Cyclic p-groups and semi-stable reduction of curves in equal characteristic $p > 0$. I

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Abstract

In this paper we study the semi-stable reduction of p and p^2-cyclic covers of curves in equal characteristic $p > 0$. The main tool we use is the classical Artin-Schreier-Witt theory for p^n-cyclic covers in characteristic p. Although the results of this paper concern only the cases of degree p and p^2-cyclic cover we develop the techniques and the framework in which the general cyclic case can be studied.

0. Introduction. This paper is a further attempt to understand the degeneration of wildly ramified Galois covers of algebraic varieties. Let R be a complete discrete valuation ring with fraction field K and residue field k of characteristic $p > 0$. Let X be a smooth (or more generally semi-stable) R-scheme and let $Y \to X$ be a finite Galois cover with group G. The basic problem we are interested in is to understand the reduction of Y and in particular its semi-stable reduction if it exists (e.g. case of curves). The case where the cardinality of G is prime to p is by now classical and well understood (cf. [SGA-1], and [S-2] for the case of semi-stable curves). Although the case of higher dimensional semi-stable schemes is not available in the literature it can be developed using the ideas in [S-2] and higher dimensional formal patching techniques. The case where the cardinality of G is divisible by p is much less understood and presents more technical difficulties du to a phenomena of wild ramification that appears. Raynaud first attempted to understand this situation in [R] where a rather general "qualitative" result is proven in the case where G is a p-group and X is a proper and smooth curve. Beside this the only case which is "explicitly" understood is the basic case where G is cyclic of order p and the ring R has inequal characteristics (cf. [G-Ma], [H], [L], [Ma], [S], [S-1], [S-3]...). Though one also has some results in inequal characteristics in the special case where p strictly divides the cardinality of G (cf. [R-1] and [We]): this case is essencially a by-product of the above discussed cases. One of the main objectives of this work is to go beyond these basic cases. In this paper we study the case where G is a cyclic p-group and R has equal characteristics. The above situation in equal characteristics p has been inexplored so far despite the important potential applications of such a theory (cf. explanations below). Though this case is in principle easier than the case of inequal characteritics, but still the two situations are well related, as it is known to experts (this is part of the philosophy that the Artin-Schreier-Witt theory can be viewed as a degeneration of the Kummer theory). Also the results in equal characteristics, beside their own interest, can shed some lights on the situation in inequal characteristics. Although the results stated in this paper concern only the cases of degree p and p^2-cyclic cover we develop, pursueing the ideas and the set-up developed in [S], [S-1], [S-3], the techniques and the framework in which the general p^n-cyclic case can be studied. In what follows we review the content and results of this paper.

In section I we recall the classical Artin-Schreier-Witt theory, which is the main tool we use in this paper, and which provides (generically) explicit equations for p^n-cyclic covers in characteristic p. In section II we consider the following situation: let X be a formal R-scheme of finite type which is normal connected and flat over R, with geometrically integral fibres, and let $f : Y \to X$ be a p^n-cyclic cover, with Y normal, which is an étale torsor on the generic fibre of X. We assume that the special fibre Y_k of Y is reduced. We are interested in describing the map f and its special fibre $f_k : Y_k \to X_k$. One of our main results is the following:

Theorem 2.2.1. *Assume* $\deg(f) = p$. *Then the cover* $f : Y \to X$ *has the structure of a torsor under a finite and flat* R*-group scheme of rank* p *over* X.

Moreover we give an explicit description of the group schemes which appear as the group of the torsor in this situation (cf. 2.1) and in particular we provide "integral" equations for the torsor f which also provide (by reduction) equations for its special fibre. Next we study the case of p^2-cyclic covers. Our main result is theorem 2.4.3. One of the main consequences of 2.4.3 is that the analog of the above theorm doesn't hold: namely one can find (generic) examples of f as above of degree p^2 which doesn't have the structure of a torsor under a finite and flat R-group scheme of rank p^2 over X (cf. 2.4.5). Such examples were not known in the literature before. We are however able in 2.4.3 to find "integral" equations for f which provide by reduction equations for its special fibre $f_k : Y_k \to X_k$. In other terms we explain how the Artin-Schreier-Witt equations of degree p^2 degenerate. The proof of 2.4.3 is rather involved and uses the technical lemma 2.4.2. Moreover we exhibit the cases as above where f has the structure of a torsor under a finite and flat R-group scheme of rank p^2 over X (cf. 2.4.3 and 2.4.4), in which case we explicit the group schemes which appear as groups of the torsor. These group schemes are basically obtained by "twisting" the Artin-Schreier-Witt theory (cf. 2.3 for more details). We are also able to associate some degeneration data to the cover f which determine explicitly the cover

f_k (cf. 2.4.8). These degeneration data will play an important role in VI were we are able to reconstruct p^2-cyclic covers of formal germs of R-curves from the " degeneration data". Finally we classify in II p and p^2-cyclic covers above boundaries of formal germs of R-curves depending on their reduction type (cf. 2.5.1 and 2.6.1).

In section III we develop in equal characteristic p the technique of computation of vanishing cycles as initiated in [S-1] (cf. also [Ma-Y]) and which is based on the technique of compactification of covers between formal germs of R-curves (cf. 3.2). Our main result is theorem 3.2.3 which gives an explicit formula which compares the dimensions of the spaces of vanishing cycles in a p-cyclic cover between formal germs of R-curves and which depends on the degeneration type of the cover on the boundaries. This formula can easily be generalised to the case of p^2-cyclic covers (cf. proposition 4.1.1). We also study examples of p and p^2-cyclic covers above formal germs of semi-stable curves (i.e. above a smooth or a double point). Theses examples play an important role in the next sections V and VI. We are for example able to classify p and p^2-cyclic covers $f : \mathcal{Y} \to \mathcal{X}$ between formal germs of double points wich are étale above the generic fibre $\mathcal{X}_K$ of $\mathcal{X}$ (cf. 3.3.9 and 4.2.11).

In section V and VI we consider the following local situation: let $f : \mathcal{Y} \to \mathcal{X}$ be a cyclic cover of degree p or p^2 above the formal germ $\mathcal{X} \simeq \operatorname{Spf} R[[T]]$ of an R-curve at a smooth point. We study the semi-stable reduction of f. We are able to exhibit in 5.2.2 and 6.2.1 "degeneration" data which completly determine the geometry of a semi-stable model of $\mathcal{Y}$. These data are of geometric and combinatorial nature and leave over the residue field k of R (cf. 5.2.2 and 6.2.1 for more details). One of the main results of this paper is that we are able to reconstruct the cover f from these data. More precisely let **Deg** be the set of isomorphism classes of the degeneration datas or rank p (resp. rank p^2) as defined in 6.2.1 (resp. 5.2.2) and let L be the function field of the geometric fibre of $\mathcal{X}$. Then we prove using the results in III and IV the following:

Theorem. 5.2.4, 6.2.3 *Let G be a cyclic p-group of order p (resp. of order p^2). Then there exists a canonical specialisation map* $\operatorname{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, G) \to \mathbf{Deg}$ *which is surjective.*

Although the above result is local the techniques and results we use in the proof of 5.2.4 and 6.2.3 can be used to prove a global result of lifting for finite covers of degree p or p^2 between semi-stable curves (such a result is proven in [M] for p-covers).

Finally let's mention some potential applications of this work. The results of this paper can be used in order to determine the semi-stable reduction of the Drinfeld modular curves and that of Galois covers of the projective line in characteristic p with Galois groups having a cyclic p-Sylow subgroup of order p or p^2. These results also can be used in the study of automorphism groups of curves in positive characteritics. One of the motivations of this work was also to explore the possibility of defining reasonable Hurwitz spaces for wildly ramified Galois covers of curves. It seems that the only case where such a definition is possible is the case of (Galois) covers of degree p: the right moduli problem to classify being torsors of rank p. Unfortunately our results show that when we degenerate Galois covers of degree higher than p then we loose in general the torsor structure. this makes it really difficult to imagine a reasonable definition of what the Hurwitz spaces should be in general. May be the right point of view to adopt here is to restrict simply to covers where one keeps the torsor structure.

Although the results of this paper concern only the cyclic groups of order p and p^2 they can be generalised, using the set-up and techniques we develop, to the general p^n-cyclic case. I hope to come back to this question in the future. This work was done during my visit to the Max-Planck-Institut Für Mathematik in Bonn. I would like very much to thank the directors of the Institut for their invitation and for the wonderful working atmosphere.

I. Artin-Schreier-Witt theory of p^n-cyclic covers in characteristic p.

1.1. In this section we review the Artin-Schreier-Witt theory (first developed in [W]) which provides explicit equations describing cyclic covers of degeree p^n in characteristic p. We refer the reader to the modern treatment of the theory as in [D-G].

Let X be a scheme of characteristic p and denote by X_{et} the étale site on X. Let $n > 0$ be an integer. We denote by $W_{n,X}$ (or simply W_n if there is no confusion) the sheaf of Witt vectors of length n on X_{et} (cf. [D-G], chapitre 5, 1). In the sequel any addition or substraction of Witt vectors will mean the addition and substarction in the sense of Witt theory. We denote by F the Frobenius endomorphism of $W_{n,X}$ which is locally defined by $F.(x_1, x_2, ..., x_n) = (x_1^p, x_2^p, ..., x_n^p)$, for a Witt vector $(x_1, x_2, ..., x_n)$ of length n, and by Id the identity automorphism of $W_{n,X}$. The following sequence is exact on X_{et}:

$$(1)\quad 0 \to (\mathbb{Z}/p^n\mathbb{Z})_X \xrightarrow{i_n} W_{n,X} \xrightarrow{\mathrm{F}-\mathrm{Id}} W_{n,X} \to 0$$

where $(\mathbb{Z}/p^n\mathbb{Z})_X$ denotes the constant sheaf $(\mathbb{Z}/p^n\mathbb{Z})$ on X_{et} and i_n is the natural monomorphism which applies $1 \in \mathbb{Z}/p^n\mathbb{Z}$ to $1 \in W_n$ (cf [G-D], chapitre 5, 5.4). From the long cohomology exact sequence associated to (1) one deduces the following exact sequence:

$$(2)\quad W_{n,X}(X) \xrightarrow{\mathrm{F}-\mathrm{Id}} W_{n,X}(X) \to H^1_{\text{et}}(X, \mathbb{Z}/p^n\mathbb{Z}) \to H^1_{\text{et}}(X, W_n) \xrightarrow{\mathrm{F}-\mathrm{Id}} H^1_{\text{et}}(X, W_n)$$

Assume that $X = \operatorname{Spec} A$ is affine in which case we have $H^1_{\text{et}}(\operatorname{Spec} A, W_n) = 0$ and hence an isomorphism: $H^1_{\text{et}}(\operatorname{Spec} A, \mathbb{Z}/p^n\mathbb{Z}) \simeq W_{n,A}(A)/(\mathrm{F}-\mathrm{Id})(W_{n,A}(A))$. This isomorphism has the following interpretation: to an étale $\mathbb{Z}/p^n\mathbb{Z}$-torsor $f : Y \to X = \operatorname{Spec} A$

above X corresponds a Witt vector $(a_1, a_2, ..., a_n) \in W_{n,A}(A)$ of length n which is uniquely determined modulo addition of elements of the form $F.(b_1, b_2, ..., b_n) - (b_1, b_2, ..., b_n)$. Moreover the equations $F.(x_1, x_2, ..., x_n) - (x_1, x_2, ..., x_n) = (a_1, a_2, ..., a_n)$ where the x_i are indeterminates are equations for the torsor f. More precisely there is a canonical factorisation of f as $Y = Y_n \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} ... \xrightarrow{f_2} Y_1 \xrightarrow{f_1} Y_0 := X$ where each $Y_i = \operatorname{Spec} B_i$ is affine and $f_i : Y_i := \operatorname{Spec} B_i \to f_{i-1} := \operatorname{Spec} B_{i-1}$ is the étale $\mathbb{Z}/p\mathbb{Z}$-torsor corresponding to the algebra extension $B_{i+1} := B_i[x_i]$. In the general case (where $H^1_{\mathrm{et}}(X, W_n) \neq 0$) the above equations provide local equations for an étale $\mathbb{Z}/p^n\mathbb{Z}$-torsor in characteristic p.

1.2. Examples. In what follows X is a scheme of characteristic p.

1.2.1. $\mathbb{Z}/p\mathbb{Z}$-Torsors. Let $f : Y \to X$ be an étale $\mathbb{Z}/p\mathbb{Z}$-torsor. Then locally f is given by an equation $x^p - x = a$ where a is a regular function on X which is uniquely defined up to addition of elements of the form $b^p - b$ for some regular function b.

1.2.2. $\mathbb{Z}/p^2\mathbb{Z}$-Torsors. Let $f : Y \to X$ be an étale $\mathbb{Z}/p^2\mathbb{Z}$-torsor. Then we have a canonical factorisation of f as: $Y =: Y_2 \xrightarrow{f_2} Y_1 \xrightarrow{f_1} X$ where f_2 and f_1 are étale $\mathbb{Z}/p\mathbb{Z}$-torsors. The torsor f is locally given by equations of the form:

$$F.(x_1, x_2) - (x_1, x_2) := (x_1^p - x_1, x_2^p - x_2 - p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} x_1^{pk} (-x_1)^{p-k}) = (a_1, a_2)$$

for some regular functions a_1 and a_2 on X and the Witt vector (a_1, a_2) is uniquely determined up to addition (in the Witt theory) of vectors of the form:

$$(b_1^p, b_2^p) - (b_1, b_2) := (b_1^p - b_1, b_2^p - b_2 - p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} b_1^{pk} (-b_1)^{p-k})$$

Thus locally the torsor f_1 is defined by the equation:

$$x_1^p - x_1 = a_1$$

and f_2 by the equation:

$$x_2^p - x_2 = a_2 + p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} x_1^{pk} (-x_1)^{p-k}$$

Moreover if we replace the vector (a_1, a_2) by the vector $(a_1, a_2) + (b_1^p, b_2^p) - (b_1, b_2)$ the above equations are replaced by:

$$x_1^p - x_1 = a_1 + b_1^p - b_1$$

and:

$$x_2^p - x_2 = a_2 + b_2^p - b_2 + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k} x_1^{pk}(-x_1)^{p-k} - p^{-1}\sum_{k=1}^{p-1}\binom{p}{k} b_1^{pk}(-b_1)^{p-k} -$$

$$p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}(b_1^p - b_1)^k (a_1)^{p-k}$$

respectively.

II. Degeneration of p and p^2-cyclic covers in equal characteristic $p > 0$.

In all this paragraph we use the following notations: R is a complete discrete valuation ring of equal characteristic $p > 0$ with residue field k and fraction field $K := \operatorname{Fr} R$. We denote by π a uniformising parameter of R.

2.1. The group schemes $\mathcal{M}_n$ (cf. also [M], 3.2). Let $n \geq 0$ be an integer and let $\mathbb{G}_{a,R} = \operatorname{Spec} R[X]$ be the additive group scheme over R. The map:

$$\phi_n : \mathbb{G}_{a,R} \to \mathbb{G}_{a,R}$$

given by:

$$X \to X^p - \pi^{(p-1)n} X$$

is an isogeny of group schemes. The kernel of ϕ_n is denoted by $\mathcal{M}_{n,R} := \mathcal{M}_n$. We have $\mathcal{M}_n := \operatorname{Spec} R[X]/(X^p - \pi^{(p-1)n}X)$ and $\mathcal{M}_n$ is a finite and flat R-group scheme of rank p. Further the following sequence is exact in the fppf topology:

$$(3) \quad 0 \to \mathcal{M}_n \to \mathbb{G}_{a,R} \xrightarrow{\phi_n} \mathbb{G}_{a,R} \to 0$$

If $n = 0$ then the sequence (3) is the Artin-Schrier sequence which is exact in the étale topology and $\mathcal{M}_0$ is the étale constant group scheme $(\mathbb{Z}/p\mathbb{Z})_R$. If $n > 0$ the sequence (3) has a generic fibre which is isomorphic to the étale Artin-Schreier sequence and a special fibre isomorphic to the radicial exact sequence:

$$(4) \quad 0 \to \alpha_{p,k} \to \mathbb{G}_{a,k} \xrightarrow{x^p} \mathbb{G}_{a,k} \to 0$$

Thus if $n > 0$ the group scheme $\mathcal{M}_n$ has a generic fibre which is étale isomorphic to $(\mathbb{Z}/p\mathbb{Z})_K$ and its special fibre is isomorphic to the infinitesimal group scheme $\alpha_{p,k}$. Let X be an R-scheme. The sequence (3) induces a long cohomology exact sequence:

$$(5)\quad \mathbb{G}_{a,R}(X) \xrightarrow{\phi_n} \mathbb{G}_{a,R}(X) \to H^1_{\mathrm{fppf}}(X, \mathcal{M}_n) \to H^1_{\mathrm{fppf}}(X, \mathbb{G}_{a,R}) \xrightarrow{\phi_n} H^1_{\mathrm{fppf}}(X, \mathbb{G}_{a,R})$$

The cohomology group $H^1_{\mathrm{fppf}}(X, \mathcal{M}_n)$ classifies the isomorphism classes of fppf-torsors with group $\mathcal{M}_n$ above X. The above sequence allows the following description of $\mathcal{M}_n$-torsors: locally a torsor $f : Y \to X$ under the group scheme $\mathcal{M}_n$ is given by an equation $T^p - \pi^{(p-1)n}T = a$ where T is an indeterminate and a is a regular function on X, which is uniquely defined up to addition of elements of the form $b^p - \pi^{(p-1)n}b$, for some regular function b. In particular if $H^1_{\mathrm{fppf}}(X, \mathbb{G}_{a,R}) = 0$ (e.g. if X is affine) then an $\mathcal{M}_n$-torsor above X is globally defined by an equation as above.

2.2. Degeneration of étale $\mathbb{Z}/p\mathbb{Z}$-torsors. In what follows let X be a formal R-scheme of finite type which is normal connected and flat over R. Let $X_K := X \times_R K$ (resp. $X_k := X \times_R k$) be the generic (resp. special) fibre of X. By generic fibre of X we mean the associated K-rigid space (cf. [B-L]). We assume further that the special fibre X_k is integral. Let η be the generic point of the special fibre X_k and let $\mathcal{O}_\eta$ be the local ring at η which is a discrete valuation ring with fraction field $K(X)$: the function field of X. Let $f_K : Y_K \to X_K$ be a non trivial étale $\mathbb{Z}/p\mathbb{Z}$-torsor with Y_K geometrically connected and let $K(X) \to L$ be the corresponding extension of function fields. The main result of this section is the following:

2.2.1. Theorem. *Assume that the ramification index above $\mathcal{O}_\eta$ in the extension $K(X) \to L$ equals 1. Then the torsor $f_K : Y_K \to X_K$ extends to a torsor $f : Y \to X$ under a finite and flat R-group scheme of rank p with Y normal. Let δ be the degree of the different above η in the extension $K(X) \to L$. Then the following cases occur:*

a) *$\delta = 0$ in which case f is an étale torsor under the group scheme $\mathcal{M}_0$ and $f_k : Y_k \to X_k$ is then an étale $\mathbb{Z}/p\mathbb{Z}$-torsor.*

b) *$\delta > 0$ in which case $\delta = n(p-1)$ for a certain integer $n \geq 1$ and f is a torsor under the group scheme $\mathcal{M}_n$. Further $f_k : Y_k \to X_k$ is a radicial torsor under the k-group scheme α_p.*

Note that starting from a torsor $f_K : Y_K \to X_K$ as in 2.2.1 the condition that the ramification index above $\mathcal{O}_\eta$ equals 1 is always satisfied after eventually a finite extension of R (cf. e.g. [E]).

Proof. We denote by v the discrete valuation of $K(X)$ corresponding to the valuation ring $\mathcal{O}_\eta$ which is normalised by $v(\pi) = 1$ (note that π is a uniformiser of $\mathcal{O}_\eta$). We first start with the special case where $H^1_{\mathrm{et}}(X_K, \mathbb{G}_{a,K}) = 0$. In this case the torsor f_K is given by an Artin-Schreier equation of the form $T^p - T = a_K$ where a_K is a regular function on X_K and we have $a_K = \pi^m a$ where $m \in \mathbb{Z}$ is an integer and a is a regular function on

X (i.e. a function with $v(a) = 0$). First note that necessarily $m \leq 0$ for if $m > 0$ then $a_K = b^p - b$ where $b = a\pi^m + (a\pi^m)^p + (a\pi^m)^{p^2} + ... + (a\pi^m)^{p^i} + ..$ (since X_K is complete for the π-adic topology the above expression converges) and this contradicts the fact that f_K is a non trivial torsor. If $m = 0$ then the equation $T^p - T = a$ defines an étale $\mathbb{Z}/p\mathbb{Z}$-torsor $f : Y \to X$ above X which coincides with f_K on the generic fibre and we are in the case a). In this case the étale torsor $f_k : Y_k \to X_k$ is given by the Artin-Schreier equation $T^p - T = \bar{a}$ where $\bar{a}$ is the image of a modulo π. Next we treat the case where $m < 0$. In this case $-m = np$ is necessarily divisible by p for otherwise the extension $K(X) \to L$ is totally ramified above $\mathcal{O}_\eta$. Assume now first that the image $\bar{a}$ of a modulo π via the canonical map $\Gamma(X, \mathcal{O}(X)) \to \Gamma(X, \mathcal{O}(X))/\pi\Gamma(X, \mathcal{O}(X))$ is not a p-power. Consider the cover $f : Y \to X$ given by the equation $\tilde{T}^p - \pi^{n(p-1)}\tilde{T} = a$. Then f is an fppf-torsor under the group scheme $\mathcal{M}_n$ which coicides with f_K on the generic fibre (consider the change of variables $T := \tilde{T}/\pi^n$) and its special fibre $f_k : Y_k \to X_k$ is the α_p-torsor given by the equation $t^p = \bar{a}$.

In the case where $\bar{a}$ is a p-power then two cases can occur: either $\bar{a}$ is a p^s-power for any positive integer s in which case necessarily $a \in R$ and the torsor f_K is just the extension of scalars of X_K from K to the extension K' of K defined by the equation $T^p - T = \pi^m a$ and this contradicts the assumption that Y_K is geometrically connected. Hence this case doesn't occur. So necessarily there exists a positive integer r such that $\bar{a}$ is a p^r power but not a p^{r+1} power. We assume for simplicity that $r = 1$ (the general case $r > 1$ is treated in a similar way). Let $\bar{a} = \bar{b}^p$ so that $a = b^p + \pi\tilde{b}$ where b and $\tilde{b}$ are functions on X_K and b is a function which reduces to $\bar{b}$ modulo π. Our equation is then of the form $T^p - T = (b/\pi^n)^p + \tilde{b}/\pi^{(pn-1)}$ and after adding $(b/\pi^n) - (b/\pi^n)^p$ to the right hand side which doesn't change the torsor f_K we get the equation $T^p - T = (b/\pi^n) + \tilde{b}/\pi^{(pn-1)}$ which can also be written in the form $T^p - T = (b/\pi^n) + b'/\pi^{n'}$ where b' is a function with $v(b') = 0$ and $n' \leq pn - 1$. If $n > n'$ then $n = ps$ is necessarily divisible by p and the equation $\tilde{T}^p - \pi^{s(p-1)}\tilde{T} = b + \pi^{n-n'}b'$ defines a torsor $f : Y \to X$ under the group scheme $\mathcal{M}_s$ which coincides with f_K on the generic fibre and its special fibre $f_k : Y_k \to X_k$ is the α_p-torsor given by the equation $\tilde{t}^p = \bar{b}$. In the case where $n' \geq n$ then $n' = s'p$ is necessarily divisible by p. In this case if $\bar{b}'$ (resp. $\bar{b}' + \bar{b}$ in case $n' = n$) is not a p power (where $\bar{b}$ and $\bar{b}'$ denote the reduction of b resp. b' modulo π) then the equation $\tilde{T}^p - \pi^{s'(p-1)}\tilde{T} = \pi^{n'-n}b + b'$ defines a torsor $f : Y \to X$ under the group scheme $\mathcal{M}_{s'}$ which coincides with f_K on the generic fibre and its special fibre $f_k : Y_k \to X_k$ is the α_p-torsor given by the equation $\tilde{t}^p = \bar{b}'$ (resp. $\tilde{t}^p = \bar{b}' + \bar{b}$ in case $n = n'$). Otherwise if $\bar{b}'$ (or $\bar{b}' + \bar{b}$ in case $n = n'$) is a p-power then we repeat the same procedure as above and since n and n' decrease at each step this process must stop at some step and we end up with an equation of the form $\tilde{T}^p - \pi^{r(p-1)}\tilde{T} = \tilde{b}$ where $\tilde{b}$ is a function whose reduction modulo π is not a p-power, for some positive integer r. Hence the required result.

The argument in the general case is similar to the one used in [S] proof of 2.4. More precisely in general there exists an open covering $(U_i)_i$ of X and regular functions $\tilde{a}_i \in \Gamma(U_{i,K}, \mathcal{O}_X)$ (where $U_{i,K} := U_i \times_R K$) which are defined up to addition of functions of the form $b_i^p - b_i$ and such that the torsor f_K is defined by the equation $T_i^p - T_i = \tilde{a}_i$ above $U_{i,K}$. Now the above discussion shows that after some modifications (of the type used above) the torsor f_K can be defined above each open $U_{i,K}$ by an equation $\tilde{T}_i - \pi^{n_i(p-1)}\tilde{T} = a_i$ for some uniquely detemined integer $n_i \geq 0$ such that in case $n_i > 0$ the image $\bar{a}_i$ of a_i modulo π is not a p-power and such that the degree of the different δ_i above the generic point η of $U_{i,k} := U_i \times_R k$ equals $n_i(p-1)$. From this we deduce that all $n_i =: n$ are equal. Then the $\mathcal{M}_n$-torsor $f : Y \to X$ given locally by the equation $\tilde{T}_i - \pi^{n(p-1)}\tilde{T}_i = a_i$ above the open U_i coincides on the generic fibre with the torsor f_K.

2.2.2. It follows from 2.2.1 that an étale $\mathbb{Z}/p\mathbb{Z}$-torsor above the generic fibre X_K of X induces canonically a *degeneration data* which is a torsor above the special fibre X_k of X under a finite and flat k-group scheme which is either étale or of type α_p. Reciprocally we have the following result of *lifting* of such a degeneration data.

2.2.3. Proposition. *Assume that X is* **affine**. *Let $f_k : Y_k \to X_k$ be a torsor under a finite and flat k-group scheme which is étale (resp. of type α_p). Then f_k can be lifted to a torsor $f : Y \to X$ under a finite and flat R-group scheme of rank p which is étale (resp. isomorphic to $\mathcal{M}_n$ where $n > 0$).*

Proof. Since X is affine the torsor f_k is given by an equation $x^p - x = \bar{a}$ where $\bar{a}$ is a regular function on X_k (resp. an equation $x^p = \bar{a}$ where $\bar{a}$ is a regular function on X_k). Let a be a regular function on X which reduces to $\bar{a}$ modulo π. The equation $X^p - X = a$ (resp. $X^p - \pi^{n(p-1)}X = a$ where $n > 0$ is an integer) defines a cover $f : Y \to X$ above X which has the structure of a torsor under the étale group scheme $(\mathbb{Z}/p\mathbb{Z})_R$ (resp. under the group scheme $\mathcal{M}_n$) and which clearly induces the torsor f_k on the special fibre X_k.

2.2.4. Remark. If X is no more affine one can find examples, where X is actually a proper and smooth R-curve, of an α_p-torsor above the special fibre X_k of X which can not be lifted to a torsor above X under a finite and flat R-group scheme of rank p which is étale above the generic fibre of X. If X_k is a proper and smooth curve over k such a lifting is however possible after eventually replacing X by another R-curve X' with special fibre X_k.

2.3. The group schemes W_{m_1,m_2}. Let m_1 and m_2 be two positive integers such that $m_2 - pm_1 \geq 0$. We define the *twisted R-Witt group scheme* W_{m_1,m_2} of length two as follows: scheme theoretically $W_{m_1,m_2} \simeq \mathbb{G}_{a,R}^2$ and the group law is defined by:

$$(x_1, x_2) + (y_1, y_2) := (x_1 + y_1, x_2 + y_2 - p^{-1}\pi^{m_2-pm_1} \sum_{k=1}^{p-1} \binom{p}{k} x_1^k y_1^{p-k})$$

The generic fibre $(W_{m_1,m_2})_K$ of W_{m_1,m_2} is isomorphic to the Witt group scheme $W_{2,K}$ via the map:

$$(W_{m_1,m_2})_K \to W_{2,K}$$

$$(x_1, x_2) \to (x_1/\pi^{m_1}, x_2/\pi^{m_2})$$

and its special fibre $(W_{m_1,m_2})_k$ is isomorphic either to the Witt group scheme $W_{2,k}$ if $m_2 - pm_1 = 0$ or to the group scheme $\mathbb{G}_{a,k}^2$ otherwise. Note finally that we have an exact sequence:

$$0 \to \mathbb{G}_a \xrightarrow{V} W_{m_1,m_2} \xrightarrow{R} \mathbb{G}_a \to 0$$

where $V : \mathbb{G}_a \to W_{m_1,m_2}$ is the *vershiebung* homomorphism defined by $V(x) = (0, x)$ and $R : W_{m_1,m_2} \to \mathbb{G}_a$ is the projection $R(x_1, x_2) = x_1$.

2.3.1. The group schemes $\mathcal{H}_{m_1,m_2}$. We use the same notations as in 2.3. The following maps I_{m_1,m_2} and F are group scheme homomorphisms:

$$I_{m_1,m_2} : W_{m_1,m_2} \to W_{pm_1,pm_2}$$

$$(x_1, x_2) \to (\pi^{m_1(p-1)}x_1, \pi^{m_2(p-1)}x_2)$$

and:

$$F : W_{m_1,m_2} \to W_{pm_1,pm_2}$$

$$(x_1, x_2) \to (x_1^p, x_2^p)$$

Consider the following isogeny:

$$\varphi_{m_1,m_2} := F - I_{m_1,m_2} : W_{m_1,m_2} \to W_{pm_1,pm_2}$$

which is given by:

$$(x_1, x_2) \to (x_1^p - \pi^{m_1(p-1)}x_1, x_2^p - \pi^{m_2(p-1)}x_2 - p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} \pi^{m_2p-m_1(pk+p-k)} x_1^{pk}(-x_1)^{p-k})$$

We define the group scheme $\mathcal{H}_{m_1,m_2}$ to be the kernel of the above isogeny. Thus we have an exact sequence:

$$(6) \quad 0 \to \mathcal{H}_{m_1,m_2} \to W_{m_1,m_2} \xrightarrow{F-I_{m_1,m_2}} W_{pm_1,pm_2} \to 0$$

and $\mathcal{H}_{m_1,m_2}$ is a finite and flat commutative R-group scheme of rank p^2. Further we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{M}_{m_2} \arrow[r] \arrow[d] & \mathbb{G}_a \arrow[r, "\phi_{m_2}"] \arrow[d] & \mathbb{G}_a \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{H}_{m_1,m_2} \arrow[r] \arrow[d] & W_{m_1,m_2} \arrow[r, "\varphi_{m_1,m_2}"] \arrow[d, "R"] & W_{pm_1,pm_2} \arrow[r] \arrow[d, "R"] & 0 \\
0 \arrow[r] & \mathcal{M}_{m_1} \arrow[r] \arrow[d] & \mathbb{G}_a \arrow[r, "\phi_{m_1}"] \arrow[d] & \mathbb{G}_a \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The group scheme $\mathcal{H}_{m_1,m_2}$ is then an extension of the group scheme $\mathcal{M}_{m_1}$ by $\mathcal{M}_{m_2}$. Its generic fibre $(\mathcal{H}_{m_1,m_2})_K$ is isomorphic to the étale constant group $\mathbb{Z}/p^2\mathbb{Z}$ and its special fibre $(\mathcal{H}_{m_1,m_2})_k$ is the group scheme H_k which is an extension of $\mathbb{Z}/p\mathbb{Z}$ by α_p if $m_1 = 0$ and $m_2 > 0$ (such an extension is necessarily split), and the group scheme G_k which is an extension of α_p by α_p if $m_1 > 0$. We have the following exact sequences:

$$(7)\ 0 \to H_k \to \mathbb{G}^2_{a,k} \xrightarrow{(x_1^p - x_1, x_2^p)} \mathbb{G}^2_{a,k} \to 0$$

and:

$$(8)\ 0 \to G_k \to \mathbb{G}^2_{a,k} \xrightarrow{(x_1^p, x_2^p)} \mathbb{G}^2_{a,k} \to 0$$

Let X be an R-scheme. The sequence (6) induces a long cohomology exact sequence (9): $W_{m_1,m_2}(X) \xrightarrow{\varphi_{m_1,m_2}} W_{pm_1,pm_2}(X) \to H^1_{\mathrm{fppf}}(X, \mathcal{H}_{m_1,m_2}) \to H^1_{\mathrm{fppf}}(X, W_{m_1,m_2}) \xrightarrow{\phi_n} H^1_{\mathrm{fppf}}(X, W_{pm_1,pm_2})$.

The cohomology group $H^1_{\mathrm{fppf}}(X, \mathcal{H}_{m_1,m_2})$ classifies the isomorphism classes of fppf-torsors with group $\mathcal{H}_{m_1,m_2}$ above X. The above exact sequence allows the following description of $\mathcal{H}_{m_1,m_2}$-torsors: locally a torsor $f : Y \to X$ under the group scheme $\mathcal{H}_{m_1,m_2}$ is given by the equations:

$$T_1^p - \pi^{m_1(p-1)} T_1 = a_1$$

and:

$$T_2^p - \pi^{m_2(p-1)} T_2 = a_2 + p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} \pi^{pm_2 - m_1(pk+p-k)} T_1^{pk} (-T_1)^{p-k}$$

where T_1 and T_2 are indeterminate and a_1, a_2 are regular functions on X. Its special fibre is the H_k-torsor given by the equations:

$$t_1^p - t_1 = \bar{a}_1$$

and:

$$t_2^p = \bar{a}_2$$

if $m_1 = 0$, and the G_k-torsor given by the equations:

$$t_1^p = \bar{a}_1$$

and:

$$t_2^p = \bar{a}_2$$

otherwise. In particular if $H^1_{\text{fppf}}(X, \mathbb{G}_{a,R}) = 0$ (e.g. if X is affine) then an $\mathcal{H}_{m_1,m_2}$-torsor above X is globally defined by an equation as above.

2.4. Degeneration of étale $\mathbb{Z}/p^2\mathbb{Z}$-torsors. Our aim in this section is to describe explicitely the degeneration of étale $\mathbb{Z}/p^2\mathbb{Z}$-torsors. In what follows we use the same notations as in 2.2. Let $f_K : Y_K \to X_K$ be a non trivial $\mathbb{Z}/p^2\mathbb{Z}$-torsor. Let $K(X) \to L$ be the cyclic p^2-extension of function fields corresponding to the torsor f_K which canonically factorises as $K(X) \to L_1 \to L_2 := L$ where $K(X) \to L_1$ is a cyclic p-extension. We assume that the ramification index above the generic point η of X_k in the extension $K(X) \to L$ equals 1. There exists a canonical factorisation $Y_K = Y_{2,K} \xrightarrow{f_{2,K}} Y_{1,K} \xrightarrow{f_{1,K}} X_K$ of f_K where $f_{i,K}$ is a $\mathbb{Z}/p\mathbb{Z}$-torsor for $i \in \{1, 2\}$. Moreover by 2.2.1 the torsor $f_{2,K}$ (resp. $f_{1,K}$) extends to a torsor $f_2 : Y_2 \to Y_1$ (resp. $f_1 : Y_1 \to X$) under a finite and flat R-group scheme of rank p and the composite $f := f_1 \circ f_2$ is a finite and flat cover which coincides on the generic fibre with f_K. We assume that the special fibre $Y_{2,k}$ of Y_2 is **irreducible**. In particular above the generic point η there exists a unique generic point η_1 in Y_1 which lies above η. We denote by δ (resp. δ_1 and δ_2) the degree of different in the extension L above the point η (resp. the degree of different in the extension L_1 above the point η and that of the different in the extension L_2 above the point η_1). Note that $\delta = \delta_1 + \delta_2$.

2.4.1. We start with the following technical lemma 2.4.2 which will be used later on. In what follows we assume that $X = \operatorname{Spec} A$ is affine and that $f_1 : Y_1 := \operatorname{Spec} B \to X$ is a torsor under the group scheme $\mathcal{M}_n$ for some integer $n > 0$. Thus f_1 is given by an equation $T^p - \pi^{n(p-1)}T = v$ where $v \in A$ is such that its image $\bar{v} \in \overline{A} := A/\pi A$ is not a p-power. In particular the special fibre $\bar{f}_1 : Y_{1,k} = \operatorname{Spec} \overline{B} \to X_k = \operatorname{Spec} \overline{A}$ of the torsor f (here $\overline{B} := B/\pi B$ and $\overline{A} := A/\pi A$) is the α_p-torsor given by the equation $t^p = \bar{v}$ and $\overline{B}$ is a free $\overline{A}$-algebra with basis $\{1, t, t^2, ..., t^{p-1}\}$. We need to characterise elements of A which become p-powers modulo π in $\overline{B}$ but are not necessarily p-powers modulo π in $\overline{A}$.

2.4.2. Lemma. *Let $u \in A$. Assume that the image $\bar{u}$ of u is a p-power in $\overline{B}$. Then $u = f(v) + \pi u'$ where $u' \in A$ and $f(v)$ belongs to the additive subgroup $A_v := A^p \oplus A^p.v \oplus .... \oplus A^p.v^{p-1}$ of A. Moreover let $f(v) := a_0^p + a_1^p v + ... + a_{p-1}^p v^{p-1} \in A_v$ and let m be an integer. Consider the element $g := f(v)\pi^{-pm} \in B_K$. Then $g = (a_o^p + a_1^p(T^p - \pi^{n(p-1)}T) + ... + a_{p-1}^p(T^p - \pi^{n(p-1)}T)^{p-1})\pi^{-pm}$ and after addition of elements of B_K of the form $b^p - b$ one can transform g in $\tilde{g} := \pi^{-m}(a_0 + a_1 T + ... + a_{p-1}T^{p-1}) + \pi^{-(pm-n(p-1))}(-\sum_{j=1}^{p-1} j a_j^p T^{p(j-1)}T) + \pi^{-(pm-2n(p-1))}h(T)$ where $h(T) \in B$. Moreover the image of $-\sum_{j=1}^{p-1} j a_j^p T^{p(j-1)}T = -a_1^p T - 2a_2^p T^{p+1} - ... - (p-1)a_{p-1}^p T^{p(p-2)+1}$ in $\overline{B}$ is not a p-power.*

Proof. We have $\overline{B} = \overline{A} \oplus \overline{A}.t \oplus ... \oplus \overline{A}.t^{p-1}$, hence $\overline{B}^p = \overline{A}^p \oplus \overline{A}^p.\bar{v} \oplus ... \oplus \overline{A}.\bar{v}^{p-1}$ and the first assertion of the lemma follows. Now let $g := (a_0^p + a_1^p v + ... + a_{p-1}^p v^{p-1})\pi^{-pm} \in B_K$ then since $T^p - \pi^{n(p-1)}T = v$ in B_K we can write $g = \pi^{-pm}(a_o^p + a_1^p(T^p - \pi^{n(p-1)}T) + ... + a_{p-1}^p(T^p - \pi^{n(p-1)}T)^{p-1})$ in B_K. After developing the terms $(T^p - \pi^{n(p-1)}T)^j$ for $j \in \{1, p-1\}$ according to the binomial expansion and rearanging the terms with the same power of π we get that $g = \pi^{-pm}(a_0^p + a_1^p T^p + ... + a_{p-1}^p T^{p(p-1)}) + \pi^{-(pm-n(p-1))}(-a_1^p T - 2a_2^p T^{p+1} - ... - (p-1)a_{p-1}^p T^{p(p-2)+1}) + \pi^{-(pm-2n(p-1))}h(T)$ where $h(T) \in B$. Finally after adding $(a_0 + a_1 T + ... + a_{p-1}T^{p-1})/\pi^m - (a_0^p + a_1^p T^p + ... + a_{p-1}^p T^{p(p-1)})/\pi^{pm}$ we get the desired expression for $\tilde{g}$.

The next theorem is the main result of this section. It describes locally and explicitely the degeneration of étale $\mathbb{Z}/p^2\mathbb{Z}$-torsors. Although we loose the structure of torsors in this case of cyclic covers of degree p^2 (cf. Proposition 2.4.5) we are able to find "canonical integral equations" which describe the reduction of p^2-cyclic covers in equal characteristic p.

2.4.3. Theorem. *We use the same notations as in 2.4. Assume that $X = \operatorname{Spec} A$ is* **affine**. *Then the torsor f_K can be described by an equation of the form:*

$$(T_1^p, T_2^p) - (T_1, T_2) = (\pi^{m_1} a_1, \pi^{m_2} a_2)$$

where a_1 and a_2 are regular functions on X_K with $v(a_1) = v(a_2) = 0$, $m_1 \leq 0$, m_2 is an integer, with the following cases which occur:

a) *$m_1 = 0$ and $m_2 \geq 0$. In this case f is an étale $\mathbb{Z}/p^2\mathbb{Z}$-torsor above X given by the equation:*

$$(T_1^p, T_2^p) - (T_1, T_2) = (a_1, \pi^{m_2} a_2)$$

Its special fibre $f_k : Y_k \to X_k$ is the étale $\mathbb{Z}/p^2\mathbb{Z}$-torsor given by the equation:

$$(t_1^p, t_2^p) - (t_1, t_2) = (\bar{a}_1, \overline{\pi^{m_2} a_2})$$

and $\delta = \delta_1 = \delta_2 = 0$.

b) $m_1 = 0$, $m_2 = -pm'_2 < 0$, *and* a_2 *is not a* p*-power modulo* π. *In this case* f *is a torsor under the* R*-group scheme* W_{0,m'_2} *given by the equations:*

$$T_1^p - T_1 = a_1$$

and

$$\tilde{T}_2^p - \pi^{m'_2(p-1)}\tilde{T}_2 = a_2 + p^{-1}\pi^{-m_2}\sum_{k=1}^{p-1}\binom{p}{k}T_1^{pk}(-T_1)^{p-k}$$

Its special fibre is the torsor under the k*-group scheme* $(W_{0,m'_2})_k \simeq H_k$ *given by the equations:* $t_1^p - t_1 = \bar{a}_1$, *and* $\tilde{t}_2^p = \bar{a}_2$, *where* $\bar{a}_1$ *(resp.* $\bar{a}_2$*) is the image of* a_1 *(resp. of* a_2*) modulo* π. *In this case* $\delta_1 = 0$ *and* $\delta = \delta_2 = m'_2(p-1)$.

c) $m_1 = -pm'_1 < 0$ *and the image* $\bar{a}_1$ *of* a_1 *modulo* π *is not a* p*-power. In this case* f_1 *is an* $\mathcal{M}_{m'_1}$*-torsor given by the equation:*

$$\tilde{T}_1^p - \pi^{m'_1(p-1)}\tilde{T}_1 = a_1$$

and its special fibre $f_{1,k} : Y_{1,k} \to X_k$ *is the* α_p*-torsor given by the equation* $\tilde{t}_1^p = \bar{a}_1$. *We have* $\delta_1 = m'_1(p-1)$. *For* f_2 *one has the following cases:*

c-1) $m'_1(p(p-1)+1) > -m_2$ *(resp.* $m'_1(p(p-1)+1) = -m_2$*) In this case* $m'_1 = pm''_1$ *is necessarily divisible by* p *and if* $\tilde{m}_1 := m''_1(p(p-1)+1)$ *then* f_2 *is a torsor under* $\mathcal{M}_{\tilde{m}_1,R}$ *given by the equation:*

$$\tilde{T}_2^p - \pi^{\tilde{m}_1(p-1)}\tilde{T}_2 = \pi^{\tilde{m}_1 p + m_2}a_2 + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}\tilde{T}_1^{pk}(-\tilde{T}_1)^{p-k}\pi^{m'_1((p(p-1)+1)-pk-p+k)}$$

Its special fibre is the α_p*-torsor given by the equation:* $\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1}$ *(resp. the equation:* $\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1} + \bar{a}_2$*).*

Otherwise $-m_2 > m'_1(p(p-1)+1)$ *in which case* $-m_2 = pm'_2$ *is necessarily divisible by* p *and we have the following description for* $\pi^{m_2}a_2$*:*

$$\pi^{m_2}a_2 = f_1(a_1)/\pi^{pm'_2} + f_2(a_1)/\pi^{pm'_2-t_1} + ... + f_r(a_1)/\pi^{pm'_2-t_1-...-t_{r-1}} + g/\pi^{pm'_2-t_1-...-t_r}$$

where $f_i(a_1)$ *belongs to the subgroup* A_{a_1} *of* A *(cf. 2.4.2),* $g \in A$, *and the* t_i *are positive integers (note that* g *and the* f_i *can be 0). Moreover the torsor* $f_{2,K}$ *is given by the equation:* $T_2^p - T_2 = f_1(a_1)/\pi^{pm'_2} + f_2(a_1)/\pi^{pm'_2-t_1} + ... + f_r(a_1)/\pi^{pm'_2-t_1-...-t_{r-1}}$ $+g/\pi^{pm'_2-t_1-...-t_r} + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}\tilde{T}_1^{pk}(-\tilde{T}_1)^{p-k}\pi^{-m'_1(pk+p-k)}$.

And the following distinct cases occur:

c-2) *$pm_2' - (p-1)m_1' > \sup(m_1'(p(p-1)+1), pm_2' - t_1 - ... - t_r)$ (resp. $pm_2' - (p-1)m_1' = m_1'(p(p-1)+1) > pm_2' - t_1 - ... - t_r$). In this case $pm_2' - m_1'(p-1) = m_2''p$ is divisible by p and $\delta_2 = m_2''(p-1)$. Let $f_1(a_1) := c_0^p + c_1^p a_1 + ... + c_{p-1}^p a_1^{p-1}$. Then f_2 is a torsor under $\mathcal{M}_{m_2''}$ and its special fibre is the α_p-torsor given by the equation: $\tilde{t}_2^p = -\bar{c}_1^p \tilde{t}_1 - 2\bar{c}_2^p \tilde{t}_1^{p+1} - ... - (p-1)\bar{c}_{p-1}^p \tilde{t}_1^{p(p-2)+1}$ (resp. the equation: $\tilde{t}_2^p = -\bar{c}_1^p \tilde{t}_1 - 2\bar{c}_2^p \tilde{t}_1^{p+1} - ... - (p-1)\bar{c}_{p-1}^p \tilde{t}_1^{p(p-2)+1} - \tilde{t}_1^{p(p-1)+1}$) where $\bar{c}_i$ is the image of c_i modulo π.*

c-3) *$pm_2' - t_1 - ... - t_r > \sup(pm_2' - (p-1)m_1', m_1'(p(p-1)+1))$ (resp. $pm_2' - (p-1)m_1' = pm_2' - t_1 - ... - t_r > m_1'(p(p-1)+1)$) and the image of g modulo π is not a p-power in $\mathcal{O}(Y_{1,k})$. In this case $pm_2' - t_1 - ... - t_r = pm_2''$ is divisible by p, $\delta_2 = m_2''(p-1)$, and f_2 is an $\mathcal{M}_{m_2''}$-torsor. Its special fibre is the α_p-torsor given by the equation: $\tilde{t}_2^p = \bar{g}$ (resp. the equation: $\tilde{t}_2^p = -\bar{c}_1^p \tilde{t}_1 - 2\bar{c}_2^p \tilde{t}_1^{p+1} - ... - (p-1)\bar{c}_{p-1}^p \tilde{t}_1^{p(p-2)+1} + \bar{g}$).*

c-4) *$m_1'(p(p-1)+1) > \sup(pm_2' - t_1 - ... - t_r, pm_2' - (p-1)m_1')$ (resp. $pm_2' - t_1 - ... - t_r = m_1'(p(p-1)+1) > pm_2' - (p-1)m_1'$) In this case $m_1' = pm_1''$ and if $\tilde{m}_1 := m_1''(p(p-1)+1)$ then f_2 is a torsor under $\mathcal{M}_{\tilde{m}_1,R}$. Its special fibre is the α_p-torsor given by the equation: $\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1}$ (resp. the equation: $\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1} + \bar{g}$).*

c-5) *$pm_2' - t_1 - ... - t_r = m_1'(p(p-1)+1) = pm_2' - (p-1)m_1'$. In this case $m_1' = pm_1''$ and if $\tilde{m}_1 := m_1''(p(p-1)+1)$ then f_2 is a torsor under $\mathcal{M}_{\tilde{m}_1,R}$. Its special fibre is the α_p-torsor given by the equation: $\tilde{t}_2^p = -\bar{c}_1^p \tilde{t}_1 - 2\bar{c}_2^p \tilde{t}_1^{p+1} - ... - (p-1)\bar{c}_{p-1}^p \tilde{t}_1^{p(p-2)+1} - \tilde{t}_1^{p(p-1)+1} + \bar{g}$.*

In all the above cases if f_1 (resp. f_2) is a torsor under the group scheme $\mathcal{M}_{\tilde{m}_1}$ (resp. $\mathcal{M}_{\tilde{m}_2}$) then necessarily $\tilde{m}_2 \geq \tilde{m}_1(p(p-1)+1)/p$. Moreover in all the cases c-2, c-3, c-4, and c-5 above the functions $\bar{c}_1$, $\bar{c}_2$,..., $\bar{c}_{p-1}$ (resp. $\bar{g}$) are uniquely determined (resp. is uniquely determined up to addition of $\bar{h}^p$ where $\bar{h}$ is a regular function on X_k). In the case c-1 the function $\bar{a}_2$ is uniquely determined up to addition of $\bar{b}^p$ where $\bar{b}$ is a regular function on X_k.

Proof. The torsor f_K is given by the Artin-Schreier-Witt theory by an equation of the form $(T_1^p, T_2^p) - (T_1, T_2) = (\tilde{a}_1, \tilde{a}_2)$ where $\tilde{a}_1$ and $\tilde{a}_2$ are regular functions on X_K. We can write $\tilde{a}_1 = \pi^{m_1} a_1$ (resp. $\tilde{a}_2 = \pi^{m_2} a_2$) with $v(a_1) = v(a_2) = 0$ and it follows from 2.2.1 that we have necessarily $m_1 \leq 0$. If $m_1 = 0$ and $m_2 \geq 0$ we are in case a) and the assertion there is clear. Assume that $m_1 = 0$ and $m_2 < 0$. Then it follows from 2.2.1 that $m_2 = -pm_2'$ is necessarily divisible by p and after eventually some modifications (as in the proof of 2.2.1) we may assume that a_2 is not a p-power modulo π (here one uses the fact that a regular function u on X which is not a p-power modulo π in X_k can not become a p-power in $Y_{1,k}$ since $f_{1,k}$ is an étale torsor hence not radicial). Then f is defined by the equations:

$$T_1^p - T_1 = a_1$$

and

$$\tilde{T}_2^p - \pi^{m_2'(p-1)}\tilde{T}_2 = a_2 + p^{-1}\pi^{-m_2}\sum_{k=1}^{p-1}\binom{p}{k}T_1^{pk}(-T_1)^{p-k}$$

where $\tilde{T}_2 := \pi^{m_2'}T_2$. The rest of the assertion in case b) follows then easily. Assume now that $m_1 < 0$. Then the assertion concerning f_1 follows from 2.2.1. Assume first that $m_2 \geq 0$. The assertion concerning f_2 follows then easily after adapting the equation defining the torsor $f_{2,K}$ to the change of variables $T_1 = \tilde{T}_1/\pi^{m_1'}$ and we are in the case c-1). In this case $m_1' = pm_1''$ is divisible by p and the cover f is given by the equations:

$$\tilde{T}_1^p - \pi^{m_1'(p-1)}\tilde{T}_1 = a_1$$

and

$$\tilde{T}_2^p - \pi^{\tilde{m}_1(p-1)}\tilde{T}_2 = \pi^{\tilde{m}_1 p + m_2}a_2 + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}\tilde{T}_1^{pk}(-\tilde{T}_1)^{p-k}\pi^{\tilde{m}_1 p - m_1'(pk+p-k)}$$

where $\tilde{m}_1 := m_1''(p(p-1)+1)$. From which we deduce that its special fibre is given by the equations: $\tilde{t}_1^p = \bar{a}_1$ and $\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1}$ where $\tilde{t}_2$ (resp. $\tilde{t}_1$) is the image of $\tilde{T}_2$ (resp. image of $\tilde{T}_1$) modulo π.

So lastly we assume that $m_2 < 0$. Then $f_{2,K}$ is given by the equation:

$$T_2^p - T_2 = \pi^{m_2}a_2 + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}\tilde{T}_1^{pk}(-\tilde{T}_1)^{p-k}\pi^{-m_1'(pk+p-k)}$$

The highest power of π in the denominators of the summands in:

$$p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}\tilde{T}_1^{pk}(-\tilde{T}_1)^{p-k}\pi^{-m_1'(pk+p-k)}$$

is $m_1'(p(p-1)+1)$ and in order to understand the reduction of the torsor f_2 we have to compare this to m_2. Assume first that $m_1'(p(p-1)+1) > -m_2$. Then it follows from 2.2.1 that $m_1' = m_1''p$ must be divisible by p. Let $\tilde{m}_1 := m_1''(p(p-1)+1)$. Then we are in case c-1) and f_2 is a torsor under the group scheme $\mathcal{M}_{\tilde{m}_1,R}$ defined by the equation:

$$\tilde{T}_2^p - \pi^{\tilde{m}_1(p-1)}\tilde{T}_2 = \pi^{\tilde{m}_1 p + m_2}a_2 + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}\pi^{\tilde{m}_1 p - m_1'(pk+p-k)}\tilde{T}_1^{pk}(-\tilde{T}_1)^{p-k}$$

Its special fibre is the α_p-torsor given by the equation: $\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1}$. Assume next that $m'_1(p(p-1)+1) < -m_2$ (the case where $m'_1(p(p-1)+1) = -m_2$ is easily treated and is left to the reader). Then it follows from 2.2.1 that $m_2 = -pm'_2$ is divisible by p and two cases occur depending on whether or not the image $\bar{a}_2$ of a_2 modulo π is or is not a p power in $\mathcal{O}(Y_{1,k})$. If $\bar{a}_2$ is not a p power in $\mathcal{O}(Y_{1,k})$ then f_2 is a torsor under $\mathcal{M}_{m'_2,R}$ given by the equation:

$$\tilde{T}_2^p - \pi^{m'_2(p-1)}\tilde{T}_2 = a_2 + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}\pi^{pm'_2-m_1(pk+p-k)}\tilde{T}_1^{pk}(-\tilde{T}_1)^{p-k}$$

Its special fibre is the α_p-torsor given by the equation $\tilde{t}_2^p = \bar{a}_2$ and we are in the case c-3). Assume that $\bar{a}_2$ is a p power in $\mathcal{O}(Y_{1,k})$. Then either a_2 is already a p-power in $\mathcal{O}(X_k)$ in which case we can transform, using the kind of transformations used in the proof of 2.2.1, the term $\pi^{m_2}a_2$ into $\pi^{\tilde{m}_2}\tilde{a}_2$ where $0 > \tilde{m}_2 > m_2$ and $\tilde{a}_2 \in A$. Or a_2 is not a p-power in $\mathcal{O}(X_1)$ but becomes a p-power in $\mathcal{O}(Y_{1,k})$. In the later case it follows from 2.4.2 that $a_2 = f_1(a_1) + \pi^{t_1}g_1$ where $f_1(a_1) := c_0^p + c_1^p a_1 + ... + c_{p-1}^p a_1^{p-1}$ belongs to the subgroup A_{a_1} of A, $t_1 > 0$, and $g_1 \in A$. Moreover the term $\pi^{m_2}a_2 = f_1(a_1)/\pi^{pm'_2} + g_1/\pi^{pm'_2-t_1}$ can be transformed to $\tilde{f}_1(T_1)/\pi^{pm'_2-m'_1(p-1)} + g_1/\pi^{pm'_2-t_1}$ where the image $\overline{\tilde{f}_1(T_1)} := -\bar{c}_1^p\tilde{t}_1 - 2\bar{c}_2^p\tilde{t}_1^{p+1} - ... - (p-1)\bar{c}_{p-1}^p\tilde{t}_1^{p(p-2)+1}$ of $\tilde{f}_1(T_1)$ modulo π is not a p-power (cf. loc. cit.). At this point we can repeat the same argument as above. Namely if in the first case the image $\bar{\tilde{a}}_2$ modulo π is not a p-power in $\mathcal{O}(Y_{1,k})$ then we conclude as above that we are either in case c-3) if $\tilde{m}_2 \geq m'_1(p(p-1)+1)$ in which case $\tilde{m}_2 = pm''_2$ and f_2 is a torsor under $\mathcal{M}_{m''_2,R}$ whose special fibre is the α_p-torosr given by the equation $\tilde{t}_2^p = \bar{\tilde{a}}_2$. Otherwise we repeat the same process as above. And in the second case if $pm'_2 - (p-1)m'_1 > \sup(pm'_2 - t_1, m'_1(p(p-1)+1))$ then $pm'_2 - (p-1)m'_1 =: pm''_2$ is divisible by p and f_2 is a torsor under the group scheme $\mathcal{M}_{m''_2,R}$ defined by the equation:

$$\tilde{T}_2^p - \pi^{m''_2(p-1)}\tilde{T}_2 = \tilde{f}_1(T_1) + \pi^{pm''_2-pm'_2+t_1}g_1 + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}\pi^{pm''_2-(pk+p-k)m_1}\tilde{T}_1^{pk}(-\tilde{T}_1)^{p-k}$$

Its special fibre is the α_p-torsor given by the equation: $\tilde{t}_2^p = -\bar{c}_1^p\tilde{t}_1 - 2\bar{c}_2^p\tilde{t}_1^{p+1} - ... - (p-1)\bar{c}_{p-1}^p\tilde{t}_1^{p(p-2)+1}$ and we are in the case c-2). If $m'_1(p(p-1)+1) > \sup(pm'_2 - t_1, pm'_2 - (p-1)m'_1)$ then $m'_1(p(p-1)+1)$ is divisible by p, f_2 is a torsor under the group scheme $\mathcal{M}_{(m'_1(p(p-1)+1))/p,R}$, its special fibre is the α_p-torsor given by the equation:
$\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1}$ and we are in the case c-4). If $pm'_2 - t_1 > \sup(m'_1(p(p-1)+1), pm'_2 - (p-1)m'_1)$ and the image $\bar{g}_1$ of g_1 in $\mathcal{O}(Y_{1,k})$ is not a p-power then $pm'_2 - t_1 =: pm''_2$ is divisible by p, f_2 is a torsor under the group scheme $\mathcal{M}_{m''_2,R}$ defined by the equation:

$$\tilde{T}_2^p - \pi^{m''_2(p-1)}\tilde{T}_2 = \pi^{\tilde{m}_2}\tilde{f}_1(T_1) + g_1 + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}\pi^{pm''_2-(pk+p-k)m_1}\tilde{T}_1^{pk}(-\tilde{T}_1)^{p-k}$$

where $\tilde{m}_2 := pm_2'' - pm_2' - (p-1)m_1'$, and its special fibre is the α_p-torsor given by the equation: $\tilde{t}_2^p = \bar{g}_1$ and we are in the case c-3). Now in the general case we repeat the same argument as above if in the first case the image $\bar{\tilde{a}}_2$ of $\tilde{a}_2$ modulo π is a p-power or if in the second case $pm_2' - t_1 > \sup(m_1'(p(p-1)+1), pm_2' - (p-1)m_1')$ and the image $\bar{g}_1$ of g_1 in $\mathcal{O}(Y_{1,k})$ is a p-power. As the exponent of π in the denominators in the equation defining $f_{2,K}$ decreases at each step we conclude that this process must stop after finitely many steps and we end up with an equation as claimed in the statement c) and the rest of the conclusion follows then easily.

2.4.4. Remark. Assume that we are in the case c-3) of 2.4.3 and that $t_1 = ... = t_r = 0$ and $f_1 = ... = f_r = 0$. Then f is a torsor under the R-group scheme $\mathcal{H}_{m_1',m_2'}$ given by the equations:

$$\tilde{T}_1^p - \pi^{m_1'(p-1)}\tilde{T}_1 = a_1$$

and:

$$\tilde{T}_2^p - \pi^{m_2'(p-1)}\tilde{T}_2 = g + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}\tilde{T}_1^{pk}(-\tilde{T}_1)^{p-k}\pi^{pm_2'-pk-p+k}$$

Its special fibre is the $(\mathcal{H}_{m_1',m_2'})_k \simeq H_k$-torsor given by the equations: $\tilde{t}_1^p = \bar{a}_1$ and $\tilde{t}_2^p = \bar{g}$.

The next proposition states that in general the cover f_2 in 2.4.3 doesn't have the structure of a torsor under a finite and flat R-group scheme of rank p^2.

2.4.5. Proposition. *We use the same notations as in 2.4.3. Assume that we are in the case c-1) and that $p > 2$. Then the finite cover f doesn't have the structure of a torsor under a finite and flat R-group scheme of rank p^2.*

Proof. Recall that in the case c-1 of 2.4.3 the étale torsor f_K is given by the equations (1): $(T_1^p, T_2^p) - (T_1, T_2) = (a_1/\pi^{pm_1'}, a_2' := \pi^{m_2}a_2)$. Let $b_1 \in A$. Then the equations: $(S_1^p, S_2^p) - (S_1, S_2) = (\pi^{-pm_1'}a_1, a_2') + (\pi^{-pm_1'}b_1^p, 0) - (\pi^{-m_1'}b_1, 0)$ are also defining equations for the torsor f_K. The cover f is then given by the equations (1):

$$\tilde{T}_1^p - \pi^{m_1'(p-1)}\tilde{T}_1 = a_1$$

and

$$\tilde{T}_2^p - \pi^{\tilde{m}_1(p-1)}\tilde{T}_2 = \pi^{\tilde{m}_1 p + m_2}a_2 + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}\tilde{T}_1^{pk}(-\tilde{T}_1)^{p-k}\pi^{\tilde{m}_1 p - m_1'(pk+p-k)}$$

And f is also given by the equations (2):

$$\tilde{S}_1^p - \pi^{m'(p-1)}\tilde{S}_1 = a_1 + b_1^p - \pi^{m(p-1)}b_1$$

and: $S_2^p - S_2 = a_2' + p^{-1}(\sum_{k=1}^{p-1} \binom{p}{k} \tilde{S}_1^{pk}(-\tilde{S}_1)^{p-k}\pi^{-m_1'(pk+p-k)}$ $-\sum_{k=1}^{p-1} \binom{p}{k} b_1^{pk}(-b_1)^{p-k}\pi^{-m_1'(pk+p-k)} - \sum_{k=1}^{p-1} \binom{p}{k}(b_1^p - \pi^{m_1'(p-1)}b_1)^k(a_1)^{p-k}\pi^{-m_1'p^2})$. By reducing modulo π both equations (1) and (2) we obtain equations (1)' and (2)' for the cover f_k on the level of special fibres. Suppose that f has the structure of a torsor under a finite and flat R-group scheme G_R of rank p^2. Then the special fibre f_k of f is a torsor under the special fibre G_k of G which is a group scheme of rank p^2 and an extension of α_p by α_p. In particular both equations (1)' and (2)' that we obtain for f_k must define the same α_p-torsor $f_{1,k}$ and $f_{2,k}$. This is the case for $f_{1,k}$ since both equations are $\tilde{t}_1^p = \bar{a}_1$ and $\tilde{s}_1^p = \bar{a}_1 + \bar{b}_1^p$ but we will see that this is not the case for $f_{2,k}$ if $p > 2$. Indeed one can see after some (easy) computations that the equations we obtain for $f_{2,k}$ are: $\tilde{t}_2^p = -\tilde{t}_1^{p(p-1)+1}$ and $\tilde{s}_2^p = -\tilde{s}_1^{p(p-1)+1} + \bar{b}_1^{p(p-1)+1} + \sum_{k=1}^{p-1}(\bar{b}_1^{p(k-1)+1}(\tilde{s}_1 - \bar{b}_1)^{p(p-k)} + \bar{b}_1^{pk}(\tilde{s}_1 - \bar{b}_1)^{p(p-k-1)+1}))$, where $\tilde{s}_2$ is the image of $\pi^{\tilde{m}_1}S_2$ modulo π, which gives $\tilde{s}_2 = -\tilde{s}_1^{p(p-1)+1} + \bar{b}_1^{p(p-1)+1} + ((\tilde{s}_1 - \bar{b}_1)^{p-1} + \bar{b}_1^{p-1}) \sum_{k=1}^{p-1} \bar{b}_1^{p(k-1)+1}(\tilde{s}_1 - \bar{b}_1)^{p(p-k-1)+1}$. An easy verification shows that these two equations define non isomorphic α_p-torsors (although they define isomorphic covers). If for example $p = 3$ then the two equations are: $\tilde{t}_2^3 = -\tilde{t}_1^7$ and $\tilde{s}_2^3 = -\tilde{s}_1^7 + \bar{b}_1^7 + ((\tilde{s}_1 - \bar{b}_1)^2 - \bar{b}_1^2)(\bar{b}_1(\tilde{s}_1 - \bar{b}_1)^4 + \bar{b}_1^4(\tilde{s}_1 - \bar{b}_1))$. Using that $\tilde{s}_1 = \tilde{t}_1 + \bar{b}_1$ we get $\tilde{s}_2^3 = -(\tilde{t}_1 + \bar{b}_1)^7 + \bar{b}_1^7 + (\tilde{t}_1^2 - \bar{b}_1^2)(\bar{b}_1\tilde{t}_1^4 + \bar{b}_1^4\tilde{t}_1)$ thus $\tilde{s}_2^3 = -\tilde{t}_1^7 - 34\tilde{t}_1^3\bar{b}_1^4 - 8\tilde{t}_1\bar{b}_1^6 = -\tilde{t}_1^7 + 2\tilde{t}_1^3\bar{b}_1^4 + \tilde{t}_1\bar{b}_1^6$ and since $2\tilde{t}_1^3\bar{b}_1^4 + \tilde{t}_1\bar{b}_1^6 = 2\bar{a}_1\bar{b}_1^4 + \tilde{t}_1\bar{b}_1^6$ is not a cube we deduce that the two equations do not define the same α_3-torsor.

2.4.6. Corollary. *Assume $p > 2$. Let X be a formal affine R-scheme of finite type with X_k irreducible. Then there exists a p^2-cyclic cover $f : Y \to X$ which is étale on the generic fibre X_K of X and such that f doesn't have the structure of a torsor under a finite and flat R-group scheme of rank p^2.*

Proof. One can easily find examples of cyclic p^2-covers $f : Y \to X$ which have the structure of an étale torsor above X_K and such that we are in the case c-1) of 2.4.3. The result follows then by 2.4.5.

2.4.7. Question. With the same notations as in 2.4.3 is it possible to find necessary and sufficient conditions, for example on δ_1 and δ_2, such that f has the structure of a torsor under a finite and flat R-group scheme of rank p^2?

Next we define the "*degeneration data*" arising from the reduction of an étale $\mathbb{Z}/p^2\mathbb{Z}$-torsor.

2.4.8. Definition. Let $f_K : Y_K \to X_K$ be an étale $\mathbb{Z}/p^2\mathbb{Z}$-torsor with $X = \operatorname{Spec} A$ affine as in 2.4.3. Then we define the *degeneration type* of the torsor f_K as follows: f_K has a degeneration of type A, or of type (étale, étale), if we are in the case a) of 2.4.3, a degeneration of type B, or of type (étale, radicial), if we are in the case b) of 2.4.3 and a degeneration of type C, or of type (radicial, radicial), if we are in the case c) of 2.4.3.

Further we define the *degeneration data* associated to a degeneration type as follows:

a) A degeneration data of type A consists of an element of $H^1_{\text{et}}(X_k, \mathbb{Z}/p^2\mathbb{Z})$.

b) A degeneration data of type B consists of an element of $H^1_{\text{fppf}}(X_k, G_k)$ where G_k is the finite commutative group scheme extension of $\mathbb{Z}/p\mathbb{Z}$ by α_p as defined in 2.3.1.

c) A degeneration data of type C consists of an element of $H^1_{\text{fppf}}(X_k, H_k) \oplus \Gamma(X_k, \mathcal{O}_{X_k})^{p-1}$. where H_k is the finite commutative group scheme extension of α_p by α_p as defined in 2.3.1

2.4.9. Proposition. *Assume that X is affine as in 2.4.3 and let $f_K : Y_K \to X_K$ be an étale $\mathbb{Z}/p^2\mathbb{Z}$-torsor which has a degeneration of type A (resp. B or C). Then f_K induces canonically a degeneration data of type A (resp. of type B or C).*

Proof. This is a direct consequence of 2.4.3. If f_K has a degeneration of type A then the special fibre f_k of f is an étale $\mathbb{Z}/p^2\mathbb{Z}$-torsor and the assertion follows in this case. Assume that f_K has a degeneration of type B. Then the special fibre f_k of f is canonically a G_k-torsor and the assertion follows in this case too. Finally assume that the torsor f_K has a degeneration of type C. Then it follows from 2.4.3 that the special fibre f_k of the cover f is defined by the equations: $\tilde{t}_1^p = \bar{a}_1$ and $\tilde{t}_2^p = -\bar{c}_1^p \tilde{t}_1 - 2\bar{c}_2^p \tilde{t}_1^{p+1} - ... - (p-1)\bar{c}_{p-1}^p \tilde{t}_1^{p(p-2)+1} - \tilde{t}_1^{p(p-1)+1} + \bar{g}$ (resp. $\tilde{t}_1^p = \bar{a}_1$ and $\tilde{t}_2^p = -\bar{c}_1^p \tilde{t}_1 - 2\bar{c}_2^p \tilde{t}_1^{p+1} - ... - (p-1)\bar{c}_{p-1}^p \tilde{t}_1^{p(p-2)+1} + \bar{g}$, or $\tilde{t}_1^p = \bar{a}_1$ and $\tilde{t}_2^p = \bar{g}$) where $\bar{c}_1$,...,$\bar{c}_{p-1}$ (resp. $\bar{g}$) are functions on X_k (eventually equal to 0) which are uniquely determined (resp. determined up to addition of element of the form $\bar{h}^p$ where $\bar{h}$ is a function on X_k). The pair $(\bar{a}_1, \bar{g})$ defines then canonically an element of $H^1_{\text{fppf}}(X_k, H_k)$ and the tuple $(\bar{c}_1, ..., \bar{c}_{p-1})$ an element of $\Gamma(X_k, \mathcal{O}_{X_k})^{p-1}$. Thus we get canonically an element of $H^1_{\text{fppf}}(X_k, H_k) \oplus \Gamma(X_k, \mathcal{O}_{X_k})^{p-1}$ associated to f_K in this case.

2.4.10. It follows from 2.4.8 that an étale $\mathbb{Z}/p^2\mathbb{Z}$-torsor above the generic fibre X_K of X induces canonically a *degeneration data* of type either A, B or C. Reciprocally we have the following result of *lifting* of such a degeneration data.

2.4.11. Proposition. *Assume given a degeneration data, say $\mathcal{D}$, of type either A, B or C as in 2.4.6. Then there exists a $\mathbb{Z}/p^2\mathbb{Z}$-torsor $f_K : Y_K \to X_K$ such that the degeneration data associated to f_K via 2.4.8 equals $\mathcal{D}$.*

Proof. The proof in the case where the degeneration data is of type A or B is similar to the proof in 2.2.3. Assume that the degeneration data is of type C and consists of the pair $(\bar{a}_1, \bar{a}_2)$, where $\bar{a}_1$ and $\bar{a}_2$ are functions on X_k which are not p-powers, and the tuple of functions $(\bar{c}_1, ..., \bar{c}_{p-1})$. Let a_1 and a_2 (resp. c_1,...,c_{p-1}) be regular functions on X which lift $\bar{a}_1$ and $\bar{a}_2$ (resp. which lifts $\bar{c}_1$,...,$\bar{c}_{p-1}$). Let $n = pn' = p^2 n'' > 0$ be an integer. Consider the $\mathbb{Z}/p^2\mathbb{Z}$-torsor given by the equations: $(T_1^p, T_2^p) - (T_1, T_2) = (a_1 \pi^{-n'p}, f(a_1)\pi^{-pm} + a_2 \pi^{-pm+n'(p-1)})$ where $f(a_1) = c_1 a_1 + ... + c_{p-1} a_1^{p-1}$ and $m = n'p$.

Then it follows easily from the proof of 2.4.3 that the degeneration data associated to f_K via 2.4.7 equals D. In this lifting we have $\delta_1 = n'(p-1)$ and $\delta_2 = n''(p(p-1)+1)(p-1)$. We have also the other following possibility for the lifting. Namely consider the $\mathbb{Z}/p^2\mathbb{Z}$-torsor given by the equations: $(T_1^p, T_2^p) - (T_1, T_2) = (a_1\pi^{-n'p}, f(a_1)\pi^{-pm} + g\pi^{-pm+n'(p-1)})$ where m is a positive integer such that $mp - n'(p-1) > n'(p(p-1)+1)$ and $mp - n'(p-1) = pm'$. In this later case we have $\delta_1 = n'(p-1)$ and $\delta_2 = m'(p-1)$.

2.5. Degeneration of $\mathbb{Z}/p\mathbb{Z}$-torsors on the boundaries of formal fibres.

In this section we assume that the *residue field k of R is algebraically closed.* In what follows we explain the degeneration of $\mathbb{Z}/p\mathbb{Z}$-torsors on the boundary $\mathcal{X} \simeq \operatorname{Spf} R[[T]]\{T^{-1}\}$ of formal fibres of germs of formal R-curves. Here $R[[T]]\{T^{-1}\}$ denotes the ring of formal power series $\sum_{i\in\mathbb{Z}} a_i T^i$ with $\lim_{i\to-\infty} |a_i| = 0$ where $|\ |$ is an absolute value of K associated to its valuation. Note that $R[[T]]\{T^{-1}\}$ is a complete discrete valuation ring with uniformising parameter π and residue field $k((t))$ where $t \equiv T \bmod \pi$. The function T is called a *parameter* of the formal fibre $\mathcal{X}$. The following result, which describes the degeneration of $\mathbb{Z}/p\mathbb{Z}$-torsors above the formal fibre $\operatorname{Spf} R[[T]]\{T^{-1}\}$, will be used in the next paragraph III in order to prove a formula comparing the dimensions of the spaces of vanishing cycles in a Galois cover of degree p between formal germs of R-curves.

2.5.1. Proposition. *Let $A := R[[T]]\{T^{-1}\}$ and let $f : \operatorname{Spf} B \to \operatorname{Spf} A$ be a non trivial Galois cover of degree p. Assume that the ramification index of the corresponding extension of discrete valuation rings equals 1. Then f is a torsor under a finite and flat R-group scheme G_R of rank p. Let δ be the degree of the different in the above extension. Then the following cases occur:*

a) $\delta = 0$. *In this case f is a torsor under the étale group $(\mathbb{Z}/p\mathbb{Z})_R$ and for a suitable choice of the parameter T of A the torsor f is given by an equation $X^p - X = T^m$ for some negative integer m which is prime to p. In this case $X^{1/m}$ is a parameter for B.*

b) $0 < \delta = n(p-1)$ *for some positive integer n. In this case f is a torsor under the group scheme $\mathcal{M}_{n,R}$. Moreover for a suitable choice of the parameter T the torsor f is given by an equation $X^p - \pi^{n(p-1)}X = T^m$ with $m \in \mathbb{Z}$ is prime to p. In this case $X^{1/m}$ is a parameter for B.*

Proof. First it follows from 2.2.1 that we are either in case a) or in case b). We start first with the case a). In this case f is an étale torsor given by an equation $X^p - X = u = \sum_{i\in\mathbb{Z}} a_i T^i \in A$. On the level of special fibres the torsor $f_k := \operatorname{Spec} B/\pi B \to \operatorname{Spec} A/\pi A$ is the étale torsor given by the equation $x^p - x = \sum_{i\geq m} \bar{a}_i t^i \in A$ where $\bar{a}_i$ is the image of a_i modulo π and $m \in \mathbb{Z}$ is some integer. Assume for example that the integer $m = pm'$ is divisible by p. Then after adding $\bar{a}_m^{1/p} t^{m'} - \bar{a}_m t^m$ into the defining equation for f_k we can replace $\bar{a}_m t^m$ by $\bar{a}_m^{1/p} t^{m'}$. Repeating this process eventually we can finally assume that the integer m is prime to p in which case $\sum_{i\geq m} \bar{a}_i t^i = t^m \bar{v}$ and $u = T^m v$ where $v \in A$ is a unit whose image modulo π equals $\bar{v}$. Further the integer m is then necessarily negative since the residue field extension $f_k := \operatorname{Spec} B/\pi B \to \operatorname{Spec} A/\pi A$ must ramify. Finally after extracting an m-th root of v and replacing T by $Tv^{1/m}$ we arrive to an equation of the form $X^p - X = T^m$. Secondly assume that we are in the case b). Then f is a torsor under the finite and flat group scheme $\mathcal{M}_{n,R}$, for some positive integer n, given by an equation $X^p - \pi^{n(p-1)} X = u = \sum_{i\in\mathbb{Z}} a_i T^i \in A$ and u is not a p-power modulo π. In this case on the level of special fibres the torsor $f_k := \operatorname{Spec} B/\pi B \to \operatorname{Spec} A/\pi A$ is the α_p-torsor given by the equation $x^p = \sum_{i\geq m} \bar{a}_i t^i \in A/\pi A$ where $\bar{a}_i$ is the image of a_i modulo π and $m \in \mathbb{Z}$ is some integer. Assume for example that the integer $m = pm'$ is divisible by p. Then after adding $\bar{a}_m^{1/p} t^{m'} - \bar{a}_m t^m$ into the defining equation for f_k we can eliminate the term $\bar{a}_m t^m$ from the defining equation for f_k and since $\sum_{i\geq m} \bar{a}_i t^i \in A/\pi A$ is not a p-power we can repeat this process eventually and assume after finitely many steps that m is prime to p. In this case $\sum_{i\geq m} \bar{a}_i t^i = t^m \bar{v}$ and $u = T^m v$ where $v \in A$ is a unit whose image modulo π equals $\bar{v}$. Finally after extracting an m-th root of v and replacing T by $Tv^{1/m}$ we arrive to an equation of the form $X^p - \pi^{n(p-1)} X = T^m$.

2.5.2. Definition. With the same notations as in 2.5.1 we define the *conductor* of the torsor f to be the integer $-m$. further we define the *degeneration type* of the torsor f to be $(0, m)$ in the case a) and (n, m) in the case b).

2.5.3. Remark. The above proposition implies in particular that Galois covers $f : \operatorname{Spf} B \to \operatorname{Spf} A$ as in 2.5.1 are classified by their degeneration type as defined in 2.5.2. More precisely given two such Galois covers which have the same degeneration type then there exists a (non canonical) Galois equivariant isomorphism between both covers.

2.6. Degeneration of $\mathbb{Z}/p^2\mathbb{Z}$-torsors on the boundaries of formal fibres.

We use the same notations and hypothesis as in 2.5. Next we explain the degeneration of étale $\mathbb{Z}/p^2\mathbb{Z}$-torsors on the boundaries of formal fibres.

2.6.1. Proposition. *Let $A := R[[T]]\{T^{-1}\}$ and let $\tilde{f} : \mathcal{Y} := \operatorname{Spf} B \to \mathcal{X} := \operatorname{Spf} A$ be a non trivial cyclic Galois cover of degree p^2 with $\mathcal{Y}_k$ irreducible. Assume that the ramification index of the corresponding extension of discrete valuation rings equals 1. Then $\tilde{f}$ factorises canonically as $\tilde{f} = f_1 \circ f_2$ with $f_2 : \mathcal{Y} \to \mathcal{Y}_1 := \operatorname{Spf} B_1$ and $f_1 : \mathcal{Y}_1 \to X$ where f_i is a Galois cover of degree p. Let δ (resp. δ_1 and δ_2) be the degree of the different in the above extension (resp. in f_1 and f_2). Then the following cases occur:*

a) $\delta = 0$. *In this case* $\tilde{f}$ *is a torsor under the étale* R*-group* $G = (\mathbb{Z}/p^2\mathbb{Z})_R$ *and for a suitable choice of the parameter* T *of* A *the torsor* $\tilde{f}$ *is given by an equation:*

$$(X_1^p, X_2^p) - (X_1, X_2) = (1/T^{m_1}, f(T))$$

where m_1 *is a positive integer prime to* p *and* $f(T) \in A$ *is such that its image* $\bar{f}(t)$ *modulo* π *equals* $\sum_{i \geq -m_2} c_i t^i$ *where* $m_2 \geq 0$ *is prime to* p. *In this case the torsor* f_1 *(resp.* f_2*) has a reduction of type* $(0, -m_1)$ *(resp.* $(0, \tilde{m}_2 := -pm_2 + m_1(p-1))$ *if* $m_2 \geq pm_1$ *and* $(0, \tilde{m}_2 := -m_1(p(p-1)+1))$ *otherwise).*

b) $\delta_1 = 0$ *and* $\delta_2 > 0$. *In this case* $\tilde{f}$ *is a torsor under the group scheme* $W_{(0,n)}$ *for some positive integer* n *and for a suitable choice of the parameter* T *of* A *the cover* $\tilde{f}$ *is generically given by the equations:*

$$X_1^p - X_1 = 1/T^{m_1}$$

and:

$$X_2^p - X_2 = f(T)/\pi^{pn} + p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} X_1^{pk} (-X_1)^{p-k}$$

with $f(T) \in A$ *is such that its image* $\bar{f}(t) = \sum_{i \in I} c_i t^i$ *modulo* π *is not a* p*-power. Let* $m_2 := \inf\{i \in I, \gcd(i,p) = 1\}$. *We call the integer* m_2 *the conductor of* $f(T)$. *Then the torsor* f_1 *(resp.* f_2*) has a reduction of type* $(0, -m_1)$ *(resp.* $(n, \tilde{m}_2)$ *where* $\tilde{m}_2 := pm_2 + m_1(p-1)$*).*

c) $\delta_1 < 0$. *In this case the cover* $\tilde{f}$ *is generically given for a suitable choice of the parameter* T *by the equations:*

$$X_1^p - X_1 = T^{m_1}/\pi^{n_1 p}$$

for some integer m_1 *prime to* p *and:*

$$X_2^p - X_2 = f(T)/\pi^{pn_2} + p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} X_1^{pk} (-X_1)^{p-k}$$

where $f(T) = \sum_{i \in I} a_i T^i \in A$ *is not a* p*-power modulo* π. *Let* $m_2 := \inf\{i \in I, \gcd(i,p) = 1,$ *and* a_i *is a unit*$\}$ *be the conductor of* f. *Then the following distinct cases occur:*

c-1) $pn_2 - (p-1)n_1 > n_1(p(p-1)+1)$. *Then* $pn_2 - (p-1)n_1 = pn_2'$ *is divisible by* p *and the torsor* f_1 *(resp.* f_2*) has a reduction of type* (n_1, m_1) *(resp.* $(n_2', m_2 p - m_1(p-1))$*).*

c-2) $pn_2 - (p-1)n_1 < n_1(p(p-1)+1)$. *Then* $n_1(p(p-1)+1) = pn_2''$ *is divisible by* p *and the torsor* f_1 *(resp.* f_2*) has a reduction of type* (n_1, m_1) *(resp.* $(n_2'', m_1(p(p-1)+1))$*).*

c-3) $pn_2 - (p-1)n_1 = n_1(p(p-1)+1)$. *Then* $n_1(p(p-1)+1) = pn_2''$ *is divisible by* p *and the torsor* f_1 *(resp.* f_2*) has a reduction of type* (n_1, m_1) *(resp.* $(n_2'', \tilde{m}_2)$ *where* $\tilde{m}_2 = \inf(m_1(p(p-1)+1), m_2 p - m_1(p-1))$*)).*

Proof. First it follows from 2.4.3 that the above cases are all the possible cases of degeneration of the $\mathbb{Z}/p^2\mathbb{Z}$-torsor f_K. We start with the case a) and assume that $\delta = 0$. In this case $\tilde{f}$ is an étale torsor given by an equation $(X_1^p, X_2^p) - (X_1, X_2) = (g(T), f(T))$ where $g(T)$ (resp. $f(T)$) are elements of A. Moreover by 2.5.1 we can assume that for a suitable choice of the parameter T we have $g(T) = 1/T^{m_1}$ for some positive integer m_1 prime to p. One has to compute the conductors of the torsors f_1 and f_2. As for f_1 we have $T = X_1^{-p/m_1}(1 - X_1^{(1-p)})^{-1/m_1}$ from which it follows that $Z := X_1^{-1/m_1}$ is a parameter for $\mathcal{Y}_1$ and we have $T = Z^p(1 - Z^{m_1(p-1)})^{-1/m_1}$. The torsor f_2 is given by the equation $X_2^p - X_2 = f(T) + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}X_1^{pk}(-X_1)^{p-k}$ and after adding elements of the form $b^p - b$ where $b \in A$ we can assume that the image $\bar{f}(t)$ of $f(T)$ modulo π equals $\sum_{i\geq -m_2} c_i t^i$ where $m_2 \geq 0$ is prime to p. Now we have $1/t^{m_2} = 1/z^{pm_2}(1 - z^{m_1(p-1)})^{m_2/m_1} = 1/z^{pm_2} - (m_2/m_1)1/z^{pm_2 - m_1(p-1)} + \ldots$, where z is the image of Z modulo π, which can be replaced after some transformation by $1/z^{m_2} - (m_2/m_1)1/z^{pm_2 - m_1(p-1)} + \ldots$. On the other hand the summand in $p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}X_1^{pk}(-X_1)^{p-k}$ which gives the highest contribution to the different is $-X_1^{p(p-1)+1} = 1/z^{m_1(p(p-1)+1)}$. from this we deduce that the conductor for the torsor f_2 equals $-pm_2 + m_1(p-1)$ if $m_2 \geq pm_1$ and $-m_1(p(p-1)+1)$ otherwise.

Assume next that we are in the case b). In this case $\tilde{f}$ is generically given by an equation $(X_1^p, X_2^p) - (X_1, X_2) = (g(T), f(T)/\pi^{pn})$ where $g(T)$ (resp. $f(T)$) are elements of A, n is a positive integer, and the image of $f(T)$ modulo π is not a p-power. Moreover by 2.5.1 we can assume that for a suitable choice of the parameter T we have $g(T) = 1/T^{m_1}$ for some positive integer m_1 prime to p. Consider the image $\bar{f}(t)$ of $f(T)$ modulo π which equals $\sum_{i\geq m_2} c_i t^i$. Then we can assume after some transformations which eliminate the p-powers that the integer m_2 is prime to p. The special fibre of the torsor f_2 is given by the equation: $\tilde{x}_2^p = \bar{f}(t) = \sum_{i\geq m_2} c_i t^i$. Now $t^{m_2} = z^{pm_2}(1 - z^{m_1(p-1)})^{-m_2/m_1} = z^{pm_2} + (m_2/m_1)z^{pm_2 + m_1(p-1)} + \ldots$ which can be transformed, after eliminating the term z^{pm_2} which is a p-power, to $(m_2/m_1)z^{pm_2+m_1(p-1)} + \ldots$. From this we deduce that the conductor of f_2 equals $pm_2 + m_1(p-1)$.

Finally assume that we are in the case c). In this case the cover $\tilde{f}$ is given by 2.4.3 and 2.5.1, and for a suitable choice of the parameter T, by the equations:

$$X_1^p - \pi^{n_1(p-1)}X_1 = T^{m_1}$$

and

$$X_2^p - X_2 = f(T)/\pi^{pn_2} + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}X_1^{pk}(-X_1)^{p-k}\pi^{-n_1(p(p-k)+k)}$$

where $f(T) = \sum_{i\in I} a_i T^i \in A$. Note that the image $\bar{f}(T) = \sum_{i\in I}\bar{a}_i t^i$ modulo π is necessarily a p-power in $B_1/\pi B_1$ since the image t of T is. We write $f(T) = f((X_1^p - \pi^{n_1(p-1)}X_1)^{1/m_1}) = \sum_{i\in I} a_i(X_1^p - \pi^{n_1(p-1)}X_1)^{i/m_1}$.

The expression $f((X_1^p - \pi^{n_1(p-1)}X_1)^{1/m_1})$ can be transformed after addition of elements of the form $b^p - b$ (where $b \in B_1$) to:

$$\sum_i a_i^{1/p} X_1^{i/m_1} \pi^{-n_2} - \sum_i (i/m_1) a_i X_1^{ip/m_1} X_1^{1-p} \pi^{-pn_2+n_1(p-1)} + g(X_1)\pi^{-n'}$$

where $g(X_1) \in B_1$ and $n' < pn_2 - n_1(p-1)$ is a positive integer. Assume first that $n_2p - n_1(p-1) > n_1(p(p-1)+1)$ then $n_2p - n_1(p-1) = n_2'p$ is necessarily divisible by p and f_2 is a torsor under the group scheme $\mathcal{M}_{n_2',R}$. Its special fibre $f_{2,k}$ is the α_p-torsor given by the equation $\tilde{x}_2^p = -\sum_{i \geq m_2} (i/m_1)\bar{a}_i x_1^{ip/m_1} x_1^{1-p}$ where $\bar{a}_i$ (resp. x_1) is the image of a_i (resp. X_1) modulo π and $m_2 := \inf\{i \in I, \gcd(i,p) = 1$, and a_i is a unit$\}$ is the conductor of f. Since x_1^{1/m_1} is a parameter for $B_1/\pi B_1$ we deduce immediately that the conductor for the torsor f_2 equals $m_2p - m_1(p-1)$. Next assume that $n_2p - n_1(p-1) < n_1(p(p-1)+1)$ then $n_1(p(p-1)+1) = n_2''p$ is necessarily divisible by p and f_2 is a torsor under the group scheme $\mathcal{M}_{n_2'',R}$. Its special fibre $f_{2,k}$ is the α_p-torsor given by the equation $\tilde{x}_2^p = -x_1^{p(p-1)+1}$ from which we deduce immediately that the conductor of f_2 equals $m_1(p(p-1)+1)$. Finally consider the last case where $n_2p - n_1(p-1) = n_1(p(p-1)+1)$ in which case $n_1(p(p-1)+1) = n_2''p$ is necessarily divisible by p and f_2 is a torsor under the group scheme $\mathcal{M}_{n_2'',R}$. Its special fibre $f_{2,k}$ is the α_p-torsor given by the equation $\tilde{x}_2^p = -x_1^{p(p-1)+1} - \sum_{i \geq m_2} (i/m_1)\bar{a}_i x_1^{ip/m_1} x_1^{1-p}$ from which we deduce that the conductor of f_2 equals $\inf(m_1(p(p-1)+1), m_2p - m_1(p-1))$ which equals $m_1(p(p-1)+1)$ if $m_2 \leq pm_1$ and $m_2p - m_1(p-1)$ otherwise.

2.6.2. Definition. With the same notations as in 2.6.1 let (n_1, m_1) (resp. (n_2, m_2)) be the degeneration type of the torsor f_1 (resp. f_2). Then we define the degeneration type of the cover $\tilde{f}$ to be $\{(n_1, m_1), (n_2, m_2)\}$. Note that the inequality $n_2 \geq n_1(p(p-1)+1)/p$ holds by 2.4.3.

2.6.3. Remark. Contrary to what happend for p-cyclic covers (cf. Remark 2.5.3) it is no more true that p^2-cyclic covers above formal boundaries are determined by their degeneration type as defined in 2.6.2.

2.6.4. Definition. Let $\{(n_1, m_1), (\tilde{n}_2, \tilde{m}_2)\}$ be a pair of pair of integers. We say that this pair is *admissible* if the following holds: either $n_1 = \tilde{n}_2 = 0$ in which case m_1 and $\tilde{m}_2$ are negative and $\tilde{m}_2 = \inf(m_1(p(p-1)+1), m_2p - m_1(p-1))$ for some negative integer m_2 prime to p or $n_1 = 0$ and $\tilde{n}_2 \neq 0$ in which case $m_1 \leq -1$ and $\tilde{m}_2 = pm_2 + m_1(p-1)$ for some integer m_2 prime to p or finally $n_1 \neq 0$ in which case either $\tilde{n}_2 > n_1(p(p-1)+1)/p$ and $\tilde{m}_2 = pm_2 - m_1(p-1)$ for some integer m_2 prime to p or $\tilde{n}_2 = n_1(p(p-1)+1)/p$ in which case $\tilde{m}_2 = m_1(p(p-1)+1)$ or $\tilde{m}_2 = m_2p - m_1(p-1))$ for some integer m_2 prime to p with $m_2 < pm_1$. It follows from 2.6.1 that the degeneration type $\{(n_1, m_1), (\tilde{n}_2, \tilde{m}_2)\}$ of the cover $\tilde{f}$ is an admissible pair.

III. Computation of vanishing cycles and examples for cyclic p-covers.

The main result of this section is Theorem 3.2.3 which gives a **formula** which compares the dimensions of the spaces of vanishing cycles in a Galois cover $\tilde{f}:\mathcal{Y}\to\mathcal{X}$ with group $\mathbb{Z}/p\mathbb{Z}$ between formal germs of R-curves where R is a complete discrete valuation ring of equal characteristic $p>0$ in terms of the degeneration type of $\tilde{f}$ above the boundaries of $\mathcal{X}$ as defined in 2.5.2. This formula enables one in principle to compare the dimensions of the spaces of vanishing cycles in a Galois cover $\tilde{f}:\mathcal{Y}\to\mathcal{X}$ with group a p-group as we will illustrate for the case of cyclic covers of degree p^2 in IV. In all this section we use the following notations: R is a complete discrete valuation ring of equal characteristic $p>0$. We denote by K the fraction field of R by π a uniformising parameter and k the residue field. We also denote by v_K the valuation of K which is normalised by $v_K(\pi)=1$. We assume also that the residue field k is *algebraically closed*.

3.1. By a (formal) R-curve we mean a (formal) R-scheme of finite type which is normal flat and whose fibres have dimension 1. For an R-scheme X we denote by $X_K := X\times_{\operatorname{Spec} R}\operatorname{Spec} K$ the *generic* fibre of X and $X_k := X\times_{\operatorname{Spec} R}\operatorname{Spec} k$ its *special* fibre. In what follows by a *(formal) germ* $\mathcal{X}$ of an R-curve we mean that $\mathcal{X} := \operatorname{Spec}\mathcal{O}_{X,x}$ is the (resp. $\mathcal{X} := \operatorname{Spf}\hat{\mathcal{O}}_{X,x}$ is the formal completion of the) spectrum of the local ring of an R-curve X at a closed point x. We refer to [S-1] 3.1 for the definition of the integers δ_x, r_x, and the genus g_x of the point x.

3.2. The compactification process. Let $\mathcal{X} := \operatorname{Spf}\hat{\mathcal{O}}_{X,x}$ be the formal germ of an R-curve at a closed point x with $\mathcal{X}_k$ reduced. Let $\tilde{f}:\mathcal{Y}\to\mathcal{X}$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z}$ and $\mathcal{Y}$ local. We assume that the special fibre of $\mathcal{Y}_k$ is reduced (this can always be achieved after a finite extension of R). We will construct a compactification of the above cover $\tilde{f}$ and as an application we will use this compactification in order to compute the arithmetic genus of the closed point of $\mathcal{Y}$. More precisely we will construct a Galois cover $f:Y\to X$ of degree p between proper algebraic R-curves, a closed point $y\in Y$ and its image $x=f(y)$, such that the formal germ of X (resp. of Y) at x (resp. at y) equals $\mathcal{X}$ (resp. $\mathcal{Y}$) and such that the Galois cover $f_x:\operatorname{Spf}\hat{\mathcal{O}}_{Y,y}\to\operatorname{Spf}\hat{\mathcal{O}}_{X,x}$ induced by f between the formal germs at y and x is isomorphic to the above given cover $\tilde{f}:\mathcal{Y}\to\mathcal{X}$. The construction of such a compactification has been done in [S-1] 3.3.1 in the inequal characteristic case. we first start with the case where the formal germ $\mathcal{X}$ has only one boundary.

3.2.1. Proposition. *Let $D := \operatorname{Spf} R < 1/T >$ be the formal closed disc centered at ∞ (cf. [B-L], 1, for the definition of $R<1/T>$). Let $\mathcal{D} := \operatorname{Spf} R[[T]]\{T^{-1}\}$ and let $\mathcal{D}\to D$ be the canonical morphism. Let $\tilde{f}:\mathcal{Y}\to\mathcal{D}$ be a non trivial torsor under a finite* *and flat R-group scheme of rank p such that the special fibre of $\mathcal{Y}$ is reduced. Then there exists a Galois cover $f : Y \to D$ with group $\mathbb{Z}/p\mathbb{Z}$ whose pull back to $\mathcal{D}$ is isomorphic to the above given torsor $\tilde{f}$. More precisely with the same notations introduced in 2.5 we have the following possibilities:*

a) *The torsor $\tilde{f}$ is étale and has a reduction of type $(0, -m)$. In this case consider the Galois cover $f : Y \to D$ given generically by the equation $Z^p - Z = 1/T^m$. This cover is an étale torsor and its special fibre $f_k : Y_k \to X_k$ is étale and Y_k is smooth. Moreover the genus of the smooth compactification of Y_k equals $(m-1)(p-1)/2$.*

b) *The cover $\tilde{f}$ is a torsor under the group scheme $\mathcal{M}_{n,R}$ for some positive integer n and has a reduction of type (n, m) for some integer m prime to p. Consider the following two cases:*

b-1) *$m > 0$. In this case consider the Galois cover $f : Y \to D$ given generically by the equation $Z^p - \pi^{n(p-1)}Z = T^m$. This cover is ramified above ∞ with conductor m and its special fibre $f_k : Y_k \to X_k$ is radicial. Moreover Y_k is smooth and its smooth compactification has genus 0.*

b-2) *$m < 0$. In this case consider the Galois cover $f : Y \to D$ given generically by the equation $Z^p - \pi^{n(p-1)}Z = T^m$. This cover is an étale torsor on the generic fibre and its special fibre $f_k : Y_k \to X_k$ is radicial. Moreover Y_k has a unique singular point y which is above ∞ and $g_y = (-m-1)(p-1)/2$.*

Proof. The proof is similar to the proof of proposition 3.3.1 in [S-1]. For the convenience of the reader we treat the case b-1). In this case consider the Galois p-cover $H \to \mathbb{P}^1_R$, with H normal, above the projective R-line with parameter T defined generically by the equation $Z^p - \pi^{n(p-1)}Z = T^m$. This cover is ramified on the generic fibre only above the point ∞ with conductor m from which we deduce that the genus of the generic fibre H_K of H equals $(m-1)(p-1)/2$. On the level of the special fibres the cover $H_k \to \mathbb{P}^1_k$ is an α_p torsor outside the point ∞ defined by the equation $z^p = t^m$. The genus of the singularity above the point $t = 0$ can be then easily calculated and equals $(m-1)(p-1)/2$ (cf. [Sa-1] 3.3.1). From this we deduce that H_k is smooth outside $t = 0$ since the arithmetic genus of H_K and H_k are equal.

In the next proposition we deal with the general case.

3.2.2. Proposition. *Let $\mathcal{X} := \operatorname{Spf} \hat{\mathcal{O}}_x$ be the formal germ of an R-curve at a closed point x and let $\{\mathcal{X}_i\}_{i=1}^n$ be the boundaries of $\mathcal{X}$. Let $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z}$ and with $\mathcal{Y}$ local. Assume that $\mathcal{Y}_k$ and $\mathcal{X}_k$ are reduced. Then there exists a Galois cover $f : Y \to X$ of degree p between proper algebraic R-curves Y and X, a closed point $y \in Y$ and its image $x = f(y)$, such that the formal germ of X (resp. of Y) at x (resp. at y) equals $\mathcal{X}$ (resp. equals $\mathcal{Y}$) and such that the Galois cover $\operatorname{Spf} \hat{\mathcal{O}}_{Y,y} \to \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$ induced by f between the formal germs at y and x is isomorphic to the above given cover*

$\tilde{f} : \mathcal{Y} \to \mathcal{X}$. *Moreover the formal completion of X along its special fibre has a covering which consists of n closed formal discs D_i which are patched with $\mathcal{X}$ along the boundaries $\mathcal{D}_i$ and the special fibre X_k of X consists of n smooth projective lines which intersect at the point x. In particular the arithmetic genus of X_K equals g_x.*

Proof. Similar to the proof of proposition 3.3.2 in [S-1].

The next result is the main one of this section. It provides an explicit **formula** which compares the dimensions of the spaces of vanishing cycles in a Galois cover of degree p between formal fibres of curves in equal characteristic $p > 0$.

3.2.3. Theorem. *Let $\mathcal{X} := \operatorname{Spf} \hat{\mathcal{O}}_x$ be the formal germ of an R-curve at a closed point x with $\mathcal{X}_k$ reduced. Let $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z}$ with $\mathcal{Y}$ local and $\mathcal{Y}_k$ reduced. Let $\{\wp_i\}_{i \in I}$ be the minimal prime ideals of $\hat{\mathcal{O}}_x$ which contain π and let $\mathcal{X}_i := \operatorname{Spf} \hat{\mathcal{O}}_{\wp_i}$ be the formal completion of the localisation of $\mathcal{X}$ at $\wp_i$. For each $i \in I$ the above cover $\tilde{f}$ induces a torsor $\tilde{f}_i : \mathcal{Y}_i \to \mathcal{X}_i$ under a finite and flat R-group scheme of rank p above the boundary $\mathcal{X}_i$ (cf. 2.5.1). Let (n_i, m_i) be the reduction type of $\tilde{f}_i$ (cf. 2.5.2). Let y be the closed point of $\mathcal{Y}$. Then one has the following* **"local Riemman-Hurwitz formula"***:*

$$2g_y - 2 = p(2g_x - 2) + d_\eta - d_s$$

Where d_η is the degree of the divisor of ramification in the morphism $\tilde{f}_K : \mathcal{Y}_K \to \mathcal{X}_K$ induced by $\tilde{f}$ on the generic fibres, where $\mathcal{X}_K := \operatorname{Spec}(\hat{\mathcal{O}}_x \otimes_R K)$ and $\mathcal{Y}_K := \operatorname{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y} \otimes_R K)$, and $d_s := \sum_{i \in I_{\text{et}}} (-m_i - 1)(p-1) + \sum_{i \in I_{\text{rad}}} (-m_i - 1)(p-1)$ where I^{rad} is the subset of I consisting of those i for which $n_i \neq 0$ and I^{et} is the subset of I consisting of those i for which $n_i = 0$ and $m_i \neq 0$.

Proof. The proof is similar, using 3.2.1, to the proof of theorem 3.4 in [S-1] with the appropriate modifications. We repeat briefly the argument for the convenience of the reader. By Proposition 3.2.2 one can compactify the above morphism $\tilde{f}$. More precisely there we constructed a Galois cover $f : Y \to X$ of degree p between proper algebraic R-curves a closed point $y \in Y$ and its image $x = f(y)$ such that the formal germ of X (resp. of Y) at x (resp. at y) equals $\mathcal{X}$ (resp. equals $\mathcal{Y}$) and such that the Galois cover $\operatorname{Spf} \hat{\mathcal{O}}_{Y,y} \to \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$ induced by f between the formal germs at y and x is isomorphic to the given cover $\tilde{f} : \mathcal{Y} \to \mathcal{X}$. The special fibre of $\mathcal{X}$ consists (by construction) of card(I)-distinct smooth projective lines which intersect at the closed point x. The formal completion of X along its special fibre has a covering which consists of card(I) formal closed unit discs which are patched with the formal fibre $\mathcal{X}$ along the boundaries $\mathcal{X}_i$. The above formula follows then by comparing the arithmetic genus of the generic fibre Y_K of Y and the arithmetic genus of its special fibre Y_k. Using the precise informations given in Proposition

3.2.1 one can easily deduce that $g(Y_K) = pg_x + (1-p) + d_\eta/2 + \sum_{i\in I_>}(m_i+1)(p-1)/2$ where $I_>$ is the subset of I consisting of those i for which the degeneration type above the boundary $\mathcal{X}_i$ is (n_i, m_i) with $n_i > 0$ and $m_i > 0$. On the other hand one has $g(Y_k) = g_y + \sum_{i\in I_<}(-m_i-1)(p-1)/2 + \sum_{i\in I_{\text{et}}}(-m_i-1)(p-1)$, where $I_<$ is the subset of I consisting of those i for which the degeneration type above the boundary $\mathcal{X}_i$ is (n_i, m_i) with $n_i > 0$ and $m_i < 0$, and I_{et} is the subset of I consisting of those i for which the degeneration type above the boundary $\mathcal{X}_i$ is $(0, m_i)$. Now since Y is flat we obtain $g(Y_K) = g(Y_k)$ and the above formula directly follows.

3.3. p-Cyclic covers above germs of semi-stable curves.

In what follows and as a consequence of theorem 3.2.3 we will deduce some results in the case of a Galois cover $\mathcal{Y} \to \mathcal{X}$ where $\mathcal{X}$ is the formal germ of a *semi-stable* R-curve at a closed point. These results will play an important role in V in order to exhibit and realise the degeneration data which describe the semi-stable reduction of Galois covers of degree p in equal characteristic p. We start with the case of a Galois cover of degree p above a germ of a smooth point.

3.3.1. Proposition. *Let $\mathcal{X} := \operatorname{Spf} R[[T]]$ be the formal germ of an R-curve at a* **smooth** *point x and let $\mathcal{X}_\eta := \operatorname{Spf} R[[T]]\{T^{-1}\}$ be the boundary of $\mathcal{X}$. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree p with $\mathcal{Y}$ local. Assume that the special fibre of $\mathcal{Y}$ is reduced. Let y be the unique closed point of $\mathcal{Y}_k$. Let d_η be the degree of the divisor of ramification in the morphism $f : \mathcal{Y}_K \to \mathcal{X}_K$. Then $d_\eta = r(p-1)$ is divisible by $p-1$. We distinguish two cases:*

1) $\mathcal{Y}_k$ is **unibranche** *at y. Let (n, m) be the degeneration type of f above the boundary $\mathcal{X}_\eta$ (cf. 2.5.2). Then necessarily $r + m - 1 \geq 0$, and $g_y = (r+m-1)(p-1)/2$.*

*2) $\mathcal{Y}_k$ has p-***branches** *at y. Then the cover f has an étale split reduction of type $(0, 0)$ on the boundary, i.e. the induced torsor above $\operatorname{Spf} R[[T]]\{T^{-1}\}$ is trivial, in which case $g_y = (r-2)(p-1)/2$.*

As an immediate consequence of 3.3.1 one can immediately see whether the point y is smooth or not. More precisely we have the following:

3.3.2. Corollary. *We use the same notation as in 3.3.1. Then y is a smooth point which is equivalent to $g_y = 0$ if and only if $r = 1 - m$ which implies that $m \leq 1$. In particular if f has a degeneration of type (n, m) on the boundary with $n > 0$ and $m > 0$ then this happend only if $r = 0$ and $m = 1$.*

Next we will give examples of Galois covers of degree p above the formal germ of a smooth point which cover all the possibilities for the genus and the degeneration type on the boundary. Both in 3.3.3 and 3.3.4 we use the same notations as in 3.3.1. We first begin with examples with genus 0.

3.3.3. Examples. The following are examples given by explicit equations of the different cases depending on the possible degeneration type over the boundary of Galois covers $f : \mathcal{Y} \to \mathcal{X}$ of degree p above $\mathcal{X} = \operatorname{Spf} R[[T]]$ and where $\mathbf{g_y} = \mathbf{0}$ (here y denotes the closed point of $\mathcal{Y}$).

1) For $m > 0$ an integer prime to p consider the cover given generically by the equation $X^p - X = T^{-m}$. Here $r = m + 1$ and this cover has a reduction of type $(0, -m)$ on the boundary.

2) For $\tilde{m} := -m$ a negative integer prime to p and a positive integer n consider the cover given generically by the equation $X^p - \pi^{n(p-1)} X = T^{\tilde{m}}$. Here $r = m + 1$ and this cover has a reduction of type $(n, \tilde{m})$ on the boundary.

3) For a positive integer n consider the cover given generically by the equation $X^p - \pi^{n(p-1)} X = T$. Here $r = 0$ and this cover has a reduction of type $(n, 1)$ on the boundary.

Next we give examples of Galois covers of degree p above formal germs of smooth points which lead to a singularity with positive genus.

3.3.4. Examples. The following are examples given by explicit equations of the different cases depending on the possible reduction type of Galois covers $f : \mathcal{Y} \to \mathcal{X}$ of degree p above $\mathcal{X} = \operatorname{Spf} R[[T]]$ and where $\mathbf{g_y} > \mathbf{0}$.

1) Let $m > 0$ and $m' > m$ be integers prime to p and consider the cover given generically by the equation $X^p - X = \pi/T^{m'} + 1/T^m$. This cover has a degeneration of type $(0, -m)$ on the boundary and the point y above x is singular and its genus equals $(m' - m)(p - 1)/2$.

2) Let m, m', and n be positive integers with m and m' prime to p and consider the cover given generically by the equation $X^p - X = T^m/\pi^{pn} + \pi/T^{m'}$. This cover has a degeneration of type (n, m) on the boundary and the point y above x is singular and its genus equals $(m' + m)(p - 1)/2$.

3) Let m, m' and n be positive integers such that m and m' are prime to p and $m' > m$. Consider the cover given generically by the equation $X^p - X = T^{-m}\pi^{-pn} + \pi/T^{m'}$. This cover has a degeneration of type $(n, -m)$ on the boundary and the point y above x is singular and its genus equals $(m' - m)(p - 1)/2$.

Next we examine the case of Galois covers of degree p above formal germs at **double points**.

3.3.5. Proposition. *Let $\mathcal{X} := \operatorname{Spf} R[[S, T]]/(ST - \pi^e)$ be the formal germ of an R-curve at an ordinary double point x of thickness e. Let $\mathcal{X}_1 := \operatorname{Spf} R[[S]]\{S^{-1}\}$ and $\mathcal{X}_2 := \operatorname{Spf} R[[T]]\{T^{-1}\}$ be the boundaries of $\mathcal{X}$. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z}$ and with $\mathcal{Y}$ local. Assume that the special fibre of $\mathcal{Y}$ is reduced. We assume that $\mathcal{Y}_k$ has* **two branches** *at the point y. Let $d_\eta := r(p-1)$ be the degree of the divisor of ramification* *in the morphism $f : Y_K \to X_K$. Let (n_i, m_i) be the degeneration type on the boundaries of $\mathcal{X}$ for $i = 1, 2$. Then necessarily $r + m_1 + m_2 \geq 0$ and $g_y = (r + m_1 + m_2)(p - 1)/2$.*

3.3.6. Proposition. *We use the same notations as in Proposition 3.3.5. We consider the remaining cases:*

1) $\mathcal{Y}_k$ **has** *$p + 1$* **branches at** *y in which case we can assume that $\mathcal{Y}$ is completely split above $\mathcal{X}_1$. Let (n_2, m_2) be the degeneration type on the second boundary $\mathcal{X}_2$ of $\mathcal{X}$. Then necessarily $r + m_2 - 1 \geq 0$ and $g_y = (r + m_2 - 1)(p - 1)/2$.*

2) $\mathcal{Y}_k$ **has** *$2p$* **branches at** *y in which case $\mathcal{Y}$ is completely split above the two boundaries of $\mathcal{X}$ and $g_y = (r - 2)(p - 2)/2$.*

With the same notations as in proposition 3.3.5 and as an immediate consequence one can recognise whether the point y is a double point or not. More precisely we have the following:

3.3.7. Corollary. *We use the same notations as in 3.3.5. Then y is an ordinary double point which is equivalent to $g_y = 0$ if and only if x is an ordinary double point of thickness divisible by p and $r = m_1 + m_2$. Moreover if $r = 0$ then $g_y = 0$ is equivalent to $m_1 + m_2 = 0$.*

Next we give examples of Galois covers of degree p above the formal germ of a double point which lead to singularities with genus 0, i.e. double points, and such that $r = 0$. These examples will be used in V in order to realise the "degeneration data" corresponding to Galois covers of degree p in equal characteristic $p > 0$.

3.3.8. Examples. The following are examples given by explicit equations of the different cases, depending on the possible degeneration type on the boundaries, of Galois covers $f : \mathcal{Y} \to \mathcal{X}$ of degree p above $\mathcal{X} = \operatorname{Spf} R[[S, T]]/(ST - \pi^e)$ with $\mathbf{r} = \mathbf{0}$ and where $\mathbf{g_y} = \mathbf{0}$ for a suitable choice of e. Note that $e = pt$ must be divisible by p. In all the following examples we have $r = 0$.

1) p-**Purity**: if f as above has an étale reduction type on the boundaries and $r = 0$ then f is necessarily étale and hence is completely split since $\mathcal{X}$ is strictly henselian.

2) Consider the cover given generically by the equation $X^p - X = 1/T^m = S^m/\pi^{mpt}$ where m is a positive integer prime to p which leads to a reduction on the boundaries of type $(0, -m)$ and (mt, m).

3) Let n and m be positive integers such that m is prime to p and $n - tm > 0$. Consider the cover given generically by the equation $X^p - X = T^m/\pi^{np} = S^{-m}/\pi^{p(n-tm)}$ which leads to a reduction on the boundaries of type (n, m) and $(n - tm, -m)$.

In fact one can describe Galois covers of degree p above formal germs of double points (in equal characteristic p) which are étale above the generic fibre and with genus 0. Namely they are all of the form given in the above examples 3.3.8. In particular these covers are uniquely determined up to isomorphism by their degeneration type on the boundaries. More precisely we have the following:

3.3.9. Proposition. *Let $\mathcal{X}$ be the formal germ of an R-curve at an ordinary double point x. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree p with $\mathcal{Y}_k$ reduced and local and with $f_K : \mathcal{Y}_K \to \mathcal{X}_K$* **étale**. *Let $\mathcal{X}_i$ for $i = 1, 2$ be the boundaries of $\mathcal{X}$. Let $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ be the torsors induced by f above $\mathcal{X}_i$ and let δ_i be the corresponding degree of the different (cf. 2.5.1). Let y be the closed point of $\mathcal{Y}$ and assume that $\mathbf{g_y} = \mathbf{0}$. Then there exists an isomorphism $\mathcal{X} \simeq \operatorname{Spf} R[[S,T]]/(ST - \pi^{tp})$ such that if say $\mathcal{X}_2$ is the boundary corresponding to the prime ideal (π, S) one of the following holds:*

a) *The cover f is generically given by the equation $X^p - X = 1/T^m = S^m/\pi^{mpt}$ where m is a positive integer prime to p. This cover leads to a reduction on the boundaries of $\mathcal{X}$ of type $(0, -m)$ and (mt, m). Here $t > 0$ can be any integer. In this case $\delta_1 = 0$ and $\delta_2 = mt(p-1)$.*

b) *The cover f is generically given by an equation $X^p - X = T^m/\pi^{np} = 1/\pi^{p(n-tm)}S^m$ where $m > 0$ is an integer prime to p and $n > 0$ such that $n - tm > 0$. This cover leads to a reduction on the boundaries of $\mathcal{X}$ of type (n, m) and $(n - tm, -m)$. In this case $\delta_1 = n(p-1)$ and $\delta_2 = (n - tm)(p-1)$.*

Proof. The proof is similar to the proof of 4.2.5 in the inequal characteristic case in [S-1].

3.3.10. variation of the different. The following result, which is a direct consequence of Proposition 3.3.9, describes the variation of the degree of the different from one boundary to another in a cover $f : \mathcal{Y} \to \mathcal{X}$ between formal germs at double points.

3.3.11. Proposition. *Let $\mathcal{X}$ be the formal germ of an R-curve at an ordinary double point x. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree p with $\mathcal{Y}_k$ reduced and local and with $f_K : \mathcal{Y}_K \to \mathcal{X}_K$* **étale**. *Let y be the closed point of $\mathcal{Y}$. Assume that $\mathbf{g_y} = \mathbf{0}$ which implies necessarily that the thickness $e = pt$ of the double point x is divisible by p. For each integer $0 < t' < t$ let $\mathcal{X}_{t'} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ at the ideal $(\pi^{pt'}, T)$. The special fibre of $\mathcal{X}_{t'}$ consists of a projective line $P_{t'}$ which meets two germs of double points x and x'. Let η be the generic point of $P_{t'}$ and let v_η be the corresponding discrete valuation of the function field of $\mathcal{X}$. Let $f_{t'} : \mathcal{Y}_{t'} \to \mathcal{X}_{t'}$ be the pull back of f which is a Galois cover of degree p and let $\delta(t')$ be the degree of the different induced by this cover above v_η (cf. 2.5.1). Also denote by $\mathcal{X}_i$ for $i = 1, 2$ the boundaries of $\mathcal{X}$. Let $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ be the torsors induced by f above $\mathcal{X}_i$. let (n_i, m_i) be their degeneration type and let δ_i be the corresponding degree of the different. Say $\delta_1 = \delta(0)$ $\delta_2 = \delta(t)$ and $\delta(0) \le \delta(t)$. We have $m := -m_1 = m_2$ say is positive. Then the following holds: for $0 \le t_1 \le t_2 \le t$ we have $\delta(t_2) = \delta(t_1) + m(p-1)(t_2 - t_1)$ and $\delta(t')$ is an increasing function of t'.*

IV. Computation of vanishing cycles and examples for p^2-cyclic covers.

4.1. In this section we use the same notations and hypothesis as in III and we compute the dimensions of the spaces of vanishing cycles arising form a cyclic p^2-cover above the formal germ of an R-curve. Further we provide examples for such covers above formal germs of semi-stable curves.

4.1.1. Proposition. *Let $\mathcal{X} := \operatorname{Spf} \hat{\mathcal{O}}_x$ be the formal germ of an R-curve at a closed point x with $\mathcal{X}_k$ reduced. Let $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group $\mathbb{Z}/p^2\mathbb{Z}$ with $\mathcal{Y}$ local and $\mathcal{Y}_k$ reduced. Let $\{\wp_i\}_{i\in I}$ be the minimal prime ideals of $\hat{\mathcal{O}}_x$ which contain π and let $\mathcal{X}_i := \operatorname{Spf} \hat{\mathcal{O}}_{\wp_i}$ be the formal completion of the localisation of $\mathcal{X}$ at $\wp_i$. For each $i \in I$ the above cover $\tilde{f}$ induces a p^2-cyclic cover $\tilde{f}_i : \mathcal{Y}_i \to \mathcal{X}_i$ above the boundary $\mathcal{X}_i$ (cf. 2.5.1). Let $\{(n_{i,1}, m_{i,1}), (n_{i,2}, m_{i,2})\}$ (resp. $\{(n_i, m_i)\}$) be the degeneration type of $\tilde{f}_i$ (cf. 2.6.2) if $\mathcal{Y}_{i,k}$ is irreducible (resp. if $\mathcal{Y}_{i,k}$ has p-components cf. 2.5.2). Let y be the closed point of $\mathcal{Y}$. Then one has the following* **"local Riemman-Hurwitz formula"**:

$$2g_y - 2 = p^2(2g_x - 2) + d_\eta - d_s$$

Where $d_\eta = pd_{\eta,1} + d_{\eta,2}$ is the degree of the divisor of ramification in the morphism $\tilde{f}_K : \mathcal{Y}_K \to \mathcal{X}_K$ induced by $\tilde{f}$ where $\mathcal{X}_K := \operatorname{Spec}(\hat{\mathcal{O}}_x \otimes_R K)$ and $\mathcal{Y}_K := \operatorname{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y} \otimes_R K)$ and $d_s := pd_{s,1} + d_{s,2}$ where $d_{s,1} = \sum_{i\in I_{\mathrm{et}}}(-m_{i,1} - 1)(p-1) + \sum_{i\in I_{\mathrm{rad}}}(-m_{i,1} - 1)(p-1)$ (resp. $d_{s,2} = \sum_{i\in I_{\mathrm{et}}}(-m_{i,2} - 1)(p-1) + \sum_{i\in \tilde{I}_{\mathrm{et}}}(-m_i - 1)(p-1) + \sum_{i\in I_{\mathrm{rad}}}(-m_{i,2} - 1)(p-1) + \sum_{i\in \tilde{I}_{\mathrm{rad}}}(-m_i - 1)(p-1)$) where I^{rad} is the subset of I consisting of those i for which $n_{i,1} \neq 0$ (resp. $n_{i,2} \neq 0$) and I^{et} is the subset of I consisting of those i for which $n_{i,1} = 0$ (resp. $n_{i,2} = 0$). Here $\tilde{I}$ denotes the subset of I consisting of those i for which $\mathcal{Y}_i$ has p-components.

Proof. Follows directly from 3.2.3

4.2. p^2-Cyclic covers above germs of semi-stable curves.

In what follows and as a consequence of 4.1.1 we will deduce some results in the case of a p^2-cyclic cover $f : \mathcal{Y} \to \mathcal{X}$ where $\mathcal{X}$ is the formal germ of a semi-stable R-curve at a closed point. We start with the case of a Galois cover of degree p^2 above a germ of a smooth point.

4.2.1. Proposition. *Let $\mathcal{X} := \operatorname{Spf} R[[T]]$ be the germ of a formal R-curve at a* **smooth** *point x and let $\mathcal{X}_\eta := \operatorname{Spf} R[[T]]\{T^{-1}\}$ be the boundary of $\mathcal{X}$. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree p^2 with $\mathcal{Y}$ local. Assume that the special fibre of $\mathcal{Y}$ is reduced. Let y be the unique closed point of $\mathcal{Y}_k$. Let d_η be the degree of the divisor of ramification in the morphism $f : \mathcal{Y}_K \to \mathcal{X}_K$. Then $d_\eta = r(p-1)$ is divisible by $p-1$. The following cases occur:*

1) $\mathcal{Y}_k$ is **unibranche** *at y. Let $\{(n_1, m_1), (n_2, m_2)\}$ be the degeneration type of f above the boundary $\mathcal{X}_\eta$ (cf. 2.5.1). Then necessarily $r + pm_1 + m_2 - p - 1 \geq 0$ and $g_y = (r + pm_1 + m_2 - p - 1)(p - 1)/2$.*

*2) $\mathcal{Y}_k$ has p-***branches*** at y. Let $\{(n, m)\}$ be the degeneration type of f above the boundary $\mathcal{X}_\eta$. Then necessarily $r + pm - p - 2 \geq 0$ and $g_y = (r + pm - p - 2)(p - 1)/2$.*

*3) $\mathcal{Y}_k$ has p^2-***branches*** at y. Then the cover f has an étale completely split reduction on the boundary, i.e. the induced torsor above $\operatorname{Spf} R[[T]]\{T^{-1}\}$ is trivial, in which case necessarily $r - 2p - 2 \geq 0$ and $g_y = (r - 2p - 2)(p - 1)/2$.*

As an immediate consequence of 4.2.1 one can immediately see whether the point y is smooth or not. More precisely we have the following:

4.2.2. Corollary. *We use the same notation as in 4.2.1. Then y is a smooth point which is equivalent to $g_y = 0$ if and only if we are in case 1) and $r + pm_1 + m_2 = p + 1$. In particular if f has a degeneration on the boundaries of type $\{(n_1, m_1), (n_2, m_2)\}$ with m_1 and m_2 positive then this happend only if $r = 0$ and $m_1 = m_2 = 1$.*

Next we will give examples of cyclic covers of degree p^2 above the formal germ of a smooth point which cover all the possibilities for the genus and the degeneration type on the boundary. Both in 4.2.3 and 4.2.4 we use the same notations as in 4.2.1. We first begin with examples with genus 0.

4.2.3. Examples. The following are examples given by explicit equations of the different cases, depending on the possible degeneration type over the boundary, of cyclic covers $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ of degree p^2 above $\mathcal{X} = \operatorname{Spf} R[[T]]$ and where $\mathbf{g_y = 0}$ (here y denotes the closed point of $\mathcal{Y}$).

1) Let m_1 and m_2 be two positive integers both prime to p and consider the cover which is generically given by the equations: $(X_1^p, X_2^p) - (X_1, X_2) = (1/T^{m_1}, f(T))$ where $f(T) = \sum_{i \geq -m_2} c_i T^i \in R[[T]][T^{-1}]$ and c_{m_2} is a unit in R. Assume that $m_2 \geq pm_1$. Then in this case $r = m_1 + 1 + pm_2 + p$ and this cover has a degeneration on the boundary of type $\{(0, -m_1), (0, \tilde{m}_2 := -pm_2 + m_1(p - 1))\}$. If $m_2 < pm_1$ then $r = m_1 + 1 + p^2 m_1 + p$ and this cover has a degeneration on the boundary of type $\{(0, -m_1), (0, -m_1(p(p - 1) + 1))\}$.

2) Let m_1 and m_2 be two positive integers both prime to p such that $m_2 \geq pm_1$ and consider the cover given generically by the equations: $(X_1^p, X_2^p) - (X_1, X_2) =$ $(1/T^{m_1}, f(T)/\pi^{pn})$ where n is a positive integer and $f(T) = \sum_{i \geq -m_2} c_i T^i \in R[[T]][T^{-1}]$ where c_{-m_2} is a unit in R. Then in this case $r = m_1 + 1 + pm_2 + p$ and this cover has a degeneration on the boundary of type $\{(0, -m_1), (n, \tilde{m}_2 := -pm_2 + m_1(p - 1))\}$. If $m_2 < pm_1$ then $\mathcal{Y}$ is not smooth.

3) Consider the cover given generically by the equations: $(X_1^p, X_2^p) - (X_1, X_2) =$ $(T/\pi^{n_1 p}, f(T)/\pi^{n_2 p})$ where $f(T) \in R[[T]][T^{-1}]$ has conductor $m_2 = 1$ (cf. 2.6.1 for the

definition of the conductor). Assume that $n_2p \geq n_1(p(p-1)+1)$. In this case $r=0$ and $m_1=m_2=1$ and this cover has a degeneration on the boundary of type $\{(n_1,1),(n_2,1)\}$.

4) Let m_1 and m_2 be two positive integers both prime to p. Consider the cover given generically by the equations $(X_1^p,X_2^p)-(X_1,X_2)=(T^{-m_1}/\pi^{n_1p}, f(T)/\pi^{pn_2})$ where $f(T)\in R[[T]][T^{-1}]$ has conductor $-m_2$ (cf. 2.6.1). First assume that $pn_2' := n_2p-n_1(p-1) \geq n_1(p(p-1)+1)$ and $m_2 \geq pm_1$. Then in this case $r=m_1+1+pm_2+p$ and this cover has a degeneration on the boundary of type $\{(n_1,-m_1),(n_2',\tilde{m}_2 := -m_2p+m_1(p-1))\}$. Further in this case if $m_2 < pm_1$ then $\mathcal{Y}$ is not smooth. Second assume that $n_2p-n_1(p-1) < pn_2' := n_1(p(p-1)+1)$ and $m_2 \leq pm_1$ then in this case $r=m_1+1+m_1p^2+p$ and this cover has a degeneration on the boundary of type $\{(n_1,-m_1),((n_2',\tilde{m}_2 := -m_1(p(p-1)+1))\}$. In this later case if $m_2 > pm_1$ then $\mathcal{Y}$ is not smooth.

4.2.4. Corollary. *Let $\mathcal{X} := \operatorname{Spf} R[[T]]$ be the germ of a formal R-curve at a* **smooth** *point x and let $\mathcal{X}_\eta := \operatorname{Spf} R[[T]]\{T^{-1}\}$ be the boundary of $\mathcal{X}$. Let $\tilde{f}:\mathcal{Y}\to\mathcal{X}$ be a Galois cover of degree p^2 with $\mathcal{Y}$ local and which ramifies above the generic fibre $\mathcal{X}_K$. Let $\{(n_1,m_1),(\tilde{n}_2,\tilde{m}_2)\}$ be the degeneration type of the cover $\tilde{f}$ above the formal boundary $\mathcal{X}_\eta$. Assume that $\mathcal{Y}$ is smooth. If $n_1=0$ and $n_2=0$ (resp. $n_1=0$ and $n_2\neq 0$) then necessarily $m_1\leq -1$ and $\tilde{m}_2=\inf(m_1(p(p-1)+1), m_2p+m_1(p-1))$ for some negative integer m_2 prime to p (resp. $\tilde{m}_2=m_2p+m_1(p-1)$ for some integer m_2 prime to p with $-m_2\geq pm_1$). If $n_1\neq 0$ then necessarily $m_1\leq -1$ and $\tilde{m}_2\leq -1$ and either $\tilde{n}_2 > n_1(p(p-1)+1)/p$ in which case $\tilde{m}_2 = pm_2-m_1(p-1)$ for some integer m_2 prime to p with $-m_2\geq -pm_1$ or $\tilde{n}_2=n_1(p(p-1)+1)/p$ in which case $\tilde{m}_2=m_1(p(p-1)+1)$ or $\tilde{m}_2=m_2p-m_1(p-1))$ for some integer m_2 prime to p such that $m_2<pm_1$.*

Proof. Follows easily from 2.6.1 and the examples in 4.2.3 which cover all the possibilities for f as above.

4.2.5. Definition. An admissible pair $\{(n_1,m_1),(\tilde{n}_2,\tilde{m}_2)\}$ as defined in 2.6.4 is said to satisfy the condition (*) if it satisfies the numerical contraints in 4.2.4 above.

Next we give examples of p^2-cyclic covers above formal germs of smooth points which lead to singularities with positive genus.

4.2.6. Examples. The following are examples given by explicit equations of the different cases, depending on the possible reduction type, of cyclic covers $\tilde{f}:\mathcal{Y}\to\mathcal{X}$ of degree p^2 above $\mathcal{X}=\operatorname{Spf} R[[T]]$ and where $\mathbf{g_y}>\mathbf{0}$.

1) Let m_1, m_1', $\tilde{m}_2$ and $\tilde{m}_2'$ be positive integers all prime to p such that $m_1<m_1'$ and $\tilde{m}_2<\tilde{m}_2'$. Consider the cover given generically by the equations $(X_1^p,X_2^p)-(X_1,X_2)=(\pi/T^{m_1'}+1/T^{m_1}, f(T))$ where $f(T)=\sum_{i\geq -\tilde{m}_2'} c_iT^i \in R[[T]][T^{-1}]$ with $c_{-\tilde{m}_2'}\in \pi R$ and $\bar{f}(t)=\sum_{i\geq -\tilde{m}_2}\bar{c}_it^i\in k[[T]]$ with $\bar{c}_{-\tilde{m}_2}\neq 0$ (i.e. $\tilde{m}_2$ is the conductor of f_2). Assume that $\tilde{m}_2' > pm_1'$. In this case $r=m_1'+pm_2'+p+1$ and this cover has a degeneration on the

boundary of type $\{(0,-m_1),(0,m_2:=-p\tilde{m}_2+m_1(p-1))\}$. Moreover $g_y=(p\tilde{m}_2'-p\tilde{m}_2+m_1'-m_1)(p-1)/2$.

2) Let m_1, m_1', $\tilde{m}_2$, and $\tilde{m}_2'$ be positive integers all prime to p such that $m_1<m_1'$ and $\tilde{m}_2<\tilde{m}_2'$. Consider the cover given generically by the equations $(X_1^p,X_2^p)-(X_1,X_2)=(\pi/T^{m_1'}+1/T^{m_1},f(T)/\pi^{pn})$ where n is a positive integer and $f(T)$ is as in 1). Assume that $\tilde{m}_2'>pm_1'$. In this case $r=m_1'+p\tilde{m}_2'+p+1$ and this cover has a degeneration on the boundary of type $\{(0,-m_1),(n,m_2:=-p\tilde{m}_2+m_1(p-1))\}$. Moreover $g_y=(p\tilde{m}_2'-p\tilde{m}_2+m_1'-m_1)(p-1)/2$.

3) Let m_1, m_1', $\tilde{m}_2$, and $\tilde{m}_2'$ be positive integers all prime to p such that $m_1<m_1'$ and $\tilde{m}_2<\tilde{m}_2'$. Consider the cover given by the equations $(X_1^p,X_2^p)-(X_1,X_2)=(\pi/T^{m_1'}+T^{-m_1}\pi^{n_1p},f(T)/\pi^{pn_2})$ where n_1 and n_2 are positive integers and $f(T)$ is as in 1). Assume that $\tilde{m}_2'>pm_1'$ in which case we have $r=m_1'+p\tilde{m}_2'+p+1$. We distinguish the following cases:

3.a) Assume that $pn_2':=n_2p-n_1(p-1)\geq n_1(p(p-1)+1)$. Then this cover has a degeneration on the boundary of type $\{(n_1,-m_1),(n_2',m_2:=-\tilde{m}_2p+m_1(p-1))\}$ and $g_y=(p\tilde{m}_2'-p\tilde{m}_2+m_1'-m_1)(p-1)/2$.

3.b) Assume that $n_2p-n_1(p-1)<pn_2':=n_1(p(p-1)+1)$ then this cover has a degeneration on the boundary of type $\{(n_1,-m_1),(n_2',m_2:=-m_1(p(p-1)+1))\}$ and $g_y=(p\tilde{m}_2'-p^2m_1+m_1'-m_1)(p-1)/2$.

4) Let m_1, m_1', $\tilde{m}_2$, and $\tilde{m}_2'$ be positive integers all prime to p such that $m_1<m_1'$ and $\tilde{m}_2<\tilde{m}_2'$. Consider the cover given generically by the equations $(X_1^p,X_2^p)-(X_1,X_2)=(\pi/T^{m_1'}+T^{-m_1}\pi^{n_1p},f(T)/\pi^{pn_2})$ where n_1 and n_2 are positive integer and $f(T)$ is as in 1). Assume that $\tilde{m}_2'<pm_1'$ in which case we have $r=m_1'+m_1'p^2+p+1$. We distinguish the following cases:

4.a) Assume that $pn_2':=n_2p-n_1(p-1)\geq n_1(p(p-1)+1)$. Then this cover has a degeneration on the boundary of type $\{(n_1,-m_1),(n_2',m_2:=-\tilde{m}_2p+m_1(p-1))\}$ and $g_y=(p^2m_1'-\tilde{m}_2p+m_1'-m_1)(p-1)/2$.

4.b) Assume that $n_2p-n_1(p-1)<pn_2':=n_1(p(p-1)+1)$ then this cover has a degeneration on the boundary of type $\{(n_1,-m_1),(n_2',m_2:=-m_1(p(p-1)+1))\}$ and $g_y=(p^2m_1'-m_1p^2+m_1'-m_1)(p-1)/2$.

Next we examine the case of cyclic covers of degree p^2 above formal germs at **double points**.

4.2.7. Proposition. *Let $\mathcal{X}:=\operatorname{Spf}R[[S,T]]/(ST-\pi^e)$ be the formal germ of an R-curve at an ordinary double point x of thickness e and let $\mathcal{X}_1:=\operatorname{Spf}R[[S]]\{S^{-1}\}$ and $\mathcal{X}_2:=\operatorname{Spf}R[[T]]\{T^{-1}\}$ be the boundaries of $\mathcal{X}$. Let $f:\mathcal{Y}\to\mathcal{X}$ be a Galois cover with group $\mathbb{Z}/p^2\mathbb{Z}$ and with $\mathcal{Y}$ local. Assume that the special fibre of $\mathcal{Y}$ is reduced. We assume that $\mathcal{Y}_k$ has* **two branches** *at the point y. Let $d_\eta:=r(p-1)$ be the degree of the* *divisor of ramification in the morphism $f: Y_K \to X_K$. Let $\{(n_{i,1}, m_{i,1}), (n_{i,2}, m_{i,2})\}$ be the degeneration type on the boundaries of $\mathcal{X}$ for $i = 1, 2$. Then necessarily $r + pm_{1,1} + pm_{2,1} + m_{1,2} + m_{2,2} \geq 0$ and $g_y = (r + pm_{1,1} + pm_{2,1} + m_{1,2} + m_{2,2})(p-1)/2$.*

Next we examine the remaining cases.

4.2.8. Proposition. *We use the same notation as in Proposition 4.2.7. We consider the remaining cases:*

1) $\mathcal{Y}_k$ **has** *$p+1$* **branches at** *y in which case we can assume that the degeneration type of f above the boundaries $\mathcal{X}_1$ is $\{(n_1, m_1)\}$. Then necessarily $r + pm_{2,1} + p + m_1 + m_{2,2} \geq 0$ and $g_y = (r + pm_{2,1} + p + m_1 + m_{2,2})(p-1)/2$.*

2) $\mathcal{Y}_k$ **has** *$2p$* **branches at** *y in which case we can assume that $n_{1,1} = n_{2,1} = m_{1,1} = m_{2,1} = 0$. Then necessarily $r + m_{2,2} + m_{2,1} - 2p \geq 0$ and $g_y = (r + m_{2,2} + m_{2,1} - 2p)(p-1)/2$.*

3) $\mathcal{Y}_k$ **has** *$p^2 + 1$* **branches at** *y and we can assume that f is completely split above the boundary $\mathcal{X}_1$. Then necessarily $r + pm_{2,1} + m_{2,2} - p - 1 \geq 0$ and $g_y = (r + pm_{2,1} + m_{2,2} - p - 1)(p-1)/2$.*

4) $\mathcal{Y}_k$ **has** *$p^2 + p$* **branches at** *y and we can assume that f is completely split above the boundary $\mathcal{X}_1$. Then necessarily $r + m_{2,2} - 2p - 1 = 0$ and $g_y = (r + m_{2,2} - 2p - 1)(p-1)/2$.*

5) $\mathcal{Y}_k$ **has** *$2p^2$* **branches at** *y and f is completely split on both boundaries of $\mathcal{X}$. In this case $g_y = (r - 2p - 2)(p-1)/2$.*

With the same notation as in proposition 4.2.7 and as an immediate consequence one can recognise whether the point y is a double point or not. More precisely we have the following:

4.2.9. Corollary. *We use the same notations as in 4.2.5. Then y is an ordinary double point which is equivalent to $g_y = 0$ if and only if x is an ordinary double point of thickness divisible by p^2 and $r + pm_{1,1} + pm_{2,1} + m_{1,2} + m_{2,2} = 0$. Moreover if $r = 0$ then $g_y = 0$ is equivalent to $m_{1,1} + m_{2,1} = 0$ and $m_{1,2} + m_{2,2} = 0$.*

Next we give examples of p^2-cyclic covers above the formal germ of a double point which lead to singularities with genus 0, i.e. double points, and such that $r = 0$. These examples will be used in VI in order to realise the "degeneration data" corresponding to cyclic covers of degree p^2 in equal characteristic $p > 0$.

4.2.10. Examples. The following are examples given by explicit equations of the different cases, depending on the possible degeneration type on the boundaries, of cyclic covers $\tilde{f}: \mathcal{Y} \to \mathcal{X}$ of degree p^2 above $\mathcal{X} = \operatorname{Spf} R[[S,T]]/(ST - \pi^e)$ with $\mathbf{r} = \mathbf{0}$ and where $\mathbf{g_y} = \mathbf{0}$ for a suitable choice of e. Note that $e = p^2 t$ must be divisible by p^2. In all the following examples we have $r = 0$.

1) p-**Purity**: if $\tilde{f}$ as above has an étale reduction type on the boundaries and $r = 0$

then f is necessarily étale and hence is completely split since $\mathcal{X}$ is strictly henselian.

2) Consider the cover given generically by the equation $(X_1^p, X_2^p) - (X_1, X_2) = (1/T^{m_1}, f(T))$ where m_1 is a positive integer prime to p and $f(T) \in R[[T]][T^{-1}]$ is such that its image $\bar{f}(t)$ modulo π equals $\sum_{i \geq -m_2} c_i t^i$ where $m_2 \geq 0$ is prime to p. This cover leads to a reduction on the boundaries of type $\{(0, -m_1), (0, \tilde{m}_2 := -pm_2 + m_1(p-1))\}$ and $\{(tm_1 p, m_1), (pm_2 t - (p-1)m_1 t, pm_2 - m_1(p-1))\}$ if $m_2 > m_1 p$, and of type $\{(0, -m_1), (0, \tilde{m}_2 := -m_1(p(p-1)+1))\}$ and $\{(tm_1 p, m_1), (tm_1(p(p-1)+1), m_1(p(p-1)+1))\}$ if $m_2 \leq m_1 p$.

3) Consider the cover given generically by the equation $X_1^p - X_1 = 1/T^{m_1}$ and $X_2^p - X_2 = f(T)/\pi^{pn} + p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} X_1^{pk} (-X_1)^{p-k}$ with $f(T) \in R[[T]][T^{-1}]$ is such that its image $\bar{f}(t) = \sum_{i \geq m'} c_i t^i$ modulo π is not a p-power, and n is a positive integer such that $pn - m_2 p^2 t > 0$. Let m_2 be the conductor of $f(T)$. If $n - m_2 pt > m_1 p^2 t$ then $\mathcal{Y}$ is smooth and this cover leads to a reduction on the boundaries of type $\{(0, -m_1), (n, pm_2 + m_1(p-1))\}$ and $\{(m_1 pt, m_1), (n - m_2 pt - (p-1)m_1 t, -pm_2 - m_1(p-1))\}$. Further in this case if $n - m_2 pt \leq m_1 p^2 t$ then $\mathcal{Y}$ is not smooth.

4) Consider the cover given generically by the equation $X_1^p - \pi^{n_1(p-1)} X_1 = T^{m_1}$ for some integer m_1 prime to p and:

$$X_2^p - X_2 = f(T)/\pi^{pn_2} + p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} X_1^{pk} (-X_1)^{p-k} / \pi^{n_1(p(p-k)+k)}$$

where $f(T) = \sum_{i \in I} a_i T^i \in R[[T]][T^{-1}]$ is not a p-power and where n_1 (resp. n_2) is a positive integer such that $n_1 - ptm_1 > 0$ (resp. $n_2 - ptm_2 > 0$). Here m_2 denotes the conductor of f_2. We distinguish the following cases:

4-1) $pn_2 - (p-1)n_1 > n_1(p(p-1)+1)$. If $pn_2 - m_2 p^2 t - (p-1)(n_1 - m_1 pt) > (n_1 - m_1 pt)(p(p-1)+1)$ which necessarily implies that $m_1 p > m_2$ then $\mathcal{Y}$ is smooth and this cover leads to a reduction on the boundaries of type $\{(n_1, m_1), ((n_2 p - n_1(p-1))/p, m_2 p - m_1(p-1))\}$ and $\{(n_1 - m_1 pt, -m_1), ((n_2 p - m_2 p^2 t - (p-1)(n_1 - m_1 pt))/p, -m_2 p + m_1(p-1))\}$. Further in this case if $pn_2 - m_2 p^2 t - (p-1)(n_1 - m_1 pt) \leq (n_1 - m_1 pt)(p(p-1)+1)$ then $\mathcal{Y}$ is not smooth.

4-2) $pn_2 - (p-1)n_1 < n_1(p(p-1)+1)$. If $pn_2 - m_2 p^2 t - (p-1)(n_1 - m_1 pt) < (n_1 - m_1 pt)(p(p-1)+1)$ (resp. $pn_2 - m_2 p^2 t - (p-1)(n_1 - m_1 pt) = (n_1 - m_1 pt)(p(p-1)+1)$) which necessarily implies that $m_1 p < m_2$ (resp. $m_1 p > m_2$) then $\mathcal{Y}$ is smooth and this cover leads to a reduction on the boundaries of type $\{(n_1, m_1), (n_1(p(p-1)+1)/p, m_1(p(p-1)+1))\}$ and $\{(n_1 - m_1 pt, -m_1), ((n_1 - m_1 pt)(p(p-1)+1)/p, -m_1(p(p-1)+1))\}$. Further in this case if $pn_2 - m_2 p^2 t - (p-1)(n_1 - m_1 pt) > (n_1 - m_1 pt)(p(p-1)+1)$ then $\mathcal{Y}$ is not smooth.

4-3) $pn_2 - (p-1)n_1 = n_1(p(p-1)+1)$. If $pn_2 - m_2 p^2 t - (p-1)(n_1 - m_1 pt) > (n_1 - m_1 pt)(p(p-1)+1)$ which is equivalent to $m_2 < m_1 p$ (resp. $pn_2 - m_2 p^2 t - (p-1)(n_1 - m_1 pt) < (n_1 - m_1 pt)(p(p-1)+1)$ which is equivalent to $m_2 > m_1 p$) then $\mathcal{Y}$ is smooth and this cover leads to a reduction on the boundaries of type $\{(n_1, m_1), (n_1(p(p-1)+1)/p, m_2p - m_1(p-1))\}$ and $\{(n_1 - m_1pt, -m_1), ((n_2p - m_2p^2t - (p-1)(n_1 - m_1pt))/p, -m_2p + m_1(p-1))\}$ (resp. $\{(n_1, m_1), (n_1(p(p-1)+1)/p, m_1(p(p-1)+1))\}$ and $\{(n_1 - m_1pt, -m_1), ((n_1 - m_1pt)(p(p-1)+1)/p, -m_1(p(p-1)+1))\}$).

In fact one can describe cyclic covers of degree p^2 above formal germs of double points (in equal characteristic p) which are étale above the generic fibre and with genus 0. Namely they are all of the form given in the above examples 4.2.10. More precisely we have the following:

4.2.11. Proposition. *Let $\mathcal{X}$ be the formal germ of an R-curve at an ordinary double point x. Let $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ be a cyclic cover of degree p^2 with $\mathcal{Y}_k$ reduced and local and with $f_K : \mathcal{Y}_K \to \mathcal{X}_K$* **étale**. *Let $\mathcal{X}_i$ for $i = 1, 2$ be the boundaries of $\mathcal{X}$. Let y be the closed point of $\mathcal{Y}$ and assume that $\mathbf{g_y} = \mathbf{0}$. Then there exists an isomorphism $\mathcal{X} \simeq \operatorname{Spf} R[[S,T]]/(ST - \pi^{tp^2})$ such that the following holds:*

a) *The cover $\tilde{f}$ is generically given by the equation:*

$$(X_1^p, X_2^p) - (X_1, X_2) = (1/T^{m_1}, f(T))$$

where m_1 is a positive integer prime to p and $f(T) \in R[[T]][T^{-1}]$ is such that its image $\bar{f}(t)$ modulo π equals $\sum_{i \geq -m_2} c_i t^i$, with $c_{-m_2} \neq 0$, where $m_2 \geq 0$ is prime to p, which leads to a reduction on the boundaries of type $\{(0, -m_1), (0, \tilde{m}_2 := -pm_2 + m_1(p-1))\}$ and $\{(tm_1p, m_1), (pm_2t - (p-1)m_1t, pm_2 - m_1(p-1))\}$ if $m_2 > m_1p$ and of type $\{(0, -m_1), (0, \tilde{m}_2 := -m_1(p(p-1)+1))\}$ and $\{(tm_1p, m_1), (tm_1(p(p-1)+1), m_1(p(p-1)+1))\}$ if $m_2 \leq m_1p$.

b) *The cover $\tilde{f}$ is given generically by the equations:*

$$X_1^p - X_1 = 1/T^{m_1}$$

and:

$$X_2^p - X_2 = f(T)/\pi^{pn} + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}X_1^{pk}(-X_1)^{p-k}$$

with $f(T) \in R[[T]][T^{-1}]$ is such that its image $\bar{f}(t) = \sum_{i \geq m'} c_i t^i$ modulo π is not a p-power and n is a positive integer such that $n - m_2pt > m_1p^2t$. Here m_2 denotes the conductor of $f(T)$. this cover leads to a reduction on the boundaries of type $\{(0, -m_1), (n, pm_2 + m_1(p-1))\}$ and $\{(m_1pt, m_1), (n - m_2pt - (p-1)m_1t, -pm_2 - m_1(p-1))\}$.

c) *The cover $\tilde{f}$ is given generically by the equations:*

$$X_1^p - \pi^{n_1(p-1)}X_1 = T^{m_1}$$

for some integer m_1 prime to p and:

$$X_2^p - X_2 = f(T)/\pi^{pn_2} + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k}X_1^{pk}(-X_1)^{p-k}/\pi^{n_1(p(p-k)+k)}$$

where $f(T) = \sum_{i\in I} a_i T^i \in R[[T]][T^{-1}]$ is not a p-power and where n_1 (resp. n_2) is a positive integer such that $n_1 - ptm_1 \geq 0$ (resp. $n_2 - ptm_2 \geq 0$). Here m_2 denotes the conductor of f_2 and the following cases occur:

c-1) *$pn_2 - (p-1)n_1 > n_1(p(p-1)+1)$ and $pn_2 - m_2p^2t - (p-1)(n_1 - m_1pt) > (n_1 - m_1pt)(p(p-1)+1)$ (which necessarily implies that $m_1p > m_2$), in which case this cover leads to a reduction on the boundaries of type $\{(n_1, m_1), ((n_2p - n_1(p-1))/p, m_2p - m_1(p-1))\}$ and $\{(n_1 - m_1pt, -m_1), ((n_2p - m_2p^2t - (p-1)(n_1 - m_1pt))/p, -m_2p + m_1(p-1))\}$.*

c-2) *$pn_2 - (p-1)n_1 < n_1(p(p-1)+1)$ and $pn_2 - m_2p^2t - (p-1)(n_1 - m_1pt) < (n_1 - m_1pt)(p(p-1)+1)$ (resp. $pn_2 - m_2p^2t - (p-1)(n_1 - m_1pt) = (n_1 - m_1pt)(p(p-1)+1)$) which necessarily implies that $m_1p < m_2$ (resp. $m_1p > m_2$). This cover leads to a reduction on the boundaries of type $\{(n_1, m_1), (n_1(p(p-1)+1)/p, m_1(p(p-1)+1))\}$ and $\{(n_1 - m_1pt, -m_1), ((n_1 - m_1pt)(p(p-1)+1)/p, -m_1(p(p-1)+1))\}$.*

c-3) *$pn_2 - (p-1)n_1 = n_1(p(p-1)+1)$ and $pn_2 - m_2p^2t - (p-1)(n_1 - m_1pt) > (n_1 - m_1pt)(p(p-1)+1)$, which is equivalent to $m_2 < m_1p$, (resp. $pn_2 - m_2p^2t - (p-1)(n_1 - m_1pt) < (n_1 - m_1pt)(p(p-1)+1)$ which is equivalent to $m_2 > m_1p$). This cover leads to a reduction on the boundaries of type $\{(n_1, m_1), (n_1(p(p-1)+1)/p, m_2p - m_1(p-1))\}$ and $\{(n_1 - m_1pt, -m_1), ((n_2p - m_2p^2t - (p-1)(n_1 - m_1pt))/p, -m_2p + m_1(p-1))\}$ (resp. $\{(n_1, m_1), (n_1(p(p-1)+1)/p, m_1(p(p-1)+1))\}$ and $\{(n_1 - m_1pt, -m_1), ((n_1 - m_1pt)(p(p-1)+1)/p, -m_1(p(p-1)+1))\}$).*

V. Semi-stable reduction of cyclic p-covers above formal germs of curves in equal characteristic $p > 0$.

In all this paragraph we use the following notations: R is a complete discrete valuation ring of equal characteristic p with residue field k which we assume to be **algebraically closed** and fraction field $K := \operatorname{Fr} R$. We denote by π a uniformising parameter of R.

5.1. Let $\mathcal{X} := \operatorname{Spec} \hat{\mathcal{O}}_{X,x}$ be the formal germ of an R-curve X at a closed point x and let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group G such that $\mathcal{Y}$ is normal and local. In this paper we are mainly concerned with the case where G is cyclic of order p or p^2. It follows then easily from the theorem of semi-stable reduction for curves (cf. [De-Mu]), as well as the compactification process in 3.2, that after eventually a finite extension R' of R with fractions field K' the formal germ $\mathcal{Y}$ has a semi-stable reduction. More precisely there exists a birational and proper morphism $\tilde{f} : \tilde{\mathcal{Y}} \to \mathcal{Y}'$ where $\mathcal{Y}'$ is the normalisation of $\mathcal{Y} \times_R R'$ such that $\tilde{\mathcal{Y}}_{K'} \simeq \mathcal{Y}'_{K'}$ and the following conditions hold:

(i) The special fibre $\tilde{\mathcal{Y}}_k := \tilde{\mathcal{Y}} \times_{\operatorname{Spec} R'} \operatorname{Spec} k$ of $\tilde{\mathcal{Y}}$ is reduced.
(ii) $\tilde{\mathcal{Y}}_k$ has only ordinary double points as singularities.

Moreover there exists such a semi-stable model $\tilde{f} : \tilde{\mathcal{Y}} \to \mathcal{Y}'$ which is **minimal** for the above properties. In particular the action of G on $\mathcal{Y}'$ extends to an action on $\tilde{\mathcal{Y}}$. Let $\tilde{\mathcal{X}}$ be the quotient of $\tilde{\mathcal{Y}}$ by G which is a semi-stable model of $\mathcal{X}$. One has the following commutative diagram:

$$\begin{array}{ccc} \tilde{\mathcal{Y}} & \xrightarrow{\tilde{f}} & \mathcal{Y}' \\ {\scriptstyle g}\downarrow & & {\scriptstyle f'}\downarrow \\ \tilde{\mathcal{X}} & \xrightarrow{\tilde{g}} & \mathcal{X}' \end{array}$$

One can moreover choose the semi-stable models $\tilde{\mathcal{Y}}$ and $\tilde{\mathcal{X}}$ such that the set of points $B_{K'} := \{x_{i,K'}\}_{1\le i\le s}$ consisting of the branch locus in the morphism $f'_K : \mathcal{Y}'_{K'} \to \mathcal{X}'_{K'}$ specialise in **smooth distincts** points of $\mathcal{X}'_k$ and one can choose such $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{Y}}$ which are minimal for these properties. The fibre $\tilde{g}^{-1}(x)$ of the closed point x in $\tilde{\mathcal{X}}$ is a tree Γ of projective lines. This tree is canonically endowed with some "*degeneration data*" that we will exhibit below and in the next section, in the cases where $G \simeq \mathbb{Z}/p\mathbb{Z}$ and $G \simeq \mathbb{Z}/p^2\mathbb{Z}$, and which take into account the geometry of the special fibre $\tilde{\mathcal{Y}}_k$ of $\tilde{\mathcal{Y}}$. This will follow mainly from the results in II, III and IV.

5.2. We will use the same notations as in 5.1. We assume that $G \simeq \mathbb{Z}/p\mathbb{Z}$. We consider the case where $\mathcal{X} \simeq \operatorname{Spf} A$ is the formal germ of a semi-stable R-curve at a **smooth** point x i.e. $A \simeq R[[T]]$. Let R' be a finite extension of R as in 5.1 and let π' be a uniformiser of R'. Below we exhibit the **degeneration data** associated to the semi-stable reduction of $\mathcal{Y}$.

Deg.1. Let $\wp := (\pi')$ be the ideal of $A' := A \otimes_R R'$ generated by π' and let $\hat{A}'_\wp$ be the completion of the localisation of A' at $\wp$. Let $\mathcal{X}'_\eta := \operatorname{Spf} \hat{A}'_\wp$ be the formal boundary of $\mathcal{X}'$ and let $\mathcal{X}'_\eta \to \mathcal{X}'$ be the canonical morphism. Consider the following cartesian diagram:

$$\begin{array}{ccc} {\mathcal{Y}'}_\eta & \xrightarrow{f_\eta} & \mathcal{X}'_\eta \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \xrightarrow{f'} & \mathcal{X}' \end{array}$$

Then $f_\eta : \mathcal{Y}'_\eta \to \mathcal{X}'_\eta$ is a torsor under a commutative finite and flat R'-group scheme $\mathcal{M}_{n,R'}$ of rank p (cf. 2.5.1) for some integer $n \ge 0$. Let (n,m) be the degeneration type of the torsor f_η (cf. 2.5.2) which is canonically associated to f. The arithmetic genus g_y of the point y equals $(r+m-1)(p-1)/2$ (cf. 3.3.1) where $d_\eta := r(p-1)$ is the degree of the divisor of ramification in the morphism $f'_{K'} : \mathcal{Y}'_{K'} \to \mathcal{X}'_{K'}$.

Deg.2. The fibre $\tilde{g}^{-1}(x)$ of the closed point x of $\mathcal{X}'$ in $\tilde{\mathcal{X}}$ is a tree Γ of projective lines. Let $\text{Vert}(\Gamma) := \{X_i\}_{i\in I}$ be the set of irreducible components of $\tilde{g}^{-1}(x)$ which are the vertices of the tree Γ. The tree Γ is canonically endowed with an origin vertex X_{i_0} which is the unique irreducible component of $\tilde{g}^{-1}(x)$ which meets the point x. We fix an orientation of the tree Γ starting from X_{i_0} in the direction of the ends.

Deg.3. For each $i \in I$ let $\{x_{i,j}\}_{j\in S_i}$ be the set of points of X_i in which specialise some point of $B_{K'}$ (S_i may be empty). Also let $\{z_{i,j}\}_{j\in D_i}$ be the set of double points of $\tilde{\mathcal{X}}_k$ supported by X_i. In particular $x_{i_0,j_0} := x$ is a double point of $\tilde{\mathcal{X}}_k$. We denote by B_k the set of all points $\cup_{i\in I}\{x_{i,j}\}_{j\in S_i}$, which is the set of specialisation of the branch locus $B_{K'}$, and by D_k the set of double points of $\tilde{\mathcal{X}}_k$.

Deg.4. Let $\mathcal{U} := \tilde{\mathcal{X}} - \{B_k \cup D_k\}$. Let $\{\mathcal{U}_i\}_{i\in I}$ be the set of connected components of $\mathcal{U}$. The restriction $f_i : \mathcal{V}_i \to \mathcal{U}_i$ of f' to $\mathcal{U}_i$ is a torsor under a commutative finite and flat R'-group scheme $\mathcal{M}_{n_i,R'}$ of rank p for some integer $n_i \geq 0$ (cf. 2.2.1) and $f_{i,k} : \mathcal{V}_{i,k} \to \mathcal{U}_{i,k} := \mathcal{U}_i \times_{R'} k$ is a torsor under the k-group scheme $\mathcal{M}_{n_i,R'} \times_{R'} k$ which is either étale isomorphic to $(\mathbb{Z}/p\mathbb{Z})_k$ or radicial isomorphic to $(\alpha_p)_k$. Further when we move in the graph Γ (following the above fixed orientation) from a fixed vertex X_i in the direction of a vertex $X_{i'}$ such that $n_{i'} = 0$ then the corresponding integers n_i decreases strictly as follows from 3.3.10.

Deg.5. Each smooth point $x_{i,j} \in B_k$ is endowed via f with a degeneration data on the boundary of the formal fibre at $x_{i,j}$, as in Deg.1 above, and which satisfy certain *compatibility conditions*. More precisely for each point $x_{i,j}$ we have the reduction type $(n_{i,j} := n_i, m_{i,j})$ on the boundary of the formal fibre at this point induced by g and such that $r_{i,j} = -m_{i,j} + 1$ where $r_{i,j}(p-1)$ is the contribution to d_η of the point which specialise into $x_{i,j}$. In particular $m_{i,j} \leq -1$ since $r_{i,j} \neq 0$.

Deg.6. Each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ of $\tilde{X}$ with origin vertex X_i and terminal vertex $X_{i'}$ is endowed with degeneration data $(n_{i,j} := n_i, m_{i,j})$ and $(n_{i',j'} := n_{i'}, m_{i',j'})$ induced by g on the two boundaries of the formal fibre at this point and we have $m_{i,j} + m_{i',j'} = 0$ (cf. 3.3.7). Let $e_{i,j}$ be the thikness of the double point $z_{i,j}$. Then $e_{i,j} = pt_{i,j}$ is necessarily divisible by p and we have $n_i - n_{i'} = t_{i,j}m_{i,j}$ as follows from 3.3.10.

Deg.7. It follows after easy calculation that:

$$g_y = \sum_{i\in I_{\text{et}}} (-2 + \sum_{j\in S_i} (m_{i,j}+1) + \sum_{j\in D_i} (m_{i,j}+1))(p-1)/2$$

where I_{et} denote the subset of I consisting of those i for which the torsor f_i is étale (i.e. $n_i = 0$).

5.2.1. Example. In the following we give an example where one can exhibit the degeneration data associated to a Galois cover $f : \mathcal{Y} \to \mathcal{X}$ of degree p where $\mathcal{X} \simeq \operatorname{Spf} R[[T]]$ is the formal germ of a smooth point. More precisely for $m > 0$ an integer such that both m and $m+1$ are prime to p consider the cover given generically by the equation $X^p - X = (T^{-m} + \pi T^{-m-1})$. Here $r = m+2$ and this cover has a reduction of type $(0, -m)$ on the boundary. In particular the arithmetic genus g_y of the closed point y of $\mathcal{Y}$ equals $(p-1)/2$. The degeneration data associated to the above cover consists necessarily of a tree with only one vertex and no edges i.e. a unique projective line X_1 with a marked point x_1 and an étale torsor $f_1 : V_1 \to U_1 := X_1 - \{x_1\}$ above U_1 with conductor 2 at the point x_1.

The above considerations lead naturally to the following abstract geometric and combinatorial definition of degeneration data.

5.2.2. Definition. A **simple degeneration data** Deg(x) of type $(r, (n, m))$ and rank p consists of the following data:

Deg.1. G_k is a commutative finite and flat k-group scheme of rank p which is either étale if $n = 0$, or radicial of type α_p otherwise, $r \geq 0$ is an integer, and m is an integer prime to p such that $r + m - 1 \geq 0$.

Deg.2. $\Gamma := X_k$ is an oriented tree of projective lines over k with vertices $\operatorname{Vert}(\Gamma) := \{X_i\}_{i \in I}$ which is endowed with an origin vertex X_{i_0} and a marked point $x := x_{i_0,j_0}$ on X_{i_0}. We denote by $\{z_{i,j}\}_{j \in D_i}$ the set of double points or (non oriented) edges of Γ which are supported by X_i. We further assume that the orientation of Γ is in the direction going form X_{i_0} towards its ends.

Deg.3. For each vertex X_i of Γ is given a set (may be empty) of smooth marked points $\{x_{i,j}\}_{j \in S_i}$.

Deg.4. For each $i \in I$ is given a torsor $f_i : V_i \to U_i := X_i - \{\{x_{i,j}\}_{j \in S_i} \cup \{z_{i,j}\}_{j \in D_i}\}$ under a commutative finite and flat k-group scheme $G_{k,i}$ of rank p, which is either étale or radicial of type α_p, with V_i **smooth**. Moreover for each $i \in I$ is given an integer n_i which equals 0 if f_i is étale and is positive otherwise.

Deg.5. For each $i \in I$ and $j \in S_i$ are given integers $m_{i,j}$ where $m_{i,j}$ is the conductor of the torsor f_i at the point $x_{i,j}$ (cf. [S] 1.5) with $m_{i_0,j_0} = -m$. We assume further that $m_{i,j} \leq -1$ if $n_i > 0$.

Deg.6. For each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ is given an integer $m_{i,j}$ (resp. $m_{i',j'}$) prime to p where $m_{i,j}$ (resp. $m_{i',j'}$) is the conductor of the torsor f_i (resp. $f_{i'}$) at the point $z_{i,j}$ (resp. $z_{i',j'}$) (cf. [S] 1.3 and 1.5). These data must satisfy $m_{i,j} + m_{i',j'} = 0$.

Deg.7. For each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ of Γ with origin vertex X_i is given an integer $e_{i,j} = pt_{i,j}$ divisible by p such that with the same notations as above we have $n_i - n_{i'} = m_{i,j} t_{i,j}$. Moreover associated to x is an integer $e = pt$ such that $n - n_{i_0} = mt$.

Deg.8. Let I_{et} denote the subset of I consisting of those i for which $G_{k,i}$ is étale. Then the following equality should hold: $(r+m-1)(p-1)/2 = \sum_{i\in I_{\text{et}}}(-2+\sum_{j\in S_i}(m_{i,j}+1)+\sum_{j\in D_i}(m_{i,j}+1))(p-1)/2$. The integer $g := (r+m-1)(p-1)/2$ is called the **genus** of the degeneration data Deg(x).

There is a natural notion of isomorphism of simple degeneration data of a given type and rank p. We will denote by $\mathbf{Deg_p}$ the set of isomorphism classes of simple degeneration data of rank p. The discussion in 5.2 can be reinterpreted as follows:

5.2.3. Proposition. *Let $\mathcal{X}$ be the germ of a formal R-curve at a smooth point x and let $f : \mathcal{Y} \to \mathcal{X}$ be a cyclic p-cover with $\mathcal{Y}$ normal and local. Then one can associate to f canonically a simple degeneration data* $\operatorname{Deg}(x) \in \mathbf{Deg_p}$ *which describes the semi-stable reduction of $\mathcal{Y}$. In other words there exists a canonical "specialisation" map* $\operatorname{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mathbb{Z}/p\mathbb{Z}) \to \mathbf{Deg_p}$ *where L is the function field of the geometric fibre of $\mathcal{X}$.*

Reciprocally we have the following result of realisation for degeneration data for such covers:

5.2.4. Theorem. *The above specialisation map* $\operatorname{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mathbb{Z}/p\mathbb{Z}) \to \mathbf{Deg_p}$ *defined in 5.2.3 is surjective.*

Proof. Consider a degeneration data $\operatorname{Deg}(x) \in \mathbf{Deg_p}$. We have to show that $\operatorname{Deg}(x)$ is associated to some cyclic p-cover above the formal germ of a smooth R-curve after eventually enlarging R. We assume that the degeneration data is of type $(r,(n,m))$. We treat only the case $n = 0$, the case where $n > 0$ is treated in a similar way. The proof is done by induction on the length of the tree Γ of $\operatorname{Deg}(x)$. Assume first that the tree Γ has minimal length and consists of one irreducible component $X = \mathbb{P}^1_k$ with one marked (double) point x and r smooth distinct marked points $\{x_i\}_{i=1}^r$. Let $U := X - \{x, x_i\}_i$ and let $\bar{f} : V \to U$ be the torsor given by the data Deg .4 which is necessarily an α_p-torsor by 3.3.8 1) (i.e. the integer $n_i := n'$ associated to the vertex X in Deg .4 is non zero). First for each $i \in \{1, ..., r\}$ consider the formal germ $\mathcal{X}_i := \operatorname{Spf} R[[T_i]]$ and the cyclic p-cover $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ given by the equation $Y_i^p - \pi^{n'} Y_i = T_i^{m_i}$ where m_i is the "conductor" associated to the point x_i in Deg .5 and n' is the positive integer associated to $\bar{f}$ in Deg .4. Let $\mathcal{X}$ be a formal projective R-line with special fibre X. Let $X' := X - \{x\}$ and let $\mathcal{X}'$ (resp $\mathcal{U}$) be the formal fibre of X' (resp. of U) in $\mathcal{X}$. The torsor $\bar{f}$ is given by an equation $t^p = \bar{u}$ where $\bar{u}$ is a regular function on U. Let u be a regular function on $\mathcal{U}$ which lifts $\bar{u}$. Then the cover $f : \mathcal{V} \to \mathcal{U}$ given by the equation $Y^p - \pi^{n'(p-1)} Y = u$ is a torsor under the R-group scheme $\mathcal{M}_n$ which lifts the α_p-torsor $\bar{f}$. By construction the torsor f has a reduction on the formal boundary at each point x_i of type (n', m_i) which coincides with the degeneration type of the cover f_i above the boundary of $\mathcal{X}_i$. The technique of formal patching (cf. [S-1], 1) allows then one to construct a p-cyclic cover $f' : \mathcal{Y}' \to \mathcal{X}'$ which restricted to $\mathcal{U}$ is isomorphic to f' and restricted to $\mathcal{X}_i$, for each $i \in \{i, ..., r\}$, is isomorphic to f_i (cf. loc. cit.). Let $\mathcal{X}_1 \to \mathcal{X}$ be the blow up of $\mathcal{X}$ at the point x and let X_1 be the exeptional fibre in $\mathcal{X}_1$ which meets X at the double point x. Let $e = pt$ be the integer associated to the marked double point x via Deg .7. After enlarging R we can assume that the double point x of $\mathcal{X}_1$ has thikness e. We have $-n' = mt$ by assumption. Let $\mathcal{X}'_1$ be the formal fibre of $X'_1 := X_1 - \{x\}$ in $\mathcal{X}_1$. Let $f'_1 : \mathcal{Y}'_1 \to \mathcal{X}'_1$ be the étale $\mathbb{Z}/p\mathbb{Z}$-torsor given by the equation $Y'^p - Y' = h^m$ where h is a "parameter" on $\mathcal{X}'_1$. Further let $\mathcal{X}_{1,x} \simeq \operatorname{Spf} R[[S,T]]/(ST - \pi^{pt})$ be the formal germ of $\mathcal{X}_1$ at the double point x. Consider the cover $f_{1,x} : \mathcal{Y}_{1,y} \to \mathcal{X}_{1,x}$ given by the equation $Y^p - Y = S^m = \pi^{ptm} T^{-m}$. Then $\mathcal{Y}_{1,y}$ is the formal germ of a double point of thikness t (cf. 3.3.9. a). Moreover the cover f'_1 (resp. f') has the same degeneration type (by construction) on the boundary corresponding to the double point x as the degeneration type of the cover $f_{1,x}$ above the formal boundary with parameter T (resp. above the formal boundary with parameter S). A second application of the formal patching techniques allows one to construct a p-cyclic cover $f_1 : \mathcal{Y}_1 \to \mathcal{X}_1$ which restricted to $\mathcal{X}'$ (resp. $\mathcal{X}'_1$ and $\mathcal{X}_{1,x}$) is isomorphic to f' (resp. to f'_1 and f_x). Let $\tilde{\mathcal{X}}$ be the R-curve obtained by contracting the irreducible component X in $\mathcal{X}_1$. We denote the image of the double point x in $\tilde{\mathcal{X}}$ simply by x. The cover $f_1 : \mathcal{Y}_1 \to \mathcal{X}_1$ induces canonically a p-cyclic cover $\tilde{f} : \tilde{\mathcal{Y}} \to \tilde{\mathcal{X}}$ above $\tilde{\mathcal{X}}$. Let $\mathcal{X}_x \simeq R[[T]]$ be the formal germ of $\tilde{\mathcal{X}}$ at the smooth point x. Then $\tilde{f}$ induces canonically a p-cyclic cover $f_x : \mathcal{Y}_y \to \mathcal{X}_x$ where y is the closed point of $\mathcal{Y}$ above x. Now it is easy to see that the degeneration data associated to f_x via 5.2.3 is isomorphic to the degeneration data $\operatorname{Deg}(x)$ we started with. Finally the proof in the general case is very similar and is left to the reader. The only modification in the proof above is that one has to considers p-cyclic covers $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ above the formal germs $\mathcal{X}_i := \operatorname{Spf} R[[T_i]]$ which one obtains by induction hypothesis as realisation of the degeneration data, induced by $\operatorname{Deg}(x)$, on the subtrees Γ_i of Γ which starts from the edge x_i in the direction of the ends and which clearly has length smaller than the length of Γ.

5.2.5. Remarks.

1. One can also define in the same way as in 5.2.2 and using the results of II, III, and IV, the set of isomorphism classes of "double" degeneration data associated to the minimal semi-stable model of p-cyclic covers $f : \mathcal{Y} \to \mathcal{X}$ above the formal germ of a double point. Moreover one can prove, in a similar way as in 5.2.4, a result of realisation for such a degeneration data.

2. In [M] Maugeais proved (in equal characteristic $p > 0$) in theorem 5.4 a global result of lifting for finite "admissible" covers of degree p between semi-stable curves. The methods used in the proof of 5.2.4 are essentially the same as he uses but more direct in the sens that the lemmas 4.2, 4.4, and corollary 4.3 he uses are avoided and we use instead our results 3.3.3 and 3.3.8 which are a direct consequence of the computation of vanishing cycles.

3. It is easy to construct examples of p-cyclic covers $f : \mathcal{Y} \to \mathcal{X}$ above the forml germ of a smooth point $\mathcal{X}$ where the special fibre $\mathcal{Y}_k$ is singular and unibranche at the closed point y of $\mathcal{Y}$ and such that the configuration of the special fibre of a semi-stable model $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$ is not a tree-like. More precisely consider the simple degeneration data $\operatorname{Deg}(x)$ of type (n, m), with $n > 0$, $m > 0$, and $n = mt$, which consists of a graph Γ with two vertices X_1 and X_2 linked by a unique edge $\tilde{x}$ with given marked points $x = x_1$ on X_1 and x_2 on X_2. Further X_1 is the original component of Γ. As part of the data are given étale torsors of rank p: $f_1 : V_1 \to U_1 := X_1 - \{x_1\}$ with conductor m at $x = x_1$ and $f_2 : V_2 \to U_2 := X_2 - \{x_2\}$ with conductor m' at x_2. Also is given the thikness $e = pt$ at the "double" point x with $n = tm$. Then it follows from 5.2.4 that there exists after eventually a finite extension of R a Galois cover $f : \mathcal{Y} \to \mathcal{X}$ of degree p above the formal germ $\mathcal{X} \simeq \operatorname{Spf} R[[T]]$ at the smooth R-point x such that the simple degeneration data associated to the above cover f is the above given one. Moreover by construction the singularity of the closed point y of $\mathcal{Y}$ is unibranche and the configuration of the semi-stable reduction of $\mathcal{Y}$ consists of two projective curves which meet at p-double points (the above cover will be étale in reduction above the double point $\tilde{y}$). In particular one has $p - 1$ cycles in this configuration.

VI. Semi-stable reduction of cyclic p^2-covers above formal germs of curves in equal characteristic $p > 0$.

6.1. In all this paragraph we use the same notations as in V.

6.2. Let $\mathcal{X} := \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$ be the formal germ of an R-curve X at a closed point x and let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group G such that $\mathcal{Y}$ is normal and local. We assume that $G \simeq \mathbb{Z}/p^2\mathbb{Z}$ is cyclic of order p^2 and we use the same notations as in 5.1 for the minimal semi-stable model $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$. We consider the case where $\mathcal{X} \simeq \operatorname{Spf} R[[T]]$ is the formal germ of a semi-stable R-curve at a **smooth** point x. Let R' be a finite extension of R as in 5.1 and let π' be a uniformiser of R'. Below we exhibit the **degeneration data** associated to the semi-stable reduction of $\mathcal{Y}$ and which are consequences of the results in II, III, and IV. In all what follows, and in order to make the statements of our results simpler, we make the following assumption (**): *the special fibre of $\mathcal{Y}$ is irreducible and for each irreducible component X_i of the special fibre $\tilde{\mathcal{X}}$ the fibre $g^{-1}(X_i)$ of X_i in $\tilde{\mathcal{Y}}$ is irreducible.*

Deg.1. Let $\wp := (\pi')$ be the ideal of $A' := A \otimes_R R'$ generated by π' and let $\hat{A}'_\wp$ be the completion of the localisation of A' at $\wp$. Let $\mathcal{X}'_\eta := \operatorname{Spf} \hat{A}'_\wp$ be the boundary of $\mathcal{X}'$ and let $\mathcal{X}'_\eta \to \mathcal{X}'$ be the canonical morphism. Consider the following cartesian diagram:

$$\begin{array}{ccc} \mathcal{Y}'_\eta & \xrightarrow{f_\eta} & \mathcal{X}'_\eta \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \xrightarrow{f'} & \mathcal{X}' \end{array}$$

Then $f_\eta : \mathcal{Y}'_\eta \to \mathcal{X}'_\eta$ is a finite cyclic cover of degree p^2. Let $\{(n_1, m_1), (n_2, m_2)\}$ be the degeneration type of the cover f (cf. 2.6.2) which is canonically associated to f and which is an admissible pair by 2.6.4. The arithmetic genus g_y of the point y equals $(r + pm_1 + m_2 - p - 1)(p-1)/2$ (cf. 4.2.1) where $d_\eta := r(p-1)$ is the degree of the divisor of ramification in the morphism $f'_{K'} : \mathcal{Y}'_{K'} \to \mathcal{X}'_{K'}$.

Deg.2. The fibre $\tilde{g}^{-1}(x)$ of the closed point x of $\mathcal{X}'$ in $\tilde{\mathcal{X}}$ is a tree Γ of projective lines. Let $\operatorname{Vert}(\Gamma) := \{X_i\}_{i \in I}$ be the set of irreducible components of $\tilde{g}^{-1}(x)$ which are the vertices of the tree Γ. The tree Γ is canonically endowed with an origin vertex X_{i_0} which is the unique irreducible component of $\tilde{g}^{-1}(x)$ which meets the point x. We fix an orientation of the tree Γ starting from X_{i_0} towards its ends.

Deg.3. For each $i \in I$ let $\{x_{i,j}\}_{j \in S_i}$ be the set of points of X_i in which specialise some point of $B_{K'}$ (S_i may be empty). Also let $\{z_{i,j}\}_{j \in D_i}$ be the set of double points of $\tilde{\mathcal{X}}_k$ supported by X_i. In particular $x_{i_0, j_0} := x$ is a double point of $\tilde{\mathcal{X}}_k$. We denote by B_k the set of all points $\cup_{i \in I} \{x_{i,j}\}_{j \in S_i}$, which is the set of specialisation of the branch locus $B_{K'}$, and by D_k the set of double points of $\tilde{\mathcal{X}}_k$.

Deg.4. Let $\mathcal{U} := \tilde{\mathcal{X}} - \{B_k \cup D_k\}$. Let $\{\mathcal{U}_i\}_{i \in I}$ be the set of connected components of $\mathcal{U}$. The restriction $f_i : \mathcal{V}_i \to \mathcal{U}_i$ of f' to $\mathcal{U}_i$ is a finite cyclic cover of degree p^2 which factorises canonically as $\mathcal{V}_i \xrightarrow{f_{i,2}} \mathcal{V}'_i \xrightarrow{f_{i,1}} \mathcal{U}_i$ where $f_{i,2}$ (resp. $f_{i,1}$) is a torsor under the R-group scheme $\mathcal{M}_{n_{i_2}}$ (resp. a torsor under the R-group scheme $\mathcal{M}_{n_{i_1}}$). Moreover to the cover f_i is canonically associated a degeneration data $\mathcal{D}_i$ of type A, B, or C and which describes the special fibre $f_{i,k} : \mathcal{V}_{i,k} \to \mathcal{U}_{i,k} := \mathcal{U}_i \times_{R'} k$ of f_i (cf. 2.4.8). The pair (n_{i_1}, n_{i_2}) satisfies $n_{i_2} \geq n_{i_1}(p(p-1)+1)$ by 2.4.3.

Deg.5. Each smooth point $x_{i,j} \in B_k$ is endowed via f with a degeneration data on the boundary of the formal fibre at $x_{i,j}$, as in Deg.1 above, and which satisfy certain *compatibility conditions*. More precisely for each point $x_{i,j}$ we have the reduction type $\{(n_{i_1,j_1} := n_{i_1}, m_{i_1,j_1}), (n_{i_2,j_2} := n_{i_2}, m_{i_2,j_2})\}$ on the boundary of the formal fibre at this point induced by g, which is an admissible pair satisfying the condition (*) in 4.2.5, and such that $r_{i,j} = -pm_{i_1,j_1} - m_{i_2,j_2} + p + 1$ where $r_{i,j}(p-1)$ is the contribution to d_η of the point which specialise into $x_{i,j}$.

Deg.6. Each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ of Γ with origin vertex X_i and terminal vertex $X_{i'}$ is endowed with degeneration data $\{(n_{i_1,j_1} := n_{i_1}, m_{i_1,j_1}), (n_{i_2,j_2} := n_{i_2}, m_{i_2,j_2})\}$ and $\{(n_{i'_1,j'_1} := n_{i'_1}, m_{i'_1,j'_1}), (n_{i'_2,j'_2} := n_{i'_2}, m_{i'_2,j'_2})\}$ induced by g on the two boundaries of the formal fibre at this point. These pairs are admissible. Moreover we have $m_{i_1,j_1} + m_{i'_1,j'_1} = 0$ and $m_{i_2,j_2} + m_{i'_2,j'_2} = 0$ (cf. 4.2.9). Let $e_{i,j}$ be the thikness of the double point $z_{i,j}$. Then $e_{i,j} = p^2 t_{i,j}$ is necessarily divisible by p^2 and we have $n_{i_1} - n_{i'_1} = pt_{i,j} m_{i_1,j_1}$ and $n_{i_2} - n_{i'_2} = t_{i,j} m_{i_2,j_2}$ as follows from 3.3.10.

Deg.7. It follows after easy calculation that:
$g_y = \sum_{i \in \tilde{I}} (-2 + \sum_{j \in S_i}(d_{i,j}) + \sum_{j \in D_i}(d_{i,j}))(p-1)/2$ where $\tilde{I}$ denote the subset of I consisting of those i for which the cover f_i has a reduction of type (étale, étale) or (étale, radicial) (cf. 2.4.8). Here $d_{i,j} = (m_{i_1} + 1) + p(pm_{i_1} + 1)$ if $pm_{i_1} > m_{i_2}$ and $d_{i,j} = (m_{i_1} + 1) + p(m_{i_2} + 1)$ otherwise in the case of (étale, étale) reduction type (resp. $d_{i,j} = (m_{i_1} + 1)$ in the case of (étale, radicial) reduction type).

The above considerations lead naturally to the following abstract geometric and combinatorial definition of degeneration data.

6.2.1. Definition. A **simple irreducible degeneration data** Deg(x) of type $(r, \{(n_1, m_1), (n_2, m_2)\})$ and rank p^2 consists of the following data:

Deg.1. The pair (n_1, n_2) satisfies $n_2 \geq n_1(p(p-1)+1)/p$, $r \geq 0$ is an integer, and m_1 (resp. m_2) is an integer prime to p such that $r + pm_1 + m_2 - p - 1 \geq 0$. We further assume that the pair $\{(n_1, m_1), (n_2, m_2)\}$ is admissible (cf. 2.6.4).

Deg.2. $\Gamma := X_k$ is an oriented tree of projective lines over k with vertices $\mathrm{Vert}(\Gamma) := \{X_i\}_{i \in I}$ which is endowed with an origin vertex X_{i_0} and a marked point $x := x_{i_0,j_0}$ on X_{i_0}. We denote by $\{z_{i,j}\}_{j \in D_i}$ the set of double points or (non oriented) edges of Γ which are supported by X_i. Further for each vertex X_i of Γ is given a set (may be empty) of smooth marked points $\{x_{i,j}\}_{j \in S_i}$. We further assume that the orientation of Γ is in the direction going form X_{i_0} towards its ends.

Deg.3. For each $i \in I$ is given a degeneration data $\mathcal{D}_i$ of type A, B, or C as in 2.4.8 associated to $U_i := X_i - \{\{x_{i,j}\}_{j \in S_i} \cup \{z_{i,j}\}_{j \in D_i}\}$ which is of type (étale, étale), (étale, radicial), or (radicial, radicial) respectively. Moreover for each $i \in I$ is given a pair of integers (n_{i_1}, n_{i_2}) which equals $(0,0)$ and $(0, n_{i_2})$ in the cases A and B respectively and which satisfies the inequality $n_{i_2} \geq n_{i_1}(p(p-1)+1)/p$.

Deg.3'. To the degeneration data $\mathcal{D}_i$ above and the pair of integers (n_{i_1}, n_{i_2}) is associated a finite cover $\bar{f}_i : V_i \to U_i$ of degeree p^2 which factorises as $V_i \xrightarrow{\bar{f}_{i,2}} V_{i,1} \xrightarrow{\bar{f}_{i,1}} U_i$ where $\bar{f}_{i,2}$ and $\bar{f}_{i,1}$ have the structure of a torsor under a finite and flat group scheme of rank p. In the case where $\mathcal{D}_i$ is of type A or B then $\bar{f}_i$ is simply the corresponding torsor. If $\mathcal{D}_i$ is of type C given by a pair $(\bar{u}_1, \bar{u}_2)$ of regular functions on U_i, defined up to addition of $(\bar{v}_1^p, \bar{v}_2^p)$, and a tuple $(\bar{c}_1, ..., \bar{c}_{p-1})$ of regular functions on U_i. Then the cover $\bar{f}_i$ is defined by the equations: $t_1^p = \bar{u}_1$ and $t_2^p = \bar{u}_2 - \bar{c}_1^p t_1 - 2\bar{c}_2^p t_1^{p+1} - ... - (p-1)\bar{c}_{p-1} t_1^{p(p-2)+1}$ if $n_{i_2} > n_{i_1}(p(p-1)+1)/p$ (resp. $t_2^p = \bar{u}_2 - \bar{c}_1^p t_1 - 2\bar{c}_2^p t_1^{p+1} - ... - (p-1)\bar{c}_{p-1} t_1^{p(p-2)+1} - t_1^{p(p-1)+1}$ if $n_{i_2} = n_{i_1}(p(p-1)+1)/p$).

Deg.4. For each $i \in I$ and $j \in S_i$ are given integers $(m_{i_1,j_1}, m_{i_2,j_2})$ such that m_{i_1,j_1} (resp. m_{i_2,j_2}) is the conductor of the torsor $\bar{f}_{i,1}$ (resp. the torsor $\bar{f}_{i,2}$) above in Deg $3'$ at the point $x_{i,j}$ (resp. the point of $V_{i,1}$ above $x_{i,j}$). we assume that the pair $\{(n_{i_1}, m_{i_1,j_1}), (n_{i_2}, m_{i_2,j_2})\}$ satisfies the condition (*) in 4.2.5. Moreover we assume that $(m_{i_{01},j_{01}}, m_{i_{02},j_{02}})$
$= (-m_1, -m_2)$.

Deg.5. For each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ is given a pair of integers $(m_{i_1,j_1}, m_{i_2,j_2})$ (resp. $(m_{i'_1,j'_1}, m_{i'_2,j'_2})$) prime to p such that m_{i_1,j_1} and m_{i_2,j_2} are the conductors of the torsor $\bar{f}_{i,1}$ (resp. the torsor $\bar{f}_{i,2}$) above at the point $z_{i,j}$ (resp. $m_{i'_1,j'_1}$ and $m_{i'_2,j'_2}$ are the conductors of the torsor $\bar{f}_{i',1}$ (resp. the torsor $\bar{f}_{i',2}$) above at the point $z_{i,j}$). We assume further that the pair $\{(n_{i_1}, m_{i_1,j_1}), (n_{i_2}, m_{i_2,j_2})\}$ (resp.$\{(n_{i'_1}, m_{i'_1,j'_1}), (n_{i'_2}, m_{i'_2,j'_2})\}$) is admissible. These data must satisfy $m_{i_1,j_1} + m_{i'_1,j'_1} = m_{i_2,j_2} + m_{i'_2,j'_2} = 0$.

Deg.6. For each double point $z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ of Γ with origin vertex X_i is given an integer $e_{i,j} = p^2 t_{i,j}$ divisible by p^2 such that with the same notations as above we have $n_{i_1} - n_{i'_1} = p t_{i,j} m_{i_1,j_1}$ and $n_{i_2} - n_{i'_2} = t_{i,j} m_{i_2,j_2}$.

Deg.7. Let $\tilde{I}$ denote the subset of I consisting of those i for which the degeneration data $\mathcal{D}_i$ f_i is of type (étale, étale) or (étale, radicial). Then the following equality should hold: $(r + pm_1 + m_2 - p - 1)(p-1)/2 = \sum_{i \in I_{\text{et}}} (-2 + \sum_{j \in S_i} (d_{i,j}) + \sum_{j \in D_i} (d_{i,j}))(p-1)/2$ where $d_{i,j} = (m_{i_1} + 1) + p(pm_{i_1} + 1)$ if $pm_{i_1} > m_{i_2}$ and $d_{i,j} = (m_{i_1} + 1) + p(m_{i_2} + 1)$ otherwise in the case of (étale, étale) type (resp. $d_{i,j} = (m_{i_1} + 1)$ in the case of (étale, radicial) type).

We further make the following important assumption as part of the degeneration data:

Deg.8. For each $i \in I$ let U_i be as in Deg .3 and let $\mathcal{U}_i$ be a formal affine R-scheme with special fibre U_i. We assume that for each $i \in I$ there exists a lifting of the degeneration data $\mathcal{D}_i$ as in 2.4.9 to a p^2-cyclic cover $f_i : \mathcal{V}_i \to \mathcal{U}_i$, which is an étale torsor above the generic fibre of $\mathcal{U}_i$, such that the following "compatibility" condition holds: for each double point $z := z_{i,j} = z_{i',j'} \in X_i \cap X_{i'}$ the restriction of f_i (resp. restriction of $f_{i'}$) to the boundary $\mathcal{X}_{z,i} \simeq \operatorname{Spf} R[[T]]\{T^{-1}\}$ corresponding to the double point z (resp. the boundary $\mathcal{X}_{z,i'} \simeq \operatorname{Spf} R[[S]]\{S^{-1}\}$ corresponding to the point z) are Galois isomorphic.

There is a natural notion of isomorphism of simple irreducible degeneration data of a given type and rank p^2. We will denote by $\mathbf{Deg_{p^2}}$ the set of isomorphism classes of simple irreducible degeneration data of rank p^2. The discussion in 6.2 can be reinterpreted as follows:

6.2.2. Proposition. *Let $\mathcal{X}$ be the germ of a formal R-curve at a smooth point x and let $f : \mathcal{Y} \to \mathcal{X}$ be a cyclic p^2-cover with $\mathcal{Y}$ normal and local. We assume that the condition $(**)$ in 6.2 holds. Then one can associate to f canonically a simple irreducible degeneration data $\operatorname{Deg}(x) \in \mathbf{Deg_{p^2}}$. In other words there exists a canonical "specialisation" map $\operatorname{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mathbb{Z}/p^2\mathbb{Z})^{\text{ir}} \to \mathbf{Deg_{p^2}}$ where L is the function field of the geometric fibre of $\mathcal{X}$ and $H^1_{\text{et}}(\operatorname{Spec} L, \mathbb{Z}/p^2\mathbb{Z})^{\text{ir}}$ is the subset of $H^1_{\text{et}}(\operatorname{Spec} L, \mathbb{Z}/p^2\mathbb{Z})$ consisting of those covers satisfying the condition $(**)$.*

Reciprocally we have the following result of realisation for degeneration data for such covers:

6.2.3. Theorem. *The above specialisation map $\operatorname{Sp} : H^1_{\text{et}}(\operatorname{Spec} L, \mathbb{Z}/p^2\mathbb{Z})^{\text{ir}} \to \mathbf{Deg_{p^2}}$ defined in 6.2.2 is surjective.*

Proof. Consider a simple irreducible degeneration data $\operatorname{Deg}(x) \in \mathbf{Deg_{p^2}}$. We have to show that $\operatorname{Deg}(x)$ is associated to some cyclic p^2-cover above the formal germ of a smooth R-curve after eventually enlarging R. We assume that the degeneration data is of type $(r, \{(n_1', m_1), (n_2', m_2)\})$. We treat only the case where $n_1' = n_2' = 0$. The remaining cases are treated in a similar way. As in the proof of 5.2.4 we proceed by induction on the length of the tree Γ of $\operatorname{Deg}(x)$. Assume first that the tree Γ has minimal length and consists of one irreducible component $X = \mathbb{P}^1_k$ with one marked (double) point $x = x_{i_0}$ and r smooth marked points $\{x_i\}_{i=1}^r$. Let $\mathcal{X}$ be a formal projective R-line with special fibre X. Let $X' := X - \{x\}$ and $U := X - \{x, x_i\}_i$. Further let $\mathcal{X}'$ (resp $\mathcal{U}$) be the formal fibre of X' (resp. of U) in $\mathcal{X}$. Let $\mathcal{D}$ be the degeneration data given by Deg .3. This data is necessarily of type C (i.e (radicial, radicial)), since $n_1' = n_2' = 0$, hence is given by an element of $H^1_{\text{fppf}}(U, H_k) \oplus \Gamma(U, \mathcal{O}_{X_k})^{p-1}$ where H_k is the finite commutative group scheme extension of α_p by α_p defined in 2.3.1. The degeneration data $\mathcal{D}$ is thus given by a pair of functions $(\bar{u}_1, \bar{u}_2)$ on U, defined up to addition of $(\bar{v}_1^p, \bar{v}_2^p)$, and a tuple $(\bar{c}_1, ..., \bar{c}_{p-1})$ of regular functions on U. Also let (n_1, n_2) be the pair of integers associated to the vertex X via Deg .3 and which satisfy $n_2 \geq n_1(p(p-1)+1)/p$. We assume for simplicity that $n_1 = p\tilde{n}_1$ and that $n_2 = \tilde{n}_1(p(p-1)+1)$. By 2.4.9 there exists a p^2-cyclic cover $f' : \mathcal{V} \to \mathcal{U}$ which gives rise to the degeneration data $\mathcal{D}$ by 2.4.8. For example we can choose the cover f' to be generically given by the equations: $(T_1^p, T_2^p) - (T_1, T_2) = (u_1 \pi^{-\tilde{n}_1 p}, f(u_1)\pi^{-p^2\tilde{n}_1} + u_2 \pi^{-p^2\tilde{n}_1 + \tilde{n}(p-1)})$ where u_1 (resp. u_2) is a regular function on $\mathcal{U}$ which lifts $\bar{u}_1$ (resp. $\bar{u}_2$) and $f(u_1) = c_1^p u_1 + c_2^p u_1^2 + ... + c_{p-1} u_1^{p-1}$ where c_i is a regular function on $\mathcal{U}$ which lifts $\bar{c}_i$.

For each $i \in \{1, ..., r\}$ let $\mathcal{X}'_i \simeq R[[T_i]]\{T_i^{-1}\}$ be the formal boundary of $\mathcal{U}$ at the point x_i. The cover f' induces canonically a cover $f'_i : \mathcal{Y}'_i \to \mathcal{X}'_i$ which by construction has a reduction of type $\{(n_1, m_{i,1}), (n_2, m_{i,2})\}$:= the admissible pair satisfying (*) which is associated to the point x_i via Deg .4. Note that we are necessarily in the case c-3) of 2.6.1 and the cover f_i is generically given by the equations:

$$X_1^p - X_1 = T_i^{m_{i,1}} / \pi^{\tilde{n}_1 p}$$

and:

$$X_2^p - X_2 = f(T_i)/\pi^{p^2 \tilde{n}_1} + p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} X_1^{pk} (-X_1)^{p-k}$$

where $f(T_i) = \sum_{j \in I} a_j T_i^j \in R[[T_i]]\{T_i^{-1}\}$ is not a p-power modulo π. Let m_2 be the conductor of f then $m_{i,2} = \inf(m_{i,1}(p(p-1)+1), m_2 p - m_{i,1}(p-1))$ (cf. loc. cit.). Further we can assume after some transformation which eliminate the p-powers that $f(T_i) = \sum_{j \geq m_2} a_j T_i^j$. Let $\mathcal{X}_i \simeq R[[T_i]]$ be the formal open disc with parameter T_i and consider the p^2-cyclic cover $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ which is generically given by the equations:

$$X_1^p - X_1 = T_i^{m_{i,1}} / \pi^{\tilde{n}_1 p}$$

and:

$$X_2^p - X_2 = f(T_i)/\pi^{p^2 \tilde{n}_1} + p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} X_1^{pk} (-X_1)^{p-k}$$

Then $\mathcal{Y}_i$ is smooth (cf. 4.2.3 4)). Now by construction the covers f_i and f' coincide when restricted to the formal fibre $\mathcal{X}'_i$. The technique of formal patching (cf. [S-1], 1) allows then one to construct a p^2-cyclic cover $\tilde{f} : \mathcal{Y}' \to \mathcal{X}'$ which restricted to $\mathcal{U}$ is isomorphic to f' and restricted to $\mathcal{X}_i$, for each $i \in \{i, ..., r\}$, is isomorphic to f_i (cf. loc. cit.).

Let $\mathcal{X}_{i_0} \simeq \operatorname{Spf} R[[T]]\{T^{-1}\}$ be the formal boundary of $\mathcal{U}$ at the point $x = x_{i_0}$. Then the cover $\tilde{f}$ induces canonically a p^2-cyclic cover $f_{i_0} : \mathcal{Y}_{i_0} \to \mathcal{X}_{i,0}$ which by construction has a reduction of type $\{(n_1, -m_1 = m_{i_{0,1}, j_{0,1}}), (n_2, -m_2 := m_{i_{0,2}, j_{0,2}})\}$:=the pair associated to the point x_{i_0} via Deg .4. Note that m_1 and m_2 must be negative since we assumed that $n_1 = n_2 = 0$. Moreover because $-n_1 = m_1 pt$ and $-n_2 = m_2 t$ by Deg .6 and because we assumed $n_2 = n_1(p(p-1)+1)$ we see that necessarily $m_2 = m_1(p(p-1)+1)$. The cover f_{i_0} is then generically given by 2.6.1 c) by equations:

$$X_1^p - X_1 = T^{-m_1} / \pi^{n_1 p}$$

and:

$$X_2^p - X_2 = f(T)/\pi^{p n''_2} + p^{-1} \sum_{k=1}^{p-1} \binom{p}{k} X_1^{pk} (-X_1)^{p-k}$$

where $f(T)=\sum_{j\in J} a_jT^j \in R[[T]]\{T^{-1}\}$ has conductor m'_2 and the following holds: either $pn''_2 - n_1(p-1) < n_1(p(p-1)+1)$ or $pn''_2 - n_1(p-1) = n_1(p(p-1)+1)$ and $m'_2 \geq m_1p$. Moreover we can assume after some transformation which eliminate the p-powers that $f(T)=\sum_{j\geq m'_2} a_jT^j$. Let $\mathcal{X}_1 \to \mathcal{X}$ be the blow up of $\mathcal{X}$ at the point x and let X_1 be the exeptional fibre in $\mathcal{X}_1$ which meets X at the double point $x=x_{i_0}$. Let $e=p^2t$ be the integer associated to the marked double point x via Deg .6. After enlarging R we can assume that the double point x of $\mathcal{X}_1$ has thikness e. Let $\mathcal{X}'_1 \simeq \operatorname{Spf} R < S^{-1} >$ be the formal fibre of $X'_1 := X_1 - \{x\}$ in $\mathcal{X}_1$. Let $f'_1 : \mathcal{Y}'_1 \to \mathcal{X}'_1$ be the étale $\mathbb{Z}/p^2\mathbb{Z}$-torsor given by the equation:

$$X_1^p - X_1 = S^{m_1}$$

and

$$X_2^p - X_2 = h'(S) + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k} X_1^{pk}(-X_1)^{p-k}$$

where $h'(S) = f(\pi^{p^2t}S^{-1})/\pi^{pn''_2} \in R < S^{-1} >$.

Further let $\mathcal{X}_{1,x} \simeq \operatorname{Spf} R[[S,T]]/(ST-\pi^{p^2t})$ be the formal germ of $\mathcal{X}_1$ at the double point x. Consider the cover $f_{1,x} : \mathcal{Y}_{1,y} \to \mathcal{X}_{1,x}$ given by the equation:

$$X_1^p - X_1 = T^{-m_1}/\pi^{n_1p}$$

and:

$$X_2^p - X_2 = f(T)/\pi^{pn''_2} + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k} X_1^{pk}(-X_1)^{p-k}$$

hen $\mathcal{Y}_{1,y}$ is the formal germ of a double point of thikness t (cf. 4.2.8). Moreover the cover $f_{1,x}$ restricted to the boundary $\operatorname{Spf} R[[S]]\{S^{-1}\}$ is given by the equations:

$$X_1^p - X_1 = S^{-m_1}$$

and

$$X_2^p - X_2 = h'(S) + p^{-1}\sum_{k=1}^{p-1}\binom{p}{k} X_1^{pk}(-X_1)^{p-k}$$

In particular the cover f'_1 (resp. f') has the same degeneration type, by construction, on the boundary corresponding to the double point x as the degeneration type of the cover $f_{1,x}$ above the formal boundary with parameter T (resp. above the formal boundary with parameter S). A second application of the formal patching techniques allows one to construct a p^2-cyclic cover $f_1 : \mathcal{Y}_1 \to \mathcal{X}_1$ which restricted to $\mathcal{X}'$ (resp. $\mathcal{X}'_1$ and $\mathcal{X}_{1,x}$) is isomorphic to f' (resp. to f'_1 and f_x). Let $\tilde{\mathcal{X}}$ be the R-curve obtained by contracting the irreducible component X in $\mathcal{X}_1$. We denote the image of the double point x in $\tilde{\mathcal{X}}$ simply by

x. The cover $f_1 : \mathcal{Y}_1 \to \mathcal{X}_1$ induces canonically a p^2-cyclic cover $\tilde{f} : \tilde{\mathcal{Y}} \to \tilde{\mathcal{X}}$ above $\tilde{\mathcal{X}}$. Let $\mathcal{X}_x \simeq R[[T]]$ be the formal germ of $\tilde{\mathcal{X}}$ at the smooth point x. Then $\tilde{f}$ induces canonically a p^2-cyclic cover $f_x : \mathcal{Y}_y \to \mathcal{X}_x$ where y is the closed point of $\mathcal{Y}$ above x. Now it is easy to see that the degeneration data associated to f_x via 6.2.2 is isomorphic to the degeneration data $\mathrm{Deg}(x)$ we started with. Now the proof in the general case is very similar and is left to the reader. The only modification is that one has to consider p^2-cyclic covers $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ above the formal germs $\mathcal{X}_i := \mathrm{Spf}\, R[[T_i]]$ which one obtains by induction hypothesis as realisation of the degeneration data, induced by $\mathrm{Deg}(x)$, on the subtrees Γ_i of Γ which starts from the edge x_i going to the ends and which have clearly length strictly less than the length of Γ. For this one still has to use the data in Deg .8 to insure that there exists a lifting f' as above of the degeneration data $\mathcal{D}$ above U which restricted to the formal boundary corresponding to the point x_i is Galois isomorphic to the restriction of f_i to the formal boundary $\mathrm{Spf}\, R[[T_i]]\{T_i^{-1}\}$.

6.2.4. Remarks.

1. The data Deg .8 in 6.2.1 seems necessary for the proof of 6.2.3. Indeed given two p^2-cyclic covers above the two boundaries of the formal germ $\mathcal{X}$ of an R-curve at a double point with "compatible" degeneration type as in 4.2.11 it is not always possible to find a p^2-cyclic cover $f : \mathcal{Y} \to \mathcal{X}$ where $\mathcal{Y}$ is the forml germ of a double point and f is étale above $\mathcal{X}_K$. Note that the later is possible for p-cyclic covers.

2. Using the same techniques as in 6.2.3 it is possible to prove global results of lifting for finite covers between proper and semi-stable k-curves of degree p^2 to a p^2-cyclic cover between R'-curve over some finite extension R' of R.

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