Manifolds of Almost Nonnegative Curvature

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§0 Introduction

In this article, we survey some results, mainly those of [Y2], [FY2], on almost nonnegatively curved manifolds.

Our concern is the topology of manifolds with almost nonnegative curvature. First recall what the topological characteristics of nonnegative curvature are. These are the first Betti number and fundamental groups at present. In fact, we know the following results on these topological invariant of nonnegatively Ricci-curved manifolds. Throughout the paper, M is a closed n-dimensional Riemannian manifolds unless otherwise stated. We denote by Ricci_M the Ricci curvature of M.

A classical result by Bochner is stated as

Theorem 0.1 ([BY]). *Suppose* $\mathrm{Ricci}_M \geq 0$. *Then the first Betti number* $b_1(M) = \mathrm{rank}\, H_1(M, \mathbf{Q})$ *is less than or equal to* n, *where* $b_1(M) = n$ *if and only if* M *is isometric to a flat torus.*

In [CGl2], Cheeger and Gromoll generalized this result as follows.

Theorem 0.2 ([CGl2]). *If* $\mathrm{Ricci}_M \geq 0$, *then* $\pi_1(M)$ *contains a finite index free abelian subgroup of rank* $\leq n$.

There still exists a uniform bound on the first Betti number due to Gromov, which generalizes Bochner's result, even if one allows M to have negative Ricci curvature somewhere:

Theorem 0.3 ([G5]). *If the diameter and curvature of* M *satisfy* $\mathrm{Ricci}_M \,\mathrm{diam}(M)^2 > -D^2$, *then* $b_1(M) \leq (n-1) + C_n^D$.

In particular, there exists a positive number ϵ_n *such that if* $\mathrm{Ricci}_M \,\mathrm{diam}(M)^2 > -\epsilon_n$, *then* $b_1(M)$ *is still less than or equal to* n.

At this stage, this is the only result known for topology of manifold whose Ricci curvature is bounded below by a *negative* constant. Gromov proposed the following conjecture in [G5].

Conjecture 0.4 (Gromov). There exists a positive number ϵ_n such that if $\mathrm{Ricci}_M \,\mathrm{diam}(M)^2 > -\epsilon_n$ and if $b_1(M) = n$, then M has the topological type of a torus.

Unfortunately, this conjecture is still too difficult to attack it. Our object here is the study of manifolds M with almost nonnegative sectional curvature in the sense:

$$K_M \,\mathrm{diam}(M)^2 > -\epsilon$$

for a small positive number ϵ. More precisely, if M satisfies the above inequality, we say that M is of *ϵ-nonnegative curvature.* We also say that a closed manifold M is of almost nonnegative curvature if M admits a metric of ϵ-nonnegative curvature for each ϵ. For instance, the product of S^2 and a nilmanifold is of almost nonnegative, and a circle bundle over an almost nonnegatively curved manifold is also of almost nonnegative. (See Theorem 2.8 for a more general example).

Another motivation to our work is the following almost flat manifold theorem due to Gromov with a modification by Ruh.

Theorem 0.5 ([G1],[Ru]). *There exists a positive number ϵ_n such that if*

$$|K_M| \operatorname{diam}(M)^2 < \epsilon,$$

then M is diffeomorphic to an infranilmanifold.

Some of main results discussed in this article are stated as follows.

Theorem A ([Y2]). *There exists a positive number ϵ_n such that*

(a) if M is of ϵ_n-nonnegative curvaure, then a finite cover of M fibers over a $b_1(M)$-dimensional torus.
(b) In the maximal case $b_1(M) = n$, M is diffeomorphic to a torus.

Next we describe the results on the fudamental groups of manifolds of almost nonnegative curvaure, obtained by the joint works with Kenji Fukaya ([FY2,3]).

A group is called almost nilpotent (abelian, solvable) if it contains a nilpotent (abelian, solvable) subgroup of finite index. Let Λ be a solvable group. The *length of polycyclicity* $\mathcal{L}(\Lambda)$ of Λ is defined as the smallest integer s for which Λ admits a filtration :

$$\Lambda = \Lambda_0 \supset \Lambda_1 \supset \cdots \supset \Lambda_s = \{1\},$$

such that each Λ_i/Λ_{i+1} is cyclic.

Theorem B ([FY2]). *There exist positive numbers ϵ_n and c_n such that if M is of ϵ_n-nonnegative curvature, then $\pi_1(M)$ is almost nilpotent and contains a solvable subgroup Λ satisfying*

$$[\pi_1(M) : \Lambda] < c_n, \tag{1}$$

$$\mathcal{L}(\Lambda) \leq n. \tag{2}$$

This extends Theorem 0.5 in the π_1-level, and settles a conjecture in [G2].

A finite index subgroup of $\pi_1(M)$ constructed in Theorem B can be generated by n elements and has the degree of nilpotency $\leq n$. However we have no uniform bounds on the index of the nilpotent subgroup in terms of dimension n, although this seems to be possible. The theorem says that we have such a uniform bound for a solvable subgroup. We remark that Theorem B is still new even for nonnegatively curved manifolds (See Remark 1.3).

A significance of the study of almost nonnegatively curved manifold is in the simple fact that a manifold collapses to a point while keeping a lower curvature bound, say $K_M \geq -1$, if and only if it is of almost nonnegative. As we see later, in the general situation that a manifold collapses to a lower dimensional space under a lower curvature bound, an "almost nonnegatively curved space" appears as fibre of some fibration (See Theorem 4.1). Theorems A and B actually hold for such a fibre.

The proof of Theorem B makes it possible to genaralize it to a class of manifolds with a lower curvature bound.

Theorem C ([FY3]). *Given n and $D > 0$, there exist only finitely many discrete groups $\Gamma_1, \dots, \Gamma_k$, which are finitely represented, such that if an n-dimensional manifold M satisfies $K_M \operatorname{diam}(M)^2 > -D^2$, then there exists an exact sequence*

$$1 \longrightarrow \Lambda \longrightarrow \pi_1(M) \longrightarrow \Gamma_i \longrightarrow 1,$$

for some $1 \leq i \leq k$, where Λ is an almost nilpotent group satisfying the conclusion of Theorem B.

In the case $D = \epsilon_n$, Γ_i should be trivial by Theorem B. Theorem C says the set of all isomorphism classes of fundamental groups of manifolds with fixed lower sectional curvature and upper diameter bounds is finite modulo almost nilpotent subgroups. In a noncollapsing case, M. Anderson [A1] proved the finiteness of the set of all isomorphism classes of fundamental groups of manifolds with fiexd lower bounds on Ricci curvature and volume and an upper diameter bound (Theorem 12.1).

Our discussion using Hausdorff convergence provides some new results even for nonnegatively curved manifolds (See Section 8), and one will recognize almost nonnegatively curved manifolds as natural objects of study.

The organization of this article is as follows.

In Section 1, we recall two basic methods in the study of nonnegative Ricci curvature, the Bochner technique and the Cheeger and Gromoll splitting theorem. Outlines of the proofs of Theorems 0.1, 0.2 are given there.

Our basic argument is the Hausdorff convergence introduced by Gromov in [G5]. We recall the fundamental properties of this notion in Section 2. We give some examples of Hausdorff convergence and provide some basic facts in the pointed equivariant Hausdorff convergence, which we need later in the study of the first Homology classes and the fundamental groups of almost nonnegatively curved manifolds.

In Section 3, we discuss Alexandrov spaces, which occur as the Hausdorff limit of manifolds with a lower sectional curvature bound. We give a proof of the splitting theorem for Alexandrov spaces with nonnegative curvature, which is one of our basic tools. As an application, we provide the Lie group property of the isometry group of an Alexandrov space, proved in [FY3].

In Section 4, we discuss the fibration theorem, which is our another basic tool. We give a proof of this theorem along the line of [Y3], and discuss the properties of fibre of the fibration.

The proof of Theorem A is given in Section 5. The fibre bundle version of Theorem A was proved in [Y2]. Here we mainly prove the case when M is almost nonnegatively curved.

In Section 6, we discuss almost nonnegative Ricci curvature under the stronger assumption $|K| \leq 1$. Under the additional assumption, one can apply the Bochner technique.

For the proof of Theorem B, we generalize the Bieberbach theorem in Section 7, and the solvability part of Theorem B is proved in Section 8.

In the proof of the nilpotency part of Theorem B, we need the notion of covering space along fibre introduced in [FY1]. After considering the three-dimensional case in Section 9 as an introduction, we give an outline of the proof of the nilpotency part in Section 10.

Theorem B is extended in Section 11 to a generalized Margulis' lemma, from which Theorem C follows.

In Section 12, we consider the special case when manifolds have a lower volume bound. In this case the structure of manifolds is similar to that of nonnegatively curved manifolds.

In the final Section 13, we provide some conjectures and related arguments.

§1 Basic Methods in Nonnegative Curvature

Before proceeding to the study of almost nonnegative curvature, we recall basic methods on the study of nonnegative curvature, and give outlines of the proofs of Theorems 0.1 and 0.2.

There are two basic methods in the study of topological structure of closed manifolds with nonnegative Ricci curvature. One is the Bochner method which is analytic, and the other is the Cheeger and Gromoll method which is more geometric.

First we recall the Bochner method. Let $\omega_1, \ldots, \omega_{b_1}$ be harmonic 1-forms on a closed Riemannian manifold M forming a basis of the first de Rham cohomology. Then the Albanese map $A : M \to T^{b_1}$ is defined by

$$A(x) = \left(\int_p^x \omega_1, \ldots, \int_p^x \omega_{b_1} \right),$$

where $p \in M$ is fixed and

$$T^{b_1} = \frac{\mathbf{R}^{b_1}}{\left\{ \left(\int_\gamma \omega_1, \ldots, \int_\gamma \omega_{b_1} \right) \mid \gamma \in \pi_1(M) \right\}}.$$

The method is based on the following Weitzenböck formula.

Theorem 1.1. *For every 1-form on M we have*

$$(\Delta\omega, \omega) = \frac{1}{2}\Delta|\omega|^2 + |D\omega|^2 + \mathrm{Ricci}(\sharp\omega, \sharp\omega).$$

The proof of Theorem 0.1 is as follows: If the Ricci cuvature of M is nonnegative and if ω is harmonic, the above formula implies

$$\int_M |D\omega|^2 \, dx \le 0.$$

Hence $|D\omega| \equiv 0$ and ω must be parallel. Thus we have $b_1 = b_1(M; \mathbf{Q}) \le n$ and the Albanese map $A : M \to T^{b_1}$ is a Riemannian submersion. In the maximal case $b_1 = n$ therefore, M is isometric to a flat torus.

Next we recall the Cheeger Gromoll method, which is based on the following splitting theorem.

Theorem 1.2 ([T],[CGl2]). *Let N be a complete manifold with nonnegative Ricci curvature, and suppose that it contains a line. Then N is isometric to a product $N_0 \times \mathbf{R}$.*

Now let M be a closed manifold of nonnegative Ricci curvature. By Theorem 1.2 M is isometric to a product $M_0 \times \mathbf{R}^k$, where we may assume that M_0 is compact. Since Γ, the fundamental group of M, preserves the splitting, we have a homomorphism $\varphi : \Gamma \to \mathrm{Isom}(\mathbf{R}^k)$, and hence an exact sequence

$$1 \longrightarrow K \longrightarrow \Gamma \longrightarrow \varphi(\Gamma) \longrightarrow 1,$$

where K is the kernel of φ. Note that K is finite. By the Bieberbach theorem (cf.[Wol]), $\varphi(\Gamma)$ contains a finite index abelian subgroup. Thus Γ must be almost abelian. (See Lemma 7.2). This completes the proof of Thorem 0.2.

Remark 1.3. Theorem 0.2 grasps only the nontorsion part of the fundamental group. Hence as a lense space $S^3/\mathbf{Z}_p$ shows, there is no bound on the index of the free abelian subgroup. However by taking a solvable subgroup Λ in place of an abelian subgroup, we can have a uniform bound on the index of Λ (Theorem B). This will be shown in Section 8 for almost nonnegatively curved manifolds (See Theorem 8.1).

§2 Hausdorff Convergence

In our argument, we shall use the notion of Hausdorff distance introduced by Gromov ([G5]) as a basic method. In this section, we present some basic properties related with the Hausdorff distance. We begin with

Definition 2.1. A (not necessarily continuous) map $f : X \to Y$ between metric spaces is called an *ϵ-Hausdorff approximation* if

(1) $|d(f(x), f(y)) - d(x, y)| < \epsilon$ for all $x, y \in X$.
(2) The ϵ-neighbohood of $f(X)$ covers Y.

Then the *Hausdorff distance* $d_H(X, Y)$ is defined as the infimum of ϵ such that there exist ϵ-Hausdorff approximations from X to Y and from Y to X.

The Hausdorff distance actually defines a distance on the set of all compact metric spaces. For unbounded spaces, this metric is not useful, but the notion of pointed Hausdorff distandce is effective. For pointed metric spaces (X, p) and (Y, q), the *pointed* Hausdorff distance $d_{p.H}((X, p), (Y, q))$ is defined as the infimum of ϵ such that there exist ϵ-Hausdorff approximations $f : B_p(1/\epsilon, X) \to B_q(1/\epsilon + \epsilon, Y)$ and $g : B_q(1/\epsilon, Y) \to B_p(1/\epsilon + \epsilon, X)$ between metric balls with $f(p) = q$ and $g(q) = p$.

Definition 2.2. The *dilatation* of a Lipschitz map $f : X \to Y$ is defined as

$$\mathrm{dil}(f) = \sup_{x \neq y \in X} \frac{d(f(x), f(y))}{d(x, y)}.$$

We say that f is an *ϵ-isometry* if it is a bilipschitz homeomorphism and if $|\log(\operatorname{dil} f)| + |\log(\operatorname{dil} f^{-1})| < \epsilon$. Then the *Lipschitz distance* $d_L(X,Y)$ between X and Y is defined as the infimum of ϵ such that there exists an ϵ-isometry between X and Y. By definition, $d_L(X,Y) = \infty$ if there are no bilipschitz homeomorphisms between X and Y.

The pointed Lipschitz distance $d_{p.L}((X,p),(Y,q))$ is defined as the infimum of ϵ such that there exist ϵ-isometries $f : B_p(1/\epsilon, X) \to Y$ and $g : B_q(1/\epsilon, Y) \to X$ onto their images such that $f(p) = q$, $g(q) = p$.

Here are some simple examples.

Example 2.3. (1) When $\epsilon \to 0$, the product $S^1(\epsilon) \times X$ converges to X with respect to the Hausdorff distance.
(2) Let $L \subset \mathbf{R}^2$ be the tree defined by

$$L = \{(x,y) \,|\, x \text{ or } y \text{ is an integer}\},$$

and d is the induced length metric on L. Then $(L, \epsilon d)$ converges to the norm space $(\mathbf{R}^2, \|\ \ \|)$ with respect to the Hausdorff distance as $\epsilon \to 0$, where $\|(x,y)\| = |x| + |y|$.
(3) For an arbitrary Riemannian manifold (M,g) of dimension n and for any $p \in M$, the scaled metrics $((M, g/\epsilon), p)$ converge to the flat Euclidean space $(\mathbf{R}^n, 0)$ as $\epsilon \to 0$ with respect to the pointed Lipschitz distance.

Why is the Hausdorff distance useful ? This is because of the Gromov precompactness theorem.

Theorem 2.4([G5]). *Let k be an arbitrary real number and $D > 0$. Then*

(1) The set of all closed n-dimensional Riemannian manifolds M with $\mathrm{Ricci}_M \geq k$ and $\mathrm{diam}_M \leq D$ is relatively compact with respect to the Hausdorff distance.
(2) The set of all pointed complete n-dimensional Riemannian manifolds (M,p) with $\mathrm{Ricci}_M \geq k$ is relatively compact with respect to the pointed Hausdorff distance.

Thus for any sequence M_i with the geometric bounds in Theorem 2.4 (1) for instance, a subsequence M_j converges to a compact metric space X. It is not difficult to see that the lmit X is a length space. However, in genaral, it is not even a topological manifold.

Let M be as in Theorem 2.4 (1). Then by the Bishop and Gromov volume comparison theorem ([G5]), we have a uniform upper bound $C(n,k,D)\epsilon^{-n}$ for the number of disjoint ϵ-balls in M. This is the key in the proof of Theorem 2.4. This argument immediately implies that

(2.5) The Hausdorff dimension, $\dim_H X$, of the limit X is less than or equal to n.

Let $\mathcal{B}$ be a set of geometric bounds on some Riemannian invariants. We say that a sequence of Riemannian n-manifolds M_i *collapses* to X under $\mathcal{B}$ if

(1) M_i satisfies $\mathcal{B}$.
(2) $\lim_{i\to\infty} d_H(M_i, X) = 0$.
(3) $\dim_H X < n$.

A smooth manifold M is called to collapse under $\mathcal{B}$ if it admits a sequence of complete Riemannian metrics g_i such that (M, g_i) collapses to a space under $\mathcal{B}$.

Here are some basic questions.

Question 2.6. (1) What can one say about the singularities of X ?
(2) What can one say about topological relations between M_i and X ?
(3) Which manifolds collapse under prescribed geometric bounds $\mathcal{B}$?

In the case when $\mathcal{B} = \{|K| \le 1, \text{diam} \le D\}$, Fukaya [F2,3] studied questions (1) and (2). In this case the limit X has a nice stratification in the $C^{1,\alpha}$-metric category. Under the bound $\mathcal{B} = \{|K| \le 1\}$, Cheeger and Gromov [CGv1,2] and Cheeger, Fukaya and Gromov [CFG] studied the question (3). If one assumes only a lower curvature bound $K \ge -1$, we know that X is an Alexandrov space, which will be discussed in the next section.

Next we exhibit an example of collapsing under a lower curvature bound. Let G be a compact connected Lie group. Clearly G with bi-invariant metrics collapses to a point under the lower curvature bound 0. This can be generalized in the following form.

Theorem 2.7 ([Y2]). *Let G act on a compact manifold M, and g an G-invariant metric on M. Then M collapses to the quotient space $(M, g)/G$ under a lower sectional curvature bound.*

For a genaralization of Theorem 2.7, see [PWZ].

To give a specific example related with the theorem above, let consider the circle action on the sphere S^{2n+2} defined as follows: We fix a great hyper sphere $S^{2n+1} \subset S^{2n+2}$. Then let the circle act on each hypersphere parallel to S^{2n+1} as Hopf fibration. This defines a smooth circle action on S^{2n+2}. By Theorem 2.7, one can find a sequence of metrics on S^{2n+1} converging to the quotient S^{2n+1}/S^1, the suspension over the complex projective space $\mathbf{C}P^n$, which is not a topological manifold.(cf.[GP2]).

When a sequence M_i of Riemannian manifolds converges to one of lower dimenson under a lower sectional curvature bound, we can describe the topological relation between M_i and the limit (See Theorem 4.1).

In a way similar to Theorem 2.7, we have the following family of almost nonnegatively curved manifolds.

Theorem 2.8 ([FY2]). *Let $F \hookrightarrow M \to N$ be a fibre bundle with structure group G, a compact Lie group, such that*

(1) *N is of almost nonnegative curvature,*
(2) *F has G-invariant metric of nonnegative curvature.*

Then M is of almost nonnegative curvatue.

Since we need to understand the convergence of isometric group actions in later sections, we now present the definition and some properties of the pointed equivariant Hausdorff distance, which was introduced by Fukaya [F1].

Definition 2.9. We say that a tripple (X, Γ, p) belongs to $\mathcal{M}_{eq}$ if every metric ball in X is relatively compact, $p \in X$ and if Γ is a closed subgroup of $\mathrm{Isom}(X)$, the group of isometries of X. For $R > 0$, we put

$$\Gamma(R) = \{\gamma \in \Gamma \mid d(\gamma p, p) < R\}.$$

For $(X, \Gamma, p), (Y, \Lambda, q) \in \mathcal{M}_{eq}$, we say that a tripple (f, φ, ψ) represent an *ϵ-pointed equivariant Hausdorff approximation* from (X, Γ, p) to (Y, Λ, q) if

(1) $f : B_p(1/\epsilon, X) \to B_q(1/\epsilon + \epsilon, Y)$ is an ϵ-Hausdorff approximation with $f(p) = q$.
(2) $\varphi : \Gamma(1/\epsilon) \to \Lambda$ and $\psi : \Lambda(1/\epsilon) \to \Gamma$ satisfy the follwing:
(2.1) If $\gamma \in \Gamma(1/\epsilon)$ and $x, \gamma x \in B_p(1/\epsilon, X)$, then $d(f(\gamma x), \varphi(\gamma)(fx)) < \epsilon$.
(2.2) If $\mu \in \Lambda(1/\epsilon)$ and $x, \psi(\mu)(x) \in B_p(1/\epsilon, X)$, then $d(f(\psi(\mu)(x)), \mu(fx)) < \epsilon$.

Now the pointed equivariant Hausdorff distance $d_{p.e.H}((X, \Gamma, p), (Y, \Lambda, q))$ is defiened as the infimum of ϵ such that there exist ϵ-pointed equivariant Hausdorff approximations from (X, Γ, p) to (Y, Λ, q) and from (Y, Λ, q) to (X, Γ, p).

When the pointed equivariant Hausdorff distance between (X, Γ, p) and (Y, Λ, q) is small, the definition says that the Γ-acton on X is close to Λ-action on Y through the Hausdorff approximation f. This implies

Proposition 2.10 ([F1]). *If a sequence (X_i, Γ_i, p_i) converges to (Y, Λ, q) with respect to the pointed equivariant Hausdorff distance, then the quotient space $(X_i/\Gamma_i, \bar{p}_i)$ converges to $(Y/\Lambda, \bar{q})$ with respect to the pointed Hausdorff distance.*

Example 2.11. Let γ_i be the isometry of $\mathbf{R}^3$ defined by $\gamma_i(x, y, z) = (R(1/i)(x, y), z + 1/i^2)$, where $R(\theta)$ denotes the rotation on the (x, y)-plane around the origin with angle θ, and let Γ_i be the group generated by γ_i. Then $(\mathbf{R}^3, \Gamma_i, 0)$ converges to $(\mathbf{R}^3, S^1 \times \mathbf{R}, 0)$. Note that the limit depends on the choice of reference points. For instance, if we take p_i with $d(0, p_i) = i$ as the reference points, then $(\mathbf{R}^3, \Gamma_i, p_i)$ converges to $(\mathbf{R}^3, \mathbf{R} \times \mathbf{Z}, 0)$.

When spaces converge, one can always construct the limit of groups as follows:

Theorem 2.12([Y2],[FY2]). *Let $(X_i, \Gamma_i, p_i) \in \mathcal{M}_{eq}$ and assume that (X_i, p_i) converges to (Y, q) with respect to the pointed Hausdorff distance. Then there exists a closed subgroup Λ of $\mathrm{Isom}(Y)$ such that for a subsequence (X_i, Γ_i, p_i) converges to (Y, Λ, q) with respect to the pointed equivariant Hausdorff distance.*

Notice that Λ may be a continuous group even if Γ_i are discrete.

Proof. For simplicity, we give the proof in the case when Y is compact. In this case, the pointed Hausdorff convergence $(X_i, p_i) \to (Y, q)$ coincides with the Hausdorff convergence $X_i \to Y$. For each positive integer j, we take a finite set $\Sigma_j \subset Y$ satisfying

(1) $\Sigma_j \subset \Sigma_{j+1}$,
(2) the union $\cup\Sigma_j$ is dense in Y.

Let $f_i : X_i \to Y$ and $g_i : Y \to X_i$ be ϵ_i-Hausdorff approximations such that $d(g_i f_i x, x) < 2\epsilon_i$ for all $x \in X_i$, where $\epsilon_i = d_H(X_i, Y) \to 0$. We consider the set $\Lambda_i(j)$ consisting of all elements of the form, $\hat{\gamma} = f_i \circ \gamma \circ g_i$ restricted to Σ_j, where γ runs over Γ_i. Since Σ_j is finite, for sufficiently large i relative to j, $\Lambda_i(j)$ is a subset of the compact metric space

$$\mathcal{L}(j) = \{\varphi : \Sigma_j \to Y \,|\, 2^{-1} \le \frac{d(\varphi x, \varphi y)}{d(x,y)} \le 2 \quad \text{for all } x, y \in \Sigma_j\},$$

equipped with L^∞-norm. Thus for each fixed j, passing to a subsequence, we may assume that (the closure of) $\Lambda_i(j)$ converges to a compact set $\Lambda(j) \subset \mathcal{L}(j)$ with respect to the (classical) Hausdorff distance in $\mathcal{L}(j)$. Remark that each element of $\Lambda(j)$ is an isometric imbedding of Σ_j into Y. By diagonal argument, we have a sequence $\Lambda(1)$, $\Lambda(2)$, ... such that for $j < k$, $\Lambda(j)$ is contained in the restriction $\Lambda(k)|_{\Sigma_j}$. Now one can define the direct limit $\Lambda = \lim_{j\to\infty} \Lambda(j)$. Since each element in Λ, an isometric imbedding of $\cup\Sigma_j$ into Y, extends to an isometry of Y, we can consider Λ as a closed set of Isom(Y). Furthermore by the choice of f_i and g_i, it is easy to see that Λ is a group. The convergence $(X_i, \Gamma_i) \to (Y, \Lambda)$ follows from construction. □

For the proof of Theorem B, we shall first prove that the fundamental group of an almost nonnegatively curved manifold is almost solvable. Recall that a solvable group is made by several extensions of abelian groups. The following result plays an essential role in finding such extensions (See Section 8).

Theorem 2.13([FY2,3]). *Let (X_i, Γ_i, p_i) converge to (Y, G, q) and G' a normal subgroup of G, and suppose that*

(1) G/G' is discrete.
(2) Y/G is compact.
(3) X_i is simply connected and the action of Γ_i is free and properly discontinuous.
(4) There exists a positive number R_0 such that G' is generated by $G'(R_0)$.

Then G/G' is finitely represented and there exists a normal subgroup Γ'_i of Γ_i such that

(5) (X_i, Γ'_i, p_i) converges to (Y, G', q) for a subsequence.
(6) Γ_i/Γ'_i is isomorphic to G/G' for sufficiently large i.
(7) For every $\epsilon > 0$, Γ'_i can be generated by $\Gamma'_i(R_0 + \epsilon)$ for sufficiently large i.

In the case when G' is the identity component of G, the group Γ'_i constructed above is called the *collapsing part* of Γ_i. For instance, in the convergence $(\mathbf{R}^3, \Gamma_i, p_i) \to (\mathbf{R}^3, \mathbf{R} \times \mathbf{Z}, 0)$ in Example 2.11, the group Γ'_i generated by γ_i^i is the collapsing part. However as the following example shows, this is not always possible if one does not suppose an assumption in Theorem 2.13.

Example 2.14. We consider the product $\mathbf{R} \times (S^3, i\, g_0)$, where g_0 is the standard metric on S^3. Let $S^1 \subset SO(4)$ be a subgroup freely acting on S^3. Define an isometry γ_i of X by $\gamma_i(x, y) = (x + i^{-2}, \theta_{i^2}\, y)$, where $\theta_t = e^{2\pi\sqrt{-1}/t} \in S^1$. Let Γ_i be the group generated by γ_i. Then one can check that (X_i, Γ_i, p_i) converges to $(\mathbf{R}^4, \mathbf{Z} \times \mathbf{R}, 0)$. However there is no sugbroups of Γ_i converging to $0 \times \mathbf{R}$.

The proof of Theorem 2.13 is too long to present here. We just describe only the construction of Γ'_i.

Let R be a large number relative to $\max\{D, R_0\}$, where D is the diameter of Y/G. Let (f_i, φ_i, ϕ_i) represent an ϵ_i-pointed equivariant Hausdorff approximation from (X_i, Γ_i, p_i) to (Y, G, q), where $\lim_{i\to\infty} \epsilon_i = 0$. Thus $\varphi_i : \Gamma_i(1/\epsilon_i) \to G(1/\epsilon_i)$. Put

$$\Gamma_i'(R) = \{\gamma \in \Gamma_i(R) \,|\, \varphi_i(\gamma) \in G'\}.$$

Then the required Γ_i' is defined to be the group generated by $\Gamma_i'(R)$.

§3 Alexandrov Spaces and Splitting Theorem

As indicated in the preceeding section, the Hausdorff limit of a sequence of Riemannian minifolds with a lower sectional curvature bound is an Alexandrov space. The properties of the limit space is likely to approximate those of manifolds in the sequence. Recently Burago, Gromov and Perelman [BGP] made an important progress in the geometry of Alexandrov spaces.

Let X be a locally compact length space. Then locally there exists a minimizing segment joining every two points nearby, which is called a *minimal geodesic*, or simply a geodeesic. A geodesic joining x and y in X is denoted by xy. For three point $x, y, z \in X$, we denote by $\Delta(x, y, z)$ a geodesic triangle consisting of three geodesic joining them. For a real number k, we use the notation $\tilde{\Delta}(x, y, z)$ to denote a geodesic triangle $\Delta(\tilde{x}, \tilde{y}, \tilde{z})$ in $M^2(k)$, the simply connected complete surface of constant curvature k, with the same side lengths as $\Delta(x, y, z)$ if it exists. We also denote by $\tilde{\angle} xyz$ the angle between $\tilde{y}\tilde{x}$ and $\tilde{y}\tilde{z}$.

Definition 3.1. Under the notation above, X is called an *Alexandrov space* with curvature $\geq k$ if it satisfies the following condition:
(1) For any point $p \in X$, there exists an open set U containing p such that for every $x, y, z \in U$ and for w on a geodesic yz, we have $d(x, z) \geq d(\tilde{x}, \tilde{w})$, where $\tilde{w}$ is the point on the side $\tilde{y}\tilde{z}$ corresponding to w.

Or equivalently,
(2) For every geodesic γ and σ from p let x_s and y_t be the points on γ and σ respectively such that $d(p, x_s) = s$, $d(p, y_t) = t$. Then $\tilde{\angle} x_s p y_t$ is monotone non-increasing in s and t.

From definition, the angle between the two geodesics γ and σ is defined as the limit:

$$\angle(\gamma, \sigma) = \lim_{(s,t)\to(0,0)} \tilde{\angle} x_s p y_t.$$

It is an important property of such spaces that geodesics do not branch, which immediately follows from the definition. Notice however that a geodesic may not be extended anymore at some point.

From now on we consider only Alexandrov spaces with curvature bounded from below and with finite Hausdorff dimension, which will be simply called Alexandrov spaces.

Example 3.2. (1) A Riemannian manifold with sectional curvature $\geq k$ is an Alexandrov space with the same lower curvature bound. More generally, let M_i be a sequence of complete Riemannian manifolds with $K_{M_i} \geq k_i$, $\lim k_i = k$ and suppose that (M_i, p_i) converges to (X, x_0) with respect to the pointed Hausdorff distance. Then X is an Alexandrov space with curvature $\geq k$.
(2) Let (X, d_0) be a length space with diameter $\leq \pi$, and consider the Euclidean cone $K(X) = X \times [0, \infty)/X \times 0$ over X with the distance:

$$d((x,s),(y,t)) = (s^2 + t^2 - 2st\cos d_0(x,y))^{1/2}, \qquad x, y \in X.$$

Then $K(X)$ is an Alexandrov space with curvature ≥ 0 if and only if (X, d_0) is an Alexandrov space with curvature ≥ 1.
(3) Let (X, d_0) be a length space with diameter $\leq \pi$. Then the spherical suspension $S(X) = X \times [0, \pi]/X \times \{0, \pi\}$ with the distance:

$$\cos d((x,s),(y,t)) = \cos s \cos t + \sin s \sin t \cos d_0(x,y),$$

is an Alexandrov space with curvature ≥ 1 if and only if (X, d_0) is an Alexandrov space with curvature ≥ 1.

A detailed argument for (2),(3) of the example above is given in [BGP].

A comparison theorem of Toponogov type is still valid in Alexandrov spaces.

Theorem 3.3 ([BGP]). *If X is a complete Alexandrov space, the triangle comparison (1) and angle comparison (2) in Definition 3.1 hold true for arbitrary triangles $\Delta(x,y,z)$ and for arbitrary minimal geodesics γ and σ in X with $\gamma(0) = \sigma(0)$.*

Before proceeding to the splitting theorem, we observe the following elementary

Splitting Principle 3.4. *Let X be a complete Alexandrov space with curvature $\geq -\kappa^2$, and $\gamma : [-\sigma, \sigma] \to X$ a minimal geodesic joining p and q. Let $\mu \gg \delta > 0$ be given. Then for any $x \in B_\gamma(\delta) \backslash (B_p(\mu) \cup B_q(\mu))$, we have*

$$\angle pxq \geq \tilde{\angle} pxq > \pi - \tau_{\kappa,\mu}(\delta),$$

where $B_\gamma(\delta)$ is the δ-neighborhood of γ, and

$$\tau_{\kappa,\mu}(\delta) = \mathrm{const}(\coth \kappa\mu \sinh \kappa\delta)^{1/2}.$$

In particular when $\kappa\mu$ is sufficiently small (e.g. $\kappa = 0$), $\tau_{\kappa,\mu}(\delta) = \mathrm{const}(\delta/\mu)^{1/2}$.

Proof. We give a proof for $\kappa = 0$. The genaral case is similar. By the law of cosine, we have

$$d(p,q)^2 = d(p,x)^2 + d(q,x)^2 - 2d(p,x)d(q,x)\cos \tilde{\angle} pxq.$$

By the assumption $x \in B_\gamma(\delta)$, we have

$$d(p,q) > d(p,x) + d(q,x) - 2\delta.$$

The conclusion follows immediately from $d(p,x) \geq \mu$, $d(q,x) \geq \mu$. $\square$

Corollary 3.5. *In addition to the assumpsion of 3.4, suppose that X is a Riemannian manifold. Then there exists a open set V_δ containinig $B_\gamma(\delta)\backslash(B_p(\mu)\cup B_q(\mu))$ such that*

(1) *V_δ is diffeomorphic to a product $W_\delta \times [0,1]$.*
(2) *V_δ is contained in a $\tau(\delta)$-neighborhood of $B_\gamma(\delta)\backslash(B_p(\mu)\cup B_q(\mu))$,*

where $\tau(\delta)=\tau_{\kappa,\mu,\sigma}(\delta)$ with $\lim_{\delta\to\infty}\tau(\delta)=0$.

Proof. Consider the distance function $f(x)=d(x,\gamma)$ to γ, which is differentiable almost everywhere. For any $x\in B_\gamma(\delta)\backslash(B_p(\mu)\cup B_q(\mu))$, let y be a point on γ closest to x. It follows from 3.4 that

$$|\angle pxy-\pi/2|<\tau(\delta),\quad |\angle qxy-\pi/2|<\tau(\delta). \tag{3.6}$$

Let us consider a C^1-approximation $\tilde{d}_p$ of the distance function $d_p(x)=d(p,x)$:

$$\tilde{d}_p(x)=\frac{1}{\operatorname{vol}B_p(\epsilon)}\int_{B_p(\epsilon)} d(x,y)\,dy.$$

By (3.6), the gradients of f and $\tilde{d}_p$ are almost parpendicular, and the integral curve of $\tilde{d}_p$ is contained in $\tau(\delta)$-neighborhood of $B_\gamma(\delta)\backslash(B_p(\mu)\cup B_q(\mu))$. Let W_δ be the set of all intersections of such integral curves of $\tilde{d}_p$ with the level $\tilde{d}_p^{-1}(\mu)$. Then the required diffeomorphism $W_\delta\times[0,1]\to V_\delta$ can be defined by using the integral curves of $\tilde{d}_p$. □

Remark 3.7. By a recent work by Perelman [Pr1], one can obtain a topological version of Corollary 3.5 for an Alexandrov space.

In the special case when $\kappa=0$ and $\sigma=\infty$, we have the following result, a genaralization of Theorem 1.2, which will play an essential role in our argument.

Splitting Theorem 3.8 ([GP3],[Y2]). *Let X be a complete Alexandrov space with curvature ≥ 0. If X contains a line, then it is isometric to a product $X_0\times\mathbf{R}$.*

Proof. Let ℓ be a line in X. We say that another line ℓ' is *biasymptotic* to ℓ if and only if $\sup d(\ell(t),\ell')<\infty$ and $\sup d(\ell'(t),\ell)<\infty$. Applying Splitting Principle 3.4 to $p=\ell(\infty)$ and $q=\ell(-\infty)$, we have

(3.9) At each point $x\in X$, there is a unique line ℓ_x biasymmptotic to ℓ with $\ell_x(0)=x$.

Let $f=f_\ell$ be the Busemann function associated with the ray $\ell|[0,\infty)$:

$$f(x)=\lim_{s\to\infty} s-d(x,\ell(s)).$$

The line ℓ_x is characterized by the equation:

$$f(\ell_x(t))=t+f(x). \tag{3.10}$$

Next we show that

(3.11) The level sets of f are totally geodesic.

We prove that $L_0 = f^{-1}(0)$ is totally geodesic. Let r be a point on a minimal geodesic joining two points $p, q \in L_0$. Put $x_s = \ell(s)$, $y_s = \ell(-s)$ for large $s > 0$. By the definition of f,

$$|d(x_s, p) - s| < o_s, \quad |d(x_s, q) - s| < o_s, \tag{3.12}$$

where $\lim_{s\to\infty} o_s = 0$. By Theorem 3.3, $d(r, x_s) > d(\tilde{r}, \tilde{x}_s)$, where $\tilde{r}$ is the point on $\tilde{p}\tilde{q} \subset \tilde{\Delta}(p, q, x_s)$ corresponding to r. It is easily verified from (3.12) that $|d(\tilde{x}_s, \tilde{r}) - s| < o_s$, and hence $d(r, x_s) > s - o_s$. Similarly, we have $d(r, y_s) > s - o_s$. On the other hand, $d(r, x_s) + d(r, y_s) - 2s = o_s$ since ℓ is a line. It follows that $|d(r, x_s) - s| < o_s$, which implies $r \in L_0$. Thus L_0 is totally geodesic.

Now we consider the following situation: Let two lines ℓ_1 and ℓ_2 biasymptotic to ℓ intersect two level sets L_1 and L_2 of f at x_i and y_i, $(i = 1, 2)$ where $x_i = \ell_1 \cap L_i$, $y_i = \ell_2 \cap L_i$. Put $a = d(L_1, L_2)$, $b_i = d(x_i, y_i)$ $c = d(x_1, y_2)$. To complete the proof of Theorem 3.4, it suffices to prove

$$b_1 = b_2 \quad \text{and} \quad c^2 = a^2 + b_i^2. \tag{3.13}$$

Let us assume $f(L_1) < f(L_2)$ and prove $b_1 = b_2$. Let z_s be the intersection of segment $y_1\ell_1(s)$ with L_2 for large $s > 0$. Then Theorem 3.3 applied to $\Delta(x_1, y_1, \ell_1(s))$ implies that $b_2 = \lim_{s\to\infty} d(z_s, y_2) \geq b_1$. Similarly, $b_1 \geq b_2$. To prove the second half, let $\tilde{x}_2$ be the point on the comparison triangle $\tilde{\Delta}(x_1, y_1, \ell_1(s))$ corresponding to x_2, and put $d_s = d(\tilde{x}_2, \tilde{y}_1)$. Since $\lim_{s\to\infty} d_s = (a^2 + b_i^2)^{1/2}$, Theorem 3.3 implies that $d(x_2, y_1) \geq (a^2 + b_i^2)^{1/2}$. The opposite inequality $d(x_2, y_1) \leq (a^2 + b_i^2)^{1/2}$ is immediate. □

A system of pairs of points $(p_i, q_i)_{i=1}^m$ is called an (m, δ)-*strainer* at p in an Alexandrov space X if it satisfies

$$\tilde{\angle} p_i p q_i > \pi - \delta, \tag{1}$$

$$\begin{cases} \tilde{\angle} p_i p p_j > \pi/2 - \delta \\ \tilde{\angle} q_i p q_j > \pi/2 - \delta \\ \tilde{\angle} p_i p q_j > \pi/2 - \delta, \quad (i \neq j) \end{cases} \tag{2}$$

Remark that when δ is small, the condition (1) above shows that the segments $p_i p$ and pq_i form an almost minimizing broken geodesic, and the condition (2) shows that those m broken geodesics are independent in a certain sense. We note that the existence of some independent lines imposes a strong restriction on the space in nonnegative curvature (Theorem 3.8). This is also the case if there exists an (m, δ)-strainer for a small δ. In fact the following result can be proved by essentially using Spritting Principle 3.4.

Theorem 3.14 ([BGP]). *Let X be an Alexandrov space. Then*

(1) The Hausdorff dimension of X is an integer, say n.
(2) There exists a positive number $\delta = \delta_n$ such that the set X_δ of all (n, δ) strained points in X is open and dense in X.

(3) Each point in X_δ has a small neighborhood which is $\tau(\delta)$-isometric to an open neighborhood in $\mathbf{R}^n$, with $\tau(\delta) = \tau_n(\delta)$, $\lim_{\delta\to 0}\tau(\delta) = 0$.

In the recent Russian version [BGP], they proved that the complement of X_δ has at least Hausdorff codimension 2. See also [OS] about this topic and C^1-differentiability (in a weak sense) of X_0.

The following result is closely related with Theorem 3.14.

Theorem 3.15 ([FY2,3]). *Let X be an Alexandrov space with curvature bounded below. Then for any $p_0 \in X$ and for any $r_i \to \infty$, there exists a sequence p_i converging to p_0 such that for a subsequence $((X, r_i d), p_i)$ converges to $(\mathbf{R}^n, 0)$, where n is the dimension of X.*

Proof. We may assume that $((X, r_i d), p_0)$ converges to (Y_0, y_0). Since Y_0 has curvature ≥ 0, by Spritting Theorem 3.8 Y_0 is isometric to a product $\mathbf{R}^k \times L$, where L does not contain a line. Let o_i denote the pointed Hausdorff distance between $((X, r_i d), p_0)$ and (Y_0, y_0). Take $\epsilon_i \to 0$ such that

$$\epsilon_i r_i < 1/o_i \qquad \epsilon_i r_i \to \infty.$$

Put $y_0 = (0, z_0) \in 0 \times L$, and let $y_i \in 0 \times L$ and $q_i \in (X, r_i d)$ be such that $d(0, y_i) = \epsilon_i r_i$ and q_i is Hausdorff close to y_i. Let p_i be the midpoint of a geodesic joining p_0 and q_i, and x_i a point in $\mathbf{R}^k \times L$ Hausdorff close to p_i. Note that

(1) $d_X(p_0, p_i) \to 0$,
(2) $r_i d_X(p_0, p_i) \to \infty$,
(3) $d_H(B_{p_i}(\epsilon_i r_i, (X, r_i d)), B_{x_i}(\epsilon_i r_i, \mathbf{R}^k \times L)) \to 0$.

We change the reference point to p_i. For a subsequence we may assume that $((x, r_i d), p_i)$ converges to (Y_1, y_1). Under this convergence, the geodesic $p_0 q_i$ converges to a line ℓ. There exist also k-independent lines in Y_1 perpendicular to ℓ comming from the $\mathbf{R}^k$-factor of Y_0. Thus by the spritting theorem again, Y_1 is isometric to a product $\mathbf{R}^{k+1} \times L_1$. If L_1 is a point, then p_i are required ones. Otherwise, after repeating a similar argument finitely many times, one can get required points p_i. □

Finally we discuss the isometry group of an Alexandrov space.

Theorem 3.16([FY3]). *Let X be an Alexandrov space. Then* Isom(X), *the group of isometries of X is a Lie group.*

Example 3.17. Let X be a union of infinitely many circles S_i with length $1/i$ such that there is the only point p at which any two of S_i have intersection. Then $\mathrm{Isom}(X) \cong \Pi \mathbf{Z}_2$, which is totally disconnected but not discrete. Hence it is not a Lie group. Notice that X has curvature $-\infty$ at p.

In the proof of Theorem 3.16, we essentially use the following result:

Theorem 3.18([Gl],[Yb]). *Let G be a topological group, and suppose that there exists a neighborhood $U \subset G$ of the identity such that there exist no non-trivial subgroups contained in U. Then G is a Lie group.*

Proof of Theorem 3.16. To avoid a technical complexity, we just give the proof for the special case when X is a Riemannian manifold. (Of course, this case is known as Myers-Steenrod's Theorem ([MS]). Then the essential point in the proof in the general case will become clear.

Suppose X is a Riemannian manifold and that $\mathrm{Isom}(X)$ is not a Lie group. For $B = B_p(1, X)$, by Theorem 3.18, we can take a sequence G_i of closed subgroup of G such that if we put

$$\delta_i = \sup\{\delta_g(x, X) \mid g \in G_i, x \in B\},$$

then $\lim_{i\to\infty} \delta_i = 0$, where $\delta_g(x, X) = d(gx, x)$. Take $p_i \in B_i$ and $g_i \in G_i$ such that $\delta_i = \delta_{g_i}(p_i)$. Since X is a Riemannian manifold,

(3.19) $((X, (1/\delta_i)d), p_i)$ coverges to $(\mathbf{R}^n, 0)$ with respect to the pointed Hausdorff distance, where n is the dimension of X.

By Theorem 2.12, we may assume that $((X, (1/\delta_i)d), G_i, p_i)$ converges to $(\mathbf{R}^n, G, 0)$ with respect to the pointed equivariant Hausdorff distance. Remark that G is nontrivial and compact. It is now easy to choose $h \in G$ and $y_0 \in \mathbf{R}^n$ such that

(1) $\delta_h(y_0, \mathbf{R}^n) \geq 2$,
(2) if y_i is a point in $(X, (1/\delta_i)d)$ converging to y_0, then y_i is contained in B_i.

Let h_i be an element of G_i converging to h under the convergence $((X, (1/\delta_i)d), G_i, p_i) \to (\mathbf{R}^n, G, 0)$. Then $\delta_{h_i}(y_i, X) > \delta_i(2 - o_i) > \delta_i$ for a sufficiently large i, where $\lim o_i = 0$. This is a contradiction. □

For an Alexandrov space X, (3.19) does not hold. Hence we need to use Theorem 3.12 to take points q_i near p_i such that $((X, (1/\delta_i)d), G_i, q_i)$ converges to $(\mathbf{R}^n, G, 0)$. Remark that in this case, the limit group G might be trivial. If G is nontrivial, the proof above would cause a contradiction. If G is trivial, then one can think of G_i like a "small subgroup" of $\mathrm{Isom}(X, (1/\delta_i)d)$ and repeat the above argument. Because of the Hausdorff closeness between $(X, (1/\delta_i)d), q_i)$ and $(\mathbf{R}^n, 0)$, if one change the reference point within some ball of fixed size around q_i, and if one rescale the metric of $(X, (1/\delta_i)d)$, it would converge to $(\mathbf{R}^n, 0)$. Thus a modification of the above proof would work to conclude the proof in the general case.

§4 Fibration Theorem

We call a C^1-map $f : M \to N$ to be an *ϵ-Riemannian submersion* if

$$\left| \frac{|df(\xi)|}{|\xi|} - 1 \right| < \epsilon,$$

for all tangent vectors ξ orthogonal to the fibers.

In this section, we give the proof of the following theorem.

Theorem 4.1([Y2,3]). *Given $n,\mu > 0$, there exists a positive number $\epsilon = \epsilon_n(\mu)$ satisfying the following: Let M and N be complete manifolds with*

(1) $K_M \geq -1$, $K_N \geq -1$,
(2) $\dim N = n$, $\mathrm{inj}(N) \geq \mu$.

If the Hausdorff distance between M and N is less than ϵ, then there exists a locally trivial fibre bundle $f : M \to N$ such that

(3) *f is a $\tau(\epsilon)$-Riemannian submersion,*
(4) *f is a $\tau(\epsilon)$-Hausdorff approximation.*

where $\tau(\epsilon) = \tau_{n,\mu}(\epsilon)$ with $\lim_{\epsilon\to 0}\tau(\epsilon) = 0$.

For simplicity, we assume $|K_N| \leq 1$ in the proof below. The general case is proved in [Y3] (See also Remark 4.21).

Let σ be a positive number such that $\epsilon \ll \sigma \ll \min\{1,\mu\}$. Both ϵ and σ will be determined in the final step. Let $h : \mathbf{R} \to [0.1]$ be a smooth cut off function such that

$$
\begin{aligned}
&h(t) = 1 \quad \text{on} \quad (-\infty, \sigma/10], \quad h(t) = 0 \quad \text{on} \quad [\sigma, \infty),\\
&h'(t) = -1/\sigma \qquad \text{on} \quad [2\sigma/10, 8\sigma/10],\\
&-1/\sigma \leq h'(t) \leq 0, \quad |h''(t)| < 100/\sigma^2.
\end{aligned}
\tag{4.2}
$$

Let $L^2(N)$ be the space of all L^2-fuctions on N with the norm normalized as

$$\|f\| = \frac{\sigma^2}{b(\sigma)}\int_N |f(x)|^2\, dx,$$

where $b(\sigma)$ is the volume of a σ-ball in $\mathbf{R}^n$. Define a smooth map $f_N : N \to L^2(N)$ by

$$f_N(p)(x) = h(d(p,x)), \qquad (x \in N).$$

Let $\varphi : N \to M$ and $\psi : M \to N$ be ϵ-Hausdorff approximations such that $d(\psi\varphi x, x) < 2\epsilon$ and $d(\varphi\psi x, x) < 2\epsilon$, where we may assume that φ is measurable. We put

$$f_M(p)(x) = h(\tilde{d}(p,\varphi(x))), \qquad (x \in N).$$

where

$$\tilde{d}(p,\varphi(x)) = \frac{1}{\mathrm{vol}\, B_\epsilon(\varphi(x))}\int_{B_\epsilon(\varphi(x))} d(p,y)\, dy.$$

Then $f_M : M \to L^2(N)$ is a C^1-map. We need this averaging since there is no lower bound for injectivity radius of M. The derivatrives of both maps are given by

$$
\begin{aligned}
&df_N(\xi)(x) = h'(d(p,x))\,\xi(d_x), \quad \xi \in U_pN,\\
&df_M(\xi)(x) = h'(d(q,\varphi(x))\,\xi(\tilde{d}_{\varphi(x)}), \quad \xi \in U_qM,
\end{aligned}
\tag{4.3}
$$

where $d_x(.) = d(., x)$, $\tilde{d}_{\varphi(x)}(.) = \tilde{d}(., \varphi(x))$ and

$$\xi(\tilde{d}_{\varphi(x)}) = \frac{1}{\operatorname{vol} B_\epsilon(\varphi(x))} \int_{B_\epsilon(\varphi(x))\setminus C(\varphi(x))} \xi(d_y)\, dy,$$

where $C(\varphi(x))$ is the cut locus of $\varphi(x)$. We have easily

$$\|f_N(\psi(p)) - f_M(p)\| < \text{const } \epsilon, \tag{4.4}$$

for every $p \in M$.

From now on, $c_1, c_2, \ldots$ denote positive constants depending only on n and μ. We denote by U_pN the set of all unit vectors at $p \in N$.

Since N has bounded geometry, it is straightfoward with (4.2), (4.3) to show

Lemma 4.5. *f_N is a smooth imbedding whose derivative is well controlled:*

(1) *For every $\xi \in U_pN$,*

$$c_1 < \|df_N(\xi)\| < c_2.$$

(2) *For every p, q with $d(p,q) \leq \sigma$,*

$$c_3 < \frac{\|f_N(p) - f_N(q)\|}{d(p,q)} < c_4.$$

Next we study the tubular neighborhood of $f_N(N)$ in $L^2(N)$ and the properties of the normal projetion. We begin with the following lemma.

Lemma 4.6. *For any points $p, q \in N$ with $d(p,q) \leq \sigma$ and for any $\xi \in U_pN$, let $\hat{\xi} \in U_qN$ be the parallel translate of ξ along the minimal geodesic from p to q. Then*

$$|\xi(d_x) - \hat{\xi}(d_x)| < \frac{c_5}{\sigma} d(p,q),$$

for every x with $\sigma/10 \leq d(p,x) \leq \sigma$.

Proof. Put $v = \exp_p^{-1} q$, $w = \exp_p^{-1} x$, and let $\hat{w}$ be the parallel translate of w along the minimal geodesic from p to q. Since $|K_N| \leq 1$, a standard comparison argument (see [BK]) shows

$$d(\exp_p(v+w), \exp_q \hat{w}) < \sigma^2 |v|,$$
$$d(\exp_p(v+w), \exp_p w) < (1+\sigma^2)|v|.$$

Hence $d(\exp_p w, \exp_q \hat{w}) < (1+\sigma^2)|v|$, which implies that the angle between $\exp_q^{-1} x$ and $\hat{w}$ is less than const $|v|/\sigma$. Thus

$$|\angle(\xi, w) - \angle(\hat{\xi}, \exp_q^{-1} x)| < \text{const } |v|/\sigma.$$

The result follows immediately. □

Bu using Lemma 4.6, we can control how rapidly the tangent spaces to $f_N(N)$ change. We put

$$T_p = T_{f_N(p)}(f_N(N)), \qquad N_p = \{f_N(p) + \xi \mid \xi \perp T_p\}.$$

Lemma 4.7. *For every p, q with $d(p,q) \leq \sigma$, we have*

$$\angle(f_N(q) - f_N(p), T_p) < \frac{c_6}{\sigma} d(p,q).$$

Proof. Let $\gamma : [0,\ell] \to N$ be a minimal geodesic joining p to q, and put $\xi_t = \dot\gamma(t)$, where $\ell = d(p,q)$. Then for each $x \in B_p(2\sigma, N)$

$$\begin{aligned}
&|h(d(q,x)) - h(d(p,x)) - \ell h'(d(p,x))\xi(d_x)| \\
&\qquad = \ell |h'(d(\gamma(t),x))\xi_t(d_x) - h'(d(p,x))\xi(d_x)| \quad (\because \text{ mean value theorem}) \\
&\qquad \leq \ell^2/\sigma^2 \quad (\because \text{ Lemma 4.6}),
\end{aligned}$$

which implies that

$$\left\| \frac{f_N(q) - f_N(p)}{\ell} - df_N(\xi) \right\| \leq \ell/\sigma.$$

The lemma follows immediately. □

Now we define the notion of angle between subspaces in a Hilbert space H. Let V, W be closed subspaces of H, and $\pi_V : H \to V$ and $\pi_W : H \to W$ the orthogonal projections. Then the angle between V and W is defined by:

$$\angle(V,W) = \begin{cases} \sup\limits_{w \neq 0 \in W} \angle(w, \pi_V(w)) & \text{if } W \cap V^\perp = \{0\} \\ \sup\limits_{v \neq 0 \in V} \angle(v, \pi_W(v)) & \text{if } V \cap W^\perp = \{0\} \\ \pi/2 & \text{otherwise.} \end{cases}$$

Obviously $\angle(V^\perp, W^\perp) = \angle(V,W)$.

lemma 4.8. *For every p, q with $d(p,q) \leq \sigma$,*

$$\angle(T_p, T_q) = \angle(N_p, N_q) < \frac{c_7}{\sigma} d(p,q).$$

Proof. For any $\xi \in U_pN$, let $\hat\xi \in U_qN$ be as in Lemma 4.6. Then Lemma 4.6 yields

$$\|df_N(\xi) - df_N(\hat\xi)\| < \text{const } d(p,q)/\sigma.$$

□

Let ν be the normal bundle of $f_N(N)$ in $L^2(N)$. For $c > 0$, we put

$$\nu(c) = \{(x,u) \in \nu \mid \|u\| < c\}, \quad W(c) = \{x + u \mid (x,u) \in \nu(c)\}.$$

Lemma 4.9. *There exists a positive number $\kappa = c_8/\sigma$ such that $W(\kappa)$ provides a tubular neighborhood of $f_N(N)$, that is, $x + u \neq y + v$ for every $(x, u) \neq (y, v)$ in $\nu(\kappa)$.*

Proof. Let $x = f_N(p)$, $y = f_N(q)$, and suppose the intersection $K = N_p \cap N_q$ contains elements $x + u = y + v$. First we consider the case $d(p, q) < \sigma^2$. Let $z \in K$ and $w \in N_q$ be such that (i) $d(x, z) = d(x, K)$, (ii) $d(x, w) = d(x, N_q)$. Then $\angle xzw \leq \angle(N_p, N_q)$, and

$$\begin{aligned} \|x - w\| &= \|x - z\| \sin \angle xzw \\ &\leq \|x - z\| \sin \angle(N_p, N_q) \\ &\leq \frac{c_7}{\sigma}\|x - z\| d(p, q) \quad (\because \text{Lemma 4.8}) \\ &\leq \frac{c_7}{c_3\sigma}\|x - z\|\|x - y\| \quad (\because \text{Lemma 4.5}). \end{aligned} \tag{4.10}$$

Lemma 4.7 implies that $|\angle(x - y, N_q) - \pi/2| < c_6\sigma$, and by the choise of w, $\angle(x - w, N_q) = \pi/2$. It follows that $\|y - w\| < \text{const}\,\sigma\|x - y\|$. Together with (4.10), this implies that $\|x - z\| > c/\sigma$, and hence $\|u\| > c/\sigma$, where $c = \text{const}$.

Next we consider the general case. It follows from the argument above that

(4.11) The normal exponential mapping $\exp^\nu : \nu \to L^2(N)$, $\exp^\nu(x, u) = x + u$, is nonsingular on $\nu(c/\sigma)$.

Let $\kappa_0 = c'/\sigma$, where the constant c' will be determined later. We suppose that $(x, u), (y, v) \in \nu(\kappa_0)$ satisfy $x + u = y + v (= z)$. Let $\gamma : [0, \ell] \to N$ be the minimal geodesic joining p to q, and put $c(s) = f_N(\gamma(s))$ and define $\alpha : [0, \ell] \times [0, 1] \to L^2(N)$ by

$$\alpha(s, t) = (1 - t)c(s) + tz.$$

Since $\|x - y\| < 2\kappa_0$, Lemma 4.5 with triangle inequality implies that $\alpha(s, t) \in W((2c_4/c_3 + 1)\kappa_0)$. Hence if c' is sufficiently small, α is contained in $W(c/2\sigma)$. Here we need the following sublemma due to Katsuda [K].

Sublemma 4.12. *There exists a smooth map $\tilde{\alpha} : [0, \ell] \times [0, 1] \to \nu(c/2\sigma)$ such that $\exp^\nu(\tilde{\alpha}(s, t)) = \alpha(s, t)$ and $\tilde{\alpha}(s, 0) = (c(s), 0)$.*

In particular we would have $\tilde{\alpha}(s, 1) \equiv z$, a contradiction to (4.11). Thus the sublemma will complete the proof of Lemma 4.9.

Proof of Sublemma 4.12. Let T be the set of $t \in [0, 1]$ for which there exists a lift $\tilde{\alpha} : [0, \ell] \times [0, t] \to \nu(c/2\sigma)$ of α. Clearly $T \ni 0$. In view of (4.11), T is open. We show that T is closed. For $t_1, t_2 \in T$, we have

$$\begin{aligned} \|\tilde{\alpha}(s, t_1) - \tilde{\alpha}(s, t_2)\| &\leq \left\|\frac{\partial \tilde{\alpha}}{\partial t}\right\| |t_1 - t_2| \\ &\leq \|d(\exp^\nu)^{-1}\| |t_1 - t_2| \\ &\leq C|t_1 - t_2|, \end{aligned}$$

where C is a constant independent of s and t. Therefore if $t_i \in T$ converges to t_0, $\tilde{\alpha}_{t_i} = \tilde{\alpha}(., t_i)$ is a Cauchy sequence in the space of continuous curves in $\nu(c/2\sigma)$ with L^∞-norm. Thus we have $t_0 \in T$. $\square$

Now let $\pi : \nu(\kappa) \to f_N(N)$ be the projection along the fibers of the normal bundle ν.

Lemma 4.13. *For every $x \in W(\kappa))$ with $\pi(x) = f_N(p)$ and for every unit vector $\xi \perp N_p$, we have*

$$\|d\pi_x(\xi) - \xi\| < \frac{c_9}{\sigma}\|x - \pi(x)\|.$$

Proof. We put $y = x + t\xi$ for small $t > 0$ and $\pi(y) = f_N(q)$. Let N_0 be the affine space in $L^2(N)$ of codimension n which is parallel to N_p and containing y, and let z and w be intersections of N_q and N_0 with the n-plane $\pi(x) + T_p$ tangent to $f_N(N)$ at $\pi(x)$. We then have in a similar way to (4.10) that

$$\begin{aligned}\|z - w\| &\le \|z - y\|\angle(N_0, N_p)\\ &\le \frac{c_7}{\sigma}\|z - y\|\|\pi(x) - \pi(y)\|\\ &\le \frac{2c_7}{\sigma}\|x - \pi(x)\|\|\pi(x) - \pi(y)\|.\end{aligned}$$

It follows from $y - x = w - \pi(x)$ and smallness of $\|z - \pi(y)\| \ll t$ that

$$\begin{aligned}\|(\pi(y) - \pi(x)) - (y - x)\| &\le \|z - w\|\\ &\le \frac{\text{const}}{\sigma}\|x - \pi(x)\|\|\pi(x) - \pi(y)\|,\end{aligned}$$

and hence

$$\left|\frac{\|\pi(y) - \pi(x)\|}{t} - 1\right| < \frac{\text{const}}{\sigma}\|x - \pi(x)\|,$$
$$\left\|\frac{\pi(y) - \pi(x)}{t} - \xi\right\| < \frac{\text{const}}{\sigma}\|x - \pi(x)\|.$$

Letting $t \to 0$, we obtain the conclusion. □

If $\epsilon \ll \sigma$, $f_M(M)$ is contained in $W(\kappa)$ by (4.4) and the map $f = f_N^{-1} \circ \pi \circ f_M : M \to N$ is defined. (4.4) also shows that $d(fx, \psi x) <$ const ϵ, and hence f is $c_{10}\epsilon$-Hausdorff approximation. To prove that f is a fibration, it suffices to show

Lemma 4.14. *For every $p \in M$ and $\bar{\xi} \in U_{f(p)}N$, let $\xi \in U_pM$ be the velocity vector of a minimal geodesic from p to $\varphi(\exp_{f(p)} \sigma\bar{\xi})$. Then we have*

$$|df(\xi) - \bar{\xi}| < \tau(\sigma) + \tau(\epsilon/\sigma).$$

Remark 4.15. The constant on the right hand side in the above inequality can be expressed in the form:

$$O(\epsilon/\sigma) + O(\sqrt{\epsilon\sigma}) + O(\sigma^2).$$

However to avoid technical complexity, we will not do such explicit calculation.

For the proof of Lemma 4.14, we need the following triangle comparison lemma.

Lemma 4.16. *Let $\Delta(x,y,z)$ and $\Delta(\bar{x}\bar{y}\bar{z})$ be triangles in M and N respectively such that $d(\varphi(\bar{x}),x) < c_{10}\epsilon$, $d(\varphi(\bar{y}),y) < c_{10}\epsilon$, $d(\varphi(\bar{z}),z) < c_{10}\epsilon$. Suppose that $\sigma/10 \le d(x,y) \le \sigma$ and $\sigma/10 \le d(x,z) \le \sigma$. Then we have*

$$|\angle yxz - \angle \bar{y}\bar{x}\bar{z}| < \tau(\sigma) + \tau(\epsilon/\sigma).$$

We assume Lemma 4.16 for a moment and prove Lemm 4.14. For every $\bar{x}$ with $\sigma/10 \le d(f(p),\bar{x}) \le \sigma$, let $\bar{\eta} \in U_{f(p)}N$ and $\eta \in U_pM$ be velocity vectors of minimal geodesics from $f(p)$ to $\bar{x}$ and from p to $\varphi(x)$ respectively. Then Lemma 4.16 yields

$$|\angle(\bar{\xi},\bar{\eta}) - \angle(\xi,\eta)| < \tau(\sigma) + \tau(\epsilon/\sigma). \tag{4.17}$$

It follows that

$$\begin{aligned}&\|df_M(\xi) - df_N(\bar{\xi})\|^2\\ &\quad = \frac{\sigma^2}{b(\sigma)}\int_N \{h'(d(p,\varphi(x)))\xi(\tilde{d}_{\varphi(x)}) - h'(d(f(p),x))\bar{\xi}(d_x)\}^2\,dx,\end{aligned}$$

where

$$\begin{aligned}&|h'(d(p,\varphi(x))) - h'(d(f(p),x))| < \frac{\text{const}}{\sigma}\epsilon, \quad (\because (4.2))\\ &|\xi(\tilde{d}_{\varphi(x)}) - \bar{\xi}(d_x)| < \tau(\sigma) + \tau(\epsilon/\sigma), \quad (\because \text{Lemma 4.16}).\end{aligned}$$

Thus we have $\|df_M(\xi) - df_N(\bar{\xi})\| < \tau(\sigma) + \tau(\epsilon/\sigma)$. Lemma 4.13 then implies that

$$\|d\pi \circ df_M(\xi) - df_N(\bar{\xi})\| < \tau(\sigma) + \tau(\epsilon/\sigma).$$

Lemma 4.14 follows from Lemma 4.5.

Proof of Lemma 4.16. We put $s = d(x,y)$, $t = d(x,z)$ and $\bar{s} = d(\bar{x},\bar{y})$, $\bar{t} = d(\bar{x},\bar{z})$ and $\theta = \angle yxz$, $\bar{\theta} = \angle\bar{y}\bar{x}\bar{z}$. From the assumption $K_M \ge -1$, Toponogov's comparison theorem implies that

$$d(y,z)^2 < s^2 + t^2 - 2st\cos\theta + O(\sigma^4).$$

Since N has bounded geometry,

$$|d(\bar{y},\bar{z}) - (\bar{s}^2 + \bar{t}^2 - 2\bar{s}\bar{t}\cos\bar{\theta})| < O(\sigma^4).$$

It follows from $|d(y,z) - d(\bar{y},\bar{z})| < 2c_{10}\epsilon$ that

$$\theta > \bar{\theta} - \tau(\sigma) - \tau(\epsilon/\sigma). \tag{4.18}$$

Next take the point $\bar{w} \in N$ such that $d(\bar{y},\bar{w}) = d(\bar{y},\bar{x}) + d(\bar{x},\bar{w})$, $d(\bar{x},\bar{w}) = \sigma$, and put $w = \varphi(\bar{w})$. Let θ^* denote the angle $\angle zxw$. Then in the same way as (4.17) we have

$$\theta^* > \pi - \bar{\theta} - \tau(\sigma) - \tau(\epsilon/\sigma), \tag{4.19}$$

and

$$\angle yxw > \pi - \tau(\sigma) - \tau(\epsilon/\sigma).$$

The last inequality implies that

$$|\theta + \theta^* - \pi| < \tau(\sigma) + \tau(\epsilon/\sigma).$$

The result follows from (4.18) and (4.19). □

Finally we prove that f is an almost Riemannian submersion.
For any $p \in M$, set

$$H_p = \{\xi \in U_pM \mid d(p, \exp_p \sigma\xi/10) = \sigma/10\}, \tag{4.20}$$

which can be thought as the set of horizontan directions. By Lemma 4.14, df_p induces a τ-Hausdorff approximation between H_p and $S^{n-1}(1) = U_{f(p)}N$, where $\tau = \tau(\sigma) + \tau(\epsilon/\sigma)$. This implies that if $\eta \in U_pM$ satisfies $|\angle(\eta, H_p) - \pi/2| < \tau$, then $|\angle(\eta, \xi) - \pi/2| < \tau$ for all $\xi \in H_p$. In view of (4.2), (4.3), we have that $|df(\eta)| < \tau$, and hence f is a τ-Riemannian submersion as required. This completes the proof of Theorem 4.1. □

Remark 4.21. The imbedding technique used here is originally due to Gromov [G5]. Katsuda [K] overcame some gaps in [G5,ch.8]. The fibration theorem for $|K_M| \le 1$ was proved by Fukaya [F2]. He proved that the fibre is diffeomorphic to an infranilmanifold in that case, which generalizes Theorem 0.5. The proof presented here comes from that in [Y3], where an extension of Theorem 4.1 to Alexandrov spaces is discussed.

Remark 4.22. It is proved in [Y2] that the fibre of f is of $\tau(\epsilon)$-nonnegative curvature in some generalized sense, which one can formulate in terms of the deviation from the totally geodesicity of the fibers. (Remark that if the fiber is totally geodesic, it is of $\tau(\epsilon)$-nonnegative in the usual sense because of 4.1 (3)). In the following we present a weaker version of this fact by making use of Splitting Theorem 3.8.

Proposition 4.23. *For given $m \ge n$, $\mu > 0$ and $0 < \rho < 1$, there exists a positive number $\epsilon = \epsilon_{m,n,\mu}(\rho)$ satisfying the following: Let M and N satisfy the assumptions of Theorem 4.1 for μ and ϵ with $\dim M = m$, and let $f : M \to N$ be the fibration constructed there. Let F be a fibre of f with diameter δ_F, and d the distance of M restricted to F. Then we have*

(1) *If γ is a minimal geodesic joining $x, y \in F$ with $d(x,y) \ge \rho\delta_F$, then the angle between γ and F is less than $\tau(\epsilon)$.*
(2) *There exists an Alexandrov space X with nonnegative curvature such that*

$$d_H((F, d/\delta_F), X) < \tau(\epsilon),$$

where $\tau(\epsilon) = \tau_{m,n,\mu,\delta}(\epsilon)$ with $\lim_{\epsilon \to 0} \tau(\epsilon) = 0$.

Proof. This is done by contradiction. If the proposition is not true, we would have sequences of m-manifolds M_i and n-manifolds N_i with $d_H(M_i, N_i) < \epsilon_i \to 0$ satisfying the assumption in Theorem 4.1 such that the $\tau(\epsilon_i)$-Riemannian submersions $f_i : M_i \to N_i$ constructed there do not satisfy the conclusion for some $\rho < 1$. Namely there exists a point $q_i \in N_i$ such that the fibre $F_i = f_i^{-1}(q_i)$ satisfies the following.

(1) For some $x_i, y_i \in F_i$ with $d(x_i, y_i) > \rho\delta_F$, the angle between F_i and a minimal geodesic γ_i joining x_i and y_i is greater than a positive number θ_0 independent of i, or
(2) There exists a positive number c such that $d_H((F_i, d/\delta_{F_i}), X) > c$ for any Alexandrov space X with nonnegative curvature.

Put $\eta = \dot\gamma_i(0)$ and let $\eta_h \in H_{x_i}$ be such that

$$\angle(\eta, \eta_h) = \angle(\eta, H_{x_i}),$$

where H_{x_i} is as in (4.20). Let $\bar\xi = df(\eta)/|df(\eta)|$, and let $\xi \in U_{x_i}M_i$ be the velocity vector of minimal geodesic c_i joining x_i to a point z_i with $f_i(z_i) = \exp \sigma_i\bar\xi$, where σ_i is a positive number, $\epsilon_i \ll \sigma_i \ll \mu$, as in the proof of Theorem 4.1. Then Lemma 4.14 implies that $\angle(\xi, \eta_h) < \tau(\epsilon_i)$.

Now we consider the scaling of metrics $g_i = g_{M_i}/\delta_{F_i}$, $h_i = g_{N_i}/\delta_{F_i}$, where g_{M_i}, g_{N_i} are the original metrics. We denote by d_i the distance function of (M_i, g_i). Let w_i be the point on c_i with $d_i(x_i, w_i) = 1$. Let suppose that $\lim d_i(x_i, y_i) = s$. Since the angle between γ_i and c_i is less than $\pi/2 - \theta_0/2$, Toponogov's theorem yields that

$$d_i(y_i, w_i) < (1 + s^2)^{1/2} - c,$$

where c is a positive constant depending only on θ_0.

For a point $p_i \in F_i$, we may assume that (M_i, g_i, p_i) and (N_i, h_i, q_i) converges to (X, x_0) and (Y, y_0) respectively. By using $\mathrm{inj}(N_i) \geq \mu$, one can verify that Y is isometric to $\mathbf{R}^n$. Noting that X is a complete Alexandrov space with curvatue ≥ 0, we see from Theorem 3.8 that X is isometric to a product $X_0 \times \mathbf{R}^k$. Notice that the Lipschitz maps $f_i : (M_i, g_i, p_i) \to (N_i, h_i, q_i)$ also converges to a Lipshitz map $f : X \to Y$ with Lipshitz constant equal to 1. It is not difficult to show that $k = n$ and $f : X_0 \times \mathbf{R}^n \to \mathbf{R}^n$ is the projection up to translation in $\mathbf{R}^n$ (See [FY2] for detail). This shows in particular that

$$\lim_{i\to\infty} d_i(y_i, w_i) = (1 + s^2)^{1/2},$$
$$\lim_{i\to\infty} d_H((F_i, d_i), X_0) = 0,$$

which is a contradiction. □

By using Proposition 4.23, one can prove the following, which is useful when studying the properties of the fibre.

Theorem 4.24([Y2]). *Given $m \geq n$ and $\mu' > 0$ there exists a positive number $\epsilon' = \epsilon_{m,n}(\mu')$ satisfying the following: Let M and N satisfy the assumptions of Theorem 4.1 for μ and $\epsilon = \epsilon_n(\mu)$ with $\dim M = m$, and let $f : M \to N$ be the fibration constructed there.*

Let F be a fibre of f with diameter δ_F, and d the distance of M restricted to F. Suppose that there exists a Riemannian manifold P such that

$$(1) \qquad K_P \geq -1, \qquad \mathrm{inj}(P) \geq \mu'.$$

$$(2) \qquad d_H(P, (F, d/\delta_F)) < \epsilon'.$$

Then then there exists a locally trivial fibre bundle $f_F : (F, d/\delta_F) \to P$ satisfying

(1) *f_F is a $\tau(\epsilon, \epsilon')$-Riemannian submersion,*
(2) *f_F is a $\tau(\epsilon')$-Hausdorff approximation,*

where $\lim_{\epsilon, \epsilon' \to 0} \tau(\epsilon, \epsilon') = 0$, $\lim_{\epsilon' \to 0} \tau(\epsilon') = 0$.

Outline of proof. Let U be a small neighborhood of F in $(M, d/\delta_F)$. Proposition 4.23 shows that for any $p \in F$, the set H_p of horizontal directions in U (See (4.20)) is almost parallel to the tangent space T_pF. Thus an almost Riemannian submersion $f_U : U \to P$, constructed in Theorem 4.1 induces an almost Riemannian submersion $f_F : F \to P$.

Remark 4.25. One can iterate Theorem 4.24 as follows: Let E be a fibre of the fibration $f_F : (F, d/\delta_F) \to P$ in Theorem 4.24, and d/δ_E the distance of E rescaled by its diameter. Then if $(E, d/\delta_E)$ is Hausdorff close to a Riemannian manifold Q, then again we have an almost Riemannian submersion $f_E : (E, d/\delta_E) \to Q$. This procedure is possible as long as the rescaled fibre is Hausdorff close to a lower dimensional Riemannian manifold.

§5 Fibering by the First Betti Number

In this section, we shall give a proof of Theorem A.

For a metric space X with $\Gamma = \pi_1(X)$, let $h : \Gamma \to \Gamma/[\Gamma, \Gamma]$ be the Hurewicz homomorphism, and Ω the torsion part of $\Gamma/[\Gamma, \Gamma]$. Then $A = \Gamma/h^{-1}(\Omega)$ is a free abelian group of rank $b_1 = b_1(X; \mathbf{Q})$. We suppose that X has a universal covering space $\widetilde{X}$ and consider the abelian covering of X :

$$\widehat{X} = \widetilde{X}/h^{-1}(\Omega) \xrightarrow{A} X.$$

For a point $p \in \widehat{X}$, we use the norm $\|\gamma\| = d(\gamma p, p)$ on A. The notation $A(R)$ is as in Definition 2.9. The following lemma is due to Gromov [G5].

Lemma 5.1. *Suppose X to be compact. Then for every $\epsilon > 0$, there exists a subgroup A_ϵ of A satisfying*

(1) *A_ϵ has rank b_1.*
(2) *$\|\gamma\| \geq \epsilon$ for every nontrivial $\gamma \in A_\epsilon$.*
(3) *There exist generators $\gamma_1, \dots, \gamma_{b_1}$ of A_ϵ such that $\|\gamma_i\| \leq 2(D + \epsilon)$ for $1 \leq i \leq b_1$, where D is the diameter of X.*

Proof. Since Γ is generated by $\Gamma(2D)$, A is also generated by $A(2D)$. Remark that $\sharp A(\epsilon) < \infty$. First we take the subgroup A_0 generated by linearly independent elements $\gamma_1, \dots, \gamma_{b_1} \in A(2D)$. If $A_0(\epsilon)$ is trivial, A_0 is the required one. If γ is a nontrivial element in $A_0(\epsilon)$, we can find an integer $m \geq 2$ such that $\epsilon \leq \|\gamma^m\| \leq 2(D+\epsilon)$. Let i, $1 \leq i \leq b_1$, be such that γ_i-component of γ is nonzero, and define A_1 to be the group generated $\gamma_1, \dots, \gamma^m, \dots, \gamma_{b_1}$ replacing γ_i by γ^m. Notice that γ is not contained in A_1 and that A_1 satisfies (1) and (3). By replacing A_0 with A_1, we repeat the argument to get A_2 such that $\sharp A_2(\epsilon) \leq \sharp A(\epsilon) - 2$. After repeating this procedure finitely many times, we get the required A_ϵ. □

Proof of Theorem A(a). We prove Theorem A(a) by contradiction. Suppose that it does not hold. Then there would exist a sequence of closed Riemannian n-manifolds M_i such that

$$K_{M_i} > -\epsilon_i \to 0, \quad \mathrm{diam}(M_i) = 1, \quad b_1 \equiv b_1(M_i) > 0,$$

and that no finite covers of M_i fiber over a b_1-dimensional torus. Let $\widehat{M_i} \xrightarrow{A_i} M_i$ be the abelian cover, and $p_i \in \widehat{M_i}$. By Theorems 2.4 and 2.12, we may assume that $(\widehat{M_i}, A_i, p_i)$ converges to (X, G, x_0) with respect to the pointed equivariant Hausdorff distance. Remark that X is a complete Alexandrov space with curvature ≥ 0. Since X is noncompact and the action of G on X is cocompact, one can find a line in X. The Splitting Theorem 3.8 then implies that X is isometric to a product $Y \times \mathbf{R}^k$, where Y does not contain a line. If Y were noncompact, it would contain a line by the same reason. Hence Y must be compact.

Let $\delta_i = \max\{\sqrt{o_i}, \sqrt{\epsilon_i}\}$, where $o_i = d_{p.H}((\widehat{M_i}, p_i), (X, x_0))$. We take the scaling of the original metric g_{M_i}: $g_i = \delta_i g_{M_i}$. Notice that $\inf K_{g_i} \to 0$, $\mathrm{diam}_{g_i} \to 0$ and that the pointed Hausdorff distance between $((\widehat{M_i}, g_i), p_i)$ and $((X, \delta_i d), x_0)$ goes to zero as $i \to \infty$. Obviously $((X, \delta_i d), x_0)$ converges to $(\mathbf{R}^k, 0)$. Thus we have

(5.2) $((\widehat{M_i}, g_i), p_i)$ converges to $(\mathbf{R}^k, 0)$.

Again we may assume that $((\widehat{M_i}, g_i), A_i, p_i)$ converges to $(\mathbf{R}^k, H, 0)$. Since H is abelian and acts on $\mathbf{R}^k$ transitively, it must be the vector group $\mathbf{R}^k$. We now consider the subgroup $\Lambda_i = (A_i)_1 \subset A_i$ constructed in Lemma 5.1 for $\epsilon = 1$. We may assume that $((\widehat{M_i}, g_i), \Lambda_i, p_i)$ converges to $(\mathbf{R}^k, \Lambda, 0)$, where $\Lambda \subset H$ is a free abelian group of rank b_1. In particular we have obtained

(5.3) $b_1 \leq k$.

Next we need to use a pseudogroup technique. For a large positive number $R \gg k$, we put $B(R) = B_0(R, \mathbf{R}^k)$ and

$$L_R = B(R) \cap \mathbf{Z}^k,$$

where $\mathbf{Z}^k$ is the integer lattice of H, and consider L_R as a pseudogroup of isometric imbeddings of $B(R)$ into $B(2R)$. Similarly, we put $B_i(R) = B_{p_i}(R, (\widehat{M_i}, g_i))$, and for the canonical basis $e_1, \dots, e_k$ of $\mathbf{Z}^k$, let $e_{1,i}, \dots, e_{k,i}$ be elements in A_i converging to $e_1, \dots, e_k$ respectively. We consider a pseudogroup $L_{i,R}$ consisting of isometric imbeddings of $B_i(R)$ into $B_i(2R)$ defined by the following form:

$$L_{i,R} = \{\gamma = e_{1,i}^{a_1} \cdots e_{k,i}^{a_k} \in A_i \,|\, e_1^{a_1} \cdots e_k^{a_k} \in L_R, \|\gamma\| \leq R\}.$$

Then one can show

(5.4) There exists R and a positive integer I_R such that there exists a bijective pseudo homomorphism $\psi_i : L_{i,R} \to L_R$ for every $i \geq I_R$. Furthermore, ψ_i induces the equivariant Hausdorff convergence: $(B_i(R), L_{i,R}) \to (B(R), L_R)$.

Remak that $M_i^* = B_i(R)/L_{i,R}$ covers M_i, and that M_i^* converges to the flat k-torus $B(R)/L_R$ ($\because$ Theorem 2.9). Fibration Theorem 4.1 then provides an fibering of M^* over the k-torus, a contradiction. This completes the proof of Theorem A(a). □

In the proof above, we have used the pseudo group $L_{i,R}$ to find the finite cover M^*. The author believes that it can be avoided. Probably, there would exist a subgroup $L_i \subset A_i$ such that $((\widehat{M}_i, g_i), L_i, p_i)$ converges to $(\mathbf{R}^k, \mathbf{Z}^k, 0)$. More generally,

Conjecture 5.5. Let an (free) abelian group A_i freely act on X_i, and suppose that (X_i, A_i, p_i) converges to (Y, G, q_0) and that Y/G is compact. Then for a given cocompact subgroup $\Lambda \subset G$, there exists a subgroup Λ_i of A_i such that (X_i, Λ_i, p_i) converges to (Y, Λ, q_0).

When X_i is simply connected, one can prove that the conjecture above is true by using an argument similar to the proof of Theorem 2.13.

Remark 5.6. (1) If one does not assume that A_i is not abelian, then the conjecture above is false. For instance, consider the three dimensional simply connected nilpotent group:

$$N = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

with the left invariant metric g_ϵ such that

$$g_\epsilon = \epsilon^2(dx^2 + dy^2) + \epsilon^4 dz^2,$$

at the unit element, and the integer laticce Γ of N. Then $((N, g_\epsilon), \Gamma, e)$ converges to $(\mathbf{R}^3, \mathbf{R}^3, 0)$. However no subgroups of Γ converge to $\mathbf{Z}^3 \subset \mathbf{R}^3$ under this convergence.
(2) If one does not assume the compactness of Y/G, then the conjecture above is false. For instance, consider the convergence $(\mathbf{R}^3, \Gamma_i, p_i) \to (\mathbf{R}^3, \mathbf{R} \times \mathbf{Z}, 0)$ in Example 2.11. Then no subgroups of Γ_i converge to $\mathbf{Z} \times \mathbf{Z}$.

From the proof of Theorem A(a), we easily have

Theorem 5.7. *Let M_i be a sequence of compact Riemannian n manifolds with* $\inf K_{M_i} \to 0$, $\mathrm{diam}_{M_i} = 1$, $b_1(M_i) \equiv b_1$, *and $\widetilde{M}_i$ the universal cover of M_i. Then for any $\tilde{p}_i \in \widetilde{M}_i$, there exists a sequence $\delta_i \to 0$ such that if $g_i = \delta_i g_{M_i}$,*

(1) $\inf K_{g_i} \to 0$,
(2) $((\widetilde{M}_i, g_i), \tilde{p}_i)$ *converges to* $(\mathbf{R}^\ell, 0)$, *where* $\ell \geq b_1$.

Proof. As before, we may assume that $(\widetilde{M}_i, \tilde{p}_i)$ and $(\widehat{M}_i, p_i)$ converges to $(X \times \mathbf{R}^\ell)$ and $(Y \times \mathbf{R}^k)$ respectively, where both X and Y are compact, and the covering map $\pi_i : \widetilde{M}_i \to \widehat{M}_i$ carries $\tilde{p}_i$ to p_i. In the same way as in (5.2), we can find $\delta_i \to 0$ such that for $g_i = \delta_i g_{M_i}$, $((\widetilde{M}_i, g_i), \tilde{p}_i)$ and $((\widehat{M}_i, g_i), p_i)$ converges to $(\mathbf{R}^\ell, 0)$ and $(\mathbf{R}^k, 0)$ respectively.

By Ascoli's Theorem, $\pi_i : ((\widetilde{M}_i, g_i), \tilde{p}_i) \to ((\widehat{M}_i, g_i), p_i)$ converges to a Lipschitz map $\pi_\infty : (\mathbf{R}^\ell, 0) \to (\mathbf{R}^k, 0)$. Thus $\ell \geq k$, and we already know $k \geq b_1$. □

Next we shall prove Theorem A(b) by contradiction. Let M_i be a sequence of closed Riemannian n-manifolds such that

$$K_{M_i} > -\epsilon_i \to 0, \quad \mathrm{diam}(M_i) \to 0, \quad b_1(M_i) \equiv n, \tag{5.8}$$

but M_i is not diffeomorphic to a torus. Theorem A(a) implies that $\widetilde{M}_i$ is diffeomorphic to $\mathbf{R}^n$. From (5.3), we see that $(\widetilde{M}_i, \tilde{p}_i)$ converges to $(\mathbf{R}^n, 0)$. Thus by Fibation Theorem 4.1,

(5.9) $(\widetilde{M}_i, \tilde{p}_i)$ converges to $(\mathbf{R}^n, 0)$ with respect to the pointed Lipschitz distance.

Lemma 5.10. *Γ_i is abelian for all suficiently large i.*

Proof. Let $\rho_i : \Gamma_i \to A_i$ be the projection, where $A_i = \Gamma_i / h^{-1}(\Omega)$ is as before. Let $\Lambda_i = (A_i)_1$ be as in Lemma 5.1 for $\epsilon = 1$. Recall that

$$\begin{aligned} &\Lambda_i \text{ has rank } n, \\ &\|\lambda\| \geq 1 \text{ for every nontrivial } \lambda \in \Lambda_i, \\ &\text{there exist genarators } \lambda_1, \dots, \lambda_n \text{ of } \Lambda_i \text{ such that } \|\lambda_j\| \leq 4. \end{aligned} \tag{5.11}$$

Take $\gamma_j \in \rho_i^{-1}(\lambda_j)$ such that $\|\gamma_j\| = \|\lambda_j\|$, where the norm is with respect to $p_i \in \widehat{M}_i$ and $\tilde{p}_i \in \widetilde{M}_i$. Suppose that the lemma does not hold. Then we would have a sequence $\delta_1, \delta_2, \dots,$ in $[\Gamma_i, \Gamma_i]$ such that $2j \leq \|\delta_j\| \leq 4j$.

Now for $\mathfrak{a} = (a_1, \dots, a_{n+1}) \in (\mathbf{Z}_+)^{n+1}$, we consider the norm: $\|\mathfrak{a}\| = \sum a_j$. We put $\gamma_{\mathfrak{a}} = \gamma_1^{a_1} \cdots \gamma_n^{a_n} \delta_{a_{n+1}} \in \Gamma_i$. Remark that

$$\begin{aligned} &\|\gamma_{\mathfrak{a}}\| \leq 4\|\mathfrak{a}\|. \\ &\|\gamma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{b}}\| \geq 1 \text{ for } \mathfrak{a} \neq \mathfrak{b}. \end{aligned} \tag{5.12}$$

For $r \gg 1$, we obtain with (5.12)

$$\begin{aligned} \mathrm{const}_n\, r^{n+1} &\sim \sharp\{\mathfrak{a} \in (\mathbf{Z}_+)^{n+1} \mid \|\mathfrak{a}\| \leq r\} \\ &\leq \sharp\{\gamma_{\mathfrak{a}} \mid \|\mathfrak{a}\| \leq 4r\} \\ &\leq \frac{\mathrm{Vol}\, B_{\tilde{p}_i}(4r + 1/2, \widetilde{M}_i)}{\mathrm{Vol}\, B_{\tilde{p}_i}(1/2, \widetilde{M}_i)}. \end{aligned}$$

In view of (5.9), the right hande side of the inequality above conveges to $(8r+1)^n$. This is a contradiction. □

As a result of the lemma above, we see that M_i is a homotopy torus. However it is known that there exist some exotic tori having different differentiable structures from the standard one, and that every homotopy torus can be covered by the standard torus (See Theorem 5.13 below).

We shall prove that M_i is diffeomorphic to the standard torus for large i. First we consider the case of higer dimensions $n \geq 5$. In this case the structure of the set $\mathcal{S}(T^n)$ of all differentiable structures whose underlying n-manifold is homotopoic to T^n is well understood.

Theorem 5.13 ([HS],[Wal],[KS]). *If $n \geq 5$, $\mathcal{S}(T^n)$ is finite and has an abelian group structure.*

The identity of $\mathcal{S}(T^n)$ corresponds to the standard torus.

It is also known that any covering $T^n \to T^n$ naturally acts on $\mathcal{S}(T^n)$, from which we obtain

Corollary 5.14. *Let d_n be a positive number that is relatively prime with the order of $\mathcal{S}(T^n)$. Let M^n be a compact differentiable homotopy torus ($n \geq 5$). Then a d_n-fold covering of M is diffeomorphic to the standard torus if and only if M is diffeomorphic to the standard torus.*

Proof of Theorem A(b) for $n \geq 5$. Let M_i be as in (5.8). Lemma 5.10 implies that $(\widetilde{M}_i, \Gamma_i, \tilde{p}_i)$ converges to $(\mathbf{R}^n, \mathbf{R}^n, 0)$. We shall find some subgroup of Γ_i having a controlled index and not "collapsing".

Since $\operatorname{diam}(M_i) \to 0$, we can find a subgroup $\Lambda_{1,i} \subset \Gamma_i$ such that

$$(5.15) \qquad \begin{aligned} &[\Gamma_i : \Lambda_{1,i}] \text{ is a power of } d_n. \\ &1 \leq \operatorname{diam}(M_i) \leq 2. \end{aligned}$$

Passing to a subsequence, we may assume that $(\widetilde{M}_i, \Lambda_{1,i}, \tilde{p}_i)$ converges to $(\mathbf{R}^n, G_1, 0)$, where G_1 is isomerphic to $\mathbf{Z}^{k_1} \oplus \mathbf{R}^{n-k_1}$, where $k_1 > 0$ by (5.15). Put $M_{1,i} = \widetilde{M}_i / \Lambda_{1,i}$. Since $M_{1,i}$ converges to the flat torus $T^{k_1} = \mathbf{R}^n / G_1$, we have a fibration

$$F_i \hookrightarrow M_{1,i} \xrightarrow{f_i} T^{k_1}.$$

In particular, we have the decomposition $\Lambda_{1,i} = H_{1,i} \oplus \Gamma_{1,i}$, where $H_{1,i}$ is isomorphic to $\mathbf{Z}^{k_1}$ and $\Gamma_{1,i}$ is the fundamental group of the fiber F_i that is the collapsing part of Γ_i. Next we choose a subgroup $\Lambda_{2,1} \subset \Gamma_{1,i}$ as follows. Let $B \subset T^{k_1}$ be a contractible ball around the reference point $f_i(\tilde{p}_i \bmod \Lambda_{1,i})$ over which F_i is the fiber of f_i. Let $U_i = f_i^{-1}(B)$ and $\widetilde{U}_i$ be the component of $\pi_i^{-1}(U_i)$ containing $\tilde{p}_i$, where $\pi_i : \widetilde{M}_i \to M_{1,i}$ is the projection. Notice that $\pi_i : \widetilde{U}_i \to U_i$ is a universal cover, and that $\widetilde{F}_i = \pi_i^{-1}(F_i)$ is also a universal cover of F_i.

Sublemma 5.16. *There exists a subgroup $\Lambda_{2,i} \subset \Gamma_{1,i}$ such that*

(1) $[\Gamma_{1,i} : \Lambda_{2,i}]$ *is a power of d_n.*
(2) $1 \leq \operatorname{diam}(\widetilde{F}_{1,i}/\Lambda_{2,i}) \leq 2$.

Proof. Under the convergence $(\widetilde{M}_i, \tilde{p}_i) \to (\mathbf{R}^n, 0)$, $(\widetilde{F}_i, \tilde{p}_i)$ converges to an $(n-k_1)$-plane. Thus it converges to $(\mathbf{R}^{n-k_1}, 0)$ with respect to the pointed Lipschitz distance (See Theorem 4.24). The assertion follows from $\operatorname{diam}(F_i) \to 0$. □

Now we may assume that $(\widetilde{M}_i, H_{1,i} \oplus \Lambda_{2,i}, \tilde{p}_i)$ converges to $(\mathbf{R}^n, G_2, 0)$, where G_2 is isomorphic to $\mathbf{Z}^{k_1+k_2} \oplus \mathbf{R}^{n-k_1-k_2}$ and $k_2 > 0$. Repeating the procedure finitely many times, we obtain subgroups $H_{1,i}, H_{2,i}, \ldots, H_{i,\ell}$ of Γ_i such that

$$\Gamma_i^* = H_{1,i} \oplus \cdots \oplus H_{\ell,i} (\subset \Gamma_i) \text{ is a direct sum.}$$
$$(5.17) \qquad [\Gamma_i : \Gamma_i^*] \text{ is a power of } d_n.$$
$$(\widetilde{M}_i, \Gamma_i^*, \tilde{p}_i) \text{ converges to } (\mathbf{R}^n, \Lambda^*, 0), \text{ where } \Lambda^* \text{ is isomorphic to } \mathbf{Z}^n.$$

Hence $M^* = \widetilde{M}_i/\Gamma_i^*$ converges to the flat torus $\mathbf{R}^n/\Lambda^*$, and it is diffeomorphic to the torus. Therefore by Corollary 5.14, M_i is diffeomorphic to a torus. This completes the of Theorem A(b) for $n \geq 5$. □

Finally we give the proof of Theorem A(b) for $n < 5$. Here we consider the metric on M_i whose diameter is equal to one. We know that M_i converges to a flat torus T^k of positive dimension, and has a fibration

$$F_i \hookrightarrow M_i \to T^k,$$

where $\pi_1(F_i) \cong \mathbf{Z}^{n-k}$.

We need some information on the space of diffeomorphisms of a torus.

Theorem 5.18 ([EE],[Wad]). (1) *The group of diffeomorphisms of any closed orientable surface which is homotopic to the identity is connected.*
(2) *The group of PL-homeomorphisms of an irreducible, sufficiently large closed PL-three manifold M^3 which are homotopic to the identity is connected.*

Remark that T^3 is irreducible and sufficiently large.

We begin with the lowest dimension.

Case 1) $n = 2$. This is trivial.

Case 2) $n = 3$.

If $k \geq 2$, M_i is clearly diffeomorphic to T^3. If $k = 1$, $F_i \cong T^2$ and M_i can be identified with the quotient $T^2 \times [0,1]/g$, where $g : T^2 \times 0 \to T^2 \times 1$ is the gluing diffeomorphism. Since Γ_i is abelian, f should be homotopic to the identity. By Theorem 5.18 (1), it is diffeotopic to the identity, and M is diffeomorphic to T^3 as required.

Case 3) $n = 4$.

If $k \geq 3$, M_i is clearly diffeomorphic to a torus. Suppose $k = 1$. It is now easy to show that the fibre F_i with metric scaled in such a way $\mathrm{diam}(F_i) = 1$ converges to a flat torus T^ℓ. By Theorem 4.24 we have a fibration $E_i \hookrightarrow F_i \to T^\ell$. The argument in Case 2) shows $F_i \cong T^3$. Now M is identified with the quotient space $T^3 \times [0,1]/g$, where $g : T^3 \times 0 \to T^3 \times 1$ is the gluing diffeomorphism. Suince Γ_i is abelian, g is homotopic to the identity. It follows from Theorem 5.18 (2) that M is PL-homeomorphic to T^4, and hence by [Mu] it is diffeomorphic to a torus. If $k = 2$, use the projection $T^2 \to S^1 \times 0$ to the first factor to get a fibration $N_i \hookrightarrow M_i \xrightarrow{f_i} S^1$. We also have a fibration $T^2 \hookrightarrow N_i \xrightarrow{f_i|N_i} 0 \times S^1$. By the argument in Case 2), we see that N_i is diffeomorphic to T^3. Thus it is reduced to the case $k = 1$:

$$T^3 \hookrightarrow M_i \to S^1.$$

The proof of Theorem A(b) is now complete.

By using Theorem 4.24, one can prove Theorem A for the fibre of a fibration as in Theorem 4.1.

Theorem 5.19([Y2]). *Given $m \geq n$, $\mu > 0$ there exists a positive number $\epsilon = \epsilon_{m,n}(\mu)$ satisfying the following: Let M^m and N^n be complete manifolds with*

$$K_M \geq -1, \quad K_N \geq -1, \quad \mathrm{inj}(N) \geq \mu,$$
$$d_H(M, N) < \epsilon.$$

Let F be a fiber of a fibration as in Theorem 4.1. Then

(a) A finite cover of F fibers over a $b_1(F)$-dimensional torus.
(b) If $b_1(M) = m - n$, F is diffeomorphic to T^{m-n}.

§6 Bounded Almost Nonnegative Curvature

In this section, we consider almost nonnegative Ricci curvature under the stronger assumption $|K| \leq 1$. By using the Bochner technique, one can generalize an argument in Section 1 as follows.

Theorem 6.1 ([Y1]). *Given n and $D > 0$, there exists a positive number $\epsilon = \epsilon_n(D)$ such that if M satisfies that*

$$|K_M| \leq 1, \quad \mathrm{diam}(M) \leq D, \quad \mathrm{Ricci}_M > -\epsilon,$$

then every harmonic 1-form on M does not vanish.

In particular, the Albanese map $A : M \to T^{b_1}$ as in Section 1 is a fibre bundle.

Proof. Let ω be a harmanic 1-form on M. By the Weitzenböck formula 1.1,

$$\int_M |D\omega|^2 + \mathrm{Ricci}(\sharp\omega, \sharp\omega) = 0.$$

Here we consider the following norm on 1-forms :

$$\|\omega\|^2 = \frac{1}{\mathrm{vol}\, M} \int_M |\omega|^2 \, dx.$$

By assuming $\|\omega\| = 1$, we have from Gallot and Li's inequalies ([Ga],[Li])

$$|\omega|_{C^0} \leq \mathrm{const}\, \|\omega\| = \mathrm{const}\,.$$

Thus $\mathrm{Ricci}_M > -\epsilon$ yields that

$$\frac{1}{\mathrm{vol}\, M} \int_M |D\omega|^2 < \mathrm{const}\, \epsilon. \tag{6.2}$$

By using the assumption $|K_M| \le 1$, we shall prove

$$|D\omega|_{C^0} < \tau(\epsilon), \tag{6.3}$$

where $\tau(\epsilon) = \tau_{n,D}(\epsilon)$ with $\lim_{\epsilon\to 0}\tau(\epsilon) = 0$.

For any $p \in M$, put $B = B_0(1, T_pM)$, $B' = B_0(1/2, T_pM)$. Let g be the metric on B pulled back by the expoential map $\exp_p : B \to M$. Since the injectivity radius of (B', g) is greater than $1/2$, it follows from Jost-Karcher ([JK]), there exists a harmonic coordinate ball of fixed size centered at any point in B' and that the eigen values of the metric tensor with respect to this coordinate have some positive uniform bounds from both below and above on B'. Consider the harmonic form $\tilde{\omega} = \exp_p^* \omega$ on B, and take a function f on B such that $df = \tilde{\omega}$. Since $\Delta f = \delta\tilde{\omega} = 0$, it follows from the Schauder estimmate (see [GT]) that

$$|f|_{B',C^{2,\alpha}} \le \text{const}\ |f|_{B,C^0}.$$

Since we may assume that $|f|_{B,C^0} \le \text{const}\,|\tilde{\omega}|_{B,C^0}$, we see

$$\begin{aligned} |\tilde{\omega}|_{B',C^{1,\alpha}} &\le \text{const}\,|\tilde{\omega}|_{B,C^0} \\ &\le \text{const}\,|\omega|_{C^0} \\ &= \text{const}\,. \end{aligned} \tag{6.4}$$

Thus we have proved that $|\omega|_{M,C^{1,\alpha}} \le$ const. Now (6.3) follows from (6.2), (6.4) and $\operatorname{diam}(M) \le D$.

Now we see that ω is almost parallel if ϵ is small. Thus if $\omega_1, \dots, \omega_k$ are harmanic 1-forms of M forming a basis of the first de Rham cohomology group, they are pointwisely linear independent. Hence the Albanese map is a fibration. □

Remark that $A : M \to T^{b_1}$ is harmonic, and by (6.3) it is a $\tau(\epsilon)$-Riemannian submersion.

Problem 6.5. Find a geometric or topological property of the fibre of A.

Recently Lacouturier and Robelt have obtained a generalization of Theorem 6.1. They replaced the uniform bound $|K_M| \le 1$ by a bound on some integral norm on K_M.

If one assumes no bounds on the sectional curvature, Theorem 6.1 dose not hold any more because of the following result due to Anderson.

Theorem 6.6 ([A2]). *For any $n \ge 4$ and $1 \le k \le n-1$, one can construct an n-dimensional closed manifold M with $b_1(M) = k$ by doing a surgery on T^n such that*

(1) *No finite cover of M fibers over b_1-dimensional torus.*
(2) *For every $\epsilon > 0$ there exists a metric g_ϵ on M such that*

$$|\operatorname{Ricci}_{g_\epsilon}| < \epsilon, \qquad \operatorname{diam}(g_\epsilon) = 1.$$

(3) *(M, g_ϵ) converges to a flat torus T^k with respect to the Hausdorff distance.*

Remark that the Gromov conjecture 0.4 is still open.

Next we present a result for aspherical manifold, i.e. $\pi_i = 0$ for $i \ge 2$, with bounded sectional and almost nonnegaive Ricci curvature. We note that every aspherical manifold with nonnegative Ricci curvature is flat ([CG2]), which directly comes from Splitting Theorem 1.2. This can be genaralized as follows.

Theorem 6.7 ([FY1]). *Given n and $D > 0$ there exists a positive number $\epsilon = \epsilon_n(D)$ such that if M is an asphirical Riemannian manifolds with*

$$|K_M| \le 1, \qquad \mathrm{diam}_M \le D, \qquad \mathrm{Ricci}_M > -\epsilon,$$

then it is diffeomorphic to an infranilmanifold.

Outline of the proof. The proof is done by contradiction. Suppose that the theorem is not true. Then we would have a sequence M_i of n-dimensional aspherical manifolds with the given sectional curvature and diameter bounds and with $\mathrm{Ricci}_{M_i} > -\epsilon_i \to 0$ such that M_i is not diffeomorphic to an infranilmanifold. By the topological assumption, we see from [F4] that the universal cover $\widetilde{M}_i$ does not collapse. Namely with the bound $|K| \le 1$, $(\widetilde{M}_i, p_i)$ converges to a pointed space (N, q) of the same dimension with rspect to $C^{1,\alpha}$ topology on compace subsets. More generally this convergence happens in the $L^{2,p}$-topology, where $p > n$ (See [N] for instance). Thus the metric tensor of N is in $L^{2,p}$, and hence has second derivatives almost everywhere. Therefore we can get the splitting theorem for N (Compare [Ca]). Since N is contractible (see [F4]), it must be isometric to $\mathbf{R}^n$. Thus in some sense, M_i is $L^{2,p}$-almost flat. If it was C^2-almost flat, Theorem 0.5 works. Since it is not the case, we use the technique of covering space along fibre introduced in [FY1]. By applying this aurgument, we have a (singular) fibration $F_i \hookrightarrow M_i \to \mathbf{R}^m/\Lambda$ over a flat orbifold $\mathbf{R}^m/\Lambda$, where the fibre F_i is an infranilmanifold and the structure group can be reduced to some particular form (similar to that in Proposition 10.1). By using that informasion on the structure group, we can construct smooth almost flat metrics on M_i for sufficient large i (See [FY1] for the detail). □

§7 Generarization of Bieberbach' Theorem

In this section, we give a generalization of Bieberbach's theorem needed in the proof of Theorem B. As indicated in Section 1, the proof of Theorem 0.2 depends on Splitting Theorem 1.2 and the Bieberbach theorem. In our case, we consider the equivariant Hausdorff convergence of the isometric action of fundamental groups on universal covering spaces. The limit group is not necessarily discrete. Thus we need the following

Theorem 7.1([FY2]). *Let G be a closed subgroup of* $\mathrm{Isom}(\mathbf{R}^n)$. *Then $\pi_0(G) = G/G_0$ is almost abelian. More precisely, $\pi_0(G)$ contains a free abelian group A of finite index such that* $\mathrm{rank}(A) \le \dim(\mathbf{R}^n/G)$.

The case when G is discrete in Theorem 7.1 is Bieberbach's theorem.

We need the following elementary lemma from group theory. The proof is omitted (See [FY2]).

lemma 7.2. *Let a group G admit an exact sequence*

$$1 \longrightarrow \Omega \longrightarrow G \longrightarrow \mathbf{Z}^k \longrightarrow 1,$$

where Ω is finite with order ℓ. Then G contains a free abelian subgroup A of rank k such that $[G : A] < C(k, \ell)$.

Proof of Theorem 7.1. The proof is done by inductoin on the dimension on G. The case $\dim G = 0$ is the Bieberbach theorem. Suppose that $\dim G \geq 1$ and let R be the radical of G. Put $N = [R, R]$. We divide the proof into the following three cases.

Case 1) $\dim N > 0$.
Case 2) $\dim R > 0$ and R is abelian.
Case 3) $R = \{1\}$.

Here we consider only Case 1). The proof for the other cases are similar. Let C be the center of N. For $g \in G$, put

$$\min(g) = \{x \in \mathbf{R}^n \mid d(gx, x) \leq d(gy, y) \quad \text{for all } y \in \mathbf{R}^n\},$$
$$L = \bigcap_{g \in C} \min(g).$$

Remark that L is nonempty and is a G-invariant convex set *without* boundary. Hence $L \cong \mathbf{R}^\ell \subset \mathbf{R}^n$, and we obtain a homomorphism $\varphi : G \to \mathrm{Isom}(\mathbf{R}^\ell)$, where $\varphi(G)$ is closed. Now putting $K = \ker(\varphi)$, we have the exact sequences

$$1 \longrightarrow K \longrightarrow G \longrightarrow \varphi(G) \longrightarrow 1,$$
$$\pi_0(K) \longrightarrow \pi_0(G) \longrightarrow \pi_0(\varphi(G)) \longrightarrow 1,$$

where $\pi_0(K)$ is finite because of the compactness of K. Hence in view of Lemma 7.2, it suffices to prove the theorem for $\varphi(G) \subset \mathrm{Isom}(\mathbf{R}^\ell)$.

Case a) $\varphi(C) = \{1\}$.

Since $\dim \varphi(G) = \dim G/K \leq \dim G/C < \dim G$, the inductive assumption works.

Case b) $\varphi(C) \neq \{1\}$.

Since $\varphi(C)$ is normal in $\varphi(G)$, $\varphi(G)$ acts on $\mathbf{R}^\ell/\varphi(C) \cong \mathbf{R}^\ell/\mathbf{R}^{\ell-m} = \mathbf{R}^m$, where $m < \ell$. Let $\psi : \varphi(G) \to \mathrm{Isom}(\mathbf{R}^m)$ be the induced homomorphism. Putting $K' = \ker(\psi)$, we have exact sequences:

$$1 \longrightarrow K' \longrightarrow \varphi(G) \longrightarrow \psi\varphi(G) \longrightarrow 1,$$
$$\pi_0(K') \longrightarrow \pi_0(\varphi(G)) \longrightarrow \pi_0(\psi\varphi(G)) \longrightarrow 1.$$

Since $\varphi(C) \subset K' \subset \mathrm{Isom}(\mathbf{R}^{\ell-m})$, we have $K'/\varphi(C) \subset O(\ell - m)$. It follows that $\pi_0(K')$ is finite. It is easy to show that $\psi\varphi(G)$ is closed. Thus it suffices to prove the theorem for $\psi\varphi(G) \subset \mathrm{Isom}(\mathbf{R}^m)$. Since

$$\dim \psi\varphi(G) \leq \dim \varphi(G)/\varphi(C) = \dim G/C < \dim G,$$

the inductive assumption now works. □

Corollary 7.3([FY2]). *Under the situation in Theorem 7.1, suppose further that $\mathbf{R}^n/G$ is compact. Then G contains a normal subgroup G' such that*

(1) $[G : G'] < c_n$.
(2) $\mathbf{R}^n/G'$ *is isometric to a flat torus.*

Proof. This is also done by induction on $\dim(G)$. The case $\dim G = 0$ is the Bieberbach's theorem. Suppose $\dim G > 0$. We use the same notation as in the proof of Theorem 7.1, and assume the case 1). Since L is G-invariant and convex and since $\mathbf{R}^n/G$ is compact, we see that $L = \mathbf{R}^n$. Hence $\varphi =$ identity. Remark that C is normal in G. Since

$$\mathbf{R}^n/G = (\mathbf{R}^n/C)/(G/C) = \mathbf{R}^m/(G/C),$$

and since $\dim(G/C) < \dim G$, the inductive assumption works. □

In Theorem 7.1, there are no uniform bounds depending only on n for the index $[\pi_0(G) : A]$ even if $\mathbf{R}^n/G$ is compact.

Example 7.4. Let G be the subgroup of $\mathrm{Isom}(\mathbf{R}^2)$ generated by the translations $\mathbf{R}^2$ and the rotation with angle $2\pi/p$ around the origin. Then $\mathbf{R}^2/G$ is a point and $\pi_0(G) = \mathbf{Z}_p$.

§8 Solvability Theorem

In this section, we give the proof of the solvability part of Theorem B.

Theorem 8.1 ([FY2]). *There exist positive numbers ϵ_n and c_n such that if M is of ϵ_n-nonnegative curature, then $\pi_1(M)$ contains a solvable subgroup Λ such that*

$$[\pi_1(M) : \Lambda] < c_n. \tag{1}$$

$$\mathcal{L}(\Lambda) \le n. \tag{2}$$

A solvable group is made of several extensions of abelian groups. In the proof below we shall see that Theorems 3.8 and 7.1 provide each building block, and that Theorems 2.12 and 4.1 provide each extension.

We begin with an algegraic lemma.

Lemma 8.2. *Consider the following exact sequence:*

$$1 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow \mathbf{Z}^k \longrightarrow 1,$$

where Λ contains a solvable subgroup Λ' such that $[\Lambda : \Lambda'] = \ell$, $\mathcal{L}(\Lambda') = m$. Then Γ contains a solvable subgroup Γ' such that $[\Gamma : \Gamma'] < C(\ell)$, $\mathcal{L}(\Gamma') \le k + m$.

Proof. Take a subgroup Λ^* of Λ' which is characteristic in Λ. (See [FY2]). Thus Λ^* is normal in Γ and we have an exact sequence :

$$1 \longrightarrow \frac{\Lambda}{\Lambda^*} \longrightarrow \frac{\Gamma}{\Lambda^*} \longrightarrow \mathbf{Z}^k \longrightarrow 1.$$

Then the conclusion follows from Lemma 7.2. □

Proof of Theorem 8.1. This is done by contradiction and induction on the dimention of M. We mainly prove only almost solvablity. Suppose that the theorem is not true. Then we would have a sequence of closed n-manifolds M_i such that $K_{M_i} > -\epsilon_i$, $\epsilon_i \to 0$ as $i \to \infty$, and that $\mathrm{diam}(M_i) = 1$ but $\Gamma_i = \pi_1(M_i)$ is not almost solvable. Passing to a subsequence, we may assume that M_i converges to a compact Alexandrov space X. Since X might not be a manifold, we need to consider the action of Γ_i on the universal cover $\widetilde{M}_i$. For $p_i \in \widetilde{M}_i$, we may assume by Theorems 2.4 and 2.12 that $(\widetilde{M}_i, \Gamma_i, p_i)$ converges to a tripple (Y, G, q) with respect to the pointed equivariant Hausdorff distance. Splitting Theorem 3.8 then shows that Y is isometric to a product $\mathbf{R}^k \times Y_0$, where Y_0 is compact.

Assertion 8.3. *G/G_0 contains a finite index free abelian subgroup of rank $\le k$.*

Proof. Since G preserves the splitting $\mathbf{R}^k \times Y_0$, we obtain a homomorphism $\varphi : G \to \mathrm{Isom}(\mathbf{R}^k)$. If $K = \ker(\varphi)$, G/K is a closed subgroup of $\mathrm{Isom}(\mathbf{R}^k)$, we see by Theorem 7.1 that

$$(G/K)/(G/K)_0 \cong G/G_0K$$

contains a finite index free abelian subgroup of rank $\le k$. In view of Lemma 7.2, the assertion follows from the exact sequence:

$$1 \longrightarrow \frac{G_0K}{G_0} \longrightarrow \frac{G}{G_0} \longrightarrow \frac{G}{G_0K} \longrightarrow 1,$$

where G_0K/G_0 is finite. □

Note that G/G_0 is discrete by Theorem 3.16, and that $(\mathbf{R}^k \times Y)/G = X$ is compact. Hence we can apply Theorem 2.12 to get a normal subgroup Γ'_i of Γ_i such that

(8.4.)
$(\widetilde{M}_i, \Gamma'_i, p_i)$ converges to (Y, G_0, q).
Γ_i/Γ'_i is isomorphic to G/G_0 for large i.
For any $\epsilon > 0$, Γ'_i is generated by $\Gamma'_i(\epsilon)$ for large $i \ge I(\epsilon)$.

Assertion 8.5. *Γ'_i is almost solvable for sufficiently large i.*

Proof. We put $x_0 = q \mod G \in X$. For a given $r_j \to \infty$, by Theorem 3.15 we can choose $x_j \in X$ such that $((X, r_jd), x_j)$ converges to $(\mathbf{R}^m, 0)$. Remark that $m \ge 1$ since $\mathrm{diam}(X) = 1$. Set $d_H(M_i, X) = o_i$ and $d_{p.H}(((X, r_jd), x_j), (\mathbf{R}^m, 0)) = o_j$, and let $p_{ij} \in M_i$ be a point which is Hausdorff close to x_j. By triangle inequality,

$$d_{p.H}(((M_i, r_jg_{M_i}), p_{ij}), (\mathbf{R}^m, 0)) < r_jo_i + o_j.$$

Hence one can make the above Hausdorff distance as small as one likes if one takes a large $j = j_o$ and any large $i \ge i_0$. Thus for some choice of such j_0 and i_0, we have an almost Riemannian submersion $f_i : B_{p_i}(1, (M_i, r_{j_0}g_{M_i})) \to B_0(2, \mathbf{R}^m)$ over its image such that the fibre $F_i = f_i^{-1}(0)$ is of almost nonnegative curvature in the generalized sense. In view of Proposition 4.23 and Theorem 4.24, we can apply the inductive assumption to the fibre. Thus $\pi_1(F_i)$ is almost solvable for all sufficiently large i. Let $\epsilon = 1/(10r_{j_0})$. Since $B_{p_i}(\epsilon, M_i)$ is included in the neighborhood $f_i^{-1}(B_0(1,\mathbf{R}^m)) \simeq B_0(1,\mathbf{R}^m) \times F_i$ for large i, (8.4) implies that Γ_i' is contained in the image of the inclusion homomorphism:

$$\Gamma_i' \subset \mathrm{Im}\,[\pi_1(F_i) \to \Gamma_i].$$

This shows that Γ_i' is also almost solvable. □

Finally, we have the exact sequence:

$$1 \longrightarrow \Gamma_i' \longrightarrow \Gamma_i \longrightarrow G/G_0 \longrightarrow 1, \tag{8.6}$$

where Γ_i' is almost solvable and G/G_0 is almost abelian. Therefore the following lemma yields that Γ_i is almost solvable, a contradiction.

This argument also gives a uniform bound on the index of a solvable subgroup of Γ_i (by contradiction).

The inequality of the length of polycyclicity follows from

$$\begin{aligned} &\mathcal{L}(\Gamma_i) = \mathcal{L}(\Gamma_i') + \mathrm{rank}(G/G_0). \\ &\mathcal{L}(\Gamma') \le \mathcal{L}(\pi_1(F_i)) \le n - m, \quad (\because \text{inductive assumption}). \\ &\mathrm{rank}(G/G_0) \le \dim \mathbf{R}^k/\varphi(G) \le \dim X = m, \quad (\because \text{Theorem 7.1}). \end{aligned} \tag{8.7}$$

This completes the proof of Theorem 8.1. □

Remark 8.8. Remark that the inductive step in the above proof, we have to pursue the fibre properties at most n times by using Proposition 4.23 and Theorem 4.24. Although this is possible, we avoided this argument in [FY2], where the theorem was proved for the fibre of the almost Riemannian submersion in Fibration Theorem 4.1 by using a reverse induction.

Next we give some consequences from Theorem 8.1. We will see below the significance of the uniform bound (2) in Theorem 8.1.

Corollary 8.9([FY2]). *There exists a positive number C_n satisfying the following: Let M be of ϵ_n-nonnegative curvature, and suppose that any solvable subgroup Λ of $\pi_1(M)$ with $[\pi_1(M), \Lambda] \le C_n$ has $\mathcal{L}(\Lambda) = n$. Then*

(1) *Λ is poly-$\mathbf{Z}$ group.*
(2) *The universal cover of M is diffeomorphic to $\mathbf{R}^n$.*

Proof. We use the notation in the proof of Theorem 8.1. By (8.7), we have

$$\mathrm{rank}(G/G_0) = \dim \mathbf{R}^k/\varphi(G) = \dim X = m,$$
$$\mathcal{L}(\Gamma_i') = n - m.$$

By Corollary 7.3, there exists a normal subgroup $G_0 \subset \hat{G} \subset G$ such that $\mathbf{R}^k/\varphi(\hat{G})$ is isometric to a flat torus T^m. We put $\hat{X} = \mathbf{R}^k \times Y/\hat{G}$. Consider the following commutative diagram:

$$\begin{array}{ccc} \mathbf{R}^k \times Y & \xrightarrow{q_1} & \mathbf{R}^k \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \hat{X} & \xrightarrow[q_2]{} & T^m \end{array}$$

where p_i and q_i are the natural projections, which do not increase distance. Let $\mathbf{R}^k = \mathbf{R}^m \times \mathbf{R}^{k-m}$ be the orthogonal decomposition for which $p_2 : \mathbf{R}^m \to T^m$ is a Riemannian covering. Since $p_2 \circ q_1$ is a local isometry on each $\mathbf{R}^m$-factor, so is q_2. It follows that $\hat{X}$ is isometric to a flat torus T^m. By Theorem 2.13, we can take a normal subgroup $\hat{\Gamma}$ converging to $\hat{G}$. Hence $\hat{M} = \widetilde{M}/\hat{\Gamma}$ converges to T^m, and we have a fibration

$$F \hookrightarrow \hat{M} \to T^m,$$

and the exact sequence :

$$1 \longrightarrow \pi_1(F) \longrightarrow \hat{\Gamma} \longrightarrow \mathbf{Z}^m \longrightarrow 1.$$

Applying the inductive assumption to the fibre F, we obtain the conslusion. □

The uniform bound on the index of a solvable subgroup in Theorem 8.1 is useful. By using this theorem essentially, we have the following corollaries.

Corollary 8.10([FY2]). *There exists a positive integer p_n such that if M is of ϵ_n-nonnegative, then $b_1(M, \mathbf{Z}_p) \le n$ for all prime number $p \ge p_n$.*

If the equality $b_1(M, \mathbf{Z}_p) = n$ happens, M is diffeomorphic to a torus.

The inequality is immediate from Theorem 8.1. The equality case follows from an argument similar to Theorem A(b).

Corollary 8.11. *There exists a positive number C_n such that if M is of ϵ_n-nonnegative curvature and if $\pi_1(M)$ is finite, then*

$$\frac{\mathrm{diam}(\widetilde{M})}{\mathrm{diam}(M)} < C_n,$$

where $\widetilde{M}$ is the universal cover of M.

Proof. Suppose that the conclusion does not hold. Then we would have a sequence of closed n-manifolds M_i with ϵ_i-nonnegative, $\epsilon_i \to 0$, with finite fundamental groups Γ_i such that the diameter quotient for M_i goes to ∞ as $i \to \infty$. By Theorem 8.1, we may assume that Γ_i is solvable with length of policyclicity $\le n$. Let

$$\Gamma_i = \Gamma_i^{(0)} \supset \Gamma_i^{(1)} \supset \cdots \supset \Gamma_i^{(n)} = \{1\},$$

be the derived series, $\Gamma_i^{(s)} = [\Gamma_i^{(s-1)}, \Gamma_i^{(s-1)}]$, and put $M_i^{(s)} = \widetilde{M}_i/\Gamma_i^{(s)}$. Then we have a tower of abelian coverings:

$$\widetilde{M}_i \longrightarrow M_i^{(n-1)} \longrightarrow \cdots \longrightarrow M_i^{(1)} \longrightarrow M_i.$$

We take a scaling of metrics here so that $\mathrm{diam}(M_i) \to 0$ and the lower bound of sectional curvature of M_i still goes to zero. Then we find some $0 \le s \le n-1$ such that for a subsequence $M_i^{(s)}$ and $M_i^{(s+1)}$ converge to a point and a space Y which is not a point, respectively. By taking the scaling carefully, we may assme that Y is noncompact and actually isometric to a Euclidean space $\mathbf{R}^k$, $k \ge 1$ (See Theorem 5.7). Then by the pseudogroup thechnique used in Section 5 (the argument after (5.3)), we can find a covering M_i^* of $M_i^{(s)}$ so that it converges to a flat torus T^k. By Fibration Theorem 4.1, we have a fibration of M_i^* over T^k, which contradicts the finiteness of Γ_i. □

Notice that we have no explicit estimates for our constant C_n in the above result. On the other hand, for any lens space S^n/Γ with constant curvature, we know

$$\frac{\mathrm{diam}(S^n)}{\mathrm{diam}(S^n/\Gamma)} = 2.$$

Thus it would be interesting to find a (realistic) explicit constant C_n.

The following corollary is immediate from the proof of Corollary 8.11.

Corollary 8.12. *Let M be of ϵ_n-nonnegative curvature with infinite fundamental group. Then a finite cover of M fibers over S^1. In particular, the Eular characteristic of M vanishes.*

Conjecture 8.13. The Pontryagin numbers of an ϵ_n-nonnegatively curved manifold with infinite fundamental group vanish as well.

§9 Three Dimensional Case

We will give an outline of the proof of Theorem B in the next section. The key point in the proof is to develope the method of covering space along fibers introduced in the study of almost nonpositively curved manifolds ([FY1]). In this section, We shall make the basic idea clear by considering the three dimensional case, where we can determine the manifold structure up to finite cover and up to the Poincare conjecture.

Theorem 9.1([FY2]). *There exists a positive number ϵ such that if a closed three-manifold M is of ϵ-nonnegative curvature, then a finite covering of M is either homotopic to S^3 or diffeomorphic to one of $S^1 \times S^2$, a nilmanifold or a torus.*

Proof. We may assume that the fundamental group of M is infinite. Then the proof of Corollary 8.11 shows that a finite cover M^* of M converges to a flat torus T^k. If $k \ge 2$, M^* is diffeomorphic to an infranilmanifold or a torus (See the augument below). Let $k = 1$ and suppose that the conclusion does not hold in this case. Then we would have a sequence of closed three manifolds M_i such that $K_{M_i} > -\epsilon_i \to 0$ and a finite cover M_i^* converges to S^1 but M_i does not satisfy the conclusion of the theorem. By Theorem 4.1, there exists a fibration

$$F_i \longrightarrow M_i^* \longrightarrow S^1,$$

where F_i is diffeomorphic to one of S^2, a projective plane P^2 , a torus T^2 or a Klein bottle K^2 by Theorem 5.19. If $F_i \simeq S^2$, M_i^* is diffeomorphic to $S^2 \times S^1$. If $F_i \simeq P^2$ or $F_i \simeq K^2$, one can take a finite cover of M_i with fibre S^2 or T^2 respectively. Thus we may assume that $F_i \simeq T^2$. Let

$$1 \longrightarrow \mathbf{Z}^2 \longrightarrow \Gamma_i \longrightarrow \mathbf{Z} \longrightarrow 1$$

be the associated exact sequence, where $\Gamma_i = \pi_1(M_i^*)$. Now take $\gamma_i \in \Gamma_i$ which projects to $1 \in \mathbf{Z}$, and define $A_{\gamma_i} \in SL(2, \mathbf{Z})$ by

$$A_{\gamma_i}(g) \equiv \gamma_i g \gamma_i^{-1}.$$

Then the following is easily verified.

Lemma 9.2. *Γ_i is nilpotent if and only if A_{γ_i} is conjugate to an element of the form*

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$

The fibre F_i collapses to a point, and it is inconvenient to analyze the properties of A_{γ_i}. Therefore we take a finite cover of F_i in order to "look at" the fibre. This is the basic idea of covering space along fibre.

For $m \in \mathbf{Z}_+$, let $\Gamma_i^{(m)}$ be the subgroup of Γ_i generated by $(m\mathbf{Z})^2 (\subset \mathbf{Z}^2)$ and γ_i, and let $M_i^{(m)}$ be the finite cover of M_i with fundamental group $\Gamma_i^{(m)}$. Let $F_i^{(m)}$ be the m^2-fold covering of F_i corresponding to $M_i^{(m)}$. Then we can find m_i such that

$$1 \le \operatorname{diam}(F_i^{(m_i)}) \le 2. \tag{9.3}$$

Passing to a subsequence, we may assume that $(\widetilde{M}_i, \Gamma_i^{(m_i)}, p_i)$ converges to $(\mathbf{R}^k, G, 0)$. Remark that $\dim(\mathbf{R}^k/G) \ge 2$ because of (9.3). By Corollary 7.3, $\mathbf{R}^k/G$ can be finitely (branched) covered by T^2 or T^3. If M_i' is the finite cover of $M_i^{(m_i)}$ corresponding to it, M_i' converges to T^ℓ, $\ell = 2$ or 3. If $\ell = 3$, $M' \cong T^3$. If $\ell = 2$, we have a fibration

$$S^1 \longrightarrow M_i' \longrightarrow T^2.$$

Hence $\mathbf{Z} = \pi_1(S^1) \subset \Gamma_i^{(m_i)}$ is a normal subgroup and is A_{γ_i}-invariant. Hence it is also normal in Γ_i. Therefore by choosing a basis of $\mathbf{Z}^2 = \pi_1(F_i)$, we see that A_{γ_i} is conjugate to an element of the form

$$\begin{pmatrix} \pm 1 & c \\ 0 & \pm 1 \end{pmatrix}.$$

Lemma 9.2 implies that Γ_i is almost nilpotent, and hence M_i^* is diffeomorphic to an infranilmanifold. □

§10 Nilpotency Theorem

In this section, we give just an outline of the proof of Theorem B. For the detail, see [FY2].

First we consider the following exact sequence:

$$1 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow A \longrightarrow 1,$$

where Λ is almost nilpotent and A is almost abelian. We need a criterion for Γ to be almost nilpotent.

Proposition 10.1 ([FY2]). *The group Γ is almost nilpotent if and only if there exists a subset $\widehat{\Gamma}$ generating Γ such that for any $\gamma \in \widehat{\Gamma}$, there exists a stratification*

$$1 = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_k = \Lambda,$$

and $N \in \mathbf{Z}_+$ such that

(1) *Λ_i is normal and γ^N-invariant.*
(2) *if $\sharp\Lambda_i/\Lambda_{i-1} = \infty$, it is abelian.*
(3) *$A_\gamma^N \in \mathrm{Aut}(\Lambda_i/\Lambda_{i-1})$ defined by*

$$A_\gamma^N(g) \equiv \gamma^N g \gamma^{-N}$$

is of finite order.

Remark that if Γ is almost nilpotent, one can take the upper central series of a normal nilpotent subgroup of Γ as stratification.

We prove Theorem B by contradiction and inductuion on $\dim M$. If the theorem does not hold, we would have a sequence of closed n-manifolds M_i with $K_{M_i} > -\epsilon_i \to 0$, $\mathrm{diam}(M_i) = 1$ but $\Gamma_i = \pi_1(M_i)$ is not almost nilpotent. As in Section 8, we may assume that $(\widetilde{M}_i, \Gamma_i, p_i)$ converges to $(\mathbf{R}^k \times Y, G, q)$. By (8.6) we have the exact sequence:

$$1 \longrightarrow \Gamma_i' \longrightarrow \Gamma_i \longrightarrow G/G_0 \longrightarrow 1,$$

where G/G_0 is almost abelian and we may assme that Γ_i' is almost nilpotent by inductive assumption (See also Remark 8.8). Now for our purpose it is better to kill the compact factor Y in the sprilliting $\mathbf{R}^k \times Y$. As in (5.2), we can choose a positive number $\theta_i \to 0$ such that for the metric $g_i = \theta_i g_{M_i}$

(10.2) $(\widetilde{M}_i, g_i, p_i)$ converges to $(\mathbf{R}^k, 0)$,

while keeping $\inf K_{g_i} \to 0$. Notice that $\mathrm{diam}(M_i, g_i) \to 0$. From now on we use the notation

$$X_i = (\widetilde{M}_i, g_i), \quad \Lambda_i = \Gamma_i', \quad A = G/G_0,$$

and verify the condition of Proposition 10.1.

Notice that Γ_i is generated by $\Gamma_i(1)$, and suppose that it does not satisfy the condition of Proposition 10.1. Then there exists $\gamma_i' \in \Gamma_i(1)$ not satisfying the condition. Remark that $[\gamma_i'] \in \Gamma_i/\Lambda_i$ must be of infinite order. Then one can prove the following.

Lemma 10.3. *There exists $\gamma_i \in \Gamma_i(1)$ of the form $\gamma_i = (\gamma_i)^m \mu_i$, $\mu_i \in \Lambda_i$ such that if $\Lambda_i(\gamma_i)$ denote the group generated by Λ_i and γ_i, then $\Lambda_i(\gamma_i)(1/3) = \Lambda_i(1/3)$.*

We may assume that Λ_i is solvable by Theorem 8.1. We may also assume that $(X_i, \Lambda_i(\gamma_i), p_i)$ converges to $(\mathbf{R}^k, \widehat{H}, 0)$, under which Λ_i and γ_i converge to $H \subset \widehat{H}$ and $\gamma_\infty \in \widehat{H}$, respectively. Let $H(\gamma_\infty)$ be the group generated by H and γ_∞. We do not know if H is connected.

By Lemma 10.3 $\widehat{H}/H$ is discrete and hence H contains the identity component of $\widehat{H}$. Summing up we have

Proposition 10.4.

(1) $\lim_{i\to\infty}(X_i, \Lambda_i, p_i) = (\mathbf{R}^k, H, 0)$ *where* $H \subset \widehat{H}_0$.

(2) Λ_i *is normal in* $\Lambda_i(\gamma_i)$.

$$1 \longrightarrow \Lambda_i \longrightarrow \Lambda_i(\gamma_i) \longrightarrow \mathbf{Z} \longrightarrow 1.$$

(3) $\gamma_i \to \gamma_\infty \in \widehat{H}$.

$$1 \longrightarrow H \longrightarrow H(\gamma_\infty) \longrightarrow \mathbf{Z} \longrightarrow 1.$$

(4) $\Lambda_i(\gamma_i)$ *is solvable with length of polycyclicity* $\leq n$.

(5) $\Lambda_i(\gamma_i)(1/3) = \Lambda_i(1/3)$, $\Lambda_i = \Lambda_i(1)$.

Our purpose is to show that Λ_i has a stratification satisfying the condition for γ_i in Proposition 10.1. From the form of γ_i, this would implies that Λ_i has a stratification satisfying the condition for γ_i' in the proposition, a contradiction.

We know that Λ_i is generated by $\Lambda_i(\delta_i)$, where $\delta_i \to 0$. However we do not know if this is the case for $[\Lambda_i, \Lambda_i]$. From this reason we need a more stronger notion.

For $(X, \Gamma, p) \in \mathcal{M}_{eq}$, we put

$$\Gamma(\epsilon; D) = \{\gamma \in \Gamma \mid d(\gamma, x) < \epsilon \quad \text{for all} \quad x \in B_p(D, X)\}.$$

Definition 10.5. We say that a sequence (X_i, K_i, p_i) is *locally generated* if for any $D > 0$, there exists $\epsilon_i(D) \to 0$ such that K_i is generated by $K_i(\epsilon_i(D); D)$.

Then we can prove

Lemma 10.6. *If (X_i, K_i, p_i) is locally generated, $(X_i, [K_i, K_i], p_i)$ is also locally generated.*

In the first step, we shall replace $\{\Lambda_i\}$ be a locally generated sequence. To do this, we need a technical lemma similar to Theorem 2.13.

lemma 10.7. *Let $L_i \subset \Lambda_i$ be normal in $\Lambda_i(\gamma_i)$ such that $\lim_{i\to\infty}(X_i, L_i, p_i) = (\mathbf{R}^k, L, 0)$, and let a sequence $\delta_i \to 0$ and $D > 0$ be given. Then for a subsequence there exists a subgroup $L_i' \subset L_i$ satisfying*

(1) *L_i' is normal in $\Lambda_i(\gamma_i)$.*
(2) *L_i' is locally generated.*
(3) *$L_i' \supset L_i(\delta_i; D)$ for each i.*
(4) *$\lim_{i\to\infty}(X_i, L_i', p_i) = (\mathbf{R}^k, L', 0)$, where $L' \supset L_0$.*

Suppose further that there exists $R_0 > 0$ such that L_i is generated by $L_i(R_0)$. Let $L(\gamma_\infty)$ and $L_i(\gamma_i)$ be the groups generated by $L \cup \gamma_\infty$ and $L_i \cup \gamma_i$ respectively. Then there exists a surjective homomorphism $L(\gamma_\infty)/L' \to L_i(\gamma_i)/L_i'$ which carries $[\gamma_\infty]$ to $[\gamma_i]$.

In the lemma above, we do not assume the compactness of $\mathbf{R}^k/L$, which is the point essentially different from Theorem 2.13.

Applying Lemma 10.7 to $L_i \equiv \Lambda_i$, $L = H$, we have the following lemma, which is the first step in construction of the stratification of Λ_i.

Lemma 10.8. *There exists $\Lambda_i' \subset \Lambda_i$ satisfying*

(1) *Λ_i' is normal in $\Lambda_i(\gamma_i)$.*
(2) *(X_i, Λ_i', p_i) is locally genarated.*
(3) *$A_{\gamma_i} \in \mathrm{Aut}(\Lambda_i/\Lambda_i')$ is of finite order.*

Proof. By Lemma 10.7, there exists $\Lambda_i' \subset \Lambda_i$ satisfying (1), (2) and such that there exists a surjective homomorphism $H(\gamma_\infty)/H_0 \to \Lambda_i(\gamma_i)/\Lambda_i'$. Theorem 7.1 shows that $H(\gamma_\infty)/H_0$ is almost abelian, which implies (3). $\square$

The following lemma is a main step in the proof.

Lemma 10.9. *Let $[E_i, E_i] \subset F_i \subset E_i \subset \Lambda_i$ be normal in $\Lambda_i(\gamma_i)$, and suppose that*

(1) *E_i and F_i are locally generated.*
(2) *$\sharp(E_i/F_i) = \infty$.*
(3) *$\lim_{i\to\infty}(X_i, E_i, p_i) = (\mathbf{R}^k, E, 0)$.*

Then there exists E_i' such that for a subsequence

(4) *$F_i \subset E_i' \subset E_i$, E_i' is normal in $\Lambda_i(\gamma_i)$.*
(5) *E_i' is locally generated.*
(6) *$A_{\gamma_i} \in \mathrm{Aut}(E_i/E_i')$ is of finite order.*
(7) *$\lim_{i\to\infty}(X_i, E_i', p_i) = (\mathbf{R}^k, E', 0)$, where $\dim E' < \dim E$.*

Remark that the conclusion (7) above is analogous to the argument in Section 9 (See (9.3)).

Now we put

$$\Lambda_{i,1} \equiv \Lambda_i', \qquad \Lambda_{i,\ell+1} \equiv [\Lambda_{i,\ell}, \Lambda_{i,\ell}].$$

By Proposition 10.4(4), there exists N such that $\Lambda_{i,N} = \{1\}$.

Applying Lemma 10.9 inductively, we obtain

Assertion 10.10. *There exists $\Lambda_{i,k}^j$ which is normal in $\Lambda_i(\gamma_i)$ satisfying*

(1) $\Lambda_{i,k} = \Lambda_{i,k}^1 \supset \Lambda_{i,k}^2 \supset \cdots \supset \Lambda_{i,k+1}$.
(2) $A_{\gamma_i} \in \mathrm{Aut}(\Lambda_{i,k}^j/\Lambda_{i,k}^{j+1})$ *is of finite order.*
(3) $\lim_{i\to\infty}(X_i, \Lambda_{i,k}^j, p_i) = (\mathbf{R}^k, H_{j,k}, 0)$,
where $\dim H_{j,k} > \dim H_{j+1,k}$ *if* $[\Lambda_{i,k}^j : \Lambda_{i,k}^{j+1}] = \infty$.

By the conclusion (3) above, there exists L independent of i such that $[\Lambda_{i,k}^L : \Lambda_{i,k+1}]$ is finite. Finally we get a stratification of Λ_i:

$$\begin{aligned} 1 = \Lambda_{i,K} \subset \Lambda_{i,K-1}^L \subset \cdots \Lambda_{i,K-1}^2 \subset \Lambda_{i,K-1} \\ \subset \Lambda_{i,K-2}^L \subset \cdots \subset \Lambda_{i,K-3} \subset \cdots \subset \Lambda_{i,1} = \Lambda_i' \subset \Lambda_i, \end{aligned}$$

satisfying the condition of Proposition 10.1. Thus the proof of Theorem B is complete. □

Remark 10.11. In our argument here, we were not able to control the torsion parts of successive quotients in the stratification. This is the main reason why we cannot get a uniform bound on the index of a nilpotent subgroup.

§11 Generalized Margulis' Lemma

Let M be a complete Riemannian n-manifold with $|K_M| \le 1$. In the proof of Theorem 0.5, Gromov [G1] essentially used Margulis' lemma, which states that the small loops at any point $p \in M$ of length $\le \epsilon_n$ generate an almost nilpotent subgroup of $\pi_1(M)$. By using Theorem B, we can drop the upper bound of curvature.

Theorem 11.1 ([FY2]). *There exists a positive number ϵ_n such that if M is a complete Riemannian n-manifold with $K_M \ge -1$, then for any $p \in M$ the image under the inclusion homomorphism:*

$$\mathrm{Im}\,[\pi_1 B_p(\epsilon_n, M) \to \pi_1 B_p(1, M)]$$

is almost nilpotent and satisfies the conclusion of Theorem B.

As the first step, we show that the conclusion holds for some $p \in M$.

Lemma 11.2. *There exists $\delta_n > 0$ such that if M and p are as in Theorem 11.1, there exists a point $q \in B_p(1/2, M)$ such that*

$$\mathrm{Im}\,[\pi_1 B_q(\delta_n, M) \to \pi_1 B_p(1, M)]$$

is almost nilpotent and satisfies the conclusion of Theorem B.

This follows from Theorem B for the fibre of an almost Riemannian submersion and the arugument in the proof of Theorem 8.1.

Outline of the proof of Theorem 11.1. Suppose the theorem does not hold. Then we would have a sequence of complete n-manifolds M_i with $K_{M_i} \geq -1$, a point $p_i \in M_i$ and $\epsilon_i \to 0$ such that

$$\Gamma_i = \mathrm{Im}\,[\pi_1 B_{p_i}(\epsilon_i, M_i) \to \pi_1 B_{p_i}(1, M_i)]$$

is not almost nilpotent. By Lemma 11.2, we have $\delta = \delta_n > 0$ and $q_i \in M_i$ with $d(p_i, q_i) < 1/2$ such that

$$\Gamma_i' = \mathrm{Im}\,[\pi_1 B_{q_i}(\delta, M_i) \to \pi_1 B_{p_i}(1, M)]$$

is almost nilpotent. In the argument below, we show that $[\Gamma_i : \Gamma_i \cap \Gamma_i']$ is finite, a contradiction.

Let $\pi_i : X_i \to B_{p_i}(1, M_i)$ be the universal covering space, and $x_i, y_i \in X_i$ such that $\pi_i(x_i) = p_i$, $\pi_i(y_i) = q_i$, and $d(x_i, y_i) = d(p_i, q_i)$. We put $G_i = \pi_1 B_{p_i}(1, M_i)$. Passing to a subsequence, we may assume that (X_i, G_i, x_i) converges to (X, G, x_∞). By [G1] (See also [BK]),

(11.3) there exists $N > 0$ such that Γ_i can be generated by N elements $\gamma_{i,1}, \ldots, \gamma_{i,N}$ with $\gamma_{i,j} \in \Gamma_i(2\epsilon_i)$.

Let $f_i : B_{x_i}(D_i, X_i) \to B_{x_\infty}(D_i + 1/D_i, X)$, $\varphi_i : G_i(D_i) \to G)$ and $\psi_i : G(D_i) \to G_i$ be a pointed equivariant Hausdorff approximation as in Definition 2.9, where $D_i \to \infty$. By Ascoli-Arzera's theorem, we may assume that

(11.4) $\varphi_i(\gamma_{i,j})$ converges to $\gamma_{\infty,j} \in G$ for each j.

By (11.3),

$$\gamma_{\infty,j}(x_\infty) = x_\infty.$$

We may also assume that Γ_i' converges to $\Gamma_\infty' \subset G_\infty$.

Sublemma 11.5. *G/Γ_∞' is discrete.*

Proof. Let $y_i \to y_\infty \in X$. By the definition of Γ_i', the open set $\{\gamma \in G \,|\, d(\gamma y_\infty, y_\infty) < \delta/2\}$ of G is contained in Γ_∞'. □

We put $x = x_\infty$ for simplicity. Let G_x be the isotoropy subgroup at x. Lemma 11.5 shows that $L = [G_x : G_x \cap \Gamma_\infty']$ is finite. In view of $\gamma_{\infty,j} \in G_x$, using (11.3) and (11.4) one can show that $[\Gamma_i : \Gamma_i \cap \Gamma_i'] < C(N, L)$ for sufficiently large i. Therefore Γ_i is almost nilpotent. □

Proof of Theorem C. Suppose that the theorem does not hold. Then there would exist a sequence of closed n-manifolds M_i with $K_{M_i} \geq -1$, $\mathrm{diam}(M_i) \leq D$, $\Gamma_i = \pi_1(M_i)$ such that

$$\Gamma_i \not\cong \Gamma_j \quad (i \neq j) \quad \text{mod almost nilpotent groups.}$$

Passing to a subsequence, we may assume that $(\widetilde{M}_i, \Gamma_i, p_i)$ converges to (Y, G, q). By Theorem 2.13, there exists a normal subgroup Γ_i' of Γ_i such that

(1) Γ_i' converges to G_0.
(2) $\Gamma_i/\Gamma_i' \cong G/G_0$ for large i.

(3) Γ_i' is generated by $\Gamma_i'(\epsilon_i)$, where $\lim_{i\to\infty}\epsilon_i = 0$.

It follows from Theorem 11.1 that Γ_i' is almost nilpotent and satisfies the conclusion of Theorem B. Theorem 2.13 also shows that G/G_0 is finitely represented. This is a contradiction. □

§12 Noncollapsing Case

In this section we observe almost nonnegatively curved manifolds with a lower volume bound. Such a manifold should have the structure similar to that of a nonnegatively curved manifold.

We begin with the following finiteness theorem for fundamental groups due to Anderson.

Theorem 12.1([A1]). *The set of all isomorphism classes of fundamental groups of closed n-manifolds with*

$$\mathrm{Ricci}_M \geq -(n-1)k^2, \quad \mathrm{diam}(M) \leq D, \quad \mathrm{vol}(M) \geq v_0,$$

is finite.

Proof. Let M satisfy the geometric bounds above, $\widetilde{M}$ the universal cover of M with a reference point $p \in \widetilde{M}$ and $\Gamma = \pi_1(M)$ the deck tranformation group. The point of the proof is to show

Assertion 12.2. *There exist positive numbers δ and N_0 depending only on the given constants such that*

$$\sharp\Gamma(\delta) = \sharp\{\gamma \in \Gamma \mid d(\gamma p, p) < \delta\} \leq N_0.$$

Proof. Let F be the fundamental domain of Γ: $F = \{x \in \widetilde{M} \mid d(p,x) \leq d(\gamma p, x) \quad \text{for all } \gamma \in \Gamma\}$. For a $\delta > 0$, we put $N = \sharp\Gamma(\delta)$, $\Gamma(\delta) = \{\gamma_1, \dots, \gamma_N\}$ and $g_i = \gamma_i \cdots \gamma_1$, $(1 \leq i \leq N)$. It follows from $d(p, g_i p) < i\delta$ that

$$\bigcup_{i=1}^{\ell} g_i(F) \subset B_p(\ell\delta + D), \quad (1 \leq \ell \leq N),$$

and hence by Bishop's volume comparison theorem

$$\ell v_0 \leq \ell\,\mathrm{vol}(M) \leq b_k(\ell\delta + D), \tag{12.3}$$

where $b_k(r)$ denotes the volume of an r-ball in the n-dimensional complete, simply connected space of constant curvature $-k^2$. Hence if we put $N_0 \equiv b_k(2D)/v_0$, $\delta \equiv D/N_0$ for instance, we have the required estimate. □

Now by a result due to Gromov [G5], Γ can be generated by those γ_i such that $d(p, \gamma_i p) \leq 2D$ and the relations are of form $\gamma_i\gamma_j\gamma_k^{-1} = 1$. Hence to prove the theorem, it suffices to evaluate $\sharp\Gamma(2D)$. Let $\{x_j\}$ be a maximal subset of $\{\gamma p \mid \gamma \in \Gamma(2D)\}$ such that $d(x_j, x_k) \geq \delta$ for $j \neq k$. Let K be the number of $\{x_j\}$. By the Bishop and Gromov volume comparison theorem [G5],

$$K \leq \frac{\mathrm{vol}\, B_p(2D + \delta/2)}{\mathrm{vol}\, B_p(\delta/2)} \leq \frac{b_k(2D + \delta/2)}{b_k(\delta/2)}.$$

This yields that

$$\sharp\Gamma(2D) \leq K \sharp\Gamma(\delta) \leq N_0 \frac{b_k(2D + \delta/2)}{b_k(\delta/2)}. \tag{12.4}$$

□

For almost nonnegative Ricci curvature, Wei gave an estimte for growth of fundamental groups by using the idea of Theorem 12.1.

Corollary 12.5([We]). *Given n and $D, v_0 > 0$ there exists a positive number $\epsilon = \epsilon_n(D, v_0)$ such that if a closed n-manifold M satisfies*

$$\mathrm{Ricci}_M > -\epsilon, \quad \mathrm{diam}(M) \leq D, \quad \mathrm{vol}(M) \geq v_0,$$

then $\pi_1(M)$ has polynomial growth of order $\leq n$.

Proof. Let M satisfy the bounds above for $\epsilon > 0$. By (12.4), we can take generators $\gamma_1, \dots, \gamma_L$ of $\Gamma = \pi_1(M)$ such that $d(\gamma_i p, p) \leq 2D$, where L is bounded by a uniform constant. Let $g(s)$ be the number of words in Γ of length $\leq s$ with respect to $\gamma_1, \dots, \gamma_L$. Similarly to (12.3), we have

$$g(s) v_0 \leq b_\epsilon(2sD + D). \tag{12.6}$$

If $g(s)$ is not of polynomial growth of order $\leq n$ for any sufficienlty small ϵ, there exists a sequence $s_i \to \infty$ such that $g(s_i) > i s_i$. Since there are only finitely many possibilities for the isomorphism class of Γ (Theorem 12.1), one can take s_i independent of M. On the other hand, by (12.6) for any large s we can find a small $\epsilon > 0$ such that $g(s) < \mathrm{const}_{n,D,v_0}\, s^n$. This is a contradiction. □

For almost nonnegative sectional curvature, by using Theorem 8.1, we have

Corollary 12.7 ([FY2]). *Given n and $D, v_0 > 0$, there exists a positive number $\epsilon = \epsilon_n(D, v_0)$ such that if a closed n-manifold M satisfies that*

$$K_M > -\epsilon, \quad \mathrm{diam}(M) \leq D, \quad \mathrm{vol}(M) \geq v_0,$$

then $\pi_1(M)$ contains a free abelian subgroup A of rank $\leq n$ such that $[\pi_1(M) : A] < c_n$.

Proof. Suppose the theorem does not hold. Then we would have a sequence of closed n-manifolds M_i with $K_{M_i} > -\epsilon_i \to 0$, $\mathrm{vol}_{M_i} > v_0$, $\mathrm{diam}(M_i) \leq D$, and that $\Gamma_i = \pi_1(M_i)$ does not satisfy the conclusion. For p_i in $\widetilde{M}_i$, the universal cover of M_i, we may assume as in the proof of Theorem 8.1 that $(\widetilde{M}_i, \Gamma_i, p_i)$ converges to $(\mathbf{R}^k \times Y, G, q)$ with respect to the pointed equivariant Hausdorff distance, where Y is compact. By a result of Cheeger [C], there exists a positive number $\delta = \delta_n(D, v_0)$ such that $\Gamma_i(\delta) = \{1\}$, which implies that G_0 is trivial, and hence G is discrete. By Assertion 8.3, G/G_0 contains a finite index, free abelian subgroup of rank $\le k$, and by Theorem 2.13, Γ_i is isomorphic to G for sufficiently large i, a contradiction. □

Remark 12.8. In Corollary 12.5, it follows from the polynomial growth theorem [G4] that $\pi_1(M)$ is almost nilpotent. Remark that for any nilmanifold N^n which is not a torus, $\pi_1(N)$ has polynomial growth of order $> n$ (See [Mi], [Wo2]). Probably, the conclusion of Corollary 12.7 should hold under the assumption of Corollary 12.5.

For a (topological) splitting property of a finite cover of an almost nonnegatively curved manifold with a lower volume bound, see [SW], [Wu], [Ca].

§13 Concluding Remarks

First we remark that the main methods in our argument was both Splitting Theorem 3.8 and Fibration Theorem 4.1. We have a generalization of Theorem 4.1 to Alexandrov spaces ([Y3],see also [Wi]). The resulting map in this case is an almost *Lipschitz submersion*, which is not not known to be a fibre bundle yet. However it is sufficient for generalizations of Theorem A(b) and the results for fundamental groups, Theorems B and C, to Alexandrov spaces (See [Y3]).

A main problem still remainning would be to extend the results to manifolds with almost nonnegative Ricci curvature. Thus we are led to

Conjecture 13.1 ([FY2]). Let (X, p) be the pointed Hausdorff limit of a sequence (M_i, p_i) of complete n-manifolds with $\mathrm{Ricci}_{M_i} > -\epsilon_i \to 0$. Then the splitting theorem holds for X.

For the fibration theorem, Anderson's theorem 6.6 shows that it does not hold for $\dim N < \dim M$. However the equality case $\dim N = \dim M$ is open:

Conjecture 13.2 ([FY2]). There exists a positive number $\epsilon = \epsilon_n(\mu_0)$ such that if the Hausdorff distance between complete n-manifolds M and N with

$$\mathrm{Ricci}_M \ge -(n-1), \quad |K_N| \le 1, \quad \mathrm{inj}(N) \ge \mu_0$$

is less than ϵ, then M and N has the same topological type.

By Perelman's recent result [Pr2], the conjecture above would be true up to homotopy if one can prove the following volume convergence:

Conjecture 13.3. Under the same situation as in Conjecture 13.2, for any $p \in M$

$$\left| \frac{\mathrm{vol}\, B_p(r, M)}{\mathrm{vol}\, B_q(r, N)} - 1 \right| < \tau(\epsilon),$$

where $q \in N$ is a point Hausdorff close to p and $\lim_{\epsilon \to 0} \tau(\epsilon) = 0$.

Let us assume that a sequence of n-dimensional complete Riemannian manifolds M_i converges to an n-dimensional Riemannian manifold N. Then the lower semicontituity of volume

$$\liminf_{i \to \infty} \operatorname{vol}(M_i) \geq \operatorname{vol}(N)$$

does *not* hold except for $n = 1$, unless no curvature assumptions are made, as the following example shows.

Example 13.4. Let $L \subset \mathbf{R}^2$ be as in Example 2.3 (2). For each positive integer k, we divide each of the four edges of $S = L \cap [0,1]^2$ into small intervals of length $\delta = 2^{-k}$. Let $S_k \subset \mathbf{R}^2$ be the tree made by joining all pairs of such partition points by segments. Then we can consider S_k as a tree on the flat torus $T^2 = \mathbf{R}^2/\mathbf{Z}^2$. For $\epsilon > 0$ much smaller than δ, let $M_k(\epsilon)$ denote the boundary of the ϵ-neighborhood of S_k in $T^3 = T^2 \times S^1$. After carrying out a smoothing procedure for $M_k(\epsilon)$, we obtain a smooth hypersurface in T^3 denoted also $M_k(\epsilon)$ such that it converges to T^2 with respect to the Hausdorff distance as both δ and $\epsilon \lll 2^{-k}$ converges to zero. Obviously, the area of $M_k(\epsilon)$ converges to zero.

The following conjecture would be affirmative if one can settle Conjectures 13.1 and 13.3.

Conjecture 13.5 (Gromov). If Ricci $\mathrm{diam}^2 > -\epsilon_n$, then the fundamental group is almost nilpotent.

For some refinements of the above conjecture, see [FY2].

The following is closely related with the Chern conjecture that every abelian subgroup of the fundamental group of a closed manifold with positive sectional curvature is cyclic.

Conjecture 13.6. There exists a positive number c_n such that if a closed n-manifold M has positive sectional curvature, then $\pi_1(M)$ contains a cyclic subgroup S such that $[\pi_1(M) : S] < c_n$. Thus the fundamental groups of positively curved manifolds would be essentially cyclic.

The following is a sort of a gap theorem conjecture.

Conjecture 13.7. There exists a positive number ϵ_n such that if M^n is of ϵ_n-nonnegaive curvature, then it is of almost nonnegative curvature.

Tha gap theorems for almost flat manifolds and almost nonpositively curved manifolds were proved in [G1] and [FY1] respectively.

So far only few results other than π_1 or b_1 are known for topology of closed manifolds of nonnegative curvature except for Gromov's Betti number theorem [G3], several sphere theorems or related results.

Question 13.8. What can one say about π_i, $(i > 1)$ for nonnegatively curved manifolds ?

See [GH] for related topics.

In this article, we considered almost nonnegative sectional or Ricci curvature. For positive scalar curvature, some topological obstruction is known ([Lc], [GL], [SY]). A question related with our work is

Question 13.9. What can one say about convergence of metrics of almost nonnegative scalar curvature on some closed manifold M ? (For instance $M = T^n$). This would be more realistic if one assumes the geometirc bounds, $|K| \leq 1$, diam $\leq D$, vol $\geq v_0$, for instance.

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