Invariants of generic plane curves via Gauss diagrams

Michael Polyak

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-53225 Bonn

Germany

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MICHAEL POLYAK

ABSTRACT. We use a notion of chord diagrams to define their representations in Gauss diagrams of plane curves. In this way we obtain elementary combinatorial formulas for the invariants $J^{\pm}$ and St of generic plane curves introduced by Arnold [A]. These formulas allow an easy computation of these invariants and enable one to answer several questions stated by Arnold. Different generalizations of the invariants and their relation to Vassiliev knot invariants are discussed.

1. INTRODUCTION

Recently V. Arnold [A] introduced in axiomatic form three basic invariants J^+, J^- and St of plane curves. Some types of formulas for these invariants were obtained by several authors, see [Ai], [S], [V]; nevertheless, these expressions are rather complicated. In this paper we present an elementary combinatorial formulas for these invariants in terms of a Gauss diagram of a curve (roughly speaking by counting, with some signs and coefficients, the number of 2-chord subdiagrams of the Gauss diagram, see Sections 3-4).

This approach allowes an easy computation of the invariants and enables one to answer several questions of Arnold, e.g. about minimal and maximal values of St and its relation with $J^{\pm}$ (see Section 4), as well as about examples (in Section 5.1) of additive non-local invariants. We also provide in Section 5.2 a similar formula for a generalization of the invariant $J = J^+ + J^-$ for curves with cusps. I would like to mention a striking similarity of these formulas with the expressions obtained in [PV] for low-degree Vassiliev invariants, which somehow explains the results of Section 5.3 where we provide a relation of $J^{\pm}$, St to the Vassiliev knot invariant of degree 2.

I am grateful to O. Viro for getting me interested in this subject and for numerous fruitful conversations.

2. PLANE CURVES AND INVARIANTS J^+, J^-, St

In this section we briefly review some of the results of Arnold (see [A] for detailes).

2.1. By a *generic plane curve* $\Gamma : S^1 \to \mathbb{R}^2$ we mean an immersion of an (oriented) circle S^1 into a plane $\mathbb{R}^2$ having only transversal double points of selfintersection. Non-generic immersions form a *discriminant hypersurface* in the space of all immersions $S^1 \to \mathbb{R}^2$. Three main (open) strata D^+, D^- and D^{St} of the discriminant consist of immersions with all transversal double points exept exactly one (a) direct selftangency point; (b) inverse selftangency point; (c) transversal triple point respectively, see Fig 1.

As shown in [A], there is a natural coorientation of the main strata $D^{\pm}$, D^{St}, i.e. a choice of one (called *positive*) of the two parts separated by a stratum in a neighborhood of any of its points. Near a singular curve with a direct or inverse selftangency point the part with a larger number of double points is positive. A coorientation of the stratum D^{St} is defined by a sign of a *vanishing triangle* formed by the three branches of a curve close to a singular curve with a triple point. The

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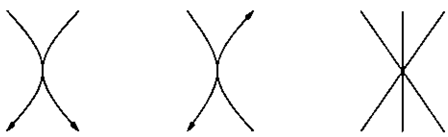


Figure 1. Strata D^+, D^-, D^{St}

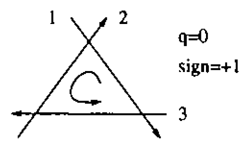


Figure 2. Signs of vanishing triangles

orientation of the curve defines an orientation of the vanishing triangle via a cyclic ordering of its sides, as illustrated in Fig 2. Denote by q a number of sides whose orientation as of a piece of the curve coincides with the orientation induced by the orientation of the triangle. The sign of a vanishing triangle is defined to be $(-1)^q$.

2.2. A (generic) regular isotopy $\Gamma_t : S^1 \times [0,1] \to \mathbb{R}^2$ of plane curves intersect the discriminant in a finite number of points of the strata D^+, D^-, D^{St} described above (Fig 1). Changes of a generic curve when it experiences such an intersection are called *perestroikas* and are illustrated in Fig 3.

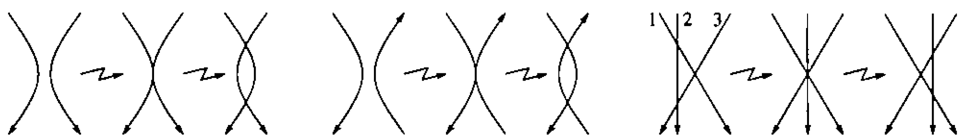


Figure 3. Perestroikas

Note that there is a natural choice of sign for each perestroika determined by the coorientations of $D^{\pm}$, D^{St}; e.g. perestroikas depicted in Fig 3 are positive.

Invariants J^+, J^-, S of regular homotopy classes of generic plane curves were introduced in [A]. These invariants are additive with respect to connected sum of curves and independent of the choice of orientation for the curve. $J^{\pm}$ and St are defined by the following properties:

(2.1) J^+ does not change under an inverse selftangency or triple point perestroikas but increases by 2 under a positive direct selftangency perestroika.

(2.2) J^- does not change under a direct selftangency or triple point perestroikas but decreases by 2 under a positive inverse selftangency perestroika.

(2.3) St does not change under selftangency perestroikas but increases by 1 under a positive triple point perestroika.

(2.4) For the curves K_i, $i = 0, 1, 2\dots$ depicted in Fig 4

$$J^+(K_0) = 0, J^-(K_0) = -1, St(K_0) = 0,$$

$$J^+(K_{i+1}) = -2i, J^-(K_{i+1}) = -3i, St(K_{i+1}) = i \quad (i = 0, 1, 2\dots).$$

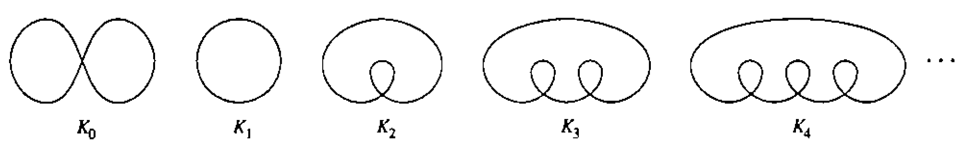


Figure 4. Standard curves of indices $0, \pm1, \pm2, \dots$

3. Chord diagrams and Gauss diagrams

3.1. A generic plane curve $\Gamma : S^1 \to \mathbb{R}^2$ may be encoded by its *Gauss diagram* G_Γ. The Gauss diagram is the immersing circle S^1 with the preimages of each double point connected with a chord. For technical reasons it will be convenient to consider *based* Gauss diagrams, i.e. to assume that there is a marked base point on S^1 (distinct from the endpoints of chords). The choice of a base point and the orientation of the circle define an ordering $(1, 2)$ of directions of outgoing branches in each double point of the curve. The corresponding chord can therefore be equipped with a positive (negative) sign if the frame $(2, 1)$ orients the plane positively (negatively), see Fig 5. Our definition differs by a sign from a similar definition of [A] and coincides with the one of [S].

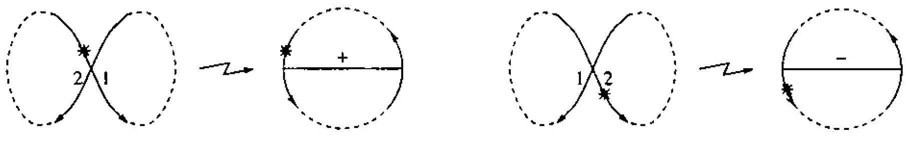


Figure 5. Signs of chords in Gauss diagrams

Remark 3.1. Note that the signs of chords depend on the choice of the base point: if we move it through a double point, the sign changes to the opposite (Fig 5). Signs change as well if we reverse the orintation of the curve.

Remark 3.2. The same information about the curve can of course be encoded without this ambiguity in the choice of the base point and orientation by means of arrows: one can orient the chords of Gauss diagram so that the frame of outgoing branches *(beginning of arrow, end of arrow)* orient the plane positively. All the formulas for the invariants may be rewritten in this way, though unfortunately this leads to much bulkier expressions. This was one of the reasons that determined our choice. Another reason is that the setup with signs fits better in the underlying interpretation of the invariants as of relative degrees of some maps, which was our starting point. We are planning to explain this in the final version of this paper.

Example 3.1. Some important classes of curves and the corresponding Gauss codes are depicted in Fig 10.

3.2. A (based, generic) *chord diagram* is an oriented circle with a base point and several chords endowed with multiplicities 1 or 2 and having distinct endpoints. By a *degree* of a chord diagram we mean a sum of multipilcities of its chords. Further we consider chord diagrams up to isomorphism (i.e. orientation-preserving homeomorphism of the circle mapping a basepoint to a basepoint and chords of one diagram to chords of another preserving multiplicities). We will depict multiplicity 2 of a chord by thickening it.

By a *representation* $\phi : A \to G$ of a chord diagram A in a Gauss diagram G we mean an embedding of A to G mapping the circle of A to the circle of G (preserving orientation), each of the chords of A to a chord of G and a basepoint to a basepoint. For such a representation we define $\mathrm{sign}(\phi) = \prod \mathrm{sign}(\phi(c))^{m(c)}$ by taking the product over all chords c of A of signs of the chords $\phi(c)$ in G with the multiplicity $m(c)$ of c. Denote by $\langle A, G\rangle$ the sum

$$\langle A, G\rangle = \sum_{\phi:A\to G} \operatorname{sign}(\phi)$$

over all representations $\phi : A \to G$.

Let A be the vector space over $\mathbb{Q}$ generated by chord diagrams. $\langle A, G\rangle$ may be extended to $A \in$ A by linearity. A degree of A is the highest degree of the diagrams in A.

3.3. Note that by its definition $\langle A, G_\Gamma\rangle$ is an invariant of a regular homotopy class of based generic curve Γ for any $A \in$ A. It is natural to study these invariants (we will call them *Gauss diagram invariants*) in the simplest cases, i.e. for low degrees of A. As we will see below, the invariant of degree 1 is well-known. Moreover, in Section 4 we show that $J^\pm$, St can be realized as Gauss diagram invariants of degree 2.

Example 3.2. There is only one chord diagram A_1 of degree 1, shown on Fig 6a. Recall that the Whitney function $w(x)$ of the base point $x \in \Gamma$ is defined as a sum of signs $w(x) = -\sum \operatorname{sign}(\gamma)$ of all double points γ of Γ (see e.g. [A]; the negative sign appears because of our sign convention). Therefore we immediately obtain

$$\langle A_1, G_\Gamma\rangle = -w(x) = ind(\Gamma) - ind(x)\ , \tag{3.1}$$

since it is well known, that $w(x) = ind(x) - ind(\Gamma)$, where $ind(\Gamma)$ is the index of Γ and $ind(x)$ is the index of the base point (defined as the number of half-twists of the vector connecting x to a point moving along the curve from x to itself or, alternatively, the sum of indices of two regions adjusent to x). We will provide a new proof of this classical equality illustating our geometrical interpretation of Gauss diagram invariants in the final version of this paper.

Remark 3.3. All the constructions of this section can be generalized to n-component based curves (with ordered components) by considering diagrams with n circles. The sign of a chord connecting i-th and j-th components, $i < j$, is defined by an orientation of the frame (j, i) of outgoing branches in the corresponding double point. The simplest example A_2 of a 2-component chord diagram is depicted in Figure 6b. It is easy to observe that $\langle A_2, G_\Gamma\rangle$ is the intersection index of two components of Γ (hence equals 0).

4. $J^\pm$, St and Gauss diagram invariants of degree 2

4.1. There exist 4 chord diagrams B_1, B_2, B_3 and B_4 of degree 2, see Fig 7 (recall that a thick chord denotes multiplicity 2).

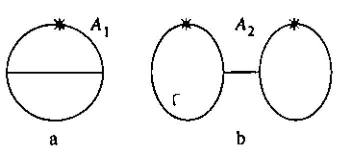


Figure 6. Diagrams of degree 1

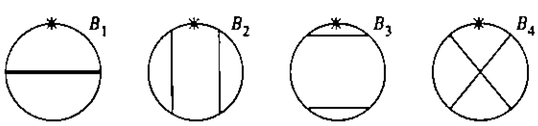


Figure 7. Diagrams of degree 2

Let Γ be a (generic) plane curve. Choose a base point on Γ and denote by G_Γ the corresponding Gauss diagram of Γ. Denote by n the number of double points of Γ and by $ind(\Gamma)$ its index. It is easy to see that $\langle B_1, G_\Gamma\rangle = n$; consideration of the other diagrams lead us to the main result of this paper:

Theorem 4.1.

$$J^+(\Gamma) = \langle B_2 - B_3 - 3B_4, G_\Gamma\rangle - \frac{n}{2} - \frac{ind(\Gamma)^2}{2} + \frac{1}{2}$$

$$J^-(\Gamma) = \langle B_2 - B_3 - 3B_4, G_\Gamma\rangle - \frac{3n}{2} - \frac{ind(\Gamma)^2}{2} + \frac{1}{2} \tag{4.1}$$

$$St(\Gamma) = \frac{1}{2}\langle -B_2 + B_3 + B_4, G_\Gamma\rangle + \frac{n}{4} + \frac{ind(\Gamma)^2}{4} - \frac{1}{4}$$

In particular, the expressions in the right hand side are independent of the choice of the base point.

Proof. Changes of Gauss diagrams under positive direct selftangency, inverse selftangency and one of the cases (others are similar) of the triple point perestroikas are depicted in Fig 8. Let us consider a direct selftangency case. As a result of this perestroika n increases by 2 and a pair of new chords (with the opposite signs) appear in G_Γ. Clearly all the representations of B_2, B_3 and B_4 which existed before the perestroika will still exist after it. All the new representations of B_2 and B_3 will be in pairs with cancelling out signs. For B_4 the situation is the same, exept for the only new representation $\phi : B_4 \to G_\Gamma$ in which both new chords of G_Γ appear in $\phi(B_4)$. Therefore, $\langle B_2, G_\Gamma\rangle$ and $\langle B_3, G_\Gamma\rangle$ do not change while $\langle B_4, G_\Gamma\rangle$ decreases by 1. This shows that the expressions in (4.1) satisfy the needed properties of $J^\pm$, St under a direct selftangency perestroika. Similar careful analysis for the rest of the cases proves that the expressions (4.1) above satisfy all the properties (2.1)-(2.4) of the invariants.

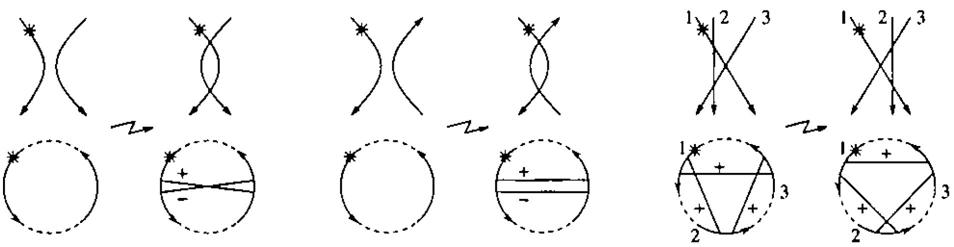


Figure 8. Perestroikas of Gauss diagrams

The only nontrivial fact is the invariance of (4.1) under the change of the base point. When the base point moves through a double point γ of Γ, the only changes occur in the terms corresponding to representations where one of the two chords of a chord diagram maps to γ. The terms corresponding to B_2 and B_3 then exchange

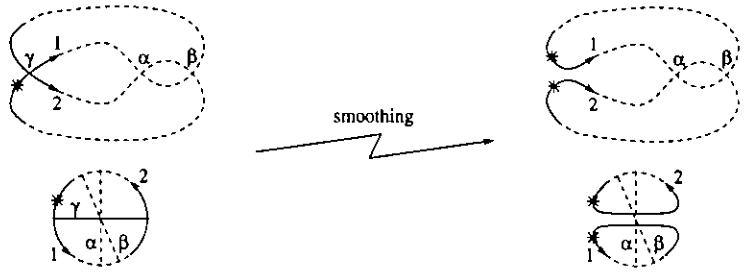


Figure 9. Smoothing the curve

(due to the change of sign of γ, see Remark 3.1), so $\langle B_2 - B_3, G_\Gamma\rangle$ is preserved. It remains to notice that the sum of signs of representations of B_4 where one of the chords maps to Γ is equal to 0. Indeed, this sum is just the intersection index of two curves obtained from Γ by smooting in γ, see Fig 9 and Remark 3.3.

Corollary 4.1. From (4.1) we immediately obtain $J^+ = J^- + n$ as expected. Moreover (answering a question posed by Arnold about a formula for $J^+ + 2St$) we obtain the following equality:

$$J^+(\Gamma) + 2St(\Gamma) = -2\langle B_4, G_\Gamma\rangle \tag{4.2}$$

In particular case of curves having planar Gauss diagrams the last term disappear and (4.2) implies the result of [Ai]. It should be also mentioned that this expression appears as well in the discussion (see Section 5.3) of relation of $J^\pm$, St with Vassiliev knot invariants.

4.2. Formula (4.1) for $St(\Gamma)$ can be significantly simplified. Note first that since $-w(x) = ind(\Gamma) - ind(x)$ is given by (3.1), its square $w(x)^2 = (ind(\Gamma) - ind(x))^2$ can be obtained from a similar formula involving a square of the chord diagram A_1, i.e. a sum of all possible superpositions of two copies of A_1. Indeed, one readily obtains

$$(ind(\Gamma) - ind(x))^2 = \langle B_1 + 2B_2 + 2B_3 + 2B_4, G_\Gamma\rangle = 2\langle B_2 + B_3 + B_4, G_\Gamma\rangle + n$$

Now, choose a base point x on the exterior contour. Then $ind(x) = \pm 1$ (see Example 3.2), so comparing the expression above with (4.1) we derive

Corollary 4.2. Let the base point x of Γ be chosen on the exterior contour (so that $ind(x) = \pm 1$). Then

$$St(\Gamma) = -\langle B_2, G_\Gamma\rangle + \frac{ind(\Gamma)^2}{2} \mp \frac{ind(\Gamma)}{2}\,. \tag{4.2}$$

Corollary 4.3. Consider the curves $K_{n,k}$ and $A_{n,k}$, $n, k \geq 0$ with n double points and $ind = \pm(n+1-2k)$ depicted in Fig 10 with the corresponding Gauss diagrams.

It was conjectured by Arnold and proved in [S], that minimal and maximal values $St_{min}(n,k)$, $St_{max}(n,k)$ of St in the class of curves with fixed number of

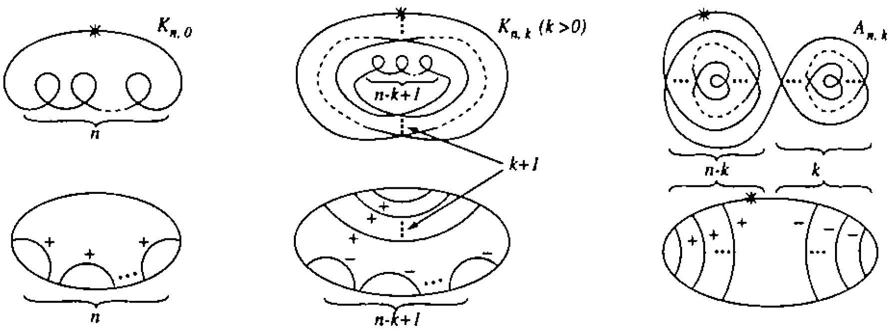


Figure 10. Curves $K_{n,k}$, $A_{n,k}$ and their Gauss diagrams

double points and fixed index are attained on the curves $K_{n,k}$ and $A_{n,k}$ respectively. Consider any curve $\Gamma_{n,k}$ with n double points and index $\pm(n+1-2k)$; choose a base point x on the exterior contour and an orientation of $\Gamma_{n,k}$ so that $ind(x) = 1$. Then by (3.1) a Gauss diagram of $\Gamma_{n,k}$ should have n chords, k of which have negative sign and $n-k$ positive if the index is $n+1-2k$; if the index is equal to $-(n+1-2k)$, there should be $n-k+1$ chords with negative sign and $k-1$ with positive (note that this can happen only if $k>0$). The result about $St_{min}(n,k)$ and $St_{max}(n,k)$ now easily follows from (4.2) and Fig 10.

5. Generalizations of basic invariants and their relation to Vassiliev invariants

5.1. As we have seen in Theorem 4.1, $J^{\pm}$, St are second degree Gauss diagram invariants. These invariants are additive under the connected summation of curves and are local in a sense that their changes after perestroikas are determined by a local picture around the point of perestroika. Additive invariants of higher degrees also exist (though are non-local), thus answering the question of Arnold about the existence of additive non-local invariants. Many examples can be provided; e.g. consider chord diagrams C_1, C_2, D_1, D_2 shown in Fig 11.

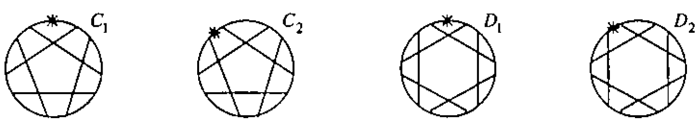


Figure 11. Some chord diagrams of degrees 5 and 6

Theorem 5.1. *Let Γ be a (generic) plane curve. Choose an arbitrary base point on Γ and denote by G_Γ the corresponding Gauss diagram of Γ. Then $V_5(\Gamma) = \langle C_1 - C_2, G_\Gamma\rangle$ and $V_6(\Gamma) = \langle D_1 - D_2, G_\Gamma\rangle$ are invariants of regular homotopy classes of generic plane curves which do not change under selftangency perestroikas. In particular, the expressions in the right hand side are independent of the choice of the base point. V_5 and V_6 are additive with respect to connected sum.*

Proof. The proof is completely similar to the proof of Theorem 4.1. Invariance under selftangency perestroikas can be observed immediately by comparing the diagrams C_1, C_2, D_1, D_2 with Fig 8. The fact that $\langle C_1 - C_2, G_\Gamma \rangle$ (and $\langle D_1 - D_2, G_\Gamma \rangle$) do not change when the base point moves through a double point $\gamma \in \Gamma$ follows again from the observation that the terms involving γ which correspond to C_1 and $-C_2$ (to D_1 and $-D_2$ respectively) interchange. A consideration of the Gauss code of connected sum assures additivity of V_5 and V_6.

5.2. An invariant $J = J^+ + J^-$ of generic immersions $S^1 \to \mathbb{R}^2$ can be generalized to curves with cusp singularities. This (additive and independent of the orientation) invariant does not change under triple point perestroika and perestroikas shown on Fig 12 while increasing (decreasing) by 2 under positive direct (respectively inverse) selftangency perestroikas. To incorporate cusps in our approach we depict them by marking the corresponding point on the immersing circle of the Gauss diagram and assigning to it a positive (negative) sign if the normal vector to the curve makes a positive (respectively negative) half-twist passing the cusp along the orientation of the curve, see Fig 13.

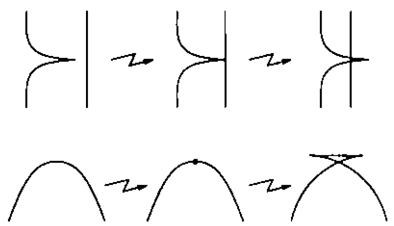


Figure 12. Additional perestroikas

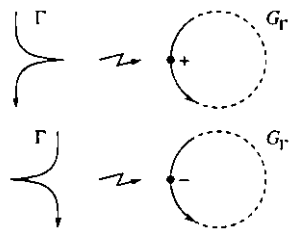


Figure 13. Signs of cusps

Similarily to Section 3.2, one can define chord diagrams A with marked points, their degree (as a sum of multiplicities of all chords plus number of marked points), their representations $\phi : A \to G$ and $\text{sign}(\phi)$ (as a product of signs over all chords and marked points in $\phi(A) \subset G$). The bracket $\langle A, G \rangle = \sum_{\phi: A \to G} \text{sign}(\phi)$ can again be extended by linearity to the vector space of chord diagrams with marked points. To formulate our next theorem consider, in addition to diagrams of Fig 8, new based diagrams B_5, B_6, B_7 of degree 2 with marked points depicted in Fig 14.

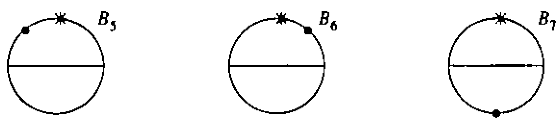


Figure 14. Chord diagrams with marked points

Theorem 5.2. *Let Γ be a plane curve (possibly with cusps) . Choose an arbitrary base point on Γ and denote by G_Γ the corresponding Gauss diagram of Γ. Let n be the number of double points of Γ. Then*

$$J(\Gamma) = \langle 2B_2 - 2B_3 - 6B_4 + B_5 + B_6 - B_7, G_\Gamma \rangle - 2n - ind(\Gamma)^2 + 1 \ .$$

Proof. The proof is completely similar to the proof of Theorem 4.1. The additional verification of invariance under perestroikas of Fig 12 is straightforward and follows from the careful analysis of the corresponding changes of Gauss diagrams shown in Fig 15 (for one of the orientations of the curve).

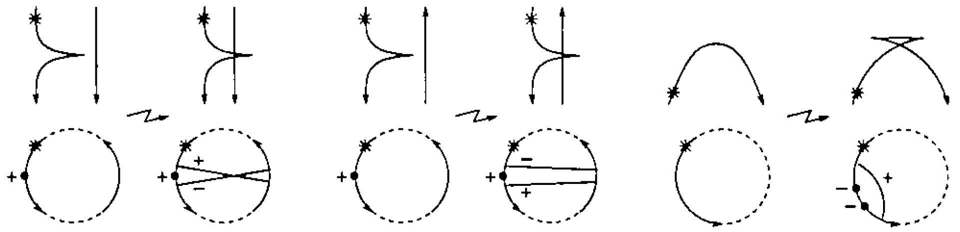
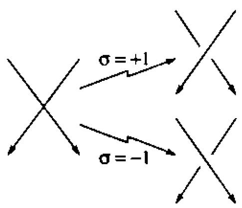


Figure 15. Perestroikas of Gauss diagrams

5.3. The original construction of $J^{\pm}$, St in [A] (by consideration of a discriminant hypersurface in the space of immersions $S^1 \to \mathbb{R}^2$) is similar to the construction of the Vassiliev knot invariants, so one is tempted to find some explicit correspondence between these objects. Indeed, as we will see below, the expression $J^+ + 2St$ (which appeared before in (4.2)) is closely related to a Vassiliev knot invariant of degree 2.

A (generic) plane curve may be considered for this purpose as a singular knot $\Gamma \subset \mathbb{R}^2 \subset \mathbb{R}^3$ with n selfintersections. Choose an ordering $1, 2, \dots n$ of selfintersections and let $\sigma = (\sigma_1, \sigma_2, \dots \sigma_n)$ be an n-tuple $\sigma_i = \pm 1$ ($i = 1, 2, \dots n$); let $s(\sigma)$ be the number of -1's in σ. Denote by Γ_σ a knot obtained from Γ by resolving i-th selfintersection, $i = 1, 2, \dots n$, in a positive way if $\sigma_i = +1$ and in a negative way otherwise, see Fig 16.



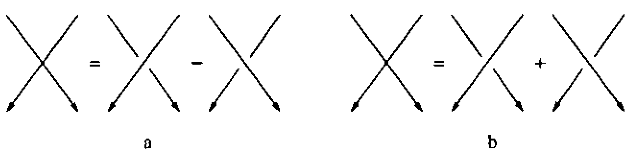


Figure 16. Resolving singularities

Figure 17. Defining V_2 and V_2^+ for singular knots

Let V_2 be Vassiliev knot invariant of degree 2 which takes values 0 on the unknot and 1 on the trefoil. One may recursively define V_2 for singular knots by the rule depicted in Fig 17a. Resolving all the singularities of Γ in this way one obtains an alternating sum $V_2(\Gamma) = \sum_\sigma (-1)^{s(\sigma)} V_2(\Gamma_\sigma)$. Unfortunately, $V_2(\Gamma) = 0$ for any Γ with $n \geq 3$ by definition of V_2. But one can use as well another resolution of singularities depicted in Fig 17b. It leads to a non-alternating sum $V_2^+(\Gamma) = \sum_\sigma V_2(\Gamma_\sigma)$. By a straightforward verification of the changes of $V_2^+(\Gamma)$ when Γ experiences different types of perestroikas one can check that $V^+(\Gamma)$ increases by 2^{n-2} under positive direct selftangency or triple point perestroikas and does not change under inverse selftangency perestroikas. Therefore, we obtain the following

Theorem 5.3.

$$V_2^+(\Gamma) = 2^{n-3}(J^+ + 2St)$$

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Max-Planck-Institut für Matematik, Gottfried-Claren-Strasse 26, D-5300 Bonn-3, Germany