of integers $n \le x$ that are coprime to D and can be represented by some primitive quadratic integral form of discriminant D. The integers so represented are known, see e.g. James [15], to form a multiplicative semigroup, S_D, generated by the primes p with $(\frac{D}{p}) = 1$ and the squares of the primes q with $(\frac{D}{q}) = -1$. James [15] showed that we have

$$B_D(x) = C(S_D)\frac{x}{\sqrt{\log x}} + O(\frac{x}{\log x}).$$

An easier proof, following closely the ideas employed by Rieger [25], was given by Williams [35]. The set of primes in S_D has density $\delta = 1/2$. By the law of quadratic reciprocity the set of primes p satisfying $(\frac{D}{p}) = 1$ is, with finitely many exceptions, precisely a union of arithmetic progressions. It thus follows that Condition B is satisfied and, moreover, that for every integer $m \ge 1$, we have an expansion of the form

$$B_D(x) = C(S_D)\frac{x}{\sqrt{\log x}}\Big(1 + \frac{b_1}{\log x} + \frac{b_2}{\log^2 x} + \cdots + O(\frac{1}{\log^m x})\Big).$$

By Theorem 2 and Theorem 3 we infer that $b_1 = (1 - \gamma_{S_D})/2$, with

$$\gamma_{S_D} = \lim_{x\to\infty}\Big(\frac{\log x}{2} - \sum_{p\le x,\ (\frac{D}{p})=1}\frac{\log p}{p-1}\Big) - \sum_{(\frac{D}{p})=-1}\frac{2\log p}{p^2-1}.$$

As remarked earlier, in order to compute γ_{S_D} with some numerical precision the above formula is not suitable and another route has to be taken.

Proposition 1. ([15].) *We have, for Re(s) > 1,*

$$L_{S_D}(s)^2 = \zeta(s)L(s,\chi_D)\prod_{(\frac{D}{p})=-1}(1-p^{-2s})^{-1}\prod_{p|D}(1-p^{-s}).$$

Proof. On noting that

$$L_{S_D}(s) = \prod_{(\frac{D}{p})=1}(1-p^{-s})^{-1}\prod_{(\frac{D}{p})=-1}(1-p^{-2s})^{-1},$$

and

$$L(s,\chi_D) = \prod_{(\frac{D}{p})=1}(1-p^{-s})^{-1}\prod_{(\frac{D}{p})=-1}(1+p^{-s})^{-1},$$

the proof follows on comparing Euler factors on both sides. □

Proposition 2. *We have*

$$2\gamma_{S_D} = \gamma + \frac{L'}{L}(1,\chi_D) - \sum_{(\frac{D}{p})=-1}\frac{2\log p}{p^2-1} + \sum_{p|D}\frac{\log p}{p-1}.$$

Proof. Follows on logarithmically differentiating the expression for $L_{S_D}(s)^2$ given in Proposition 1, invoking (5) and recalling that $L(1,\chi_D)\neq 0$. □

The latter result together with $b_1=(1-\gamma_{S_D})/2$ leads to a formula first proved by Heupel [13] in a different way.

The first sum appearing in Proposition 2 can be evaluated with high numerical precision by using the identity

$$\sum_{(\frac{D}{p})=-1}\frac{2\log p}{p^2-1}=\sum_{k=1}^{\infty}\Big(\frac{L'}{L}(2^k,\chi_D)-\frac{\zeta'}{\zeta}(2^k)-\sum_{p|D}\frac{\log p}{p^{2^k}-1}\Big). \tag{15}$$

This identity in case $D=-3$ was established in [18, p. 436]. The proof given there is easily generalized. An alternative proof follows on combining Proposition 3 with Proposition 4.

Proposition 3. *We have*

$$\sum_p\frac{(\frac{D}{p})\log p}{p-1}=-\frac{L'}{L}(1,\chi_D)+\sum_{k=1}^{\infty}\Big(-\frac{L'}{L}(2^k,\chi_D)+\frac{\zeta'}{\zeta}(2^k)+\sum_{p|D}\frac{\log p}{p^{2^k}-1}\Big).$$

Proof. This is Lemma 12 in Cilleruelo [5]. □

Proposition 4. *We have*

$$-\sum_p\frac{(\frac{D}{p})\log p}{p-1}=\frac{L'}{L}(1,\chi_D)+\sum_{(\frac{D}{p})=-1}\frac{2\log p}{p^2-1}.$$

Proof. Put $G_d(s)=\prod_p(1-p^{-s})^{(D/p)}$. We have

$$\frac{1}{G_d(s)}=L(s,\chi_D)\prod_{(\frac{D}{p})=-1}(1-p^{-2s}).$$

The result then follows on logarithmic differentiation of both sides of the identity and the fact that $L(1,\chi_D)\neq 0$. □

The terms in (15) can be calculated with MAGMA with high precision and the series involved converge very fast. Cilleruelo [5] claims that

$$\sum_{k=1}^{\infty}\frac{L'}{L}(2^k,\chi_D)=\sum_{k=1}^{6}\frac{L'}{L}(2^k,\chi_D)+\text{Error},\ |\text{Error}|\le 10^{-40}.$$

We will now rederive Shanks' result (14). Since there is only one primitive quadratic form of discriminant -4, we see that S_{-4} is precisely the set of odd integers that can be written as a sum of two squares. If m is an odd integer that can be written as a sum of two squares, then so can $2^e m$ with $e\ge 0$ arbitrary. It follows that $L_{S_B}(s)=(1-2^{-s})^{-1}L_{S_{-4}}(s)$ and hence $\gamma_{S_B}=\gamma_{S_{-4}}-\log 2$. On invoking Proposition 2 one then finds the identity (14).

5. Integers composed only of primes in a prescribed arithmetic progression

Consider an arithmetic progression having infinitely many primes in it, that is consider the progression $a, a+d, a+2d, \ldots$ with a and d coprime. Let $S'_{d;a}$ be the multiplicative set of integers composed only of primes $p \equiv a(\text{mod } d)$. Here we will only consider the simple case where $a = 1$ and $d = q$ is a prime number. This problem is very closely related to that in Section 3.2. One has $L_{S'_{\varphi;q}}(s) = (1+q^{-s})\prod_{\substack{p\equiv 1(\text{mod } q)\\ p\neq q}}(1-p^{-s})^{-1}$. Since $L_{S'_{q;1}}(s) = \prod_{p\equiv 1(\text{mod } q)}(1-p^{-s})^{-1}$, we then infer that

$$L_{S'_{\varphi;q}}(s)L_{S'_{q;1}}(s) = \zeta(s)(1-q^{-2s})$$

and hence

$$\begin{aligned}\gamma_{S'_{q;1}} &= \gamma - \gamma_{S'_{\varphi;q}} + \frac{2\log q}{q^2-1} \\ &= \frac{\log q}{(q-1)^2} + S(q) + \frac{\mathcal{E}\mathcal{K}_{\mathbb{Q}(\zeta_q)}}{q-1},\end{aligned} \tag{16}$$

where the latter equality follows by identity (13). By Theorem 5, (16) and the Table in Ford et al. [10], we then arrive after some easy analysis at the following result.

Theorem 7. *For $q \le 7$ we have $\gamma_{S'_{q;1}} > 0.5247$. For $q > 7$ we have $\gamma_{S'_{q;1}} < 0.2862$. Furthermore we have $\gamma_{S'_{q;1}} = O(\log^2 q/\sqrt{q})$, unconditionally with an effective constant, $\gamma_{S'_{q;1}} = O(q^{\epsilon-1})$, unconditionally with an ineffective constant and $\gamma_{S'_{q;1}} = O((\log q)(\log\log q)/q)$ if ERH holds true.*

6. Multiplicative set races

Given two multiplicative sets S_1 and S_2, one can wonder whether for every $x \ge 0$ we have $S_1(x) \ge S_2(x)$. We give an example showing that this question is not as far-fetched as one might think at first sight. Schmutz Schaller [27, p. 201], motivated by considerations from hyperbolic geometry, conjectured that the hexagonal lattice is better than the square lattice, by which he means that $S_B(x) \ge S_H(x)$ for every x, where S_H is the set of squared distances occurring in the hexagonal lattices, that is the integers represented by the quadratic form $X^2 + XY + Y^2$. It is well-known that the numbers represented by this form are the integers generated by the primes $p \equiv 1(\text{mod } 3)$, 3 and the numbers p^2 with $p \equiv 2(\text{mod } 3)$. Thus S_H is a multplicative set. If $0 < h_1 < h_2 < \ldots$ are the elements in ascending order in S_H and $0 < q_1 < q_2 < \ldots$ the elements in ascending order in S_B, then the conjecture can also be formulated (as Schmutz Schaller did) as $q_j \le h_j$ for every $j \ge 1$. Asymptotically one easily finds that

$$S_B(x) \sim C_0(S_B)\frac{x}{\sqrt{\log x}},\ S_H(x) \sim C_0(S_H)\frac{x}{\sqrt{\log x}},$$

with $C_0(S_B) \approx 0.764$ the Landau-Ramanujan constant (see Finch [8, Section 2.3]) and $C_0(S_H) \approx 0.639\cdots$. It is thus clear that asymptotically the conjecture holds true. However, if one wishes to make the above estimates effective, matters become much more complicated. Nonetheless, the author, with computational help of H. te Riele, managed to establish the conjecture of Schmutz Schaller.

Theorem 8. [22]. *If S_B races against S_H, S_B is permanently ahead, that is, we have $S_B(x) \geq S_H(x)$ for every $x \geq 0$.*

Many of the ideas used to establish the above result were first developed in [18]. There some other multiplicative set races where considered. Given coprime positive integers a and d, let $S'_{d;a}$ be the multiplicative set of integers composed only of primes $p \equiv a(\mathrm{mod}\ d)$. The author established the following result as a precursor to Theorem 8.

Theorem 9. [18]. *For every $x \geq 0$ we have $S'_{3;2}(x) \geq S'_{3;1}(x)$, $S'_{4;3}(x) \geq S'_{3;1}(x)$, $S'_{3;2}(x) \geq S'_{4;1}(x)$ and $S'_{4;3} \geq S'_{4;1}(x)$.*

We like to point out that in every race mentioned in the latter result, the associated prime number races have no ultimate winner. For example, already Littlewood [16] in 1914 showed that $\pi_{S'_{3;2}}(x) - \pi_{S'_{3;1}}(x)$ has infinitely many sign changes. Note that trivially if $\pi_{S'_{d_1;a_1}}(x) \geq \pi_{S'_{d_2;a_2}}(x)$ for every $x \geq 0$, then $S'_{d_1;a_1}(x) \geq S'_{d_2;a_2}(x)$ for every $x \geq 0$.

See Granville and Martin [11] for a nice introduction to prime number races.

7. Exercises

`Exercise 1`. The non-hypotenuse numberss $n = 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14, 16, \ldots$ are those natural numbers for which there is no solution of $n^2 = u^2 + v^2$ with $u > v > 0$ integers. The set S_{NH} of non-hypotenuse numbers forms a multiplicative set that is generated by 2 and all the primes $p \equiv 3(\mathrm{mod}\ 4)$. Show that $L_{NH}(s) = L_{S_B}(s)/L(s, \chi_{-4})$ and hence

$$2\gamma_{NH} = 2\gamma_{S_B} - 2\frac{L'}{L}(1, \chi_{-4}) = \gamma - \log 2 + \sum_{p>2} \frac{(\frac{-1}{p}) \log p}{p-1}.$$

`Remark`. Put $f(x) = X^2 + 1$. Cilleruelo [5] showed that, as n tends to infinity,

$$\log \mathrm{l.c.m.}(f(1), \ldots, f(n)) = n \log n + Jn + o(n),$$

with

$$J = \gamma - 1 - \frac{\log 2}{2} - \sum_{p>3} \frac{(\frac{-1}{p}) \log p}{p-1} = -0.0662756342\ldots$$

We have $J = 2\gamma - 1 - \frac{3}{2}\log 2 - 2\gamma_{NH}$.

Recently the error term $o(n)$ has been improved by Rué et al. [26] to

$$O_\epsilon\left(\frac{n}{\log^{4/9-\epsilon} n}\right),$$

with $\epsilon > 0$.

Exercise 2. Let S'_D be the semigroup generated by the primes p with $(\frac{D}{p}) = -1$. It is easy to see that $L_{S'_D}(s)^2 = L_{S_D}(s)^2 L(s,\chi_D)^{-2}$ and hence, by Proposition 2, we obtain

$$\begin{aligned} 2\gamma_{S'_D} &= 2\gamma_{S_D} - 2\frac{L'}{L}(1,\chi_D) \\ &= \gamma - \frac{L'}{L}(1,\chi_D) - \sum_{(\frac{D}{p})=1} \frac{2\log p}{p^2-1} + \sum_{p|D} \frac{\log p}{p-1} \\ &= \gamma + \sum_{p} \frac{(\frac{D}{p})\log p}{p-1} + \sum_{p|D} \frac{\log p}{p-1}. \end{aligned}$$

Table : Overview of Euler-Kronecker constants discussed in this paper

set	γ_{set}	winner	reference
$n = a^2 + b^2$	$-0.1638\ldots$	Ramanujan	[29]
non-hypotenuse	$-0.4095\ldots$	Ramanujan	[30]
$3 \nmid \tau$	$+0.5349\ldots$	Landau	[17]
$5 \nmid \tau$	$+0.3995\ldots$	Ramanujan	[17]
$7 \nmid \tau$	$+0.2316\ldots$	Ramanujan	[17]
$23 \nmid \tau$	$+0.2166\ldots$	Ramanujan	[17]
$691 \nmid \tau$	$+0.5717\ldots$	Landau	[17]
$q \nmid \varphi$, $q \le 67$	< 0.4977	Ramanujan	[10]
$q \nmid \varphi$, $q \ge 71$	> 0.5023	Landau	[10]
$S'_{q;1}$, $q \le 7$	> 0.5247	Landau	Theorem 7
$S'_{q;1}$, $q > 7$	< 0.2862	Ramanujan	Theorem 7

Acknowledgement. I like to thank Andrew Earnest and John Voight for helpful information concerning qudaratic forms having a value set that is multiplicative, and Ana Zumalacárregui for sending me [26]. Bruce Berndt kindly commented on an earlier version and sent me an early version of [1].

This paper was written for the Newsletter of the Ramanujan Mathematical Society on request of B. Sury.

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