L^p-estimates for functions of elliptic operators on manifolds of bounded geometry

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Abstract

We study L^p-continuity for functions of uniformly elliptic, properly supported pseudodifferential operators with uniformly bounded symbols on manifolds of bounded geometry.

1 Introduction

Let M be a (connected) manifold of bounded geometry of dimension n, E be an Hermitian vector bundle of bounded geometry on M. Let us introduce also the bundle $p : O(M,E) = O(M) \times_M O(E) \to M$, whose fiber $O(M,E)_m$ at a point $m \in M$ is a set of pairs $e = (e_1, e_2)$, where e_1 is an orthonormal frame in $T_m M$ and e_2 is an orthonormal frame in E_m. The bounded geometry conditions for M and E provide a family of synchronous trivializations of E, $\gamma_e : B \times \mathbf{C}^r \to E$ parametrized by $e \in O(M,E)$ with B, being a fixed ball in $\mathbf{R}^n$ centered at the origin, such that the transition functions $\gamma_{e'}^{-1} \circ \gamma_e$ are C^∞-bounded on e and e' (see [3] for details). Such trivializations will be called in the sequel standard coordinate systems on E.

Let A be a pseudodifferential operator of order q, acting in the space $C_c^\infty(M,E)$ of smooth, compactly supported sections of the bundle E. We recall [2, 3] that A belongs to the class $B\Psi^q(M,E)$ of properly supported pseudodifferential operators with uniformly bounded symbols, if,

(1) in any standard coordinate system γ_e on E, A is of the form

$$A_e = a_e(x,D) + R_e, \tag{1}$$

where:

(a) the operator $a_e(x,D)$ is given via the standard formula

$$a_e(x,D)u(x) = \int\int e^{i<x-y,\xi>} a_e(x,\xi) u(y)\, dy\, d\xi, u \in C_c^\infty(B), \tag{2}$$

by a complete symbol $a_e \in C^\infty(B \times \mathbf{R}^n)$, satisfying the estimates

$$|\partial_x^\beta \partial_\xi^\alpha a_e(x,\xi)| < C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}, x \in B, \xi \in \mathbf{R}^n, \quad (3)$$

for all the multi-indices α, β with a constant $C_{\alpha,\beta} > 0$, not depending on e,

(b) the operator R_e is a smoothing operator on B, uniformly bounded on e.

(2) there exists a constant $c_A > 0$ such that $K_A(m_1, m_2) = 0$, if $\rho(m_1, m_2) > c_A$ (K_A is the Schwartz kernel of the operator A, ρ is the (geodesic) distance function on M).

(3) for any $\delta > 0$, covariant derivatives of K_A of any order are uniformly bounded on $M \times M \backslash U_\delta$, where U_δ is a δ-neighborhood of the diagonal in $M \times M$.

Let $\sigma(x,\xi) : E_x \to E_x, (x,\xi) \in T^*M\backslash\{0\}$, be the principal symbol of A. An uniform ellipticity of the operator A means that there exists a constant $\varepsilon > 0$ such that

$$|(\sigma(x,\xi)v, v)| \geq \varepsilon|\xi|^q|v|, (x,\xi) \in T^*M\backslash\{0\}, v \in E_x. \quad (4)$$

In the sequel, we will consider only pseudodifferential operators with polyhomogeneous symbols. Recall that an operator A belongs to the class $B\Psi^q_{phg}(M,E)$, if, $A \in B\Psi^q(M,E)$, and, in any standard coordinate system, given by $e \in O(M,E)$, the complete symbol a of A is represented as an asymptotic sum $a \sim a_q + a_{q-1} + \ldots$ uniformly on e, where $a_{q-j}(x,\xi) \in BS^{q-j}(B \times \mathbf{R}^n)$ is homogeneous of degree $q-j$ in ξ for $|\xi| \geq 1$, $j = 1, 2, \ldots$.

Let $A \in B\Psi^q_{phg}(M,E)$ be an uniformly elliptic operator with the positive definite principal symbol. By [2], A defines an unbounded closed operator A in the Hilbert space $L^2(M,E)$, and, for any function f, holomorphic in some neighborhood of the spectrum of A and rapidly decreasing at the infinity, an operator $f(A)$ can be defined by the Cauchy integral formula (see Section 2 for more details). The main results of the paper give sufficient conditions for the operator $f(A)$ to be bounded in L^p spaces.

We say that a function f on $\mathbf{R}$ belongs to the class $\mathcal{F}^k_1(\mathbf{R})$, $k \in \mathbf{R}$, if

$$||f||_{k,1} = \int |F[(1+|\ |)^k f](t)|dt < +\infty. \quad (5)$$

Here F denotes the Fourier transform.

Further, we say that a function f on $\mathbf{R}$ belongs to the class $\mathcal{F}^k_1(\mathbf{R}, W)$, $k \in \mathbf{R}$, $W \in \mathbf{R}$, if f is a holomorphic function in the strip $\{z \in \mathbf{C} : |Im\ z| < W\}$ and, for any $\eta \in \mathbf{R}$ with $|\eta| < W$, a function $f(\ +i\eta)$ belongs to $\mathcal{F}^k_1(\mathbf{R})$ with the norm, finite on compacts in the interval $|\eta| < W$.

Theorem 1 *Let $A \in B\Psi^q_{phg}(M,E)$ is an uniformly elliptic, pseudodifferential operator with the positive definite principal symbol. Then there exist constants $W > 0$ and $c > 0$ such that for any function f on the complex plane such that a function $g(t) = f(t^q - c)$, $t \in \mathbf{R}$, belongs to the space $\mathcal{F}^k_1(\mathbf{R}, W)$ with $k > n/2$, the operator $f(A)$ defines a continuous mapping from $L^p(M,E)$ to $L^p(M,E)$, $p \in [1,+\infty]$.*

In a case when the principal symbol σ of the operator A is positive and scalar, that is, $\sigma(x,\xi) = p(x,\xi)I_x$ (I_x is the identity operator in E_x) with $p \in C_c^\infty(T^*M\backslash\{0\}), p > 0$, this assertion can be extended in the following way.

Recall that a smooth function f on $\mathbf{R}$ belongs to the class $S^m_{\rho,0}(\mathbf{R})$, $m \in \mathbf{R}$, $0 < \rho \leq 1$, if, for any integer j, there exists a constant $C_j > 0$ such that

$$|f^{(j)}(t)| \leq C_j(1+|t|)^{m-\rho j}, t \in \mathbf{R}. \tag{6}$$

Further, we say that a function f belongs to the class $S^m_{\rho,0}(\mathbf{R}, W)$, $m \in \mathbf{R}$, $0 < \rho \leq 1$, $W > 0$, if f is a holomorphic function in the strip $\{z \in \mathbf{C} : |Im\ z| < W\}$ such that, for any $\eta \in \mathbf{R}$ with $|\eta| < W$, a function $f(\ +i\eta)$ belongs to $S^m_{\rho,0}(\mathbf{R})$ with the norm, finite on compacts in the interval $|\eta| < W$.

Theorem 2 *Let $A \in B\Psi^q_{phg}(M,E)$ is an uniformly elliptic, pseudodifferential operator with the positive, scalar principal symbol. Then there exist constants $W > 0$ and $c > 0$ such that for any $p \in (1,+\infty)$ and for any function f on the complex plane such that a function $g(t) = f(t^q - c)$, $t \in \mathbf{R}$, belongs to the space $S^0_{\rho,0}(\mathbf{R}, W_p)$, where $W_p = |1/2 - 1/p|W$, $1/2 < \rho \leq 1$, the operator $f(A)$ defines a continuous mapping*

$$f(A) : L^p(M,E) \to L^p(M,E). \tag{7}$$

The similar results were obtained in [10] for functions of the Laplace operator on manifolds of bounded geometry. We refer the reader to [10] for further comments and historical remarks on this problem (see, also [1]). The proof of [10] is based essentially on finite propagation speed arguments, which cannot be extended to the case of higher order differential and pseudodifferential operators. In this work, following, essentially, to the lines of [10], we utilize methods of weighted Sobolev estimates, developed in [2], to control the large distance behaviour of the wave kernel. Briefly speaking, the main point is that the distributional kernels of operators $\exp(it(A+c)^{1/q})$ decrease exponentially when we are moving away from wave front, and this exponential decreasing is getting more and more fast when c tends to infinity. It should be noted, however, that the results of this work applied to the Laplace operator is not so sharp as results obtained by the finite propagation speed arguments (see Section 9 for detailed discussions). We meet here a phenomenon like the well-known local cancellation property, which allows to obtain results for Dirac operators, more refined than for general elliptic differential operators.

Let us say some words about the organization of the paper. In Section 2, we develop a holomorphic functional calculus for an operator A under consideration. In Sections 3, we recall some necessary facts about weighted Sobolev spaces and obtain weighted Sobolev estimates for operators $f(A)$ for holomorphic functions f. The considerations of Sections 2 and 3 allow us to be restricted by a study of functions of first-order pseudodifferential operator P, replacing the operator A in question by the operator $P = (A+c)^{1/q}$. Then we make use of Fourier inversion formula for studying of functions $f(P)$. In Section 4, we state the existence of the wave semigroup and norm estimates of operators of the semigroup e^{itP}, and, in Section 5, we formulate the final result on weighted estimates for operators $f(A)$. In Section 6, we transfer our weight estimates for operators $f(A)$ into L^p-estimates and complete a prove of Theorem 1. In Section 7, we develop geometrical optics constructions for operators with the scalar principal symbol, and, in Section 8, we complete a proof of Theorem 2. Finally, as mentioned above, in Section 9 we consider our results in the case when A is the Laplace operator).

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2 Holomorphic functional calculus

Throughout in the paper, M denotes a (connected) manifold of bounded geometry ($\dim M = n$), E denotes an Hermitian vector bundle of bounded geometry. We make extensive use of Sobolev spaces $H^s(M,E)$(see [2, 7] for definitions). We will fix a Hilbert norm $\| \ \|_s$ in the space $H^s(M,E)$.

It is convenient to consider classes of pseudodifferential operators with uniformly bounded symbols, which are not properly supported, but satisfy to some integral conditions on its kernel. A class $B\tilde{\Psi}^{-\infty}(M,E)$ consists of all operators $K : C_c^\infty(M,E) \to C^\infty(M,E)$, which define a continuous map $K : H^s(M,E) \to H^{+\infty}(M,E)$ for any s. Finally, a class $B\tilde{\Psi}^q(M,E)$ consists of all operators $A : C_c^\infty(M,E) \to C^\infty(M,E)$, which can be represented as $A = A_1 + K$, where $A_1 \in B\Psi^q(M,E)$ and $K \in B\tilde{\Psi}^{-\infty}(M,E)$.

Let A be an uniformly elliptic pseudodifferential operator of the class $B\Psi^q_{phg}(M,E)$ with the positive definite principal symbol. In this Section, we will discuss some aspects of the holomorphic functional calculus for the operator A. First of all, we turn to a construction of a parametrix with a parameter, that is, of a parametrix for the operator $A - \lambda I$, having a right behaviour in λ, when λ tends to the infinity. It is an operator $B(\lambda) \in B\Psi^{-q}(M,E)$, $\lambda \notin \mathbf{R}_+$, such that

$$(A - \lambda I)B(\lambda) = I - R(\lambda), \lambda \notin \mathbf{R}_+, \tag{8}$$

where $R(\lambda) \in B\Psi^{-\infty}(M,E)$, and, for any $\delta \in (0, \pi/2)$, we have the estimates

$$\|R(\lambda) : H^s(M,E) \to H^t(M,E)\| \leq C(1 + |\lambda|)^{-1}, \lambda \in \Lambda_\delta, \tag{9}$$

$$\|B(\lambda) : H^s(M,E) \to H^{s+\alpha q}(M,E)\| \leq C(1 + |\lambda|)^{-1+\alpha}, \lambda \in \Lambda_\delta, \tag{10}$$

where $\Lambda_\delta = \{\lambda \in \mathbf{C} : |\arg \lambda| > \delta\}$,$s, t$ are any real numbers, $\alpha \in (0,1)$.

Recall that the operator $B(\lambda)$ can be constructed in the following way. Assume that $a \sim \sum_{j=0}^{\infty} a_{m-j}$ is an asymptotic expansion of the complete symbol of the operator A in some standard coordinate system. For any $\delta > 0$, define the functions $b_{-m-l}(\lambda), \lambda \in \Lambda_\delta, l = 0, 1, \ldots$, by the following system

$$(a_m - \lambda)b_{-m} = 1, \tag{11}$$

$$(a_m - \lambda)b_{-m-l} + \textstyle\sum_{j<l, j+k+|\alpha|=l} \partial_\xi^\alpha b_{-m-j} D_x^\alpha a_{m-k}/\alpha! = 0, \quad l > 0. \tag{12}$$

Let $B(\lambda)$ be an operator on B with the complete symbol $b(\lambda)$ given by an asymptotical sum $b(\lambda) \sim \sum_{j=0}^{\infty} b_{-m-j}(\lambda)$. Then the global parametrix $B(\lambda)$ is obtained by gluing together these local parametrices, using a partition of unity.

By (9) and (10), for any $\delta \in (0, \pi/2)$, there exists a constant $R > 0$ such that, for any $\lambda \in \Lambda_\delta$, $|\lambda| > R$, and $s \in \mathbf{R}$, an operator $A - \lambda I$ is invertible as an unbounded operator in $H^s(M, E)$, and an inverse operator $(A - \lambda I)^{-1}$ can be represented in the form

$$(A - \lambda I)^{-1} = B(\lambda) + (A - \lambda I)^{-1} R(\lambda) \tag{13}$$

and satisfies the following norm estimate

$$\|(A - \lambda I)^{-1} : H^s(M, E) \to H^s(M, E)\| \le C_s(1 + |\lambda|)^{-1}, \lambda \in \Lambda_\delta, |\lambda| > R, \tag{14}$$

$$\|(A - \lambda I)^{-1} : H^s(M, E) \to H^{s+q}(M, E)\| \le C_s, \lambda \in \Lambda_\delta, |\lambda| > R. \tag{15}$$

Moreover, the formula (13) implies that $(A - \lambda I)^{-1}$ is a pseudodifferential operator of the class $B\tilde{\Psi}^{-q}(M, E)$.

Basing on these estimates, we can define a bounded linear operator $f(A)$ for any entire function f on the complex plane such that, for any $\eta \in \mathbf{R}$, a function $f(\ +i\eta)$ on $\mathbf{R}$ belongs to the space $S(\mathbf{R}_+)$ with the norm, uniformly bounded on compacts in $\mathbf{R}$. Further, using a method of hyperbolic equation, we will essentially extend this assertion, but now we need to point out two particular cases.

1. As a direct consequence of (14) and Hille-Phillips-Yosida theorem, we have the following result on the parabolic semigroup e^{-tA}.

Proposition 1 *Let A be a uniformly elliptic pseudodifferential operator of the class $B\Psi^q_{phg}(M, E)$ with the positive definite principal symbol. Then A generates a holomorphic semigroup $\exp(-zA), Re\ z > 0$, in $L^2(M, E)$. Moreover, the operators of this semigroup satisfy the following estimate*

$$\|e^{-tA}u\|_r \le Ct^{(s-r)/q}\|u\|_s, u \in C_c^\infty(M, E), 0 < t < T, \tag{16}$$

for any $T > 0$ with a constant $C > 0$, not depending on r and s.

2. Now let us say some words about a construction of complex powers A^z (see, for instance, [8, 6]). Replacing the operator A by $A + cI$, where $c > 0$ is a constant, we can assume that the operator $A - \lambda I$ is invertible in $H^s(M, E)$ for any λ , $Re\ \lambda < 0$, and $s \in \mathbf{R}$. An operator A^z , $Re\ z < 0$, is defined by the formula

$$A^z = (i/2\pi) \int_\Gamma \lambda^z (A - \lambda I)^{-1} d\lambda, \tag{17}$$

where Γ is a contour in the complex plane of the form $\Gamma = \Gamma_1 \bigcup \Gamma_2 \bigcup \Gamma_3$, where $\lambda = re^{i\alpha}, +\infty > r > \rho$, on Γ_1, $\lambda = \rho e^{i\phi}, \pi > \phi > -\pi$, on Γ_2, $\lambda = re^{-i\alpha}, \rho < r < +\infty$, on Γ_3, ($\alpha \in (0, \pi)$ is arbitrary), a branch of an analytic function λ^z is chosen so that $\lambda^z = e^{z \ln \lambda}$ for $\lambda > 0$.

This definition is extended to all z by the formula

$$A^z = A^k A^{z-k} \tag{18}$$

for $z \in \mathbf{C}$, where $Re\ z < k$, k is any natural number such that $Re\ z < k$. It is easy to see that A^z is a pseudodifferential operator of a class $B\Psi^{qRe\ z}_{phg}(M, E)$

3 Holomorphic functional calculus and weighted Sobolev spaces

Now we recall some facts from [2] about weighted Sobolev spaces. Let ρ denote a distance function on M. In [2, Prop. 4,1], we constructed a "smoothed distance" function $\tilde{\rho}$ on M. It is a function $\tilde{\rho} \in C^\infty(M \times M)$, satisfying the following conditions:

(1) there is a constant $r > 0$ such that

$$|\tilde{\rho}(m,p) - \rho(m,p)| < r \tag{19}$$

for any $m, p \in M$;

(2)

$$|\partial_p^\alpha \tilde{\rho}(m,p)| < C_\alpha \tag{20}$$

for any multi-index α with $|\alpha| > 0$.

We will also write $\tilde{\rho}_m$, $m \in M$, for a smooth function on M, given by the formula $\tilde{\rho}_m(p) = \tilde{\rho}(m,p)$, $p \in M$.

Let $f_{\varepsilon,m} \in C^\infty(M)$ be given by a formula $f_{\varepsilon,m}(p) = \exp(\varepsilon\tilde{\rho}_m(p))$, $p \in M$. A weighted Sobolev space is defined as follows:

$$H^s_\varepsilon(M,E) = \{u \in \mathcal{D}'(M,E) : f_{\varepsilon,m}u \in H^s(M,E)\}, s \in \mathbf{R}, \tag{21}$$

where $\varepsilon \in \mathbf{R}$ and $m \in M$ be any fixed point. The space $H^s_\varepsilon(M,E)$ is a Hilbert space with the norm

$$\|u\|_{s,\varepsilon,m} = \|f_{\varepsilon,m}u\|_s, u \in H^s_\varepsilon(M,E). \tag{22}$$

The norms (22) obtained by use of different points m are equivalent but the dependence on m will be essential in the sequel. We will write $H^s_{\varepsilon,m}(M,E)$ for the space $H^s_\varepsilon(M,E)$, equipped with the norm $\|u\|_{s,\varepsilon,m}$.

As above, we extend the classes $B\Psi^q(M,E)$, adding pseudodifferential operators with uniformly bounded symbols, which are not properly supported, but satisfy to some weighted integral conditions on its kernel. For any ε, a class $B\tilde{\Psi}^{-\infty}_\varepsilon(M,E)$ consists of all operators $K : C^\infty_c(M,E) \to C^\infty(M,E)$, which define a continuous map $K : H^s_{\varepsilon,m}(M,E) \to H^{+\infty}_{\varepsilon,m}(M,E)$ for any s with the norm, uniformly bounded on m. Finally, a class $B\tilde{\Psi}^q_\varepsilon(M,E)$ consists of all operators $A : C^\infty_c(M,E) \to C^\infty(M,E)$, which can be represented as $A = A_1 + K$, where $A_1 \in B\Psi^q(M,E)$ and $K \in B\tilde{\Psi}^{-\infty}_\varepsilon(M,E)$. It is clear that $B\tilde{\Psi}^q_\varepsilon(M,E)$ coincides with $B\tilde{\Psi}^q(M,E)$ when $\varepsilon = 0$.

To study operators $f(A)$ in weighted Sobolev spaces for the operator A in question, we can make use of results of the Section 2 due to the following lemma (see also [4, 2]):

Lemma 2 *Let A be a pseudodifferential operator of the class $B\Psi^q_{phg}(M,E)$. For any ε and $m \in M$, the operator $A_{\varepsilon,m} = F^{-1}_{\varepsilon,m}AF_{\varepsilon,m}$ is a pseudodifferential operator of the class $B\Psi^q_{phg}(M,E)$ with the same principal symbol.*

Moreover, for any ε, a family $\{A_{\varepsilon,m} : m \in M\}$ is uniformly bounded in $B\Psi^q(M,E)$

Proof. Let us represent the operator A as $A = A_1 + R$, where $R \in B\Psi^{-\infty}(M,E)$, and the kernel of the operator $A_1 \in B\Psi^q(M,E)$ is supported in a small neighborhood of the diagonal in $M \times M$. Then we have $A_{\varepsilon,m} = A_{1,\varepsilon,m} + R_{\varepsilon,m}$, where $A_{1,\varepsilon,m} = F_{\varepsilon,m}^{-1} A_1 F_{\varepsilon,m}$, $R_{\varepsilon,m} = F_{\varepsilon,m}^{-1} R F_{\varepsilon,m}$.

The kernel $K_{1,\varepsilon,m}$ of the operator $R_{\varepsilon,m}$ is given by the formula

$$K_{1,\varepsilon,m}(m_1,m_2) = e^{-\varepsilon\tilde{\rho}(m,m_1)} K_R(m_1,m_2) e^{\varepsilon\tilde{\rho}(m,m_2)},$$

and it can be easily checked that the operator $R_{\varepsilon,m}$ belongs to the class $B\Psi^{-\infty}(M,E)$.

In a similar way, if restrictions of A_1 on standard coordinate systems are given by a uniformly bounded family $\{a \in S^q(B \times \mathbf{R}^n)\}$ due to the formula (2), then restrictions of $A_{1,\varepsilon,m}$ on standard coordinate systems are given by a uniformly bounded family of amplitudes $\{a_{\varepsilon,m} \in S^q(B \times B \times \mathbf{R}^n)\}$, where

$$a_{\varepsilon,m}(x,y,\xi) = e^{-\varepsilon\tilde{\rho}(m,x)} a(x,\xi) e^{\varepsilon\tilde{\rho}(m,y)},$$

and, by a slight modification of the arguments of the standard theory of pseudodifferential operators (see, for instance, [8]), it can be easily checked that $A_{1,\varepsilon,m}$ is given by a uniformly bounded family of symbols from S^q.

Now let us apply the resuls of Section 2 to a family $\{A_{\varepsilon,m} : m \in M\}$. At first, we obtain that, for any $\varepsilon > 0$ and $\delta \in (0,\pi/2)$, there exists a constant $R > 0$ such that, for any $\lambda \in \Lambda_\delta$, $|\lambda| > R$, and $s \in \mathbf{R}$, an operator $A - \lambda I$ is invertible as an unbounded operator in $H_\varepsilon^s(M,E)$, and an inverse operator $(A-\lambda I)^{-1}$ belongs to $B\tilde{\Psi}_\varepsilon^{-q}(M,E)$ and satisfies the estimates

$$\begin{aligned} \|(A-\lambda I)^{-1} : H_{\varepsilon,m}^s(M,E) \to H_{\varepsilon,m}^s(M,E)\| \ &\leq \ C_s(1+|\lambda|)^{-1}, \\ &\lambda \in \Lambda_\delta, |\lambda| > R, \end{aligned} \tag{23}$$

$$\|(A-\lambda I)^{-1} : H_{\varepsilon,m}^s(M,E) \to H_{\varepsilon,m}^{s+q}(M,E)\| \leq C_s, \lambda \in \Lambda_\delta, |\lambda| > R, \tag{24}$$

for any $m \in M$, with constants $C_s > 0$, not depending on m. It is worthwhile to note that, by the uniform boundedness of the family $\{A_{\varepsilon,m} : m \in M\}$ in $B\Psi^q(M,E)$, all of weighted Sobolev estimates obtained here, are uniform on $m \in M$ with respect to the norm $\|u\|_{s,\varepsilon,m}$.

Then we turn to weighted estimates for complex powers A^z. For this, we apply the above construction of complex powers to the operator $A_{\varepsilon,m}$ (see the formulae (17) and (18)). The unique difference is connected with the fact that, if the operator A was invertible in the space $L^2(M,E)$, the operators $A_{\varepsilon,m}$ are, in general, not invertible in $L^2(M,E)$, therefore, we must replace the operators $A_{\varepsilon,m}$ by $A_{\varepsilon,m} + cI$, where $c > 0$ is a constant, to reach an invertibility of the operator $A_{\varepsilon,m} - \lambda I$ in $L^2(M,E)$ for any λ , $Re\ \lambda < 0$. After that, we can immediately establish the following result.

Proposition 3 *Let A be an uniformly elliptic pseudodifferential operator of the class $B\Psi_{phg}^q(M,E)$ with the positive definite principal symbol. Then, for any $\varepsilon > 0$, there exists a constant $c > 0$ such that an operator $(A+cI)^z$ belongs to $B\tilde{\Psi}_\varepsilon^{q\ Re\ z}(M,E)$. Moreover, the norm of the operator $(A+cI)^z$ as an operator from $H_{\varepsilon,m}^s(M,E)$ to $H_{\varepsilon,m}^{s-q Re\ z}(M,E)$ is uniformly bounded on m.*

4 The wave semigroup and energy estimates

Now we turn to the investigation of the wave semigroup $\exp(itP)$, generated by a uniformly elliptic pseudodifferential operator $P \in B\tilde{\Psi}^1_\varepsilon(M,E)$ with the Hermitian principal symbol ($\varepsilon > 0$ is some fixed number), keeping in mind its further applications to the operator P of the form $P = (A+c)^{1/q}$ with A as in Proposition 3.

Proposition 4 *Let $P \in B\tilde{\Psi}^1_\varepsilon(M,E)$ be as above. Then P generates a strongly continuous semigroup $\exp(itP)$ in the space $L^2_{\varepsilon,m}(M,E)$, $m \in M$, that is, for any $u_0 \in L^2_\varepsilon(M,E)$, the function $u(t) = e^{itP}u_0$ gives a unique solution of a Cauchy problem*

$$\begin{aligned} d/dtu(t) &= iPu(t), t > 0, \\ u(0) &= u_0, \end{aligned} \tag{25}$$

in $L^2_\varepsilon(M,E)$. Moreover, for any $s \in \mathbf{R}$, there exists a constant α such that

$$\|e^{itP} : H^s_{\varepsilon,m}(M,E) \to H^s_{\varepsilon,m}(M,E)\| \le Ce^{\alpha t}, t \in \mathbf{R}, m \in M. \tag{26}$$

Proof. This Proposition can be, probably, proved, using the technique of approximation by Friedrichs'mollifier as described in [9, Chapter IV]. Here we utilize the heat semigroup regularization.

At first, let us note that, replacing the operator P by $P_{\varepsilon,m}$, we may assume that $\varepsilon = 0$. By Proposition 1, the operator P^2 generates a holomorphic semigroup $\exp(-zP^2), Re\ z > 0$, in $L^2(M,E)$. Therefore, the space $D = \bigcup_{t>0} \mathrm{Im}\ \exp(-tP^2)$ is contained in $H^{+\infty}(M,E)$ and is dense in $L^2(M,E)$. Moreover, we have the estimate

$$\|P^k u\| \le C/R^{k/2}, u \in D, \tag{27}$$

for any even natural k. By interpolation arguments, the estimate (27) can be extended to any natural k.

Now we define the wave operator $\exp(itP)$ on D by the Taylor expansion

$$e^{itP} = \sum_{k=0}^{+\infty} (it)^k\ P^k u/k!, u \in D. \tag{28}$$

The operator $\exp(itP)$ defined on D by (28) extends to a bounded operator in the space $L^2(M,E)$ by the standard energy estimate:

$$\begin{aligned} \frac{d}{dt}\|e^{itP}u\|^2 &= i((P-P^*)e^{itP}u, e^{itP}u) \\ &\le C\|e^{itP}u\|^2, u \in D, \end{aligned}$$

(Here we used that $P - P^* \in B\tilde{\Psi}^0(M,E)$). From where, the proposition follows immediately.

5 Weighted estimates for operators $f(A)$

Now, using the Fourier inversion formula, we extend the results of previous Section on weighted energy estimates for wave semigroup to weighted estimates for operators $f(A)$ for more general classes of operators. Recall that the function classes $\mathcal{F}_1^k(\mathbf{R}, W)$ are defined in Section 1.

Proposition 5 *Let $P \in B\tilde{\Psi}^1_\varepsilon(M, E), \varepsilon > 0$ be an uniformly elliptic pseudodifferential operator with the Hermitian principal symbol. Then, for any $s \in \mathbf{R}$, there exists $W_s > 0$ such that for any $f \in \mathcal{F}_1^k(\mathbf{R}, W_s)$ the operator $f(P)$ defines a continuous mapping*

$$f(P) : H^s_{\varepsilon,m}(M, E) \to H^{s+k}_{\varepsilon,m}(M, E) \tag{29}$$

with the norm, uniformly bounded on m.

Proof. As mentioned above, we will make use of the Fourier inversion formula

$$f(P) = (1/2\pi) \int \tilde{f}(t) e^{itP} dt, \tag{30}$$

By Paley-Wiener theorem, the condition $f \in \mathcal{F}_1^k(\mathbf{R}, W)$ is equivalent to the following condition:

$$\int |F[(1 + |\ |)^k f](t)| e^{Wt} dt < +\infty. \tag{31}$$

Therefore, the proposition is a direct consequence of Proposition 4.

Now we establish a result on weighted Sobolev estimates for functions of pseudodifferential operators of arbitrary order.

Proposition 6 *Let $A \in B\Psi^q_{phg}(M, E)$ is an uniformly elliptic pseudodifferential operator with the positive definite principal symbol. Then, for any $\varepsilon \in \mathbf{R}$ and $s \in \mathbf{R}$, there exist constants $W > 0$ and $c > 0$ such that, for any function f on the real line such that a function $g(t) = f(t^q - c), t \in \mathbf{R}$, belongs to the space $\mathcal{F}_1^k(\mathbf{R}, W)$, the operator $f(A)$ defines a continuous mapping*

$$f(A) : H^s_{\varepsilon,m}(M, E) \to H^{s+k}_{\varepsilon,m}(M, E) \tag{32}$$

with the norm, uniformly bounded on m.

Proof. Note that $f(A) = g(P)$, where $P = (A + cI)^{1/q}$, therefore, the proposition follows immediately from Proposition 3 and Proposition 5 applied to the operator P.

6 From weighted estimates to L^p-estimates

In this Section, we translate weighted Sobolev estimates of Section 3 into L^p-estimates and complete the proof of Theorem 1, concerning to L^p-continuity of operators $f(A)$ for an uniformly elliptic pseudodifferential operator $A \in B\Psi^q_{phg}(M,E)$ with *the positive definite* principal symbol.

Proof of Theorem 1. Let μ denote the exponential growth of the manifold M, that is,

$$\mu = \limsup_{r\to+\infty} volB(m,r), \tag{33}$$

where $B(m,r)$ is a (geodesic) ball with radius $r>0$ centered at a point $m \in M$, m is an arbitrary point of M (it is well-known that μ is finite and doesn't depend on a choice of m).

Let $K : C_c^\infty(M,E) \to C_c^\infty(M,E)$ be an operator with a kernel $k \in C_c^\infty(M\times M)$. To estimate the norm of the operator K in the space $L^1(M,E)$, it suffices to obtain bounds for

$$\sup_{m\in M}\int |k(m,m')|dm' \text{ and } \sup_{m\in M}\int |k(m,m')|dm'. \tag{34}$$

By the Cauchy-Schwartz inequality, we have

$$\int |k(m',m)|dm' \leq (\int |k(m',m)|^2 e^{2\varepsilon\tilde{\rho}(m,m')}dm')^{1/2}(\int e^{-2\varepsilon\tilde{\rho}(m,m')})dm')^{1/2}, \tag{35}$$

where the last integral is finite and (uniformly) bounded if $\varepsilon > \mu/2$. Since $k(m',m) = [K\delta_m](m')$, $m, m' \in M$, and the delta-function δ_m at the point m belongs to the space $H^s_\varepsilon(M,E)$ for any $s < -n/2$ and $\varepsilon \in \mathbf{R}$ with $\sup_{m\in M}\|\delta_m\|_{s,\varepsilon,m} < +\infty$, we obtain

$$\begin{aligned}\|K : L^1(M,E) \to L^1(M,E)\| \;\leq\; & C\max(\sup_{m\in M}\|K : H^s_{\varepsilon,m}(M,E) \to L^2_{\varepsilon,m}(M,E)\|, \\ & \sup_{m\in M}\|K^+ : H^s_{\varepsilon,m}(M,E) \to L^2_{\varepsilon,m}(M,E)\|). \end{aligned}\tag{36}$$

where $s < -n/2$, $\varepsilon > \mu/2$. Theorem 1 immediately follows from Proposition 6 and the estimate (36) applied to the operator $f(A)$.

7 Geometrical optics construction and estimates of kernels near the diagonal

All considerations of previous sections, based on weighted estimates, are, essentially, concerned with a behaviour of kernels of the operators $f(P)$ away of the diagonal and, in its turn, with a behaviour of a Fourier transform $\tilde{f}(t)$ when t tends to the infinity. So weighted estimates allow us to study operators $f(P)$ with the kernels, being continuous near the diagonal.

Here we will make use of some pseudodifferential technique, namely, geometrical optics constructions, to involve in our considerations operators $f(P)$ with singular kernels. But, for this, it is necessary to impose some additional condition on the operator in question. Namely, in this Section, we will suppose that the principal symbol of the operator P is scalar.

Here we make use also of the technique of weighted Sobolev spaces developed above to estimate the difference between the exact wave operator $\exp itP$ and its geometrical optics approximation $W(t)$.

Let $P \in B\tilde{\Psi}^1_\varepsilon(M, E)$, $\varepsilon > 0$, be an uniformly elliptic pseudodifferential operator on M with the positive, scalar principal symbol. Decompose P in the sum $P = P_1 + P_2$, where $P_1 \in B\Psi^1(M, E)$ is supported in a small neighborhood of the diagonal, $P_2 \in B\tilde{\Psi}^{-\infty}_\varepsilon(M, E)$. Let us fix a standard coordinate system, given by a frame $e \in O(M, E)$. Let $p(x, \xi)$ be a complete symbol of P_1 in this coordinate system, $p(x,\xi) = p(x,\xi)I_x, p \in C^\infty(B \times \mathbf{R}^n \backslash \{0\}, \mathcal{L}(\mathbf{C}^r))$. We can assume that the parametrix $W(t)$ for e^{itP_1} is of the following form

$$(W(t)u)(x) = \int a(t, x, \xi) e^{i\phi(t,x,\xi)} \tilde{u}(\xi) d\xi, u \in C_c^\infty(B). \tag{37}$$

Here ϕ is a solution of the eikonal equation

$$\partial\phi/\partial t = p(x, \nabla_x \phi), \tag{38}$$

$$\phi(0, x, \xi) = x\xi \tag{39}$$

for $|t| \leq r$, $x \in B$; $a(t, x, \xi)$ is determined by the usual transport equations of geometrical optics (see, for instance, [9, Chapter VIII]) with $a(0, x, \xi) = 1$.

Further, using a partition of unity, we can glue together these local parametrices into a global one, and obtain a Fourier integral operator $W(t) : C_c^\infty(M, E) \to C_c^\infty(M, E)$, such that

$$(D_t - P_1)W(t) = K(t), |t| < r, \tag{40}$$

$$W(0) = I + K_0, \tag{41}$$

where $K_0 \in B\Psi^{-\infty}(M, E)$, $K(t), |t| < r$, is a smooth family of operators from $B\Psi^{-\infty}(M, E)$.

To estimate the difference between the operator e^{itP} and its geometrical optics approximation, we make use of the Duhamel formula

$$W(t) - e^{itP} = e^{itP} K_0 + \int_0^t e^{i(t-\tau)P}(K(\tau) - P_2 W(\tau)) d\tau. \tag{42}$$

By (42) and Proposition 4, we immediately obtain that the operator $W(t) - e^{itP}$ belongs to $B\tilde{\Psi}^{-\infty}_\varepsilon(M, E)$ with the following estimate for its norm

$$\| W(t) - e^{itP} : H^s_{\varepsilon,m}(M, E) \to H^{s+k}_{\varepsilon,m}(M, E) \| \leq C e^{\alpha t}, |t| < r. \tag{43}$$

After that, we can establish the final result of this Section.

Proposition 7 *Under current hypotheses on the operator P, there exists a constant $r > 0$ such that, for any function $f \in S^q(\mathbf{R})$ such that its Fourier transform is supported in the interval $(-r, r)$, the operator $f(P)$ belongs to $B\tilde{\Psi}^q_\varepsilon(M, E)$.*

Proof. The proof of this proposition can be obtained by word by word repeatition of the arguments of [9].

8 L^p-estimates in a case of the scalar principal symbol

Now we combine the results of Section 7 and Theorem 1 to complete the proof of Theorem 2.

Proposition 8 *Let $A \in B\Psi^q_{phg}(M,E)$ is a uniformly elliptic pseudodifferential operator with the positive, scalar principal symbol. Then there exist constants $W > 0$ and $c > 0$ such that for any $p \in [1,+\infty]$ and for any function f on the real line such that a function $g(t) = f(t^q - c)$, $t \in \mathbf{R}$, belongs to the space $S^{-\varepsilon}_{\rho,0}(\mathbf{R}, W)$, where $\varepsilon > 0$, $1/2 < \rho \leq 1$, the operator $f(A)$ defines a continuous mapping*

$$f(A) : L^p(M,E) \to L^p(M,E). \tag{44}$$

Proof. Let W and c be given by Theorem 1, and $g(t) = f(t^q - c)$. Let us write the function g as a sum $g = g_1 + g_2$, where $\tilde{g}_1$ is supported in a interval $(-r, r)$, where $r > 0$ given by Proposition 7, and $\tilde{g}_2$ vanishes in some neighborhood of zero. Then we have also $f = f_1 + f_2$, where $g_i(t) = f_i(t^q - c)$, $t \in \mathbf{R}$, $i = 1, 2$.

L^p-continuity of an operator $f_1(A)$ follows from Proposition 7. As well-known (see, for instance, [7]), for any function $f \in S^m_{\rho,0}(\mathbf{R})$, $0 < \rho \leq 1$, a function $\tilde{f}(t)$ and all its derivatives are rapidly descreasing on $\mathbf{R}\backslash(-r, r)$ for any $r > 0$, therefore, $g_2 \in \mathcal{F}_1^{+\infty}(\mathbf{R}, W)$, and L^p-continuity of an operator $f_2(A)$ is a direct consequence of Theorem 1.

Proof of Theorem 2. Theorem 2 can be derived from Proposition 8, using Stein interpolation theorem (cf. [5]), by the same arguments as in [10].

From Theorem 2 we can deduce an information on the spectrum of an uniformly elliptic differential operator $A \in B\Psi^q(M,E)$ with the positive, scalar principal symbol on $L^p(M,E)$. Let A_p denote the minimal operator in $L^p(M,E)$, generated by A , $1 \leq p < +\infty$. By [2], the operator A_p coincides with the maximal operator, defined by A in $L^p(M,E)$.

Proposition 9 *Let W and c be as in Proposition 8. For any $\zeta \in \mathbf{C}$ of the form*

$$\zeta = z^q - c, |Im\ z| > W_p, \tag{45}$$

ζ belongs to the resolvent set of the operator A_p.

Proof. If (45) holds, then, by Theorem 2 for $p \in (1, +\infty)$, Proposition 8 for $p = 1$, the operator $R_\zeta = (A - \zeta)^{-1}$, defined a priori on $L^2(M,E)$, extends uniquely to be continuous on $L^p(M,E)$. We must state that such R_ζ is actually the resolvent of A_p, i.e.,

$$R_\zeta : L^p(M,E) \to Dom(A_p) \tag{46}$$

and

$$R_\zeta \text{ is a two-sided inverse to } A - \zeta : Dom(A_p) \to L^p(M,E). \tag{47}$$

This fact follows from the resolvent identity

$$R_\zeta = R_\mu + (\mu - \zeta) R_\mu R_\zeta, \tag{48}$$

valid a priori on $L^2(M, E)$. If $\mu \in \mathbf{C}$ such that $Re\ \mu > 0$ is big enough, then, by results of [2], μ belongs to the resolvent set of A_p and the resolvent $(A_p - \mu)^{-1}$ coincides with R_μ. For such ζ and μ, the identity (48) extends to $L^p(M, E)$, and then (46) and (47) easily follow.

9 The case of Laplace operator

In this Section, we compare the results of previous Sections with the results of [10] on L^p-continuity of functions of the Laplace operator on a manifold of bounded geometry.

Let $A = -\Delta - E$, where Δ is the Laplace-Beltrami operator on M and $E \geq 0$ is a constant such that the spectrum of $-\Delta$ is contained in $[E, +\infty)$. Then, by Theorem 2, there exist constants $W > 0$ and $c > 0$ such that, for any function $g \in S^0_{\rho,0}(\mathbf{R}, W_p), W_p = |1/2 - 1/p|W$, the operator $g((A + c)^{1/2})$ defines a bounded operator from $L^p(M, E)$ to $L^p(M, E)$, $1 \leq p < +\infty$. From the proof of this Theorem, we can derive the following estimates on W and c:

1. The constant $c > 0$ is chosen so that the spectrum of the operator $A + c$ in the weighted L^2 space $L^2_\varepsilon(M, E)$ for some $\varepsilon > 0$ doesn't contain the semiaxis $(-\infty, 0]$. For instance, we can take c so that

$$c > - \inf_{\|u\|_{\varepsilon,m}=1} Re(Au, u)_{\varepsilon,m}, m \in M. \tag{49}$$

By a direct calculation, it can be seen that

$$Re\ A_{\varepsilon,m} = A - \varepsilon^2 |\nabla \tilde{\rho}_m|^2. \tag{50}$$

By [2, Prop. 4.1], for any $\alpha > 0$, there exists a smoothed distance function $\tilde{\rho}$, satisfying (19), (20) and the following condition

$$|\nabla \tilde{\rho}_m(m_1)| < 1 + \alpha, m \in M, m_1 \in M. \tag{51}$$

Since the weighted Sobolev spaces $H^s_\varepsilon(M, E)$ don't depend on a choice of a function $\tilde{\rho}$ under conditions (19), (20), by (50) and (51), we obtain that

$$\inf_{\|u\|_{\varepsilon,m}=1} Re\ (Au, u)_{\varepsilon,m} \leq -\varepsilon^2, m \in M. \tag{52}$$

So we can take $c > \mu^2/4$ in Theorem 2.

2. From the proof of the Proposition 5, it can be easily seen that

$$W > \sup_{\|u\|=1} Im\ ((A_{\varepsilon,m} + c)^{1/2} u, u). \tag{53}$$

Let us remark that a right-hand side of (53) is finite, since $Im\ (A_{\varepsilon,m}+c)^{1/2}) \in B\tilde{\Psi}^0(M,E)$. Indeed, it is not easy to calculate this quantity, and we can only obtain an estimate of the form

$$Im\ ((A_{\varepsilon,m}+c)^{1/2}u,u) \leq C\varepsilon\|u\|^2, u \in L^2(M,E), \tag{54}$$

with a constant $C > 0$, not depending on ε and m.

Now let us turn to results on L^p-spectra. By Proposition 9, L^p-spectrum of the operator $-\Delta$ is contained in a parabolic region

$$\{\lambda \in \mathbf{C} : \lambda = E - \frac{\mu^2}{4} - z^2, |Re\ z| \leq |1/p - 1/2|W\}, \tag{55}$$

where W is given by (53).

This result is weaker than the following estimate of L^p spectrum of the operator $-\Delta$

$$\sigma(-\Delta) \subset \{\lambda \in \mathbf{C} : \lambda = E - z^2, |Re\ z| < |1/p - 1/2|\mu\}, \tag{56}$$

obtained in [10].

Comparing the proofs of these estimates, it is easy to find the essential point of the finite propagation speed arguments in this context. Namely, by the finite propagation speed arguments, the operator $\cos(tA^{1/2})$ is properly supported and, therefore, defines a continuous mapping from $H^s_\varepsilon(M,E)$ to $H^s_\varepsilon(M,E)$ *for any* $\varepsilon > 0$, while the operator $A^{1/2}$ (and even $e^{itA^{1/2}}$) doesn't act, in general, in the weighted Sobolev space $H^s_\varepsilon(M,E)$ for any ε (it can be easily seen in the simplest case of the Laplace operator on the Eucledean space). This fact makes unnecessary to replace the operator $A_{\varepsilon,m}$ by $A_{\varepsilon,m} + cI$ in our considerations and gives an explanation of the fact that expressions for λ in (55) and (56) differ by the term $-\mu^2/4$.

Further, by a slight modification of the proof of [10, Prop. 1.4] (also using essentially finite propagation speed arguments), we can calculate the estimate for W, namely, $W = \mu$. We don't know if it is possible to derive tis estimate from our estimate (53).

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